Order-Invariant First-Order Logic over Hollow Trees

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Abstract

We show that the expressive power of order-invariant first-order logic collapses to first-order logic over hollow trees. A hollow tree is an unranked ordered tree where every non leaf node has at most four adjacent nodes: two siblings (left and right) and its first and last children. In particular there is no predicate for the linear order among siblings nor for the descendant relation. Moreover only the first and last nodes of a siblinghood are linked to their parent node, and the parent-child relation cannot be completely reconstructed in first-order.

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1 Introduction

First-order logic (FO) is a classical formalism for expressing properties over finite structures. It is the building block of many other formalisms that are highly expressive such as MSO or logics using fixpoints such as LFP. An important and desirable feature of FO, and of all its extensions mentioned above, is that it expresses only intrinsic properties of the structure, i.e. properties invariant under isomorphisms. A limitation of FO is that it cannot express some simple properties. In particular, as it cannot distinguish between nodes that are related via some automorphism, it cannot always go through all the nodes of a structure in order to perform simple tasks such as counting them.

In many scenarios, in particular in computer science, the structures under investigation are stored on a disk: this yields an implicit order among the elements of the structure. It is then reasonable to use this order within the logical formalism. In the case of FO this means adding a new binary predicate that is interpreted as a linear order. However, we want to do this in such a way that closure under isomorphisms is retained: the expressible properties should only depend on the structure and not on the way it is stored on the disk, the latter being arbitrary and subject to change. When this property is verified we say that the formula is **order-invariant** and we denote by $<$-inv FO the set of first-order formulas that are order-invariant. We stress that being order-invariant is not a decidable property [5] hence $<$-inv FO is not a recursive set of formulas.

Obtaining a "real" logic (in the sense of Gurevich, in particular with a recursive syntax) that has exactly the same expressive power as $<$-inv FO is a challenging question. Solving the same question for $<$-inv LFP would solve the longstanding quest of finding a logic for PTime as it follows from Immermann-Vardi Theorem that $<$-inv LFP captures PTime.

In order to find a logic for $<$-inv FO, it is useful to understand a bit better its expressive power; such is the goal of this paper.

An example, attributed to Gurevich, shows that $<$-inv FO is in general strictly more

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expressive than FO [1]. Another key result shows that $<$-inv FO retains the local property of FO [8]. It seems that it requires dense structures for $<$-inv FO to express strictly more than FO. For instance when the structures are trees it has been shown that $<$-inv FO has exactly the same expressive power than FO [5]. In [5] a “tree” is either a binary tree, where every node has at most three neighbors: its parent, its left child and its right child or, an unranked unordered tree where every node is related to its parent and all of its children, but no order is assumed among siblings.

The question of whether $<$ -inv FO $=$ FO over any class of structures of bounded treewidth was left open in [5], where it is only shown that, over structures of bounded treewidth, $<$-inv FO can only express properties definable in MSO.

In order to show that $<$-inv FO collapses to FO over a class of structures of bounded treewidth, it is tempting to reduce the case of bounded treewidth to the case of trees, using tree decompositions. When trying this strategy one immediately faces two difficulties. The first one is, given two FO similar structures (in this introduction we informally say that two structures are “FO similar” if they satisfy the same FO sentences of quantifier rank k for some k sufficiently large and depending on the context), to exhibit a tree decomposition for each of them such that the resulting tree decompositions are FO similar. Once this is done, we can apply the known result over trees showing that the tree decompositions actually agree on all order-invariant properties of a given quantifier rank: they are $<$-inv FO similar. The second difficulty is then to lift the order-invariance similarity from the tree decompositions to the original structures.

The second difficulty could be solved easily if we could interpret the original structure within its tree decomposition. Unfortunately this cannot be done in first-order (this requires reachability as an element of the structure could appear in bags arbitrarily far away within the tree decomposition). This problem can be eliminated by assuming “domino treewidth”, i.e. that an element appears in a bounded number of bags, which is equivalent to assuming bounded degree of the structure on top of bounded treewidth [6].

Even when assuming bounded degree, the first difficulty remains and we still do not know the precise expressive power of $<$-inv FO over structures of bounded degree and pathwidth 2! This paper is an attempt toward solving the pathwidth 2 case.

We show that $<$-inv FO collapses to FO over the class of hollow trees. Hollow trees are first-order structures with two binary relations that are interpreted so that the resulting structure is a tree with the following features: each node has at most four neighbors: its first child, its last child and possibly a left and a right sibling. One of the binary relation denotes the sibling relation while the other one denotes the partial parent-child relation. This model strictly extends the case of binary trees as a node may have arbitrarily many children. However it is less powerful than the unranked ordered model as a node is not directly related to its parent, unless it is the first or last of its children. Note that because of its locality, FO cannot reconstruct the complete parent-child relation of every node within a hollow tree (this can be done in MSO or using the transitive closure of the sibling relation).

It is not immediate to see how hollow trees are related to structures of pathwidth 2 and of bounded degree. It turns out that if in the model of hollow trees we only had one binary relation and could not distinguishing between the (partial) parent-child relation and the sibling one, then we would be able to encode in FO the basic blocks, called *tracks*, of structures of bounded degree and pathwidth 2 in this model. This operation is described in Section 2.2. In particular the collapse of $<$-inv FO to FO on hollow trees would imply the collapse on tracks, as we explain in Section 2.4. We leave the extension of our result to this class of structures as an open problem.

Our proof follows a strategy similar to the case of binary trees: we first exhibit a set of operations over hollow trees (actually over structures FO similar to hollow trees) that preserve order-invariance similarity. We then show that if two hollow trees are FO similar then one of them can be transformed using our set of operations into the other, lifting FO similarity to $<$-inv FO similarity. The first part is standard and makes use of the locality of $<$-inv FO [8]. The second part is more combinatorial and forms the main technical contribution of this paper.

Related work. Besides the papers already mentioned above, there exist several other publications related to our work. We will make use in our proof of the fact that $<$-inv FO $\subseteq$ MSO over classes of graphs of bounded treewidth, which has been initially claimed in [5]. Another proof of this result, extended to a broader class called "decomposable structures", can be found in [7].

If testing order invariance is undecidable for FO it is decidable for its two variable fragment [15].

Several authors considered order-invariance for more expressive logics (first-order with modulo predicates [13], MSO [7]) or with more expressive numerical predicates [10, 9, 2, 14]. Our proof technique follows lines similar to [5, 13] but is mildly related to the others.

The proofs that are missing or just sketched in the main part of the paper are given in greater details in the appendix.

2 Preliminaries

2.1 General notations

We consider relational structures and use classical terminology for them. We use Σ to denote a relational schema and Σ-structure to denote a structure over Σ. Our structures are always finite and are denoted through calligraphic upper-case letters and their domain through the corresponding standard upper-case letter. For instance, A would denote the domain of the structure $\mathcal{A}$. For a relation symbol $R \in \Sigma$ and a Σ-structure $\mathcal{A}$, we denote by $R^{\mathcal{A}}$ the interpretation of R in $\mathcal{A}$.

Given a relational signature Σ, first-order logic, FO(Σ), and monadic second-order logic, MSO(Σ), are defined in the standard way (see, e.g., [11]). The main formalism of interest here is order-invariant first-order logic, denoted $<$-inv FO(Σ). A sentence φ in $\mathrm{FO}(\Sigma \cup \{<\})$ belongs to $<$-inv FO(Σ) if for every Σ-structure $\mathcal{A}$, whether $(\mathcal{A}, <^{\mathcal{A}}) \models \varphi$ is independent of the choice of the linear order $<^{\mathcal{A}}$ on A. In that case, we write $\mathcal{A} \models \varphi$. For any $\mathcal{L} \in \{\mathrm{FO}(\Sigma), \mathrm{MSO}(\Sigma), <\text{-inv FO}(\Sigma)\}$ and two Σ-structures $\mathcal{A}$ and $\mathcal{B}$, we write $\mathcal{A} \equiv_k^{\mathcal{L}} \mathcal{B}$ to mean that $\mathcal{A}$ and $\mathcal{B}$ satisfy the same sentences of $\mathcal{L}$ of quantifier rank at most k. As usual we omit Σ when it is clear from the context.

We use the standard notion of FO-interpretations in order to define a new structure from an existing one. Given a FO-interpretation $\mathcal{I}$, we call **arity** of $\mathcal{I}$ the number of free variables in the formula of $\mathcal{I}$ which defines the domain of the new structure, and **depth** of $\mathcal{I}$ the maximum among the quantifier ranks of the formulas defining the domain and the new relations. It is a well known result that for every $\mathcal{A}, \mathcal{B}$, and $\mathcal{I}$ of arity a and depth d, and for every $k \in \mathbb{N}$, if $\mathcal{A} \equiv_{ak+d}^{\mathcal{L}} \mathcal{B}$ then $\mathcal{I}(\mathcal{A}) \equiv_k^{\mathcal{L}} \mathcal{I}(\mathcal{B})$.

Let $\mathcal{A}$ be a structure over a vocabulary containing the binary relation symbol R. We say that $U \subseteq A$ is **R-stable** if $\forall x \in U, \forall y \in A, (R(x, y) \vee R(y, x)) \to y \in U$.

For a set σ of symbols, we define the vocabulary $P_\sigma := \{P_s : s \in \sigma\}$, where every P_s is a unary relation symbol.

As usual the Gaifman graph of a relational structure $\mathcal{A}$ is the (unoriented) graph whose vertices are the elements of the domain of the structure and the edges relate two vertices that appear in the same tuple of a relation of $\mathcal{A}$. We denote by $\text{dist}_{\mathcal{A}}(x, y)$ the distance between x and y in the Gaifman graph of $\mathcal{A}$. Given two sets S and T of elements of A and $m \in \mathbb{N}$, we say that S and T are **m-distant** in $\mathcal{A}$, if $\text{dist}_{\mathcal{A}}(x, y) \geq m$ for all $x \in S$ and all $y \in T$. The **k-neighborhood** $\mathcal{N}^k_{\mathcal{A}}(x)$ of some $x \in A$ is the substructure of $\mathcal{A}$ induced by $\{y \in A : \text{dist}_{\mathcal{A}}(x, y) \leq k\}$ together with an additional constant interpreted as x. The **k-type** $\text{tp}^k_{\mathcal{A}}(x)$ of x in $\mathcal{A}$ is the isomorphism class of its k-neighborhood. We extend those definitions to tuples of elements in the usual way, fixing the tuples pointwise.

For $k \in \mathbb{N}$, we define the **k-enrichment** $\mathcal{E}_k(\mathcal{A})$ of a Σ-structure $\mathcal{A}$ as $\mathcal{A}$ itself where each element has been recolored with its k-type. $\mathcal{E}_k(\mathcal{A})$ is a structure over the vocabulary Σ augmented with a unary predicate for every k-type over Σ: there are a finite number of them as long as we consider classes of structures of bounded degree.

2.2 Hollow trees

Barát, Hajnal, Lin and Yang [3] proved that any graph of pathwidth at most 2 can be decomposed in a series of what they called *tracks*. Thus, a first step towards proving the collapse of $<$-inv FO to FO on classes of pathwidth at most 2 is to show that $<$-inv FO = FO on the class of tracks.

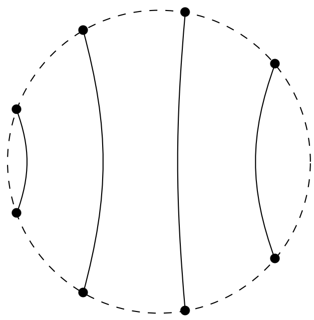

Figure 1 Example of track of degree 3. Each dashed arc represent a path.

A typical example of track of degree 3 is depicted in Figure 1, where the dashed arcs are colored paths, and all the chords are single edges. Each chord could actually be a single edge or the juxtaposition of two edges with a single vertex in the middle; we can however ignore that case, since the middle vertices can be encoded in FO by coloring the chords according to whether they are simple or double edges.

We show in Figure 2 how such a track can be turned into a structure resembling a tree. We add color and number identifiers to clarify the translation.

Note that those two transformations, as well as their inverse, are definable as FO-interpretations as soon as the square edge is part of the track. We will thus see in Lemma 2 that the collapse of $<$-inv FO to FO on any of these classes of structures amounts to the collapse on the other.

This remark motivates the definition of hollow trees. Informally, hollow trees resemble the aforementioned tree-like structure with the key difference that the vertical edges (i.e. the parent-child edges) and the horizontal one are distinguishable. In return for that specification, we do not restrict the complexity of the underlying tree, while the tree-like structures resulting

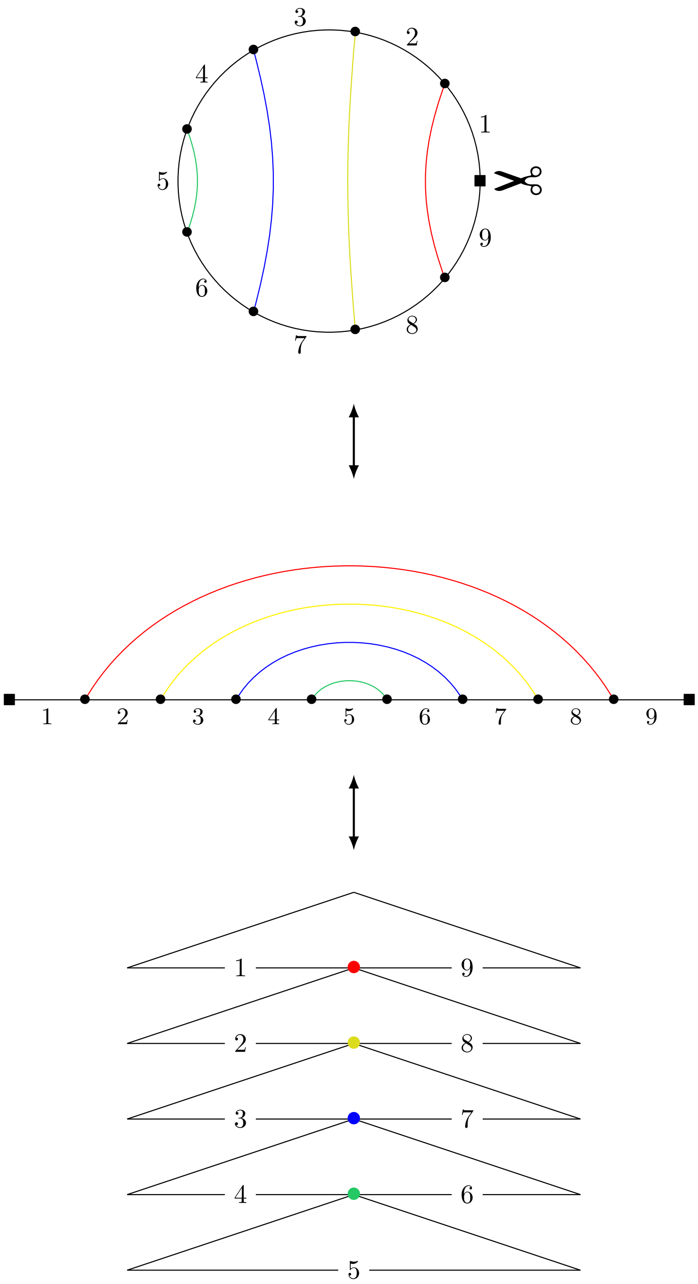

Figure 2 Turning a track of degree 3 into a tree-like structure. One goes from the track to the intermediate structure by cutting the edge represented as a square, and from the intermediate structure to the tree-like structure by contracting each chord into a vertex.

from the transformation of a track are very constrained. In particular, the class of hollow trees has unbounded pathwidth.

Let's now formally define hollow trees.

An unranked ordered tree is a tree with a successor relation among the children of any node. We see unranked ordered trees as structures over the signature composed of two binary relation symbols S and S', where S is interpreted as the parent-child relation, and S' as the horizontal successor. A set of nodes that share the same parent is called a siblinghood.

We define a mapping H from the set of unranked ordered trees to structures over two binary predicates S and E. Given an unranked ordered tree $\mathcal{T}$, $H(\mathcal{T})$ is defined as follows:

- its domain is T
- $H(\mathcal{T}) \models S(x, y)$ iff $\mathcal{T} \models S(x, y)$ and y is either the first or the last of its siblings
- E is interpreted as the symmetrical closure of S'

The image of H is the set of **hollow trees**, denoted $\mathbb{H}$. If $\mathcal{P} = H(\mathcal{T})$ then $\mathcal{T}$ is the underlying tree structure of $\mathcal{P}$.

In other words, within a hollow tree, only the two children at the endpoints of a siblinghood know their parent. Notice that we do not distinguish between the first and last child, nor do we between the left and right sibling. This makes the model more general, as explained in Section 2.4. An example of hollow tree is given in the left part of Figure 3.

Figure 3 An example of hollow tree (left) and of hollow quasitree (right). The dotted arrows represent S and the plain (symmetrical) lines represent E.

Given a finite alphabet σ, we define $\mathbb{H}_\sigma$, the set of **hollow trees over** σ, as the set of colored extensions of hollow trees using the vocabulary P_σ, where the interpretations of the predicates of P_σ partition the domain.

2.3 Main result

If $\mathcal{C}$ is a class of structures, we say that $<$-inv FO $=$ FO over $\mathcal{C}$ if for each property definable in $<$-inv FO, there exists a first-order formula expressing this property over all structures of $\mathcal{C}$. Notice that for every σ, $\mathbb{H}_\sigma$ is a class of structures of treewidth 2. Therefore $<$-inv FO $\subseteq$ MSO over $\mathbb{H}_\sigma$ [5]. The main result we prove in this paper is:

▶ **Theorem 1.** *For all σ, $<$-inv FO $=$ FO over $\mathbb{H}_\sigma$*

We outline the proof here, and give more details in the rest of this paper.

Proof sketch. Our goal is to find some function f such that, $\forall \alpha \in \mathbb{N}, \forall \mathcal{P}, \mathcal{Q} \in \mathbb{H}_\sigma$, if $\mathcal{P} \equiv^{\text{FO}}_{f(\alpha)} \mathcal{Q}$ then $\mathcal{P} \equiv^{<\text{-inv FO}}_{\alpha} \mathcal{Q}$. This means that the equivalence relation $\equiv^{\text{FO}}_{f(\alpha)}$ refines $\equiv^{<\text{-inv FO}}_{\alpha}$. Both equivalence relations being of finite index and the former being definable in FO for every fixed α, the result follows.

To show this we fix some $\alpha \in \mathbb{N}$ and consider two hollow trees $\mathcal{P}$ and $\mathcal{Q}$, such that $\mathcal{P} \equiv^{\text{FO}}_{f(\alpha)} \mathcal{Q}$ for a large enough $f(\alpha)$. The general idea is to modify $\mathcal{Q}$ through some operations that are invisible to all formulas of $<$-inv FO of quantifier rank less than α, until we reach $\mathcal{P}$. This will ensure that $\mathcal{P} \equiv^{<\text{-inv FO}}_{\alpha} \mathcal{Q}$.

We will use two kinds of operations as described in Section 3: "swap operations", which preserve $<$-inv FO, and one which preserves MSO (and *a fortiori* $<$-inv FO as $<$-inv FO $\subseteq$ MSO over $\mathbb{H}_\sigma$ by [5]).

The MSO-preserving operation will be used in Section 3.3, in order to pump $\mathcal{Q}$ to make sure that every neighborhood type is present at least as many times in $\mathcal{Q}$ as in $\mathcal{P}$.

Once this is done, we explain in Section 4 how to transform $\mathcal{Q}$ with swap operations in order to include $\mathcal{P}$ into it. Since $\mathcal{Q}$ may be larger than $\mathcal{P}$, there could be some extra material in $\mathcal{Q}$ that we call "loops". The last step is to remove those loops and this is the goal of Section 6.

When performing the swap operations, there will be a constant need for reorganizing the S-edges (in particular to make sure that the loops are S-stable). Section 5 and Section 6.3 compile the results that allow us to do so. ◀

2.4 Bi-FO-interpretations and corollaries

Before we give more details about the proof of our main result, we recall in this section a classical tool for reducing the collapse of $<$-inv FO to FO from one class of structures to another. We then state a few corollaries of Theorem 1.

Let $\mathcal{C}_1, \mathcal{C}_2$ be two classes of structures over the respective vocabularies τ_1 and τ_2.

We say that $\mathcal{C}_1$ is **bi-FO-interpretable** through $\mathcal{C}_2$ if there exist two FO-interpretations $\mathcal{I}_{12}$ and $\mathcal{I}_{21}$, respectively from τ_1 to τ_2, and from τ_2 to τ_1, such that for every $\mathcal{A} \in \mathcal{C}_1$, $\mathcal{I}_{12}(\mathcal{A}) \in \mathcal{C}_2$ and $\mathcal{I}_{21}(\mathcal{I}_{12}(\mathcal{A})) \simeq \mathcal{A}$, where $\simeq$ denotes the existence of an isomorphism between two structures. The following result is rather straightforward:

▶ **Lemma 2.** *If $\mathcal{C}_1$ is bi-*FO*-interpretable through $\mathcal{C}_2$ and $<$-inv FO $=$ FO over $\mathcal{C}_2$, then $<$-inv FO $=$ FO over $\mathcal{C}_1$*

Recall that in the definition of hollow trees the relation E is symmetric. This turns out to be more general than choosing E as an arbitrary directed binary relation as shown in the following result where a **directed hollow tree** is defined as for hollow trees but with a directed binary relation E. Note that we do not assume that E is a successor relation among siblings, the direction of E could be arbitrary, but the result below works in particular when E is a successor relation. Via a simple bi-FO-interpretation which uses extra colors to encode the direction of the edges, we get the following result:

▶ **Corollary 3.** *For every σ, $<$-inv FO $=$ FO on the class of σ directed hollow trees*

Define a path over σ as a word over the alphabet σ, where the successor edges are symmetrical (the argument used in the proof of Corollary 3 guarantees that paths are a more general model than words). The class of paths over σ is obviously bi-FO-interpretable through $\mathbb{H}_\sigma$: just add a S-parent to the endpoints of the path, and then forget about it. Thus we get:

▶ **Corollary 4.** *For every alphabet σ, $<$-inv FO $=$ FO on the class of paths over σ.*

Similarly, a straightforward bi-FO-interpretation together with Theorem 1 give us back the result from [5] that $<$-inv FO $=$ FO on ranked trees.

3 Swaps and pumping

In this section we provide a few operations, denoted swaps, that preserve $\equiv_k^{<\text{-inv FO}}$. Although the k-type of every element will be left unchanged, applying these operations may break the somewhat rigid structure of hollow trees. In order to work with the intermediate structures, we loosen the definition of hollow trees and define hollow quasitrees as follows:

▶ **Definition 5.** *For $k > 0$ and σ a set of colors, we define the set of* ***hollow k-quasitrees on*** *σ, quasi-$\mathbb{H}^k_\sigma$, as the set of all finite structures over $\{E, S\} \cup P_\sigma$ such that the k-type of any of their elements is the k-type of some element in some hollow tree in $\mathbb{H}_\sigma$, and which are such that their relation E is acyclic.*

In other words a hollow quasitree differs from a hollow tree by its relation S which may not induce a tree structure: a node may have its S-children in two distinct siblinghoods and a hollow quasitree may have cycles using the relation S (but not using only the relation E). Note that by definition $\mathbb{H}_\sigma \subseteq$ quasi-$\mathbb{H}^k_\sigma$ for every k. An example of what a hollow quasitree could look like is given in the right part of Figure 3. Note that locally, it looks like a hollow tree.

Let $\mathcal{T} \in$ quasi-$\mathbb{H}^k_\sigma$. We define the **support** of $\mathcal{T}$ as its restriction to the vocabulary $P_\sigma \cup \{E\}$. The **n-enriched support** of $\mathcal{T}$, denoted $\mathcal{S}\text{upp}_n(\mathcal{T})$, is the support of its n-enrichment (and not the other way around). Hence, it keeps in memory the local behavior within $\mathcal{T}$. The set $\text{End}(\mathcal{T})$ of **endpoints** of $\mathcal{T}$ is the set of elements of the support having degree one. A connected component of the support of $\mathcal{T}$ is called a **thread**[1]. Note that by E-acyclicity of $\mathcal{T}$, each of its threads is a path, hence contains exactly two endpoints. We say that a hollow k-quasitree has the **matching endpoints property** if the two endpoints of each thread have the same S-parent. Note that a hollow tree has the matching endpoints property. Notice also that in a hollow k-quasitree, any thread of length less than $2k + 1$ has matching endpoints. For $x, y \in T$ belonging to the same thread, $[x, y]$ denotes the set of elements that lie between them (formally, those who disconnect x from y in $\mathcal{S}\text{upp}_0(\mathcal{T})$), including x and y. We naturally define $[x, y[$ as $[x, y] \setminus \{y\}$.

The following lemma, implicit in the proof of locality of $<$ -inv FO by Grohe and Schwentick [8], will allow us to prove that our operations preserve order-invariance equivalence:

▶ **Lemma 6.** *Let Σ be a relational vocabulary and let $p, \alpha \in \mathbb{N}$. There exists $o^\Sigma_p(\alpha) \in \mathbb{N}$ such that for every structure $\mathcal{A}$ over Σ, and for every p-tuples of elements $\bar{a}, \bar{b} \in A^p$ that have the same $o^\Sigma_p(\alpha)$-type in $\mathcal{A}$, there are two orders $<_{\bar{a}\bar{b}}$ and $<_{\bar{b}\bar{a}}$ on A such that*

- $(\mathcal{A}, <_{\bar{a}\bar{b}}) \equiv^{\text{FO}}_\alpha (\mathcal{A}, <_{\bar{b}\bar{a}})$
- *$\bar{a}\bar{b}$ is an initial segment of $<_{\bar{a}\bar{b}}$*
- *$\bar{b}\bar{a}$ is an initial segment of $<_{\bar{b}\bar{a}}$*

Our operations are divided into three families depending on whether we modify the relation S, the relation E, or whether we do a global pumping,

In the following, $\mathcal{R}$ is a hollow $(m + 1)$-quasitree on σ.

3.1 crossing-S-swaps

Let $a, a', a'', b, b', b'' \in R$ be such that $S(a, a'), S(a, a''), S(b, b'), S(b, b'')$ and such that $\text{tp}^m_{\mathcal{R}}(a, a', a'') = \text{tp}^m_{\mathcal{R}}(b, b', b'')$. Let $\mathcal{R}^- := \mathcal{R} \setminus \{S(a, a'), S(a, a''), S(b, b'), S(b, b'')\}$ and assume that the sets $\{a', a''\}, \{b', b''\}$ and $\{a, b\}$ are pairwise $(2m + 3)$-distant in $\mathcal{R}^-$.

Then $\mathcal{R}' := \mathcal{R}^- \cup \{S(a, b'), S(a, b''), S(b, a'), S(b, a'')\}$ is called the **m-guarded crossing-S-swap between a and b in $\mathcal{R}$** (see Figure 4).

▶ **Note 7.** *A particular case where the distance condition is met is when $dist_{\mathcal{R}}(a, b) \geq 2m+5$.*

[1] A thread is nothing other than a siblinghood when the quasitree is a tree.

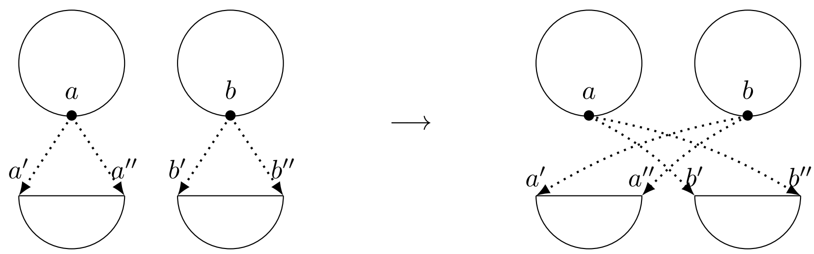

Figure 4 The crossing-S-swap between a and b

▶ **Lemma 8.** *For all $\alpha \in \mathbb{N}$ there exists $s(\alpha) \in \mathbb{N}$ such that for all $m \geq s(\alpha)$, and every hollow $(m+1)$-quasitree $\mathcal{R}$,*

if $\mathcal{R}'$ is the m-guarded crossing-S-swap between a and b in $\mathcal{R}$,

then $\mathcal{R}' \equiv_\alpha^{<\text{inv FO}} \mathcal{R}$, and $\forall x \in R$, $tp_{\mathcal{R}'}^{m+1}(x) = tp_{\mathcal{R}}^{m+1}(x)$. Moreover $\mathcal{R}' \in$ quasi-$\mathbb{H}_\sigma^{m+1}$ and $\mathcal{S}upp_{m+1}(\mathcal{R}') = \mathcal{S}upp_{m+1}(\mathcal{R})$.

Proof sketch. In order to prove that $\mathcal{R}' \equiv_\alpha^{<\text{inv FO}} \mathcal{R}$ we need to exhibit a linear order over $\mathcal{R}$ and one over $\mathcal{R}'$ such that we can play an α-round Ehrenfeucht-Fraïssé game between the resulting ordered structures. The linear orders are constructed using Lemma 6 applied to (a', a'') and (b', b'') and the structure $\mathcal{R}^-$. A simple FO-interpretation is then used to transfer the corresponding orders onto $\mathcal{R}$ and $\mathcal{R}'$. Proving that the type of an element is unchanged is straightforward. ◀

3.2 E-swaps

We define four different kinds of E-swaps.

Let $a, b, a', b' \in R$ be such that $E(a,b), E(a',b')$, a, b and a', b' appear in two different threads of $\mathcal{R}$ and such that $\{a, b, a', b'\}$ and $\text{End}(\mathcal{R})$ are $(2m+3)$-distant in $\mathcal{S}\text{upp}_0(\mathcal{R})$. Furthermore, assume that $\text{tp}_{\mathcal{R}}^m(a,b) = \text{tp}_{\mathcal{R}}^m(a',b')$. Let $\mathcal{R}' := \mathcal{R} \setminus \{E(a,b), E(a',b')\} \cup \{E(a,b'), E(a',b)\}$.

Then $\mathcal{R}'$ is called the **m-guarded crossing-E-swap between ab and $a'b'$ in $\mathcal{R}$** (c.f. Figure 5).

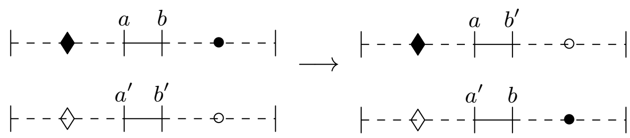

Figure 5 Illustration of the m-guarded crossing-E-swap between ab and $a'b'$ in $\mathcal{R}$

Let $a, b, b', a' \in R$ appear in that order in a single thread of $\mathcal{R}$, such that $E(a,b), E(a',b')$, and such that $\{a, b, a', b'\}$ and $\text{End}(\mathcal{R})$ are $(2m+3)$-distant in $\mathcal{S}\text{upp}_0(\mathcal{R})$. Furthermore, assume that $\text{tp}_{\mathcal{R}}^m(a,b) = \text{tp}_{\mathcal{R}}^m(a',b')$. Let $\mathcal{R}' := \mathcal{R} \setminus \{E(a,b), E(a',b')\} \cup \{E(a,b'), E(a',b)\}$. Then $\mathcal{R}'$ is called the **m-guarded mirror-E-swap at $[b,b']$ in $\mathcal{R}$** (c.f. Figure 6).

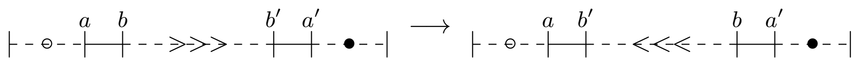

Figure 6 Illustration of the m-guarded mirror-E-swap at $[b,b']$ in $\mathcal{R}$

Consider now $a, b, c, d, a', b', c', d' \in R$ appearing in that order in a single thread of $\mathcal{R}$ such that $E(a,b), E(c,d), E(a',b'), E(c',d')$ and such that $\{a,b,c,d,a',b',c',d'\}$ and $\mathrm{End}(\mathcal{R})$ are $(2m+3)$-distant in $\mathcal{S}\mathrm{upp}_0(\mathcal{R})$. Furthermore, assume that $\mathrm{tp}^m_{\mathcal{R}}(a,b) = \mathrm{tp}^m_{\mathcal{R}}(a',b')$ and $\mathrm{tp}^m_{\mathcal{R}}(c,d) = \mathrm{tp}^m_{\mathcal{R}}(c',d')$.
Let $\mathcal{R}' := \mathcal{R} \setminus \{E(a,b), E(a',b'), E(c,d), E(c',d')\} \cup \{E(a,b'), E(a',b), E(c,d'), E(c',d)\}$. $\mathcal{R}'$ is called the **m-guarded segment-E-swap between $[b,c]$ and $[b',c']$ in $\mathcal{R}$** (c.f. Figure 7).

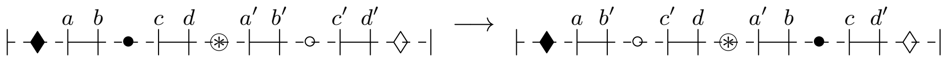

Figure 7 Illustration of the m-guarded segment-E-swap between $[b,c]$ and $[b',c']$ in $\mathcal{R}$

Finally, let a, b, a', b', a'', b'' be elements of R appearing in that order in a single thread of $\mathcal{R}$, such that $E(a,b), E(a',b')$ and $E(a'',b'')$ and $\{a,b,a',b',a'',b''\}$ and $\mathrm{End}(\mathcal{R})$ are $(2m+3)$-distant in $\mathcal{S}\mathrm{upp}_0(\mathcal{R})$. Furthermore, suppose that $\mathrm{tp}^m_{\mathcal{R}}(a,b) = \mathrm{tp}^m_{\mathcal{R}}(a',b') = \mathrm{tp}^m_{\mathcal{R}}(a'',b'')$.

Let $\mathcal{R}' := \mathcal{R} \setminus \{E(a,b), E(a',b'), E(a'',b'')\} \cup \{E(a,b'), E(a',b''), E(a'',b)\}$. $\mathcal{R}'$ is called the **m-guarded contiguous-segment-E-swap between $[b,a']$ and $[b',a'']$ in $\mathcal{R}$** (c.f. Figure 8).

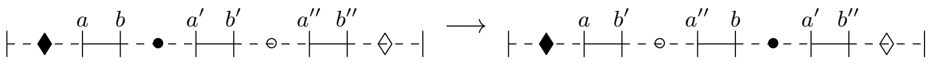

Figure 8 Illustration of the m-guarded contiguous-segment-E-swap between $[b,a']$ and $[b',a'']$ in $\mathcal{R}$

As long as m is large enough, all the m-guarded E-swaps preserve $\equiv_\alpha^{<\text{-inv FO}}$ and the $(m+1)$-type of every element:

▶ **Lemma 9.** *For all $\alpha \in \mathbb{N}$ there exists $s(\alpha) \in \mathbb{N}$ such that for every $m \geq s(\alpha)$ and every hollow $(m+1)$-quasitree $\mathcal{R}$, if $\mathcal{R}'$ is either*

- *the m-guarded crossing-E-swap between ab and $a'b'$ in $\mathcal{R}$*
- *the m-guarded mirror-E-swap at $[b,b']$ in $\mathcal{R}$*
- *the m-guarded contiguous-segment-E-swap between $[b,a']$ and $[b',a'']$ in $\mathcal{R}$*
- *the m-guarded segment-E-swap between $[b,c]$ and $[b',c']$ in $\mathcal{R}$*

then $\mathcal{R}' \equiv_\alpha^{<\text{-inv FO}} \mathcal{R}$, $\forall x \in R$, $tp^{m+1}_{\mathcal{R}'}(x) = tp^{m+1}_{\mathcal{R}}(x)$ and $\mathcal{R}' \in$ quasi-$\mathbb{H}^{m+1}_\sigma$.

Proof sketch. The proof is a tedious case analysis. Basically it amounts to the following idea: if the elements involved in the swap are far away from each other then we can use Lemma 6 in the structure $\mathcal{R}$ minus the E-edges of interest, and get orders on $\mathcal{R}$ and $\mathcal{R}'$ which make these structures similar as in the proof of Lemma 8.

On the other hand, if the elements are close to each other, then the fact that they share the same type induces some periodicity on their neighborhoods. These neighborhoods can therefore be decomposed into several consecutive similar pieces. We can then apply Lemma 6 to these smaller components to conclude. ◀

3.3 Pumping

The next operation makes use of the fact that $<$-inv FO $\subsetneq$ MSO over hollow trees. Hence our hollow trees can be "pumped" in order to duplicate some of their parts.

Given a structure $\mathcal{A}$ and a k-type τ, we denote by $|\mathcal{A}|_\tau$ the number of elements of A whose k-type is τ. We will essentially use 0-types as our structures will be enriched by recoloring each element by its k-type. In view of this we denote by $[\![\mathcal{A}]\!]$ the function $\tau \mapsto |\mathcal{A}|_\tau$ whose domain is the set of 0-types over the considered vocabulary.

Let $d, D \in \mathbb{N}$, and f, g be functions from a same domain to $\mathbb{N}$. We say that $f \leq_d^D g$ if for every x in the domain:

- if $f(x) \leq d$, then $f(x) = g(x)$
- if $f(x) \neq g(x)$, then $g(x) \geq f(x) + D$

By $f < g$, we mean that $\forall x, f(x) < g(x)$ or $f(x) = g(x) = 0$.

In the following proposition $<$-inv FO can be replaced by MSO.

▶ **Proposition 10.** *$\forall \alpha, n, d \in \mathbb{N}, \exists M \in \mathbb{N}, \forall D \in \mathbb{N}, \forall \mathcal{P}, \mathcal{Q} \in \mathbb{H}_\sigma$, if $\mathcal{P} \equiv_M^{\text{FO}} \mathcal{Q}$, then there exists $\mathcal{Q}' \in \mathbb{H}_\sigma$ such that $\mathcal{Q}' \equiv_\alpha^{<\text{inv FO}} \mathcal{Q}$ and $[\![\mathcal{E}_{n+1}(\mathcal{P})]\!] \leq_d^D [\![\mathcal{E}_{n+1}(\mathcal{Q}')]\!]$.*

Proof sketch. This is a pumping argument: by setting M large enough, we make sure in FO that if a $(n+1)$-type has more occurrences in $\mathcal{P}$ than in $\mathcal{Q}$, then it has enough occurrences in $\mathcal{Q}$ so that we can find a context in $\mathcal{Q}$ containing at least one occurrence, and no occurrence of a rare type, such that we can duplicate this context inside $\mathcal{Q}$ without changing its MSO-type. ◀

4 Inclusion and pseudo-inclusion

Recall that our ultimate goal is to show that if two hollow trees agree on the same FO sentences of quantifier rank $f(\alpha)$ then they agree on all $<$-inv FO sentences of quantifier rank α. For this, we will show that if $\mathcal{P}$ and $\mathcal{Q}$ are hollow trees that agree on all FO sentences of quantifier rank $f(\alpha)$ then we can use operations such as the swap operations described in Section 3 to transform $\mathcal{Q}$ into $\mathcal{P}$. As these operations preserve $<$-inv FO we get the desired result.

In this section we perform the first step towards transforming $\mathcal{Q}$ into $\mathcal{P}$. We show that using the swap operations we can transform $\mathcal{Q}$ into $\mathcal{Q}'$ so that $\mathcal{Q}'$ "includes" $\mathcal{P}$. The resulting structure $\mathcal{Q}'$ will be a hollow quasitree. In the next sections we will continue the transformation and remove from $\mathcal{Q}'$ all the extra material it contains, deriving $\mathcal{P}$.

In order to define what we mean by "inclusion" we need the notion of a n-abstract context of a hollow quasitree. Intuitively this is a S-stable n-enriched substructure. More formally, given a hollow quasitree $\mathcal{T} \in$ quasi-$\mathbb{H}_\sigma^n$ and a set U of its domain that is S-stable, then $\mathcal{C} := \mathcal{T}_{|U}$, together with the function $\text{tp}^n(.)$ that maps $x \in U$ to its n-type in $\mathcal{T}$, is called a **n-abstract context** denoted $\mathcal{C} = \text{Ctxt}_n(\mathcal{T}|_U)$. The set of n-abstract contexts is denoted Ctxt_σ^n. Note that $\text{tp}^n(x)$ denotes $\text{tp}_\mathcal{T}^n(x)$ and not $\text{tp}_\mathcal{C}^n(x)$. We need to remember, at least locally, how $\mathcal{C}$ was glued to the rest of $\mathcal{T}$ in order to preserve n-types when moving $\mathcal{C}$ to some other place.

We are now ready to define the notion of "inclusion". We actually define both "inclusions" and "pseudo-inclusions". We will need to pseudo-include a hollow quasitree into another (Proposition 12), and then to include an abstract context into a hollow quasitree (Proposition 13). Since a hollow k-quasitree $\mathcal{T} \in$ quasi-$\mathbb{H}_\sigma^k$ can be seen as a k-abstract context ($\mathcal{T} = \text{Ctxt}_k(\mathcal{T}|_T)$), we only need to define (pseudo-)inclusions from an abstract context into a hollow quasitree.

▶ **Definition 11.** *Let $k \in \mathbb{N}$, $\mathcal{U} \in Ctxt_\sigma^k$ and $\mathcal{Q} \in$ quasi-$\mathbb{H}_\sigma^k$. We say that $h : U \to Q$ is a **k-pseudo-inclusion** if h is injective and for all $x, y, z \in U$ the following is verified:*

1. $tp_\mathcal{Q}^k(h(x)) = tp^k(x)$,

2. *if x and y are in the same thread of $\mathcal{U}$ then $h(x)$ and $h(y)$ are also on the same thread of $\mathcal{Q}$ and if moreover $z \in [x, y]$ then $h(z) \in [h(x), h(y)]$,*
3. *if $\mathcal{U} \models E(x, y)$ and t is the E-neighbor of $h(x)$ in $[h(x), h(y)]$ then t is the image of y by an isomorphism (induced by the fact that they share the same k-type) between the n-neighborhood of x and that of $h(x)$.*

*If $\mathcal{U} \models E(x, y)$ and $\mathcal{Q} \not\models E(h(x), h(y))$ then $\{x, y\}$ is said to be a **jumping pair** for h, and $tp^{k-1}_{\mathcal{Q}}(h(x), t)$, where t is the E-neighbor of $h(x)$ in $[h(x), h(y)]$, is called its type.*[2]

*A k-pseudo-inclusion is said to be **reduced** if there is at most one jumping pair of a given type.*

*A k-pseudo-inclusion is called a k**-inclusion** if it has no jumping pairs, that is if it preserves E.*

The last condition of pseudo-inclusion is a complication induced by the fact that E is not oriented and that we thus cannot distinguish between the two siblings of a node. It ensures that h preserves the neighborhoods in the right order. We can now state the main result of this section. Note that the precondition that $\mathcal{Q}$ has more realizations for each type than $\mathcal{U}$ or $\mathcal{P}$ will not be a problem in view of Proposition 10. The second proposition is stronger than the first one as it derives inclusion instead of pseudo-inclusion, but it requires the stronger hypothesis that every occurring type has strictly more realizations in $\mathcal{Q}$ than in $\mathcal{U}$.

▶ **Proposition 12.** *For every $\alpha, m \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that $\forall \mathcal{P}, \mathcal{Q} \in$ quasi-$\mathbb{H}^{N+1}_{\sigma}$, if $[\![\mathcal{E}_{N+1}(\mathcal{P})]\!] \leq [\![\mathcal{E}_{N+1}(\mathcal{Q})]\!]$, then there exists $\mathcal{Q}' \in$ quasi-$\mathbb{H}^{m+1}_{\sigma}$ such that $\mathcal{Q}' \equiv^{<\text{-inv FO}}_{\alpha} \mathcal{Q}$, $[\![\mathcal{E}_{m+1}(\mathcal{Q}')]\!] = [\![\mathcal{E}_{m+1}(\mathcal{Q})]\!]$ and h that is a $(m+1)$-pseudo-inclusion from $\mathcal{P}$ into $\mathcal{Q}'$.*

▶ **Proposition 13.** *For every $\alpha, m \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that $\forall \mathcal{U} \in Ctxt^{N+1}_{\sigma}$, $\forall \mathcal{Q} \in$ quasi-$\mathbb{H}^{N+1}_{\sigma}$, if $[\![\mathcal{E}_{N+1}(\mathcal{U})]\!] < [\![\mathcal{E}_{N+1}(\mathcal{Q})]\!]$, then there exists $\mathcal{Q}' \in$ quasi-$\mathbb{H}^{m+1}_{\sigma}$ such that $\mathcal{Q}' \equiv^{<\text{-inv FO}}_{\alpha} \mathcal{Q}$, $[\![\mathcal{E}_{m+1}(\mathcal{Q}')]\!] = [\![\mathcal{E}_{m+1}(\mathcal{Q})]\!]$ and $\mathcal{U}$ is $(m+1)$-included in $\mathcal{Q}'$.*

Proof sketch. Both propositions have a similar proof: we first prove Proposition 12, and explain afterwards how to move from pseudo-inclusions to inclusions.

We define the pseudo-inclusion h step by step, extending the domain of h thread by thread and, inside each thread, from one of its endpoint to the other. At each step we modify $\mathcal{Q}$ using E-swaps, if necessary.

We give a special treatment to short threads and portions of the long threads that are close to the endpoints: in that case, no modification of $\mathcal{Q}$ is required as the cardinality precondition ensures the presence of the necessary sequences within $\mathcal{Q}$. We then move to the parts of the long threads that are far from the endpoints, adding them one node at a time to the domain of the pseudo-inclusion. Note that as all the elements involved in the E-swaps to come are distant from the endpoints, the E-swaps involved are guarded.

Let x' be the last node of the current thread t that has been given an image by h, and let x be the next node to which we want to extend the domain of h. By hypothesis, we know that there exists a node $y \notin \mathrm{Im}(h)$ far from any endpoint, that has the same $(m+1)$-type as x. We denote by y' the neighbor of y that has the same m-type as x', and by $\hat{x}$ the neighbor of $h(x')$ having the same m-type as x.

We proceed to a case analysis depending on the relative position of $y, y', h(x')$ and $\hat{x}$. If y', y are on the same thread as $h(x'), \hat{x}$ and in the same direction (in particular when $y = \hat{x}$),

[2] This is an ease of notation; to be more precise, we should make the type of a jumping pair symmetrical.

we simply set $h(x)$ to y and we are done. If not, one of the E-swaps will place y to the desired position.

For instance, if y', y are on the same thread as $h(x'), \hat{x}$ but in the reverse direction (c.f. Figure 9, where the double line represents $\mathrm{Im}(h)$), then we consider the m-guarded mirror-E-swap at $[\hat{x}, y]$ in $\mathcal{Q}$ and extend h by setting $h(x)$ to y.

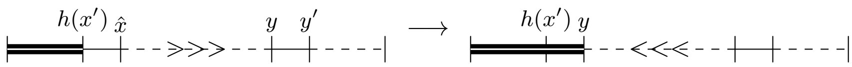

Figure 9 $h(x'), \hat{x}$ and y', y are in the same thread, but in reverse order: we use a mirror-E-swap

Now, if y is on a thread that does not intersect $\mathrm{Im}(h)$ (c.f. Figure 10), we consider the m-guarded crossing-E-swap between $h(x')\hat{x}$ and $y'y$ in $\mathcal{Q}$, and extend h by setting $h(x)$ to y.

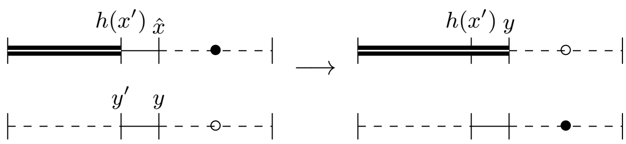

Figure 10 y is on a thread disjoint from $\mathrm{Im}(h)$: we use a crossing-E-swap

If y', y are in the same direction as $h(x'), \hat{x}$, and are between $h(z)$ and $h(z')$ where z and z' are consecutive node of the current thread (c.f. Figure 11).

Then we consider the m-guarded segment-E-swap between $[u', y']$ and $[h(z'), h(x')]$ in $\mathcal{Q}$, and extend h by setting $h(x)$ to y.

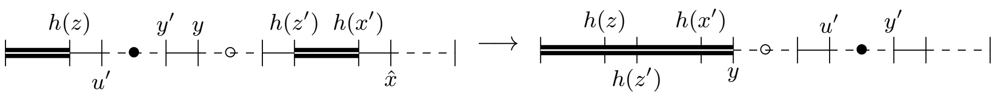

Figure 11 y', y are between the images of two already included neighbors: we use a segment-E-swap

There are a few other cases that are treated similarly. This concludes the proof for pseudo-inclusion.

For Proposition 13, as we wish to construct an inclusion, we need to make sure that there is no “jump” in the mapping.

Note that among all the previously mentioned cases, only one didn’t guarantee the absence of a jump, namely when y', y are on the same thread as $h(x'), \hat{x}$ and in the right direction, but when $y \neq \hat{x}$. We then use the stronger hypothesis on the number of types in $\mathcal{Q}$, which guarantees that there also exist z, z' verifying the same conditions as y, y' (cf. Figure 12). We consider the m-guarded contiguous-segment-E-swap between $[\hat{x}, y']$ and $[y, z']$ in $\mathcal{Q}$, and extend h by setting $h(x)$ to y. h is now an inclusion.

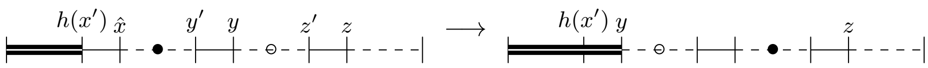

Figure 12 y', y, z', z and $h(x'), \hat{x}$ are on the same thread, in the same order: we use a contiguous-segment-E-swap to avoid a jump in the inclusion

◀

5 Tools for reorganizing S-edges

In the previous section, we have seen how to “rewrite” $\mathcal{Q}$ using E-swap operations in order to pseudo-include $\mathcal{P}$ into the resulting quasitree. By definition, the pseudo-inclusion h of $\mathcal{P}$ into $\mathcal{Q}$ respects the enriched support but can be completely wild relatively to the S-edges. For instance, in $\mathcal{Q}$, the endpoints of a thread may not have the same S-parent. In this section we show how to use S-swaps in order to ensure that our pseudo-inclusion mapping takes into account (to various degrees) the S-edges. We say that two nodes of a quasitree are S-siblings if they share the same S-parent.

In Section 5.1, we show how to make sure that the pseudo-inclusion respects the S-siblings relation. In Section 5.2 we show how to ensure that the image of a pseudo-inclusion is S-stable. S-stability is required to define and operate on the loops, as will be established in Section 6.

5.1 S-siblings re-association

The following Lemma shows how to modify a pseudo-inclusion in order for it to preserve the S-siblings relation. Note that it doesn't necessarily mean that the image structure has the matching endpoint property because the initial structure itself may not have this property as it is derived from a quasitree.

▶ **Lemma 14.** *$\forall \alpha, m \in \mathbb{N}, \exists N \in \mathbb{N}, \forall \mathcal{W} \in Ctxt^N_\sigma, \forall \mathcal{Q} \in quasi\text{-}\mathbb{H}^N_\sigma$, if $h : W \to Q$ is a N-pseudo-inclusion, then there exists some $\mathcal{Q}' \in quasi\text{-}\mathbb{H}^{m+1}_\sigma$ and some $(m+1)$-pseudo-inclusion $h' : W \to Q'$ such that $\mathcal{Q}' \equiv^{<\text{-inv FO}}_\alpha \mathcal{Q}$, $\mathcal{S}upp_{m+1}(\mathcal{Q}') \simeq \mathcal{S}upp_{m+1}(\mathcal{Q})$ and, if x and y are S-siblings in $\mathcal{W}$, then so are $h'(x)$ and $h'(y)$ in $\mathcal{Q}'$.*

Proof sketch. We correct the S-edges two by two: let x, y be two S-siblings in $\mathcal{W}$ such that $h(x), h(y)$ are not S-siblings in $\mathcal{Q}$, and let $z \in Q$ be the S-sibling of $h(x)$.

z and $h(y)$ must have the same $(N-2)$-type: we can use a crossing-E-swap or a mirror-E-swap (depending on whether they are the endpoints of a same thread) to exchange their positions and make sure $h(x)$ and $h(y)$ are S-siblings.

However, for these swaps to be guarded, we must operate far enough from the endpoints. This can be done as long as we choose N large enough. ◀

A particular case of the previous lemma is when $\mathcal{W}$ is a hollow tree and h is surjective: then $\mathcal{Q}'$ has the matching endpoints property. This result will be useful in the proof of Proposition 18.

5.2 S-stabilization

The image of a pseudo-inclusion has no reason to be S-stable, thus neither has its complement. However, this is a crucial requirement to apply the results presented in the next section, Section 6, in order to remove the extra material not in the image of the pseudo-inclusion.

The next result provides a method to ensure that the image (and its complement) of a pseudo-inclusion is S-stable.

Recall that a pseudo-inclusion is said to be reduced if there is at most one jumping pair of a given type. At the end of this process, we get a reduced pseudo-inclusion, which will allow us to minimize the complement of its image in Section 6.1.

▶ **Proposition 15.** *For every $\alpha, m \in \mathbb{N}$, there exist $N, d, D \in \mathbb{N}$ such that, for every $\mathcal{P} \in \mathbb{H}_\sigma$, $\mathcal{Q} \in quasi\text{-}\mathbb{H}^{N+1}_\sigma$ such that $[\![\mathcal{E}_{N+1}(\mathcal{P})]\!] \leq^D_d [\![\mathcal{E}_{N+1}(\mathcal{Q})]\!]$ and $\mathcal{P}$ is $(N+1)$-pseudo-included in $\mathcal{Q}$ through some h, there are some h' and $\mathcal{Q}' \in quasi\text{-}\mathbb{H}^{m+1}_\sigma$ such that $\mathcal{Q}' \equiv^{<\text{-inv FO}}_\alpha \mathcal{Q}$,*

$\mathcal{S}upp_{m+1}(\mathcal{Q}') \simeq \mathcal{S}upp_{m+1}(\mathcal{Q})$, h' is a reduced $(m+1)$-pseudo-inclusion of $\mathcal{P}$ in $\mathcal{Q}'$ and $Q' \setminus \mathrm{Im}(h')$ is S-stable in $\mathcal{Q}'$.

Proof sketch. We consider all the pairs of elements x, y which break the S-stability of $\mathrm{Im}(h)$, i.e. such that $S(x,y)$, $x \in \mathrm{Im}(h)$ and $y \notin \mathrm{Im}(h)$. If there are many of them, then at least two of them are far from each other and we can apply a crossing-S-swap to correct the mapping h. We end up with a bounded number of problematic pairs that can be corrected separately. ◀

6 Removing unnecessary material

In this section we show how to remove the material in $\mathcal{Q}$ that is not present in the image of the pseudo-inclusion of $\mathcal{P}$. From the previous section we can assume that the pseudo-inclusion mapping preserves the S-siblings relation and that its image is S-stable. The remaining part of $\mathcal{Q}$ is then a union of "loops" in the sense that they connect nodes that have the same type. After defining properly the notion of loop, we will use in Section 6.1 a pumping argument in order to reduce the size of the loop to some constant while preserving $\equiv_\alpha^{<\text{-inv FO}}$. In Section 6.2 we then show how to remove small loops without affecting the order-invariant equivalence class. Finally, in Section 6.3 we show that if a hollow tree and a hollow quasitree have the same enriched support, then they are $\equiv_\alpha^{<\text{-inv FO}}$: this concludes the proof of Theorem 1.

We start with the definition of an abstract loop.

Let $n \in \mathbb{N}$. Let $\mathrm{Type}_\sigma^n[2]$ denote the set of $(n-1)$-types for pairs over the vocabulary $P_\sigma \cup \{E, S\}$, of degree ≤ 4. Let Σ_n be the vocabulary enriching $P_\sigma \cup \{E, S\}$ with two unary symbols J_τ^1 and J_τ^2 for every $\tau \in \mathrm{Type}_\sigma^n[2]$.

Let h be a reduced n-pseudo-inclusion from $\mathcal{P} \in \mathbb{H}_\sigma$ to $\mathcal{Q} \in \text{quasi-}\mathbb{H}_\sigma^n$, such that $V := Q \setminus \mathrm{Im}(h)$ is S-stable.

Let $\mathcal{Q}_+$ be an extension of $\mathcal{Q}$ to Σ_n obtained in the following way. Since h is reduced, for every $\tau \in \mathrm{Type}_\sigma^n[2]$, there is at most one jumping pair of type τ. If there isn't, J_τ^1 and J_τ^2 are interpreted as the empty set. Else, let $\{x, x'\}$ be this pair, and u' (resp. u) be the E-neighbor of $h(x)$ (resp. $h(x')$) in $[h(x), h(x')]$. Interpret J_τ^1 as $\{h(x), u'\}$ and J_τ^2 as $\{h(x'), u\}$ (the assignments $x \mapsto 1$ and $x' \mapsto 2$ are arbitrary). This is illustrated on the left part of Figure 13, where the double line represents $\mathrm{Im}(h)$. We say that $\mathcal{Q}_+$ is a **h-jump-extension of $\mathcal{Q}$**.

We define $\mathcal{V}_+ = \mathrm{Ctxt}_n(\mathcal{Q}_+|_V)$ as the extension of $\mathrm{Ctxt}_n(\mathcal{Q}|_V)$ to Σ_n where every J_τ^i is defined consistently with $\mathcal{Q}_+$ (i.e. $\forall x \in V, \mathcal{V}_+ \models J_\tau^i(x)$ iff $\mathcal{Q}_+ \models J_\tau^i(x)$). This process is illustrated in Figure 13. $\mathcal{V}_+$ is called an **n-abstract loop**. Let $\mathbb{L}_\sigma^n$ be the set of n-abstract loops.

Figure 13 Example of a h-jump-extension $\mathcal{Q}_+$ of $\mathcal{Q}$ (on the left), and its associated abstract loop $\mathcal{V}_+$ of support $V := Q \setminus \mathrm{Im}(h)$ (on the right)

Every Σ_n-structure will have a '+' symbol in its name. When we omit it, we mean the reduction of the structure to $P_\sigma \cup \{E, S\}$ (for instance, from $\mathcal{V}_+ \in \mathbb{L}_\sigma^n$, we get $\mathcal{V} := \mathrm{Ctxt}_n(\mathcal{Q}|_V) \in \mathrm{Ctxt}_\sigma^n$).

6.1 Loop minimization

It will be crucial to bound the size of the loops left by a pseudo-inclusion. The following result does this using a simple pumping argument.

▶ **Proposition 16.** *For every $\alpha, n \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that for every $\mathcal{P} \in \mathbb{H}_\sigma$, $\mathcal{Q} \in$ quasi-$\mathbb{H}_\sigma^n$ and reduced n-pseudo-inclusion $h : P \to Q$, if $V := Q \setminus \mathrm{Im}(h)$ is S-stable then there exists some $\mathcal{Q}' \in$ quasi-$\mathbb{H}_\sigma^n$ and a reduced n-pseudo-inclusion $h' : P \to Q'$ such that $\mathcal{Q}' \equiv_\alpha^{<\text{-inv FO}} \mathcal{Q}$, $U := Q' \setminus \mathrm{Im}(h')$ is S-stable and $|U| \leq N$.*

6.2 Loop elimination

It now remains to get rid of the small loops. This is a consequence of the "aperiodicity" of $<$-inv FO: we cannot distinguish in $<$-inv FO between k and $k+1$ copies of the same object if k is sufficiently large. Starting from a small loop, we can use the inclusion results of Section 4 to recreate many copies of the loop within $\mathcal{Q}$, then, according to the following proposition, get rid of one copy using aperiodicity.

▶ **Proposition 17.** $\forall \alpha \in \mathbb{N}, \exists l \in \mathbb{N}, \forall m \in \mathbb{N}, \exists n \in \mathbb{N}, \forall M \in \mathbb{N}, \exists K \in \mathbb{N}$ *such that for every abstract loop $\mathcal{U}_+ \in \mathbb{L}_\sigma^{n+1}$ and every $\mathcal{Q} \in$ quasi-$\mathbb{H}_\sigma^{n+1}$ such that $|U| \leq M$, $(l+1) \cdot [\![\mathcal{E}_{n+1}(\mathcal{U})]\!] < [\![\mathcal{E}_{n+1}(\mathcal{Q})]\!]$ and such that for every $(n+1)$-type χ that occurs in $\mathcal{U}$, $|\mathcal{Q}|_\chi \geq K$, there exists $\mathcal{Q}' \in$ quasi-$\mathbb{H}_\sigma^m$ such that $\mathcal{Q}' \equiv_\alpha^{<\text{-inv FO}} \mathcal{Q}$ and $[\![\mathcal{E}_m(\mathcal{Q})]\!] = [\![\mathcal{E}_m(\mathcal{Q}')]\!] + [\![\mathcal{E}_m(\mathcal{U})]\!]$*

Proof sketch. The proof is based on the well known result that first-order formulas of quantifier-rank k cannot distinguish between a linear order of length 2^k and a linear order of length $2^k + 1$ (see, for instance, [11]). Hence if a loop is repeated at least $2^k + 1$ times, we can eliminate one instance without changing the $\equiv_k^{<\text{-inv FO}}$ class of the structure.

First, we include many copies of the loop in $\mathcal{Q}$. The inclusion may not preserve S-edges: the next step is to re-associate these S-edges with crossing-S-swaps in order for these copies to be isomorphic. This is made possible by the hypothesis on the number of occurrences of types appearing in $\mathcal{U}$: it gives us room to make sure the crossing-S-swaps are guarded.

Once this is done, we can remove one copy in a $<$-inv FO-indistinguishable way. ◀

6.3 S-parents re-association

We now turn to the last step of the proof of Theorem 1.

After the removal of the extra material in $\mathcal{Q}$, we have transformed our initial hollow tree $\mathcal{Q}$ into a hollow quasitree having the same number of occurrences of any type as the initial $\mathcal{P}$. They both have the same threads but may differ with their S-edges. The following proposition states that they are $\equiv_\alpha^{<\text{-inv FO}}$, thus ending the proof of Theorem 1.

The techniques used in the proof of the following proposition are strongly reminiscent of those used in [4]; it requires a notion of vertical-S-swaps adapted to hollow trees.

▶ **Proposition 18.** $\forall \alpha \in \mathbb{N}$, *there exists $n_1 \in \mathbb{N}$ such that $\forall \mathcal{P} \in \mathbb{H}_\sigma, \forall \mathcal{Q} \in$ quasi-$\mathbb{H}_\sigma^{n_1}$, if $\mathcal{S}upp_{n_1}(\mathcal{P}) \simeq \mathcal{S}upp_{n_1}(\mathcal{Q})$ then $\mathcal{P} \equiv_\alpha^{<\text{-inv FO}} \mathcal{Q}$.*

7 Conclusion

We have shown that $<$-inv FO $=$ FO over hollow trees. As we have discussed in Section 2.2, in order to lift this result to tracks of bounded degree, i.e. the basic blocks of structures of pathwidth 2 and bounded degree, it suffices to show that $<$-inv FO $=$ FO over structures that have the same underlying graph than hollows trees, but without the possibility to distinguish a sibling from a child; in other words, there is only one binary relation that is the union of E and S. This is because there exists a bi-FO-interpretation from tracks of bounded degree through this class of structures.

Unfortunately our proof does not extend to this class of structures as it was crucial in our proof to distinguish between E-swaps and S-swaps. We leave this generalization as an open problem.

We also have no idea yet on what to do when the degree is not assumed to be bounded, as we are then also facing the second difficulty mentioned in the introduction, namely reinterpreting the initial structure within its tree representation.

In this paper we bypassed the first problem mentioned in the introduction, finding similar tree decompositions given similar structures, by working directly on trees. This problem seems unavoidable when working with graphs. There are examples of similar structures of treewidth 2 that do not have any similar tree decompositions of width 2. It might even be the case that for all k there are two similar structures of treewidth 2 that do not have similar tree decomposition of width k. If that were true, completely new ideas would be needed to solve the treewidth 2 case.

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A Proofs for Section 2 (Preliminaries)

▶ **Theorem 1.** *For all* σ*,* $<$-inv FO $=$ FO *over* $\mathbb{H}_\sigma$

Proof. Let $\alpha \in \mathbb{N}$. Recall that we want to find $f(\alpha)$ such that $\forall \mathcal{P}, \mathcal{Q} \in \mathbb{H}_\sigma$, if $\mathcal{P} \equiv^{\text{FO}}_{f(\alpha)} \mathcal{Q}$ then $\mathcal{P} \equiv^{<\text{-inv FO}}_{\alpha} \mathcal{Q}$.

We set, in that order:

- l as in Proposition 17 (loop elimination), that is such that FO$[\alpha]$ cannot distinguish the linear order on $\{1, \cdots, l\}$ from the linear order on $\{1, \cdots, l+1\}$
- n_1 as in Proposition 18 (S-parents re-association)
- n_2 as in Proposition 12 (pseudo-inclusion) for $n_1 - 1$
- n_3 as in Proposition 17 (loop elimination) for n_2
- M as in Proposition 16 (loop minimization) for $n_3 + 1$
- K as in Proposition 17 (loop minimization) for n_2 and M
- n_4, d_1, D as in Proposition 15 (S-stabilization of the image of a pseudo-inclusion) for n_3
- n_5 as in Proposition 12 (pseudo-inclusion) for n_4
- $f(\alpha)$ as in Proposition 10 (pumping) for n_5 and $d := \max(d_1, K)$

Starting from $\mathcal{P} \equiv^{\text{FO}}_{f(\alpha)} \mathcal{Q}$, we unfold the previously set indexes to apply the corresponding propositions in the reverse order: we transform $\mathcal{Q}$ into $\mathcal{P}$ along a sequence of $\equiv^{<\text{-inv FO}}_{\alpha}$ hollow quasitrees (with smaller and smaller radius) $\mathcal{Q}_i$ as follows:

According to Proposition 10, we can pump inside $\mathcal{Q}$ to get $\mathcal{Q}_0 \equiv^{<\text{-inv FO}}_{\alpha} \mathcal{Q}$ such that $[\![\mathcal{E}_{n_5+1}(\mathcal{P})]\!] \leq^D_d [\![\mathcal{E}_{n_5+1}(\mathcal{Q}_0)]\!]$.

Now that we've made sure there were at least as many occurrences of every type in $\mathcal{Q}_0$ as in $\mathcal{P}$, Proposition 12 yields a $\mathcal{Q}_1 \equiv^{<\text{-inv FO}}_{\alpha} \mathcal{Q}_0$ such that $[\![\mathcal{E}_{n_4+1}(\mathcal{Q}_1)]\!] = [\![\mathcal{E}_{n_4+1}(\mathcal{Q}_0)]\!]$ and $\mathcal{P}$ is (n_4+1)-pseudo-included in $\mathcal{Q}_1$ by some h.

Since $[\![\mathcal{E}_{n_4+1}(\mathcal{P})]\!] \leq^D_{d_1} [\![\mathcal{E}_{n_4+1}(\mathcal{Q}_1)]\!]$, Proposition 15 gives some $\mathcal{Q}_2 \equiv^{<\text{-inv FO}}_{\alpha} \mathcal{Q}_1$ such that $\mathcal{S}\text{upp}_{n_3+1}(\mathcal{Q}_2) \simeq \mathcal{S}\text{upp}_{n_3+1}(\mathcal{Q}_1)$, and some reduced h' which (n_3+1)-pseudo-includes $\mathcal{P}$ in $\mathcal{Q}_2$, where $V := Q_2 \setminus \text{Im}(h')$ is S-stable in $\mathcal{Q}_2$

Proposition 16 gives us some $\mathcal{Q}_3$ and $\mathcal{U}_+ \in \mathbb{L}^{n_3+1}_\sigma$ such that $\mathcal{Q}_3 \equiv^{<\text{-inv FO}}_{\alpha} \mathcal{Q}_2$, $|U| \leq M$ and $[\![\mathcal{E}_{n_3+1}(\mathcal{Q}_3)]\!] = [\![\mathcal{E}_{n_3+1}(\mathcal{P})]\!] + [\![\mathcal{E}_{n_3+1}(\mathcal{U})]\!]$

Since $|U| \leq M$ and for every type (n_3+1)-type ξ that occurs in $\mathcal{U}$, $|\mathcal{Q}_3|_\xi \geq K$ (indeed: since ξ occurs in $\mathcal{U}$, $|\mathcal{Q}_0|_\xi \neq |\mathcal{P}|_\xi$ and $|\mathcal{Q}_3|_\xi > |\mathcal{P}|_\xi > d \geq K$), we can remove the extra elements by applying Proposition 17, which gives $\mathcal{Q}_4 \equiv^{<\text{-inv FO}}_{\alpha} \mathcal{Q}_3$ such that $[\![\mathcal{E}_{n_2+1}(\mathcal{Q}_4)]\!] = [\![\mathcal{E}_{n_2+1}(\mathcal{Q}_3)]\!] - [\![\mathcal{E}_{n_2+1}(\mathcal{U})]\!] = [\![\mathcal{E}_{n_2+1}(\mathcal{P})]\!]$

$\mathcal{P}$ and $\mathcal{Q}_4$ having the same number of occurrences of every (n_2+1)-type, we can pseudo-include one into another according to Proposition 12, which gives $\mathcal{Q}_5 \equiv^{<\text{-inv FO}}_{\alpha} \mathcal{Q}_4$ such that $\mathcal{S}\text{upp}_{n_1}(\mathcal{Q}_5) \simeq \mathcal{S}\text{upp}_{n_1}(\mathcal{P})$ (indeed, the pseudo-inclusion cannot have any jumping pair)

Finally, Proposition 18 allows us to conclude that $\mathcal{Q}_5 \equiv^{<\text{-inv FO}}_{\alpha} \mathcal{P}$, completing our sequence of transformations.

This concludes the proof that $\mathcal{P} \equiv^{<\text{-inv FO}}_{\alpha} \mathcal{Q}$.

◀

▶ **Lemma 2.** *If* $\mathcal{C}_1$ *is bi-*FO*-interpretable through* $\mathcal{C}_2$ *and* $<$-inv FO $=$ FO *over* $\mathcal{C}_2$*, then* $<$-inv FO $=$ FO *over* $\mathcal{C}_1$

Proof. We show that for some function f, for every $k \in \mathbb{N}$ and $\mathcal{A}, \mathcal{B} \in \mathcal{C}_1$, if $\mathcal{A} \equiv^{\text{FO}}_{f(k)} \mathcal{B}$, then $\mathcal{A} \equiv^{<\text{-inv FO}}_{k} \mathcal{B}$.

As $<$-inv FO = FO over $\mathcal{C}_2$ we know that there is a function g such that for all $k \in \mathbb{N}$ and $\mathcal{A}, \mathcal{B} \in \mathcal{C}_2$, if $\mathcal{A} \equiv^{\text{FO}}_{g(k)} \mathcal{B}$, then $\mathcal{A} \equiv^{<\text{-inv FO}}_{k} \mathcal{B}$: to any order-invariant formula with a quantifier rank less than k, choose an arbitrary first-order equivalent formula over $\mathcal{C}_2$ and take $g(k)$ as the max of their quantifier rank.

Let a_{12}, d_{12} (resp. a_{21}, d_{21}) be the arity and depth of $\mathcal{I}_{12}$ (resp. $\mathcal{I}_{21}$), and set $f(k) := a_{12} g(a_{21}k + d_{21}) + d_{12}$.

Assume now that $\mathcal{A}, \mathcal{B} \in \mathcal{C}_1$ are such that $\mathcal{A} \equiv^{\text{FO}}_{f(k)} \mathcal{B}$.

Applying $\mathcal{I}_{12}$ to both structures gives us $\mathcal{I}_{12}(\mathcal{A}) \equiv^{\text{FO}}_{g(a_{21}k+d_{21})} \mathcal{I}_{12}(\mathcal{B})$.

Hence $\mathcal{I}_{12}(\mathcal{A}) \equiv^{<\text{-inv FO}}_{a_{21}k+d_{21}} \mathcal{I}_{12}(\mathcal{B})$, which yields $\mathcal{A} \equiv^{<\text{-inv FO}}_{k} \mathcal{B}$ after applying $\mathcal{I}_{21}$. ◀

The first consequence of this result is that we can assume a normal form over hollow trees without loss of generality. Namely that each S-parent has exactly two S-children, and that no element is at the same time a S-parent and a S-child. Indeed, there is a simple bi-FO-interpretation that transforms a general hollow tree into one having the desired properties (by duplicating nodes that are only child, and those that are simultaneously S-parent and S-child, and marking them with new unary predicates), and back to the initial one.

▶ **Corollary 3.** *For every σ, $<$-inv FO = FO on the class of σ directed hollow trees*

Proof. We use Lemma 2 and exhibit a bi-FO-interpretation from directed hollow trees over σ through hollow trees over $\sigma \cup \{—, |\}$.

We give the first FO-interpretation $\mathcal{I}$ (from directed hollow trees to hollow trees), and leave the reverse one to the reader.

To avoid confusion in the notations, let's rename the directed binary relation E as F in the vocabulary of directed hollow trees: hence $\mathcal{I}$ goes from the vocabulary $\{F, S\} \cup P_\sigma$ to $\{E, S\} \cup P_{\sigma \cup \{—,|\}}$.

Given a σ directed hollow tree $\mathcal{T}$, $\mathcal{I}(\mathcal{T})$ is defined as follow:

- its domain is T, plus two new elements v_{xy} and v_{yx} for every $x, y \in T$ such that $\mathcal{T} \models F(x, y)$
- the interpretation of S is unchanged
- E is interpreted as the union of $\{(x, v_{xy}), (v_{xy}, x), (v_{xy}, v_{yx}), (v_{yx}, v_{xy}), (v_{yx}, y), (y, v_{yx})\}$ for every $x, y \in T$ such that $\mathcal{T} \models F(x, y)$
- the interpretation of every $P \in P_\sigma$ is unchanged
- $P_—$ is interpreted as $\{v_{xy} : x, y \in T, \mathcal{T} \models F(x, y)\}$
- $P_|$ is interpreted as $\{v_{yx} : x, y \in T, \mathcal{T} \models F(x, y)\}$

Intuitively, xFy has been transformed into the (symmetrical) gadget on four elements $xE—E|Ey$. This encoding allows the converse FO-interpretation to recover the orientation of the F-edges of $\mathcal{T}$ from $\mathcal{I}(\mathcal{T})$ in a straightforward way. ◀

B Proofs for Section 3 (Swaps and pumping)

▶ **Lemma 8.** *For all $\alpha \in \mathbb{N}$ there exists $s(\alpha) \in \mathbb{N}$ such that for all $m \geq s(\alpha)$, and every hollow $(m+1)$-quasitree $\mathcal{R}$,*

if $\mathcal{R}'$ is the m-guarded crossing-S-swap between a and b in $\mathcal{R}$,

then $\mathcal{R}' \equiv^{<\text{-inv FO}}_{\alpha} \mathcal{R}$, and $\forall x \in R$, $tp^{m+1}_{\mathcal{R}'}(x) = tp^{m+1}_{\mathcal{R}}(x)$. Moreover $\mathcal{R}' \in$ quasi-$\mathbb{H}^{m+1}_\sigma$ and $\mathcal{S}upp_{m+1}(\mathcal{R}') = \mathcal{S}upp_{m+1}(\mathcal{R})$.

Proof. We first show that $\mathcal{R}' \equiv_{\alpha}^{<\text{inv FO}} \mathcal{R}$. This is essentially a reduction to Lemma 6 using FO-interpretations. Recall the function o_p^{Σ} given by Lemma 6. We use it with $p = 2$ and $\Sigma := P_\sigma \cup \{E, S, P_{1/2}, P_{3/4}\}$ where $P_{1/2}$ and $P_{3/4}$ are unary. Assume now that $m \geq o_2^{\Sigma}(\alpha + c)$ where c is a constant to be chosen later on.

Consider the extension $\mathcal{R}^\star$ of $\mathcal{R}^-$ to Σ where the interpretation of $P_{1/2}$ is $\{a\}$ and that of $P_{3/4}$ is $\{b\}$. Since $P_{1/2}^{\mathcal{R}^\star}$ and $P_{3/4}^{\mathcal{R}^\star}$ are at distance $> m$ from a', a'', b' and b'' in $\mathcal{R}^\star$, we have that $\text{tp}_{\mathcal{R}^\star}^m(a', a'') = \text{tp}_{\mathcal{R}^\star}^m(b', b'')$.

We can therefore apply Lemma 6, and get two orders $<_{a'a''b'b''}$ and $<_{b'b''a'a''}$ such that $(\mathcal{R}^\star, <_{a'a''b'b''}) \equiv_{\alpha+c}^{\text{FO}} (\mathcal{R}^\star, <_{b'b''a'a''})$. Now, consider the FO-interpretation that adds a S-edge between u and v if either:

- $P_{1/2}(u)$ and v is the first or the second element of $<$
- $P_{3/4}(u)$ and v is the third or the fourth element of $<$

and then forgets about $P_{1/2}$ and $P_{3/4}$.

Take c to be the depth of this FO-interpretation (which has arity 1).

Note that the result of this FO-interpretation on $(\mathcal{R}^\star, <_{a'a''b'b''})$ is an ordered extension of $\mathcal{R}$ and its result on $(\mathcal{R}^\star, <_{b'b''a'a''})$ is an ordered extension of $\mathcal{R}'$.

This entails $\mathcal{R}' \equiv_{\alpha}^{<\text{inv FO}} \mathcal{R}$.

Now, let $x \in R$, and let's show that $\text{tp}_{\mathcal{R}'}^{m+1}(x) = \text{tp}_{\mathcal{R}}^{m+1}(x)$.

First, if x is at distance $> m + 1$ of $\{a, a', a'', b, b', b''\}$ in $\mathcal{R}$, there isn't any change in its $(m + 1)$-neighborhood.

Otherwise, there are several cases, according to whether x belongs to $\mathcal{N}_{\mathcal{R}^-}^{m+1}(a)$, $\mathcal{N}_{\mathcal{R}^-}^{m+1}(a')$, $\mathcal{N}_{\mathcal{R}^-}^{m+1}(a'')$, $\mathcal{N}_{\mathcal{R}^-}^{m+1}(b)$, $\mathcal{N}_{\mathcal{R}^-}^{m+1}(b')$, or $\mathcal{N}_{\mathcal{R}^-}^{m+1}(b')$; we treat the first one of them, the others being similar.

Set $d_a := \text{dist}_{\mathcal{R}^-}(x, a)$ and $d_b := \text{dist}_{\mathcal{R}^-}(x, b)$. By hypothesis, $d_a \leq m + 1$.

We distinguish two cases:

- if $d_b > m+1$: because of the distance constraint, we can partition $\mathcal{N}_{\mathcal{R}}^{m+1}(x)$ into $\mathcal{N}_{\mathcal{R}^-}^{m+1}(x)$ and $\mathcal{N}_{\mathcal{R}^-}^{m-d_a}(a', a'')$, with two S-edges joining a in the first and a', a'' in the second. These two parts are at distance ≥ 2 in R^-, hence they are fully independent (no overlap, and no edge between the two except for $S(a, a')$ and $S(a, a'')$).
  Likewise, we can partition $\mathcal{N}_{\mathcal{R}'}^{m+1}(x)$ into $\mathcal{N}_{\mathcal{R}^-}^{m+1}(x)$ and $\mathcal{N}_{\mathcal{R}^-}^{m-d_a}(b', b'')$.
  $\mathcal{N}_{\mathcal{R}^-}^{m-d_a}(a', a'') \simeq \mathcal{N}_{\mathcal{R}^-}^{m-d_a}(b', b'')$, hence $\mathcal{N}_{\mathcal{R}}^{m+1}(x) \simeq \mathcal{N}_{\mathcal{R}'}^{m+1}(x)$.
- if $d_b \leq m + 1$: now, we can partition $\mathcal{N}_{\mathcal{R}}^{m+1}(x)$ into $\mathcal{N}_{\mathcal{R}^-}^{m+1}(x)$, $\mathcal{N}_{\mathcal{R}^-}^{m-d_a}(a', a'')$ and $\mathcal{N}_{\mathcal{R}^-}^{m-d_b}(b', b'')$, with two S-edges joining a in the first to a', a'' in the second and two S-edges joining b in the first to b', b'' in the third, as depicted in Figure 15. These three parts are at distance ≥ 2 in $\mathcal{R}^-$, hence they are fully independent (no overlap, and no edge between the two except for $S(a, a'), S(a, a''), S(b, b')$ and $S(b, b'')$).
  Likewise, we can partition $\mathcal{N}_{\mathcal{R}'}^{m+1}(x)$ into $\mathcal{N}_{\mathcal{R}^-}^{m+1}(x)$, $\mathcal{N}_{\mathcal{R}^-}^{m-d_a}(b', b'')$ and $\mathcal{N}_{\mathcal{R}^-}^{m-d_b}(a', a'')$, as shown in Figure 15.
  $\mathcal{N}_{\mathcal{R}^-}^{m-d_a}(a', a'') \simeq \mathcal{N}_{\mathcal{R}^-}^{m-d_a}(b', b'')$ and $\mathcal{N}_{\mathcal{R}^-}^{m-d_b}(a', a'') \simeq \mathcal{N}_{\mathcal{R}^-}^{m-d_b}(b', b'')$, hence $\mathcal{N}_{\mathcal{R}}^{m+1}(x) \simeq \mathcal{N}_{\mathcal{R}'}^{m+1}(x)$.

◀

▶ **Lemma 9.** *For all $\alpha \in \mathbb{N}$ there exists $s(\alpha) \in \mathbb{N}$ such that for every $m \geq s(\alpha)$ and every hollow $(m + 1)$-quasitree $\mathcal{R}$, if $\mathcal{R}'$ is either*

- *the m-guarded crossing-E-swap between ab and $a'b'$ in $\mathcal{R}$*
- *the m-guarded mirror-E-swap at $[b, b']$ in $\mathcal{R}$*
- *the m-guarded contiguous-segment-E-swap between $[b, a']$ and $[b', a'']$ in $\mathcal{R}$*

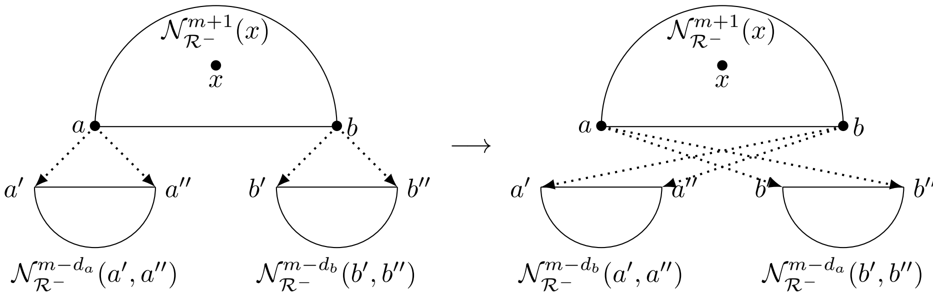

Figure 14 Evolution of the neighborhood of x before and after a crossing-S-swap in the proof of Lemma 8. We see $\mathcal{N}^{m+1}_{\mathcal{R}}(x)$ on the left and $\mathcal{N}^{m+1}_{\mathcal{R}'}(x)$ on the right.

- *the m-guarded segment-E-swap between $[b, c]$ and $[b', c']$ in $\mathcal{R}$*

then $\mathcal{R}' \equiv_{\alpha}^{<\text{-inv FO}} \mathcal{R}$, $\forall x \in R$, $tp^{m+1}_{\mathcal{R}'}(x) = tp^{m+1}_{\mathcal{R}}(x)$ and $\mathcal{R}' \in$ quasi-$\mathbb{H}^{m+1}_{\sigma}$.

▶ Note. In each of these cases, the swap doesn't introduce any E-loop. Hence, once we've shown that every elements keeps its $(m+1)$-type, we immediately get that $\mathcal{R}' \in$ quasi-$\mathbb{H}^{m+1}_{\sigma}$.

However, these operations do not preserve hollowtreeness. This is the reason why we introduced the notion of quasitree.

We prove Lemma 9 separately for every type of E-swap. We will need the following lemmas:

▶ **Lemma 19.** *Let $\mathcal{Q} \in$ quasi-$\mathbb{H}^{m+1}_{\sigma}$ and $x, y \in Q$ such that the sets $\{x\}, \{y\}$ and $End(\mathcal{Q})$ are pairwise $(2m+3)$-distant in $\mathcal{S}upp_0(\mathcal{Q})$. Then $dist_{\mathcal{Q}}(x, y) \geq 2m+3$.*

Proof. This is a consequence of the fact that in a hollow $(m+1)$-quasitree, a thread of length less than $2m+1$ must have matching endpoints.

Suppose that there exists path of length $\leq 2m+2$ in $\mathcal{Q}$ from x to y, and let p be such a path of shortest length.

The path p may use either an E-edge or an S-edges. We divide p into segments between two consecutive S-edges. Let $t_1, \cdots, t_r$ be the corresponding threads in that order (with possible repetitions).

We know that $x \in t_1$ and $y \in t_r$. As x and y are at distance at least $2m+3$ when using only E-edges, we must have $r \geq 2$.

There are two ways for p to go from t_i to t_{i+1}: either (1) using and S-edge $S(a, e_{i+1})$ with $a \in t_i$ and $e_{i+1} \in \text{End}(t_{i+1})$, or (2) using an S-edge $S(b, e_i)$ with $b \in t_{i+1}$ and $e_i \in \text{End}(t_i)$.

As x is far from the endpoints of t_1, p must go from t_1 to t_2 using case (1). Similarly, p must go from t_{r-1} to t_r using case (2). Hence, there must exist some $1 < i < r$ such that p moves from t_{i-1} to t_i following (1), and from t_i to t_{i+1} following (2).

Since p is a shortest path, the two endpoints of t_i involved in (1) and (2) cannot be the same; hence p goes from one endpoint of t_i to the other and the length of t_i must be $\leq 2m+1$. Since $Q \in$ quasi-$\mathbb{H}^{m+1}_{\sigma}$, branches of length $\leq 2m+1$ must have matching endpoints. This contradicts the minimality of p, since p could have avoided t_i completely. ◀

▶ **Lemma 20.** *Let $\mathcal{Q} \in$ quasi-$\mathbb{H}^{m+1}_{\sigma}$ and $x \neq y \in Q$ belonging to the same thread, such that $\{x, y\}$ and $End(\mathcal{Q})$ are $(2m+3)$-distant in $\mathcal{S}upp_0(\mathcal{Q})$.*

Every path of length $< 2m+3$ between x and y goes through every E-edge of $[x, y]$.

In other words, if $\mathcal{Q}^-$ is $\mathcal{Q}$ minus any E-edge of $[x, y]$, $\text{dist}_{\mathcal{Q}^-}(x, y) \geq 2m+3$.

Proof. The proof is identical as the one of Lemma 19, by considering a shortest path of length $\leq 2m+2$ from x to y that doesn't go through every E-edge of $[x, y]$: we arrive at the same contradiction. ◀

The following lemma will only be needed in Section 5; however, we state it here as its proof is very similar to the previous ones.

▶ **Lemma 21.** *Let $\mathcal{Q} \in$ quasi-$\mathbb{H}_\sigma^{m+1}$, and $x, y \in End(\mathcal{Q})$ such that $dist_{\mathcal{S}upp_0(\mathcal{Q})}(x, y) \geq 2m+3$. Then any path of length $\leq 2m+3$ from x to y goes through at least one of their S-parents.*

Proof. We proceed similarly as in Lemma 19; let's use the same notations.

Suppose that p doesn't go through x's parent neither y's. Let's show that p goes from t_1 to t_2 using case (1): if not, it uses case (2) through the other endpoint of t_1. Hence, t_1 would be of length $\leq 2m+1$ (since it takes at least 2 to reach y from there), and would have matching endpoints; this is absurd, since p would go through x's parent.

Similarly, we show that p moves from t_{r-1} to t_r following (2).

We can conclude exactly as in Lemma 19. ◀

Proof of Lemma 9 for crossing-E-swaps

We let $\mathcal{R}^- := \mathcal{R} \setminus \{E(a, b), E(a', b')\}$.

It follows from Lemma 19 and Lemma 20 that a, b, a' and b' are at distance at least $2m+3$ from each other in $\mathcal{R}^-$.

First, we show that $\mathcal{R}' \equiv_\alpha^{<\text{-inv FO}} \mathcal{R}$. This is essentially a reduction to Lemma 6 using FO-interpretations. Recall the function o_p^Σ given by Lemma 6. We use it with $p = 1$ and $\Sigma := P_\sigma \cup \{E, S, P_1, P_2\}$ where P_1 and P_2 are unary. Assume now that $m \geq o_1^\Sigma(\alpha + c)$ where c is a constant to be chosen later on.

Consider the extension $\mathcal{R}^\star$ of $\mathcal{R}^-$ to Σ where the interpretation of P_1 is $\{b\}$ and that of $\mathcal{P}_2$ is $\{b'\}$. Since $P_1^{\mathcal{R}^\star}$ and $P_2^{\mathcal{R}^\star}$ are at distance $> m$ from a and a' in $\mathcal{R}^\star$, we have that $\text{tp}_{\mathcal{R}^\star}^m(a) = \text{tp}_{\mathcal{R}^\star}^m(a')$.

We can therefore apply Lemma 6, and get two orders $<_{aa'}$ and $<_{a'a}$ such that $(\mathcal{R}^\star, <_{aa'}) \equiv_{\alpha+c}^{\text{FO}} (\mathcal{R}^\star, <_{a'a})$. Now, consider the FO-interpretation that adds a (symmetrical) E-edge between u and v if either:

- $P_1(u)$ and v is the first element of $<$
- $P_2(u)$ and v is the second element of $<$

and then forgets about P_1 and P_2.

Take c to be the depth of this FO-interpretation (which has arity 1). Note that the result of this FO-interpretation on $(\mathcal{R}^\star, <_{aa'})$ is an ordered extension of $\mathcal{R}$ and that its result on $(\mathcal{R}^\star, <_{a'a})$ is an ordered extension of $\mathcal{R}'$. This entails $\mathcal{R}' \equiv_\alpha^{<\text{-inv FO}} \mathcal{R}$.

Now, let $x \in R$, and let's show that $\text{tp}_{\mathcal{R}'}^{m+1}(x) = \text{tp}_{\mathcal{R}}^{m+1}(x)$.

First, if x is at distance $> m+1$ of $\{a, b, a', b'\}$ in $\mathcal{R}$, there isn't any change in its $(m+1)$-neighborhood.

Otherwise, there are several cases, according to whether x belongs to $\mathcal{N}_{\mathcal{R}^-}^{m+1}(a)$, $\mathcal{N}_{\mathcal{R}^-}^{m+1}(a')$, $\mathcal{N}_{\mathcal{R}^-}^{m+1}(b)$ or $\mathcal{N}_{\mathcal{R}^-}^{m+1}(b')$; we treat the first one, the others being similar.

Set $d := \text{dist}_{\mathcal{R}}(x, a)$.

We can partition $\mathcal{N}_{\mathcal{R}}^{m+1}(x)$ into $\mathcal{N}_{\mathcal{R}^-}^{m+1}(x)$ and $\mathcal{N}_{\mathcal{R}^-}^{m-d}(b)$, with an E-edge joining a in the first and b in the second.

Because of the distance condition, these two parts are at distance ≥ 2 in $\mathcal{R}^-$, hence they are fully independent (no overlap, and no edge between the two except for $E(a, b)$).

Likewise, we can partition $\mathcal{N}_{\mathcal{R}'}^{m+1}(x)$ into $\mathcal{N}_{\mathcal{R}^-}^{m+1}(x)$ and $\mathcal{N}_{\mathcal{R}^-}^{m-d}(b')$.

$\mathcal{N}^{m-d}_{\mathcal{R}^-}(b) \simeq \mathcal{N}^{m-d}_{\mathcal{R}^-}(b')$, hence $\mathcal{N}^{m+1}_{\mathcal{R}}(x) \simeq \mathcal{N}^{m+1}_{\mathcal{R}'}(x)$.

Proof of Lemma 9 for mirror-E-swaps

Let $\mathcal{R}^- := \mathcal{R} \setminus \{E(a,b), E(a',b')\}$. It follows from Lemma 20 that the three sets $\{a\}, \{a'\}$ and $\{b,b'\}$ are $(2m+3)$-distant in $\mathcal{R}^-$.

The proof that $\mathcal{R}' \equiv^{<\text{-inv FO}}_{\alpha} \mathcal{R}$ is done exactly as in the case of crossing-E-swaps.

Now, let $x \in R$, and let's show that $\text{tp}^{m+1}_{\mathcal{R}'}(x) = \text{tp}^{m+1}_{\mathcal{R}}(x)$.

First, if x is at distance $> m+1$ of $\{a,b,a',b'\}$ in $\mathcal{R}$, there isn't any change in its $(m+1)$-neighborhood.

Otherwise, there are several cases, according to whether x belongs to $\mathcal{N}^{m+1}_{\mathcal{R}^-}(a)$, $\mathcal{N}^{m+1}_{\mathcal{R}^-}(a')$, $\mathcal{N}^{m+1}_{\mathcal{R}^-}(b)$ or $\mathcal{N}^{m+1}_{\mathcal{R}^-}(b')$; the first two are similar to the cases appearing in the proof for crossing-E-swaps.

We treat the third one, the fourth being symmetrical.

Set $d := \text{dist}_{\mathcal{R}^-}(x,b)$ and $d' := \text{dist}_{\mathcal{R}^-}(x,b')$. By hypothesis, $d \leq m+1$.

$\mathcal{N}^{m+1}_{\mathcal{R}}(x)$ can partitioned into $\mathcal{N}^{m+1}_{\mathcal{R}^-}(x)$, $\mathcal{N}^{m-d}_{\mathcal{R}^-}(a)$ and the possibly empty $\mathcal{N}^{m-d'}_{\mathcal{R}^-}(a')$, with an E-edge joining b in the first to a in the second, and an E-edge joining b' in the first to a' in the third (if it is nonempty).

We claim that any two of these three neighborhoods are at distance ≥ 2 in $\mathcal{R}^-$, hence they are fully independent: no overlap, and no edge between any two of them, except (possibly) for $E(a,b)$ and (possibly) $E(a',b')$.

Indeed, suppose (the other pairs of neighborhoods are treated similarly) that $\mathcal{N}^{m+1}_{\mathcal{R}^-}(x)$ and $\mathcal{N}^{m-d}_{\mathcal{R}^-}(a)$ are at distance ≤ 1. Then $\text{dist}_{\mathcal{R}^-}(a,x) \leq 2m+2-d$, hence $\text{dist}_{\mathcal{R}^-}(a,b) \leq 2m+2$, which contradicts Lemma 20 for a and b (recall that $\{a,b\}$ and $\text{End}(\mathcal{R})$ are $(2m+3)$-distant in $\mathcal{S}\text{upp}_0(\mathcal{R})$).

Likewise, we can partition $\mathcal{N}^{m+1}_{\mathcal{R}'}(x)$ into $\mathcal{N}^{m+1}_{\mathcal{R}^-}(x)$, $\mathcal{N}^{m-d}_{\mathcal{R}^-}(a')$ and $\mathcal{N}^{m-d'}_{\mathcal{R}^-}(a)$.

$\mathcal{N}^{m-d}_{\mathcal{R}^-}(a) \simeq \mathcal{N}^{m-d}_{\mathcal{R}^-}(a')$ and $\mathcal{N}^{m-d'}_{\mathcal{R}^-}(a') \simeq \mathcal{N}^{m-d'}_{\mathcal{R}^-}(a)$, hence $\mathcal{N}^{m+1}_{\mathcal{R}}(x) \simeq \mathcal{N}^{m+1}_{\mathcal{R}'}(x)$.

Proof of Lemma 9 for contiguous-segment-E-swaps

Let $\mathcal{R}^- := \mathcal{R} \setminus \{E(a,b), E(a',b'), E(a'',b'')\}$.

Let x, y be non-endpoint elements of the same thread of some $\mathcal{Q} \in \text{quasi-}\mathbb{H}^{n+1}_{\sigma}$. Let $\mathcal{Q}^- := \mathcal{Q} \setminus \{E(x',x), E(y,y')\}$, where x' (resp. y') is the E-neighbor of x (resp. y) that doesn't belong to $[x,y]$.

We denote by $[x,y]^{\mathcal{Q}}_n$ the substructure of $\mathcal{Q}^-$ induced by the set of nodes at distance $\leq n$ in $\mathcal{Q}^-$ from $[x,y]$, together with a new color marking x as the left endpoint.

We define concatenation as follows: if x, x_1, y_1, y appear in the same thread in that order, and $E(x_1,y_1)$, then we write $[x,y]^{\mathcal{Q}}_n =: [x,x_1]^{\mathcal{Q}}_n \cdot [y_1,y]^{\mathcal{Q}}_n$.

Let us abbreviate $\text{dist}_{\mathcal{S}\text{upp}_0(\mathcal{Q})}(x,y)$ as $|[x,y]|$ (that is, the distance from x to y if we are only allowed E-edges).

First, let's prove that $(m+1)$-types are unchanged by a m-guarded contiguous-segment-E-swap.

Let $x \in R$, and let's show that $\text{tp}^{m+1}_{\mathcal{R}'}(x) = \text{tp}^{m+1}_{\mathcal{R}}(x)$

If x is at distance $> m+1$ of $\{a,b,a',b',a'',b''\}$ in $\mathcal{R}$, there isn't any change in its $(m+1)$-neighborhood.

Otherwise, there are several cases, according to whether x belongs to $\mathcal{N}^{m+1}_{\mathcal{R}^-}(a)$, $\mathcal{N}^{m+1}_{\mathcal{R}^-}(b)$, $\mathcal{N}^{m+1}_{\mathcal{R}^-}(a')$, $\mathcal{N}^{m+1}_{\mathcal{R}^-}(b')$, $\mathcal{N}^{m+1}_{\mathcal{R}^-}(a'')$ or $\mathcal{N}^{m+1}_{\mathcal{R}^-}(b'')$. We treat the second one, the other ones being similar or simpler.

Set $d_b := \mathrm{dist}_{\mathcal{R}^-}(x,b)$, $d_{a'} := \mathrm{dist}_{\mathcal{R}^-}(x,a')$, and $d_{b'a''} := \mathrm{dist}_{\mathcal{R}^-}(b',a'')$. By hypothesis, $d_b \leq m+1$.

We can partition $\mathcal{N}^{m+1}_{\mathcal{R}}(x)$ into $\mathcal{N}^{m+1}_{\mathcal{R}^-}(x)$, $\mathcal{N}^{m-d_b}_{\mathcal{R}^-}(a)$, and the possibly empty $\mathcal{N}^{m-d_{a'}}_{\mathcal{R}^-}(b')$ and $\mathcal{N}^{m-1-d_{a'}-d_{b'a''}}_{\mathcal{R}^-}(b'')$, with an E-edge joining b in the first to a in the second, an E-edge joining a' in the first to b' in the third, and an E-edge joining a'' in the third to b'' in the fourth (in the case they are non-empty).

We claim that any two of these four neighborhoods are at distance ≥ 2 in $\mathcal{R}^-$, hence they are fully independent: no overlap, and no edge between any two of them except (possibly) for $E(a,b)$, $E(a',b')$ and $E(a'',b'')$.

Indeed, suppose (the other pairs of neighborhoods are treated similarly) that $\mathcal{N}^{m+1}_{\mathcal{R}^-}(x)$ and $\mathcal{N}^{m-d_b}_{\mathcal{R}^-}(a)$ are at distance ≤ 1. Then $\mathrm{dist}_{\mathcal{R}^-}(a,x) \leq 2m+2-d_b$, hence $\mathrm{dist}_{\mathcal{R}^-}(a,b) \leq 2m+2$, which contradicts Lemma 20 for a and b (recall that $\{a,b\}$ and $\mathrm{End}(\mathcal{R})$ are $(2m+3)$-distant in $\mathcal{S}\mathrm{upp}_0(\mathcal{R})$).

Likewise, we can partition $\mathcal{N}^{m+1}_{\mathcal{R}'}(x)$ into $\mathcal{N}^{m+1}_{\mathcal{R}^-}(x)$, $\mathcal{N}^{m-d_b}_{\mathcal{R}^-}(a'')$, $\mathcal{N}^{m-1-d_b-d_{b'a''}}_{\mathcal{R}^-}(a)$ and $\mathcal{N}^{m-d_{a'}}_{\mathcal{R}^-}(b'')$.

Because $\mathrm{tp}^m_{\mathcal{R}}(a,b) = \mathrm{tp}^m_{\mathcal{R}}(a',b') = \mathrm{tp}^m_{\mathcal{R}}(a'',b'')$, $\mathcal{N}^{m-d_b}_{\mathcal{R}^-}(a)$ is isomorphic to the union of $\mathcal{N}^{m-d_b}_{\mathcal{R}^-}(a'')$ and $\mathcal{N}^{m-1-d_b-d_{b'a''}}_{\mathcal{R}^-}(a)$ with an E-edge joining b' in the first and a in the second (if they are both nonempty).

Similarly, the union of $\mathcal{N}^{m-d_{a'}}_{\mathcal{R}^-}(b')$ and $\mathcal{N}^{m-1-d_{a'}-d_{b'a''}}_{\mathcal{R}^-}(b'')$ with an E-edge joining a'' in the first and b'' in the second (if they are both nonempty) is isomorphic to $\mathcal{N}^{m-d_{a'}}_{\mathcal{R}^-}(b'')$.

Hence $\mathcal{N}^{m+1}_{\mathcal{R}}(x) \simeq \mathcal{N}^{m+1}_{\mathcal{R}'}(x)$.

Now, let's exhibit a $s(\alpha)$ such that for every $m \geq s(\alpha)$, m-guarded contiguous-segment-E-swaps preserve $\equiv^{<\text{-inv FO}}_{\alpha}$.

We will first set $N \in \mathbb{N}$ instead of $s(\alpha)$, that will be sufficient for most cases. Then, we will define $s(\alpha) \geq N$ which will work for all cases.

Recall the function o^{Σ}_p needed for Lemma 6, and consider $n := o^{\Sigma}_2(\alpha + c)$ where c is to be chosen later on, and $\Sigma := P_\sigma \cup \{E, S, P_1, P_4\}$ where P_1 and P_4 are unary. We distinguish between several cases depending on whether a, a' and a'' are close or not, where "close" is relative to n:

1. Assume first that $\mathrm{tp}^n_{\mathcal{R}^-}(b,a') = \mathrm{tp}^n_{\mathcal{R}^-}(b',a'')$.
   This case covers the instances where $[b,a']^{\mathcal{R}}_n \simeq [b',a'']^{\mathcal{R}}_n$, as well as those where $|[a,a']|$ and $|[a',a'']|$ are both $> 2n+2$.
   Consider the extension $\mathcal{R}^\star$ of $\mathcal{R}^-$ to Σ where $P_1^{\mathcal{R}^\star} := \{a\}$ and $P_4^{\mathcal{R}^\star} := \{b''\}$. Since $P_1^{\mathcal{R}^\star}$ and $P_4^{\mathcal{R}^\star}$ are at distance $> n$ from $\{b, a', b', a''\}$ (this is guaranteed by Lemma 20, because we will make sure that $s(\alpha) \geq n$), $\mathrm{tp}^n_{\mathcal{R}^\star}(b,a') = \mathrm{tp}^n_{\mathcal{R}^\star}(b',a'')$.
   Hence, we can apply Lemma 6, and get two orders $<_{ba'b'a''}$ and $<_{b'a''ba'}$ such that $(\mathcal{R}^\star, <_{ba'b'a''}) \equiv^{\mathrm{FO}}_{\alpha+c} (\mathcal{R}^\star, <_{b'a''ba'})$.
   Now, consider the FO-interpretation that adds a symmetrical E-edge between u and v if either:

   - $P_1(u)$ and v is the first element of $<$
   - u is the second element of $<$ and v is either its third one
   - u is the fourth element of $<$ and $P_4(v)$

   and then forgets about P_1 and P_4.
   Take c to be the depth of this FO-interpretation (which has arity 1).

Note that the result of this FO-interpretation on $(\mathcal{R}^\star, <_{ba'b'a''})$ is an ordered extension of $\mathcal{R}$ and that its result on $(\mathcal{R}^\star, <_{b'a''ba'})$ is an ordered extension of $\mathcal{R}'$.
This entails $\mathcal{R}' \equiv_\alpha^{\text{<-inv FO}} \mathcal{R}$.

2. Assume next that $[b', a'']_n^{\mathcal{R}}$ can be decomposed as $[b', a_1]_n^{\mathcal{R}} \cdot [b_1, a_2]_n^{\mathcal{R}} \cdots [b_k, a'']_n^{\mathcal{R}}$, where each of these enriched segments is isomorphic to $[b, a']_n^{\mathcal{R}}$.
We can then apply $k+1$ times Case 1 and obtain $\mathcal{R}' \equiv_\alpha^{\text{<-inv FO}} \mathcal{R}$ as desired.
3. From now on, $N \geq l(2n+2)+n$ for a large enough l to be chosen later on.
As we are not in Case 1, we can restrict our study to the cases where $|[a, a']| \leq 2n+2$ (the cases where $|[a', a'']| \leq 2n+2$ can be treated similarly).
We will need the following claim, which is based on the Lyndon-Schützenberger Theorem.
▶ **Claim 22.** *Let $n \in \mathbb{N}$ and $\mathcal{R} \in$ quasi-$\mathbb{H}_\sigma^{n+1}$.*
Let $a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4$ appear in that order in a single thread of $\mathcal{R}$, such that $E(a_1, b_1), E(a_2, b_2), E(a_3, b_3)$ and $E(a_4, b_4)$.
Suppose that $[b_1, a_2]_n^{\mathcal{R}} \simeq [b_3, a_4]_n^{\mathcal{R}}$ and $[b_1, a_3]_n^{\mathcal{R}} \simeq [b_2, a_4]_n^{\mathcal{R}}$.
Then there exist decompositions $[b_1, a_2]_n^{\mathcal{R}} = \mathcal{U}_1 \cdots \mathcal{U}_p$, $[b_2, a_3]_n^{\mathcal{R}} = \mathcal{V}_1 \cdots \mathcal{V}_q$, and $[b_3, a_4]_n^{\mathcal{R}} = \mathcal{W}_1 \cdots \mathcal{W}_p$, where all the $\mathcal{U}_i, \mathcal{V}_i$ and $\mathcal{W}_i$ are isomorphic.

Proof. Consider Θ_n which maps $[x, y]_n^{\mathcal{R}}$ to the word $[x, y]$ where each element is colored with its n-type in $[x, y]_n^{\mathcal{R}}$.
Let $u := \Theta_n([b_1, a_2]_n^{\mathcal{R}})$, $v := \Theta_n([b_2, a_3]_n^{\mathcal{R}})$ and $w := \Theta_n([b_3, a_4]_n^{\mathcal{R}})$.
The hypothesis guarantee $u = w$ and $uv = vw$. Hence $uv = vu$.
By Lyndon-Schützenberger Theorem, there must exist a word a and integers p, q such that $u = w = a^p$ and $v = a^q$ [12].
We can decompose $[b_1, a_2]_n^{\mathcal{R}}, [b_2, a_3]_n^{\mathcal{R}}$ and $[b_3, a_4]_n^{\mathcal{R}}$ alongside those decompositions of u, v and w, to get $[b_1, a_2]_n^{\mathcal{R}} = \mathcal{U}_1 \cdots \mathcal{U}_p$, $[b_2, a_3]_n^{\mathcal{R}} = \mathcal{V}_1 \cdots \mathcal{V}_q$, and $[b_3, a_4]_n^{\mathcal{R}} = \mathcal{W}_1 \cdots \mathcal{W}_p$, where all the $\mathcal{U}_i, \mathcal{V}_i$ and $\mathcal{W}_i$ are mapped to a by Θ_n, hence are isomorphic. ◁

Let ϕ be an isomorphism between the N-neighborhood of (a, b) and that of (a', b').
As $|[a, a']| \leq 2n+2$, a' and b' are in the N-neighborhood of (a, b): set $x_0 := a'$ and $y_0 := b'$.
Construct by induction $x_{i+1} := \phi(x_i)$ and $y_{i+1} := \phi(y_i)$ until $i > l$. Our choice of N ensures that x_i and y_i are well defined as x_{i-1} and y_{i-1} remain in the N-neighborhood of (a, b). For all $j \leq l$, $\mathcal{X}_j := [y_{j-1}, x_j]_n^{\mathcal{R}}$ is isomorphic to $[b, a']_n^{\mathcal{R}}$.
Likewise, starting from (a'', b'') instead of (a', b'), we show that there exist $x'_1, y'_1, \cdots, x'_l, y'_l$ such that for $j \in [1, l]$ (and with the convention that $x'_0 = a''$), $\mathcal{X}'_j := [y'_j, x'_{j-1}]_n^{\mathcal{R}}$ is isomorphic to $[b, a']_n^{\mathcal{R}}$.
We distinguish several cases:

a. Suppose that $|[a', a'']| \geq 2N$. This ensures that all the $(x_i)_{i\geq 1}$, $(y_i)_{i\geq 0}$, $(x'_i)_{i\geq 0}$ and $(y'_i)_{i\geq 1}$ belong to $[b', a'']$.
We can decompose $[b, a'']_n^{\mathcal{R}}$ as

$$[b, a']_n^{\mathcal{R}} \cdot \underbrace{[y_0, x_1]_n^{\mathcal{R}}}_{\mathcal{X}_1} \cdots \underbrace{[y_{l-1}, x_l]_n^{\mathcal{R}}}_{\mathcal{X}_l} \cdot [y_l, x'_l]_n^{\mathcal{R}} \cdot \underbrace{[y'_l, x'_{l-1}]_n^{\mathcal{R}}}_{\mathcal{X}'_l} \cdot \underbrace{[y'_{l-1}, x'_{l-2}]_n^{\mathcal{R}}}_{\mathcal{X}'_{l-1}} \cdots \underbrace{[y'_1, x'_0]_n^{\mathcal{R}}}_{\mathcal{X}'_1}$$

Let $\mathcal{R}_1$ be the n-guarded contiguous-segment-E-swap between $[b, x'_{l-1}]$ and $[y'_{l-1}, a'']$ in $\mathcal{R}$ (recall that $a'' = x'_0$).
If l is chosen large enough, namely $l \geq 2n+4$, this swap falls in Case 1 of this Lemma and therefore $\mathcal{R}_1 \equiv_\alpha^{\text{<-inv FO}} \mathcal{R}$.

In $\mathcal{R}_1$, $[y'_{l-1}, x'_{l-1}]_n^{\mathcal{R}_1}$ (that is, the segment strictly between a and b'') is decomposed as

$$\underbrace{[y'_{l-1}, x'_{l-2}]_n^{\mathcal{R}}}_{\mathcal{X}'_{l-1}} \cdots \underbrace{[y'_1, x'_0]_n^{\mathcal{R}}}_{\mathcal{X}'_1} \cdot [b, a']_n^{\mathcal{R}} \cdot \underbrace{[y_0, x_1]_n^{\mathcal{R}}}_{\mathcal{X}_1} \cdots \underbrace{[y_{l-1}, x_l]_n^{\mathcal{R}}}_{\mathcal{X}_l} \cdot [y_l, x'_l]_n^{\mathcal{R}} \cdot \underbrace{[y'_l, x'_{l-1}]_n^{\mathcal{R}}}_{\mathcal{X}'_l}$$

Now, let $\mathcal{R}_2$ be the n-guarded contiguous-segment-E-swap between $[y'_{l-1}, a']$ and $[y_0, x'_{l-1}]$ in $\mathcal{R}_1$.
By choice of l, this swap falls again in Case 1 of this Lemma. Thus, $\mathcal{R}_2 \equiv_\alpha^{<\text{-inv FO}} \mathcal{R}_1$.
Observe now that in $\mathcal{R}_2$, $[b, a'']_n^{\mathcal{R}_2}$ (the segment strictly between a and b'') is decomposed as

$$\underbrace{[y_0, x_1]_n^{\mathcal{R}}}_{\mathcal{X}_1} \cdots \underbrace{[y_{l-1}, x_l]_n^{\mathcal{R}}}_{\mathcal{X}_l} \cdot [y_l, x'_l]_n^{\mathcal{R}} \cdot \underbrace{[y'_l, x'_{l-1}]_n^{\mathcal{R}}}_{\mathcal{X}'_l} \cdot \underbrace{[y'_{l-1}, x'_{l-2}]_n^{\mathcal{R}}}_{\mathcal{X}'_{l-1}} \cdots \underbrace{[y'_1, x'_0]_n^{\mathcal{R}}}_{\mathcal{X}'_1} \cdot [b, a']_n^{\mathcal{R}}$$

that is, $[b', a'']_n^{\mathcal{R}} \cdot [b, a']_n^{\mathcal{R}}$. Hence $\mathcal{R}_2 = \mathcal{R}'$, and we get $\mathcal{R}' \equiv_\alpha^{<\text{-inv FO}} \mathcal{R}$.

b. Suppose now that $|[a', a'']| < 2N$. Set $s(\alpha) := (2N+1)(2n+2)+n$. Just as before (by replacing l with $2N+1$), we define $x_0, y_0, \cdots, x_{2N+1}, y_{2N+1}$ and $x'_0, y'_0, \cdots, x'_{2N+1}, y'_{2N+1}$, and accordingly, $\mathcal{X}_1, \cdots, \mathcal{X}_{2N+1}$ and $\mathcal{X}'_1, \cdots, \mathcal{X}'_{2N+1}$ that all are isomorphic to $[b, a']_n^{\mathcal{R}}$.
Not all of the $(x'_i)_{0 \le i \le 2N}$ can be in $[b', a'']$. Let k be the smallest index such that $x'_i \notin [b', a'']$ (we know that $1 \le k \le 2N$).
If $x'_k = a'$, we can conclude using Case 2.
Otherwise, $a, b, x'_k, y'_k, a', b', x'_{k-1}, y'_{k-1}$ must appear in that order in the thread.
$[b, a']_n^{\mathcal{R}} \simeq [y'_k, x'_{k-1}]_n^{\mathcal{R}}$ by definition.
To see that $[b, x'_k]_n^{\mathcal{R}} \simeq [b', x'_{k-1}]_n^{\mathcal{R}}$, consider the restriction of an isomorphism between $\mathcal{X}'_k$ and $\mathcal{X}'_{k+1}$ to the final segments of length $|[b', x'_{k-1}]| = |[b, x'_k]|$.
We can now apply Claim 22, and get decompositions $[b, x'_k]_n^{\mathcal{R}} = \mathcal{U}_1 \cdots \mathcal{U}_p$, $[y'_k, a']_n^{\mathcal{R}} = \mathcal{V}_1 \cdots \mathcal{V}_q$, and $[b', x'_{k-1}]_n^{\mathcal{R}} = \mathcal{W}_1 \cdots \mathcal{W}_p$, where all the $\mathcal{U}_i, \mathcal{V}_i$ and $\mathcal{W}_i$ are isomorphic.
Hence, $[b, a']_n^{\mathcal{R}}$ can be decomposed as $\mathcal{U}_1 \cdots \mathcal{U}_p \cdot \mathcal{V}_1 \cdots \mathcal{V}_q$, and such a decomposition can be transposed onto each $\mathcal{X}'_i, 0 < i < k$, as $\mathcal{X}'_i = \mathcal{Y}_1^i \cdots \mathcal{Y}_{p+q}^i$, where all the $\mathcal{Y}_j^i$, the $\mathcal{U}_i$, the $\mathcal{V}_i$ and the $\mathcal{W}_i$ are isomorphic.
We can now decompose $[b, a'']_n^{\mathcal{R}}$ as

$$\underbrace{\mathcal{U}_1 \cdots \mathcal{U}_p \cdot \mathcal{V}_1 \cdots \mathcal{V}_q}_{[b,a']_n^{\mathcal{R}}} \cdot \underbrace{\underbrace{\mathcal{W}_1 \cdots \mathcal{W}_p}_{[b',x'_{k-1}]_n^{\mathcal{R}}} \cdot \underbrace{\mathcal{Y}_1^{k-1} \cdots \mathcal{Y}_{p+q}^{k-1}}_{\mathcal{X}'_{k-1}} \cdots \underbrace{\mathcal{Y}_1^1 \cdots \mathcal{Y}_{p+q}^1}_{\mathcal{X}'_1}}_{[b',a'']_n^{\mathcal{R}}}$$

Now, we can use Case 2 to swap $\mathcal{V}_q$ with $[b', a'']_n^{\mathcal{R}}$: $\mathcal{R}_1 \equiv_\alpha^{<\text{-inv FO}} \mathcal{R}$, where in $\mathcal{R}_1$, the segment strictly between a and b'' is

$$\mathcal{U}_1 \cdots \mathcal{U}_p \cdot \mathcal{V}_1 \cdots \mathcal{V}_{q-1} \cdot \mathcal{W}_1 \cdots \mathcal{W}_p \cdot \mathcal{Y}_1^{k-1} \cdots \mathcal{Y}_{p+q}^{k-1} \cdots \mathcal{Y}_1^1 \cdots \mathcal{Y}_{p+q}^1 \cdot \mathcal{V}_q$$

Repeating this operation $p+q-1$ times allows us to conclude that $\mathcal{R}' \equiv_\alpha^{<\text{-inv FO}} \mathcal{R}$.

Proof of Lemma 9 for segment-E-swaps

The proof that $\forall x \in R, \mathrm{tp}^{m+1}_{\mathcal{R}'}(x) = \mathrm{tp}^{m+1}_{\mathcal{R}'}(x)$ is done as for the contiguous-segment-E-swaps.

Let's now find $s(\alpha) \in \mathbb{N}$ that guarantees the $\equiv^{<\text{-inv FO}}_{\alpha}$ invariance of any m-guarded segment-E-swaps, for $m \geq s(\alpha)$.

Let $n := o^{\Sigma}_{2}(\alpha + c)$ where c is the depth of some FO-interpretation to be specified later on, and $\Sigma := P_\sigma \cup \{E, S, P_1, P_2, P_3, P_4\}$ where P_1, P_2, P_3 and P_4 are unary.

Let $\mathcal{R}^- := \mathcal{R} \setminus \{E(a,b), E(a',b'), E(c,d), E(c',d')\}$.

1. Assume first that $\mathrm{tp}^n_{\mathcal{R}^-}(b,c) = \mathrm{tp}^n_{\mathcal{R}^-}(b',c')$.
   This case covers the instances where $[b,c]^{\mathcal{R}}_n \simeq [b',c']^{\mathcal{R}}_n$, as well as those where $|[a,c]|$ and $|[a',c']|$ both are $> 2n+2$.
   Consider the extension $\mathcal{R}^\star$ of $\mathcal{R}^-$ to Σ where $P_1^{\mathcal{R}^\star} := \{a\}$, $P_2^{\mathcal{R}^\star} := \{d\}$, $P_3^{\mathcal{R}^\star} := \{a'\}$ and $P_4^{\mathcal{R}^\star} := \{d'\}$. Since $P_1^{\mathcal{R}^\star}, P_2^{\mathcal{R}^\star}, P_3^{\mathcal{R}^\star}$ and $P_4^{\mathcal{R}^\star}$ are at distance $> n$ from $\{b, c, b', c'\}$ (this is guaranteed by Lemma 20, because we will make sure that $s(\alpha) \geq n$), $\mathrm{tp}^n_{\mathcal{R}^\star}(b,c) = \mathrm{tp}^n_{\mathcal{R}^\star}(b',c')$.
   Hence, we can apply Lemma 6, and get two orders $<_{bcb'c'}$ and $<_{b'c'bc}$ such that $(\mathcal{R}^\star, <_{bcb'c'}) \equiv^{\text{FO}}_{\alpha+c} (\mathcal{R}^\star, <_{b'c'bc})$.
   Now, consider the FO-interpretation that adds a symmetrical E-edge between u and v if either:
   - $P_1(u)$ and v is the first element of $<$
   - u is the second element of $<$ and $P_2(v)$
   - $P_3(u)$ and v is the third element of $<$
   - u is the fourth element of $<$ and $P_4(v)$

   and then forgets about P_1, P_2, P_3 and P_4.
   Take c to be the depth of this FO-interpretation (which has arity 1).
   Note that the result of this FO-interpretation on $(\mathcal{R}^\star, <_{bcb'c'})$ is an ordered extension of $\mathcal{R}$ and that its result on $(\mathcal{R}^\star, <_{b'c'bc})$ is an ordered extension of $\mathcal{R}'$.
   This entails $\mathcal{R}' \equiv^{<\text{-inv FO}}_{\alpha} \mathcal{R}$.
2. We can now, without loss of generality, assume that $|[a,c]| \leq 2n+2$.
   Let s be the threshold $s(\alpha)$ from the proof of contiguous-segment-E-swaps. Let us increase that threshold for it to account for segment-E-swaps: set $s(\alpha) := (2n+2) + M$ with $M := \max(s,n)$.
   Consider an isomorphism φ from $\mathcal{N}^{s(\alpha)}_{\mathcal{R}}(a,b)$ to $\mathcal{N}^{s(\alpha)}_{\mathcal{R}}(a',b')$.
   By choice of $s(\alpha)$, $\mathrm{tp}^M_{\mathcal{R}}(c,d) = \mathrm{tp}^M_{\mathcal{R}}(\varphi(c), \varphi(d))$.
   Since $(\varphi(c), \varphi(d)) \neq (c', d')$ (for otherwise, we would be in the Case 1), there are only two subcases to consider:

   - if $a', b', \varphi(c), \varphi(d), c', d'$ appear in that order, i.e. the segment strictly between a and d' can be decomposed as:

     $$[b,c]^{\mathcal{R}}_n \cdot [d,a']^{\mathcal{R}}_n \cdot [b', \varphi(c)]^{\mathcal{R}}_n \cdot [\varphi(d), c']^{\mathcal{R}}_n$$

     Let $\mathcal{R}_1$ be the M-guarded segment-E-swap between $[b,c]$ and $[b', \varphi(c)]$ in $\mathcal{R}$.
     This swap falls under the scope of Case 1 since $M \geq n$, hence $\mathcal{R}_1 \equiv^{<\text{-inv FO}}_{\alpha} \mathcal{R}$ and $\forall z \in R$, $\mathrm{tp}^{M+1}_{\mathcal{R}_1}(z) = \mathrm{tp}^{M+1}_{\mathcal{R}}(z)$. In $\mathcal{R}_1$, the segment strictly between a and d' can be decomposed as:

     $$[b', \varphi(c)]^{\mathcal{R}}_n \cdot [d,a']^{\mathcal{R}}_n \cdot [b,c]^{\mathcal{R}}_n \cdot [\varphi(d), c']^{\mathcal{R}}_n$$

Hence we are in the conditions (since $M \geq s$) to apply Lemma 9 in the case of the M-guarded contiguous-segment-E-swap between $[d, c]$ and $[\varphi(d), c']$ in $\mathcal{R}_1$.
We get $\mathcal{R}_2 \equiv_\alpha^{<\text{-inv FO}} \mathcal{R}_1$. In $\mathcal{R}_2$, the segment strictly between a and d' can be decomposed as:

$$[b', \varphi(c)]_n^{\mathcal{R}} \cdot [\varphi(d), c']_n^{\mathcal{R}} \cdot [d, a']_n^{\mathcal{R}} \cdot [b, c]_n^{\mathcal{R}} = [b', c']_n^{\mathcal{R}} \cdot [d, a']_n^{\mathcal{R}} \cdot [b, c]_n^{\mathcal{R}}$$

That is, $\mathcal{R}_2 = \mathcal{R}'$, and we get $\mathcal{R}' \equiv_\alpha^{<\text{-inv FO}} \mathcal{R}$ as desired.

- if $a', b', c', d', \varphi(c), \varphi(d)$ appear in that order, i.e. the segment strictly between a and $\varphi(d)$ can be decomposed as:

  $$[b, c]_n^{\mathcal{R}} \cdot [d, a']_n^{\mathcal{R}} \cdot [b', c']_n^{\mathcal{R}} \cdot [d', \varphi(c)]_n^{\mathcal{R}}$$

  Let $\mathcal{R}_1$ be the M-guarded segment-E-swap between $[b, c]$ and $[b', \varphi(c)]$ in $\mathcal{R}$.
  This swap falls under the scope of Case 1 since $M \geq n$, hence $\mathcal{R}_1 \equiv_\alpha^{<\text{-inv FO}} \mathcal{R}$ and $\forall z \in R$, $\text{tp}_{\mathcal{R}_1}^{M+1}(z) = \text{tp}_{\mathcal{R}}^{M+1}(z)$. In $\mathcal{R}_1$, the segment strictly between a and $\varphi(d)$ can be decomposed as:

  $$[b', c']_n^{\mathcal{R}} \cdot [d', \varphi(c)]_n^{\mathcal{R}} \cdot [d, a']_n^{\mathcal{R}} \cdot [b, c]_n^{\mathcal{R}}$$

  Hence we are in the conditions (since $M \geq s$) to apply Lemma 9 in the case of the M-guarded contiguous-segment-E-swap between $[d', \varphi(c)]$ and $[d, c]$ in $\mathcal{R}_1$.
  We get $\mathcal{R}_2 \equiv_\alpha^{<\text{-inv FO}} \mathcal{R}_1$. In $\mathcal{R}_2$, the segment strictly between a and $\varphi(d)$ can be decomposed as:

  $$[b', c']_n^{\mathcal{R}} \cdot [d, a']_n^{\mathcal{R}} \cdot [b, c]_n^{\mathcal{R}} \cdot [d', \varphi(c)]_n^{\mathcal{R}}$$

  That is, the segment strictly between a and d' is

  $$[b', c']_n^{\mathcal{R}} \cdot [d, a']_n^{\mathcal{R}} \cdot [b, c]_n^{\mathcal{R}}$$

  Hence $\mathcal{R}_2 = \mathcal{R}'$, and we get $\mathcal{R}' \equiv_\alpha^{<\text{-inv FO}} \mathcal{R}$ as desired.

▶ **Proposition 10.** *$\forall \alpha, n, d \in \mathbb{N}, \exists M \in \mathbb{N}, \forall D \in \mathbb{N}, \forall \mathcal{P}, \mathcal{Q} \in \mathbb{H}_\sigma$, if $\mathcal{P} \equiv_M^{\text{FO}} \mathcal{Q}$, then there exists $\mathcal{Q}' \in \mathbb{H}_\sigma$ such that $\mathcal{Q}' \equiv_\alpha^{<\text{-inv FO}} \mathcal{Q}$ and $[\![\mathcal{E}_{n+1}(\mathcal{P})]\!] \leq_d^D [\![\mathcal{E}_{n+1}(\mathcal{Q}')]\!]$.*

Proof. The proof is a simple pumping argument. We rely on the fact that hollow trees have bounded treewidth hence $<$-inv FO $\subseteq$ MSO on $\mathbb{H}_\sigma$. In particular there is a $\beta \in \mathbb{N}$ such that $\equiv_\beta^{\text{MSO}}$ subsumes $\equiv_\alpha^{<\text{-inv FO}}$. We will construct $\mathcal{Q}'$ such that $\mathcal{Q}' \equiv_\beta^{\text{MSO}} \mathcal{Q}$.

Let $d' > d$ be a number that will be specified during the proof. We choose M large enough to make sure that every $(n+1)$-type has the same number of occurrences in $\mathcal{P}$ and in $\mathcal{Q}$ up to a threshold d' (this can be expressed in FO).

We prove the proposition by induction on the number κ of $(n+1)$-types τ such that $|\mathcal{P}|_\tau \neq |\mathcal{Q}|_\tau$ and $|\mathcal{Q}|_\tau < |\mathcal{P}|_\tau + D$.

If $\kappa = 0$, there is nothing to do as $\mathcal{Q}' := \mathcal{Q}$ fits. Otherwise, let τ be such a type. Notice that because $\mathcal{P} \equiv_M^{\text{FO}} \mathcal{Q}$ we must have $|\mathcal{Q}|_\tau > d'$.

There are two cases to consider:

- Assume there exists a thread in $\mathcal{Q}$ which contains at least l nodes $x_1, x_2, \cdots, x_l$ (in that order) having the same $(n+1)$-type, whose subtrees each contains at least one node of type τ in $\mathcal{Q}$, and such that for every $i < l$, duplicating within the thread the forest below $[x_i, x_{i+1}[$ does not affect the $\equiv_\beta^{\text{MSO}}$ of $\mathcal{Q}$, where l is chosen large enough so that

there exists $i < l$ such that the forest below $[x_i, x_{i+1}[$ doesn't contain any occurrence of a $(n+1)$-type τ' such that $|\mathcal{Q}|_{\tau'} \leq d$.
Then we construct $\mathcal{Q}'$ from $\mathcal{Q}$ by duplicating the forest below $[x_i, x_{i+1}[$ as many times as necessary to have enough nodes of type τ. This decreases κ and guarantees that $\mathcal{Q}' \equiv_{\beta}^{\text{MSO}} \mathcal{Q}$ and we can conclude by induction.

- Assume now that there is a chain for the ancestor relation $x_1, x_2, \cdots, x_l$ having the same $(n+1)$-type such that each of the contexts $\mathcal{C}_{\mathcal{P}}(x_i, x_{i+1})$ (we use here the notations introduced for Lemma 26 to denote the context between x_i and x_{i+1}) contains at least one node of type τ and $\forall i, j, \mathcal{S}_{\mathcal{P}}(x_i) \equiv_{\beta}^{\text{MSO}} \mathcal{S}_{\mathcal{P}}(x_j)$ (the subtrees at x_i and x_j), where l is large enough to guarantee the existence of some $i < l$ such that $\mathcal{C}_{\mathcal{P}}(x_i, x_{i+1})$ contains no node of any type τ' such that $|\mathcal{Q}|_{\tau'} \leq d$
Let $\mathcal{Q}' := \mathcal{P}_{\mathcal{P}}(x_i) \cdot \mathcal{C}_{\mathcal{P}}(x_i, x_{i+1})^k \cdot \mathcal{S}_{\mathcal{P}}(x_{i+1})$ (that is, we've duplicated k times the context between x_i and x_{i+1}) with k large enough so we have enough nodes of type τ. We have $\mathcal{Q}' \equiv_{\beta}^{\text{MSO}} \mathcal{Q}$ and κ has decreased by 1: we can conclude by induction.

It remains to fix d' large enough so that one of the two cases above must hold. ◀

C Proofs for Section 4 (Inclusion and pseudo-inclusion)

▶ **Proposition 12.** *For every $\alpha, m \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that $\forall \mathcal{P}, \mathcal{Q} \in$ quasi-$\mathbb{H}_{\sigma}^{N+1}$, if $[\![\mathcal{E}_{N+1}(\mathcal{P})]\!] \leq [\![\mathcal{E}_{N+1}(\mathcal{Q})]\!]$, then there exists $\mathcal{Q}' \in$ quasi-$\mathbb{H}_{\sigma}^{m+1}$ such that $\mathcal{Q}' \equiv_{\alpha}^{<\text{-inv FO}} \mathcal{Q}$, $[\![\mathcal{E}_{m+1}(\mathcal{Q}')]\!] = [\![\mathcal{E}_{m+1}(\mathcal{Q})]\!]$ and h that is a $(m+1)$-pseudo-inclusion from $\mathcal{P}$ into $\mathcal{Q}'$.*

▶ **Proposition 13.** *For every $\alpha, m \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that $\forall \mathcal{U} \in Ctxt_{\sigma}^{N+1}$, $\forall \mathcal{Q} \in$ quasi-$\mathbb{H}_{\sigma}^{N+1}$, if $[\![\mathcal{E}_{N+1}(\mathcal{U})]\!] < [\![\mathcal{E}_{N+1}(\mathcal{Q})]\!]$, then there exists $\mathcal{Q}' \in$ quasi-$\mathbb{H}_{\sigma}^{m+1}$ such that $\mathcal{Q}' \equiv_{\alpha}^{<\text{-inv FO}} \mathcal{Q}$, $[\![\mathcal{E}_{m+1}(\mathcal{Q}')]\!] = [\![\mathcal{E}_{m+1}(\mathcal{Q})]\!]$ and $\mathcal{U}$ is $(m+1)$-included in $\mathcal{Q}'$.*

Proof. We mainly focus on the proof of Proposition 12. We will then explain how to modify the proof using the extra hypothesis in order to get inclusion instead of pseudo-inclusion thus proving Proposition 13.

We modify $\mathcal{Q}$ using E-swaps in order to construct a pseudo-inclusion h from $\mathcal{P}$. This is done step by step, extending the domain of h thread by thread and, inside each thread, from one of its endpoint to the other.

We distinguish between two kinds of threads of $\mathcal{P}$. The short ones will be easily taken care of as they can be completely described in first-order. The long ones will require more work.

In view of Lemma 9, we assume that $m \geq s(\alpha)$. We set $n := 3m+3$. We will only perform swaps involving nodes at distance (along E) $\geq n-m$ from the endpoints; hence, the "distant from endpoints" conditions of m-guarded E-swaps will always be satisfied.

A thread is short if its length (the distance along E between its two endpoints) is at most $2(n-m)$. By taking N large enough, our hypothesis $[\![\mathcal{E}_{N+1}(\mathcal{P})]\!] \leq [\![\mathcal{E}_{N+1}(\mathcal{Q})]\!]$ guarantees that we can find a injective mapping from the short threads of $\mathcal{P}$ to that of $\mathcal{Q}$, which sends each short thread to one having an isomorphic $(m+1)$-enrichment. We initialize h according to this mapping. It is clear that h is a partial $(m+1)$-pseudo-inclusion mapping.

It remains to extend the domain of h to the long threads.

Let a be an endpoint of a long thread of $\mathcal{P}$: $\text{segtype}_{m+1,\mathcal{P}}^{n-m}(a)$ denotes the isomorphism type of the segment $[a, b]$, where b is the element at distance $n-m$ of a in its thread, and every element is colored with its $(m+1)$-type in $\mathcal{P}$. By $\text{End}_{m+1}^{n-m}(\mathcal{P}) \leq \text{End}_{m+1}^{n-m}(\mathcal{Q})$, we mean that every $\text{segtype}_{m+1,.}^{n-m}(.)$ has at least as many occurrences in $\mathcal{P}$ as in $\mathcal{Q}$.

Let $\mathring{\mathcal{S}}^{n-m}_{m+1}(\mathcal{P})$ be the restriction of $\mathcal{S}\text{upp}_{m+1}(\mathcal{P})$ to elements that are at distance $> n-m$ from $\text{End}(\mathcal{P})$

Every intermediate structure $\mathcal{Q}'$ will verify the following invariant:

$$\begin{cases} \mathcal{Q}' \equiv_\alpha^{<\text{-inv FO}} \mathcal{Q} \\ [\![\mathcal{E}_{m+1}(\mathcal{Q}')]\!] = [\![\mathcal{E}_{m+1}(\mathcal{Q})]\!] \\ \text{End}^{n-m}_{m+1}(\mathcal{P}) \leq \text{End}^{n-m}_{m+1}(\mathcal{Q}') \text{ and } h \text{ preserves } \text{segtype}^{n-m}_{m+1,.}(.) \\ [\![\mathring{\mathcal{S}}^{n-m}_{m+1}(\mathcal{P})]\!] \leq [\![\mathring{\mathcal{S}}^{n-m}_{m+1}(\mathcal{Q}')]\!] \end{cases} \tag{1}$$

As long as N is large enough, the hypothesis guarantees that $\mathcal{Q}$ verifies (1).

Assume we have already constructed a partial $(m+1)$-pseudo-inclusion h from $\mathcal{P}$ to $\mathcal{Q}'$ where $\mathcal{Q}'$ verifies (1). We show that we can construct a new hollow quasitree $\mathcal{Q}'' \in \text{quasi-}\mathbb{H}^{m+1}_\sigma$ verifying (1), using a sequence of E-swaps applied to $\mathcal{Q}'$ in such a way that h can be extended by at least one element of P.

To this end, assume first that the domain of h is the union of a number of threads, each contained in its entirety (this is the case at the beginning). Let $\text{Im}(h)$ denote the image of h. Let t be any thread of $\mathcal{P}$ not in the domain of h and let x be an endpoint of t. We want to extend h in order for its domain to contain x.

(1) ensures that there exists an endpoint $y \notin \text{Im}(h)$ of a long thread of $\mathcal{Q}'$ such that $\text{segtype}^{n-m}_{m+1,\mathcal{Q}'}(y) = \text{segtype}^{n-m}_{m+1,\mathcal{P}}(x)$. We don't modify $\mathcal{Q}'$ and extend h by sending every $z \in [x, x']$ to the corresponding $h(z) \in [y, y']$ (where x', y' are the elements at distance $n-m$ of x, y in their threads). Every z and $h(z)$ have the same $(m+1)$-type, and h preserves E on $[x, x']$.

By construction, h is a partial $(m+1)$-pseudo-inclusion mapping as desired.

Suppose now that the domain of h contains a set of (entire) threads and the initial segment of a thread t of $\mathcal{P}$, that includes at least the points of t at distance $\leq n-m$ from its endpoint in the domain of h. Let x' be the last element of t in the domain of h and x be the first element of t not in the domain of h. In particular we have $E(x', x)$. Assume furthermore that x is at distance $> n-m$ from the other endpoint of t.

Since $[\![\mathring{\mathcal{S}}^{n-m}_{m+1}(\mathcal{P})]\!] \leq [\![\mathring{\mathcal{S}}^{n-m}_{m+1}(\mathcal{Q}')]\!]$, there exists an element $y \in \mathring{\mathcal{S}}^{n-m}_{m+1}(\mathcal{Q}') \setminus \text{Im}(h)$ having the same $(m+1)$-type as x. Let y' be the image of x' by an isomorphism between $\mathcal{N}^{m+1}_{\mathcal{P}}(x)$ and $\mathcal{N}^{m+1}_{\mathcal{Q}'}(y)$.

Let $\hat{x}$ be the image of x by an isomorphism between $\mathcal{N}^{m+1}_{\mathcal{P}}(x')$ and $\mathcal{N}^{m+1}_{\mathcal{Q}'}(h(x'))$

By definition, $\text{tp}^m_{\mathcal{Q}'}(h(x'), \hat{x}) = \text{tp}^m_{\mathcal{Q}'}(y', y)$.

If $y = \hat{x}$, leave $\mathcal{Q}'$ unchanged and let h map x to $\hat{x}$. Otherwise, there are several cases to consider depending on the positions of y and y'.

1. if y is on a thread that does not intersect $\text{Im}(h)$.
   Let $\mathcal{Q}''$ be the m-guarded crossing-E-swap between $h(x')\hat{x}$ and $y'y$ in $\mathcal{Q}'$. Extend h by setting $h(x)$ to y (c.f. Figure 16, in which $\text{Im}(h)$ is represented as double lines).

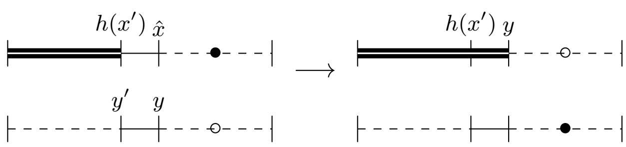

Figure 15 An illustration of the progression in the (pseudo-)inclusion (case 1)

2. if y is between $h(z)$ and $h(z')$ where z and z' are consecutive node of the current thread t already in the domain of h and such that y' is between $h(z)$ and y (that is, they are in the right order for a segment-E-swap), c.f. Figure 17.
   Let u' and u be the respective E-neighbors of $h(z)$ and $h(z')$ in $[h(z), h(z')]$. h being a pseudo-inclusion, $\mathrm{tp}^m_{\mathcal{Q}'}(h(z), u') = \mathrm{tp}^m_{\mathcal{Q}'}(u, h(z'))$.
   Let $\mathcal{Q}''$ be the m-guarded segment-E-swap between $[u', y']$ and $[h(z'), h(x')]$ in $\mathcal{Q}'$, and extend h by setting $h(x)$ to y

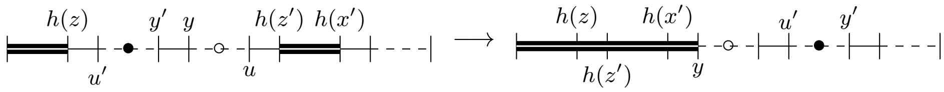

Figure 16 An illustration of the progression in the (pseudo-)inclusion (case 2)

3. if y is between $h(z)$ and $h(z')$ where z and z' are consecutive nodes of the current thread t already in the domain of h and such that y' is between y and $h(z')$ (that is, they are not in the right order for a segment-E-swap), c.f. Figure 18. This means that $y, y', h(x'), \hat{x}$ appear in that order.
   Let $\mathcal{R}$ be the m-guarded mirror-E-swap at $[y', h(x')]$ in $\mathcal{Q}'$
   In $\mathcal{R}$, $h(z), u', h(z'), u$ now appear in that order.
   Let $\mathcal{Q}''$ be the m-guarded mirror-E-swap at $[u', h(z')]$ in $\mathcal{R}$ and extend h by setting $h(x)$ to y.

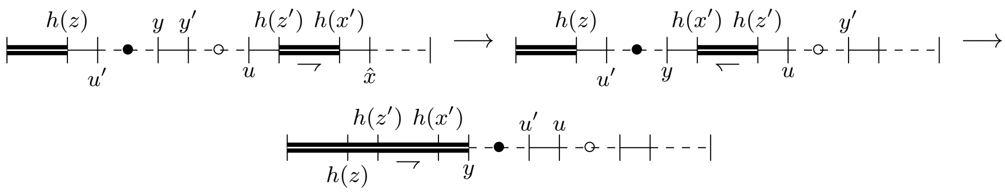

Figure 17 An illustration of the progression in the (pseudo-)inclusion (case 3)

4. if y is between $h(z)$ and $h(z')$ where z and z' are consecutive node in some thread different from t already in the domain of h (c.f. Figure 19).
   Let $\mathcal{R}$ be the m-guarded crossing-E-swap between $y'y$ and $h(x')\hat{x}$ in $\mathcal{Q}'$
   Let $\mathcal{Q}''$ be the m-guarded crossing-E-swap between $h(z)u'$ and $uh(z')$ in $\mathcal{R}$, and extend h by setting $h(x)$ to y.

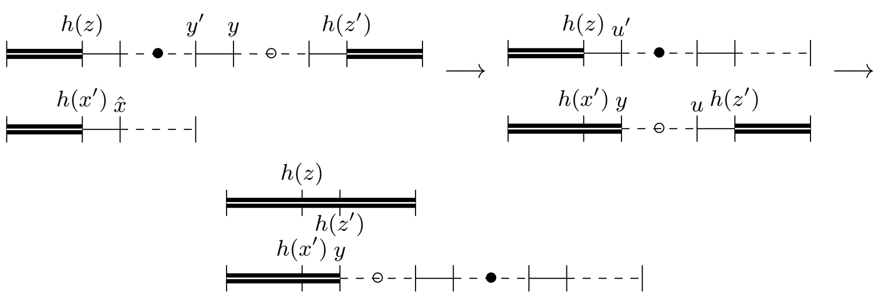

Figure 18 An illustration of the progression in the (pseudo-)inclusion (case 4)

5. if y is on the same thread as $h(x')$, such that $h(x'), \hat{x}, y, y'$ appear in that order (c.f. Figure 20).
   Then let $\mathcal{Q}''$ be the m-guarded mirror-E-swap at $[\hat{x}, y]$ in $\mathcal{Q}'$ and extend h by setting $h(x)$ to y.

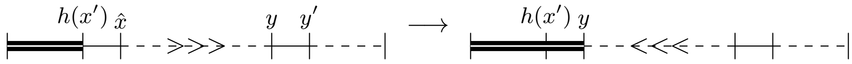

Figure 19 An illustration of the progression in the (pseudo-)inclusion (case 5)

6. finally if y is on the same thread as $h(x')$ but $h(x'), \hat{x}, y', y$ appear in that order.
   This is the case where we cannot achieve inclusion without extra hypothesis. For Proposition 12, we simply allow a "jump" and set $h(x)$ to y without changing $\mathcal{Q}'$.

In the previous case analysis, in order to perform E-swaps, it was important for x (and therefore y) to be far away from the endpoint e of t that is not already in the domain of h. In order to conclude the proof of Proposition 13, it remains to consider the case where x is at distance $n-m$ from e.

By hypothesis, there exists an endpoint a outside of $\mathrm{Im}(h)$ of a long thread such that $\mathrm{segtype}^{n-m}_{m+1,\mathcal{Q}'}(a) = \mathrm{segtype}^{n-m}_{m+1,\mathcal{P}}(e)$. Let ξ be the isomorphism between $[e, x]$ and $[a, y]$, where y is the element at distance $n-m$ of a in its thread.

If a is not on the same thread as $h(x')$, let y' be the E-neighbor of y not in $[y, a]$. We let $\mathcal{Q}''$ be the m-guarded crossing-E-swap between $h(x')\hat{x}$ and $y'y$ in $\mathcal{Q}'$ and extend h by setting $h(u)$ to $\xi(u)$ for all u in $[x, e]$.

Otherwise, we don't modify $\mathcal{Q}'$ and simply extend h by setting $h(u)$ to $\xi(u)$ for every u in $[x, e]$. Notice that there may be a jump between $h(x')$ and $h(x)$.

This concludes the proof of Proposition 12. We now move to the proof of Proposition 13.

We decompose U as $P \uplus V$, where P is the union of the threads of $\mathcal{U}$ whose endpoints were endpoints in the structure from which $\mathcal{U}$ is derived (that is, their type is a type of endpoint). We let $\mathcal{P}$ be $(U)_{|P}$ and proceed as above with the threads of $\mathcal{P}$.

It all works as above except for the two cases where we introduced a jump. Consider again the situation of Case 6. Our extra cardinality hypothesis ensures that there is a $z \neq y$ verifying the same conditions as y (otherwise we would be in a previous case). Assume WLOG that $h(x'), y, z$ appear in that order (c.f. Figure 21).. Set $\mathcal{Q}''$ to be the m-guarded contiguous-segment-E-swap between $[\hat{x}, y']$ and $[y, z']$ in $\mathcal{Q}'$, and extend h by setting $h(x)$ to y. h is now an inclusion.

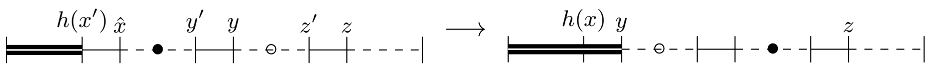

Figure 20 An illustration of the progression in the inclusion (case 6, for Proposition 13)

We also introduced a jump when extending h to the endpoint of some thread. But the cardinality condition ensures that we have two endpoints $a_1 \neq a_2$ outside of $\mathrm{Im}(h)$ such that $\mathrm{segtype}^{n-m}_{m+1,\mathcal{Q}'}(a_1) = \mathrm{segtype}^{n-m}_{m+1,\mathcal{Q}'}(a_2) = \mathrm{segtype}^{n-m}_{m+1,\mathcal{P}}(e)$. Hence at least one of them is on a different thread than $h(x')$ and the procedure described above yields an inclusion.

In order to conclude the proof of Proposition 13 it remains to extend the domain of h to V. This done in the exact same way but, as the threads of V may not include the endpoints, it gives rise to new cases. We use the same notations. Let t be the thread under investigation and let u be its first element in V. Note that u doesn't have to be an endpoint of t.

The first difference is in Case 6: it may be the case that there is no z verifying the same conditions as y. In this case, and if no previous case is applicable, it must be the case that such a z appear "before" $h(u)$: that is, $z, h(u), h(x'), y$ appear in that order. There are now two possibilities:

- as described in Figure 22, $z', z, h(x')$ are in that order, where z' is the image of x' by an isomorphism mapping the neighborhood of x to that of z. Then set $\mathcal{Q}''$ to be the m-guarded contiguous-segment-E-swap between $[z, h(x')]$ and $[\hat{x}, y']$ in $\mathcal{Q}'$, and extend h by setting $h(x)$ to y.

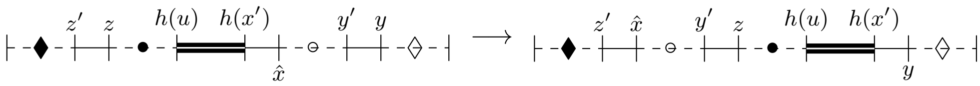

Figure 21 An illustration of the progression in the inclusion of $\mathcal{V}$, first completion of case 6

- otherwise, $z, z', h(x')$ appear in that order (c.f. Figure 23). Set $\mathcal{Q}''$ to be the m-guarded mirror-E-swap at $[z', h(x')]$ in $\mathcal{Q}'$, and extend h by setting $h(x)$ to z. Notice that we have "reversed" the direction on the inclusion of the current thread but this isn't an issue since E is not oriented.

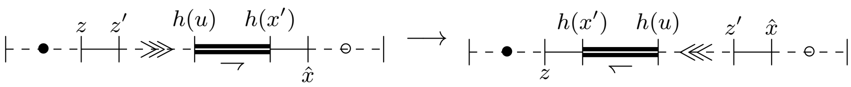

Figure 22 An illustration of the progression in the inclusion of $\mathcal{V}$, second completion of case 6

The second difference is that it is now possible that none of the cases described above are applicable. In that situation, there must exist two nodes y and z "before" $h(u)$ having the same type as x. If at least one of them (say z) is in reverse order (i.e. $z, z', h(x')$ appear in that order, c.f. Figure 23) we proceed exactly as before.

Otherwise, it means that we can set $\mathcal{Q}''$ to be (assuming WLOG that $y, z, h(x')$ appear in that order, c.f. Figure 24) the m-guarded contiguous-segment-E-swap between $[y, z']$ and $[z, h(x')]$ in $\mathcal{Q}'$ and extend h by setting $h(x)$ to y.

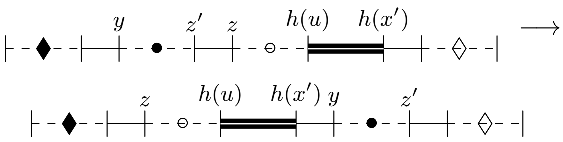

Figure 23 An illustration of the progression in the inclusion of $\mathcal{V}$, if no previous case is applicable

◀

D Proofs for Section 5 (Tools for reorganizing S-edges)

▶ **Lemma 14.** $\forall \alpha, m \in \mathbb{N}, \exists N \in \mathbb{N}, \forall \mathcal{W} \in Ctxt^N_\sigma, \forall \mathcal{Q} \in quasi\text{-}\mathbb{H}^N_\sigma$*, if* $h : W \to Q$ *is a* N*-pseudo-inclusion, then there exists some* $\mathcal{Q}' \in quasi\text{-}\mathbb{H}^{m+1}_\sigma$ *and some* $(m+1)$*-pseudo-inclusion*

$h' : W \to Q'$ such that $\mathcal{Q}' \equiv_\alpha^{<\text{inv FO}} \mathcal{Q}$, $\mathcal{S}upp_{m+1}(\mathcal{Q}') \simeq \mathcal{S}upp_{m+1}(\mathcal{Q})$ and, if x and y are S-siblings in $\mathcal{W}$, then so are $h'(x)$ and $h'(y)$ in $\mathcal{Q}'$.

Proof. We can assume that $m \geq s(\alpha)$. N is to be fixed later, and will chosen such that $2N \geq 2(2m+3)+1$.

Let $(x_1, y_1), \cdots, (x_r, y_r)$ denote all the pairs of endpoints of threads of length $\leq 2N-1$ of $\mathcal{W}$ (they must be S-siblings), and let $(x_{r+1}, y_{r+1}), \cdots, (x_s, y_s)$ denote the other pairs of S-siblings of $\mathcal{W}$, in an arbitrary order.

We are going to construct a sequence of structures $\mathcal{Q} = \mathcal{Q}_r \equiv_\alpha^{<\text{inv FO}} \cdots \equiv_\alpha^{<\text{inv FO}} \mathcal{Q}_s$ of same $(m+1)$-enriched support, and functions $f_r, \cdots, f_s$ such that

- f_i $(m+1)$-pseudo-includes $\mathcal{W}$ in $\mathcal{Q}_i$
- $\forall j \leq i$, $f_i(x_j)$ and $f_i(y_j)$ are S-siblings in $\mathcal{Q}_i$
- $\forall j > i$, let z be the S-sibling of $f_i(x_j)$ in $\mathcal{Q}_i$. Let Z (resp. Y) be the element at distance $2m+3$ of z (resp. $f_i(y_j)$) in $\mathcal{S}\text{upp}_0(\mathcal{Q}_i)$ (Z and Y exist since their threads are of length $\geq 2N$). Then $\mathcal{S}\text{upp}_{m+1}(\mathcal{Q}_i)|_{[z,Z]} \simeq \mathcal{S}\text{upp}_{m+1}(\mathcal{Q}_i)|_{[f_i(y_j),Y]}$

For $i = r$, set $\mathcal{Q}_r := \mathcal{Q}$ and $f_r := h$. Note that threads of $\mathcal{Q}$ of length $\leq 2N-1$ must have matching endpoints. N is chosen large enough so that the last property holds ($N := 2 + (2m+3) + (m+1)$ is enough).

Assume now that we have constructed $\mathcal{Q}_i$ and f_i as required. If $f_i(x_{i+1})$ and $f_i(y_{i+1})$ are S-siblings in $\mathcal{Q}_i$, set $\mathcal{Q}_{i+1} := \mathcal{Q}_i$ and $f_{i+1} := f_i$.

Otherwise, let z be the S-sibling of $f_i(x_{i+1})$, Z (resp. Y) be the element at distance $2m+3$ of z (resp. $f_i(y_{i+1})$) in $\mathcal{S}\text{upp}_0(\mathcal{Q}_i)$, and Z' (resp. Y') be the element at distance $2m+4$ of z (resp. $f_i(y_{i+1})$) in $\mathcal{S}\text{upp}_0(\mathcal{Q}_i)$. We know that $\mathcal{S}\text{upp}_{m+1}(\mathcal{Q}_i)|_{[z,Z]} \simeq \mathcal{S}\text{upp}_{m+1}(\mathcal{Q}_i)|_{[f_i(y_{i+1}),Y]}$ (witnessed by an isomorphism ϕ).

In particular, $\text{tp}^m_{\mathcal{Q}_i}(Z, Z') = \text{tp}^m_{\mathcal{Q}_i}(Y, Y')$, and $\{Y, Y', Z, Z'\}$ and $\text{End}(\mathcal{Q}_i)$ are $(2m+3)$-distant in $\mathcal{S}\text{upp}_0(\mathcal{Q}_i)$ by choice of N.

We distinguish between two cases:

- if Y, Y' and Z, Z' are in different threads.
  Let $\mathcal{Q}_{i+1}$ be the m-guarded crossing-E-swap between ZZ' and YY' in $\mathcal{Q}_i$ (c.f. Figure 25).

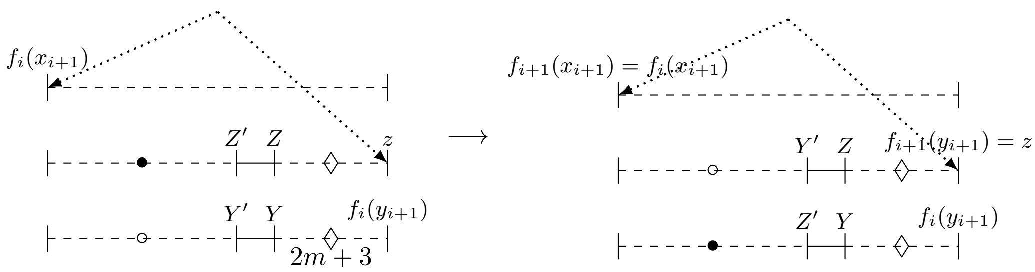

Figure 24 After the m-guarded crossing-E-swap between ZZ' and YY' in $\mathcal{Q}_i$, $f_{i+1}(x_{i+1})$ and $f_{i+1}(y_{i+1})$ are S-siblings

- if Y, Y' and Z, Z' are in the same thread.
  Let $\mathcal{Q}_{i+1}$ be the m-guarded mirror-E-swap at $[Z', Y']$ in $\mathcal{Q}_i$ (c.f. Figure 26).

In both cases, we define f_{i+1} as $\Phi \circ f_i$ where Φ the permutation of Q_i defined as ϕ on $[z, Z]$, ϕ^{-1} on $[f_i(y_{i+1}), Y]$, and the identity elsewhere.

Lemma 9 guarantees that in both cases, $\mathcal{Q}_{i+1} \equiv_\alpha^{<\text{inv FO}} \mathcal{Q}_i$ and, together with the invariant $\mathcal{S}\text{upp}_{m+1}(\mathcal{Q}_i)|_{[z,Z]} \simeq \mathcal{S}\text{upp}_{m+1}(\mathcal{Q}_i)|_{[f_i(y_{i+1}),Y]}$, that $\mathcal{S}\text{upp}_{m+1}(\mathcal{Q}_{i+1}) \simeq \mathcal{S}\text{upp}_{m+1}(\mathcal{Q}_i)$.

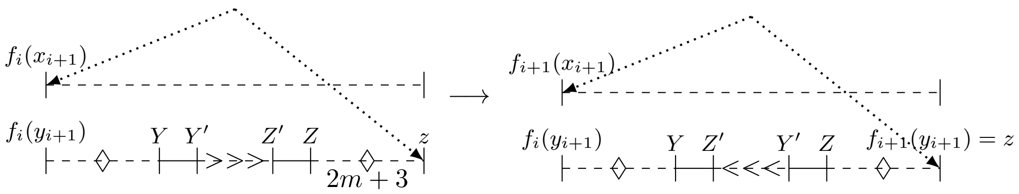

Figure 25 After the m-guarded mirror-E-swap at $[Z', Y']$ in $\mathcal{Q}_i$, $f_{i+1}(x_{i+1})$ and $f_{i+1}(y_{i+1})$ are S-siblings

Furthermore, $f_{i+1}(x_{i+1})$ and $f_{i+1}(y_{i+1})$ are S-siblings, and it is straightforward to see that f_{i+1} is a $(m+1)$-pseudo-inclusion, and that the other conditions are still respected.

In the end, $\mathcal{Q}' := \mathcal{Q}_s$ and $h' := f_s$ fit. ◀

We will often need to state that several sets are far from each other. To this end we introduce the notion of scattering, which is a compact way of saying that. For a subset A of a structure $\mathcal{R}$ whose vocabulary contains the binary relation S, define $\mathcal{R} \setminus S(A)$ to be $\mathcal{R}$ minus all the S-edges adjacent to any element of A. If $A = \{z\}$, we note $\mathcal{R} \setminus S(z)$ instead of $\mathcal{R} \setminus S(\{z\})$.

▶ **Definition 23.** *Let $A_1, \cdots, A_k, B$ be subsets of R, and $\delta \in \mathbb{N}$.*

*We say that $A_1, \cdots, A_k$ **are δ-scattered wrt.** B if $A_1, \cdots, A_k, B$ are pairwise δ-distant in $\mathcal{R} \setminus S(A_1 \cup \cdots \cup A_k)$.*

The following lemma will be useful in a couple of proofs. It gives a setting in which we can apply simultaneous crossing-S-swaps:

▶ **Lemma 24.** *Let $\alpha, s \in \mathbb{N}$, $m \geq s(\alpha)$,$\mathcal{R} \in$ quasi-$\mathbb{H}_\sigma^{m+1}$, and distinct elements a_1, a_1', a_1'', b_1, b_1', b_1'',$\cdots$, a_s, a_s', a_s'', b_s, b_s', $b_s'' \in R$ such that a_i', a_i'' (resp. b_i', b_i'') are the S-children of a_i (resp. b_i) and $B \supseteq \{b_1, b_1', b_1'', \cdots, b_s, b_s', b_s''\}$ such that $\{a_1\}, \cdots, \{a_s\}, B$ are pairwise $(2m+5)$-distant and for every i, $tp^m_{\mathcal{R}}(a_i, a_i', a_i'') = tp^m_{\mathcal{R}}(b_i, b_i', b_i'')$.*

Let $\mathcal{R}'$ be $\mathcal{R}$ where all the $S(a_i, a_i')$, $S(a_i, a_i'')$, $S(b_i, b_i')$ and $S(b_i, b_i'')$ have been replaced by $S(a_i, b_i')$, $S(a_i, b_i'')$, $S(b_i, a_i')$ and $S(b_i, a_i'')$: in other words, $\mathcal{R}'$ is the simultaneous crossing-S-swap between a_i and b_i in $\mathcal{R}$.

Then $\mathcal{R}' \equiv_\alpha^{<\text{inv FO}} \mathcal{R}$, $\mathcal{S}upp_{m+1}(\mathcal{R}') = \mathcal{S}upp_{m+1}(\mathcal{R})$ (in particular, $\mathcal{R}' \in$ quasi-$\mathbb{H}_\sigma^{m+1}$) and $\{a_1\}, \{a_1', a_1''\}, \cdots, \{a_s\}, \{a_s', a_s''\}$ are m-scattered wrt. B in $\mathcal{R}'$.

Proof. Recall from Note 7 that $2m+5$ provides a sufficient distance condition to apply a m-guarded crossing-S-swap.

We construct a sequence of structures $\mathcal{R} = \mathcal{R}_0 \equiv_\alpha^{<\text{inv FO}} \cdots \equiv_\alpha^{<\text{inv FO}} \mathcal{R}_s$ having the same $(m+1)$-enriched support, where $\mathcal{R}_i$ is the simultaneous crossing-S-swap between a_j and b_j in $\mathcal{R}$ for every $j \leq i$.

Let's show by induction that for every i, $\mathcal{R}_i$ verifies (P_i):

1. $\{a_{i+1}\}, \cdots, \{a_s\}, B$ are pairwise $(2m+5)$-distant
2. $\{a_1\}, \{a_1', a_1''\}, \cdots, \{a_i\}, \{a_i', a_i''\}$ are m-scattered wrt. B

$\mathcal{R}$ verifies (P_0). Suppose that we have constructed $\mathcal{R}_i$ and let $\mathcal{R}_{i+1}$ be the m-guarded crossing-S-swap between a_{i+1} and b_{i+1} in $\mathcal{R}_i$: (P_i).1 ensures that $\text{dist}_{\mathcal{R}_i}(a_{i+1}, b_{i+1}) \geq 2m+5$. Lemma 8 gives $\mathcal{R}_{i+1} \equiv_\alpha^{<\text{inv FO}} \mathcal{R}_i$ and $\mathcal{S}\text{upp}_{m+1}(\mathcal{R}_{i+1}) = \mathcal{S}\text{upp}_{m+1}(\mathcal{R}_i)$. Let's show that $\mathcal{R}_{i+1}$ verifies (P_{i+1}):

(P_{i+1}).1 holds since the $(2m+4)$-neighborhoods of the $(a_j)_{j>i+1}$ haven't seen any change, because of (P_i).1.

(P_{i+1}).2 : Let $\mathcal{R}_i^-$ denote $\mathcal{R}_i \setminus S(\{a_1, a_1', a_1'', \cdots, a_i, a_i', a_i''\})$.
Let $\mathcal{R}_{i+1}^-$ denote $\mathcal{R}_{i+1} \setminus S(\{a_1, a_1', a_1'', \cdots, a_{i+1}, a_{i+1}', a_{i+1}''\})$.

Let $x, y \in \{a_1\} \cup \{a_1', a_1''\} \cup \cdots \cup \{a_i\} \cup \{a_i', a_i''\}$ be elements of two different sets. (P_i).2 entails that x and y are each at distance $\geq m$ in $\mathcal{R}_i^-$ from each other and from B, and (P_i).1 implies (since x and y are at distance 1 of B) that they are at distance $\geq m$ in $\mathcal{R}_i$ (hence in $\mathcal{R}_i^-$) from $\{a_{i+1}, a_{i+1}', a_{i+1}''\}$.
Hence, the swap doesn't affect their m-neighborhoods in $\mathcal{R}_i^-$, and they are still at distance $\geq m$ from each other and from B in $\mathcal{R}_{i+1} \setminus S(\{a_1, a_1', a_1'', \cdots, a_i, a_i', a_i''\})$, hence in $\mathcal{R}_{i+1}^-$

Let $a \in \{a_{i+1}, a_{i+1}', a_{i+1}''\}$ and $b \in B$. A path in $\mathcal{R}_{i+1}^-$ of length $\leq m-1$ from a to b or from a to x doesn't go through the new S-edges, hence is valid in $\mathcal{R}_i$ and contradicts (P_i).1 (in the second case, because x is at distance 1 from B in $\mathcal{R}_i$). This entails $\text{dist}_{\mathcal{R}_{i+1}^-}(a, b) \geq m$ and $\text{dist}_{\mathcal{R}_{i+1}^-}(a, x) \geq m$

It remains to show that $\text{dist}_{\mathcal{R}_{i+1}^-}(a_{i+1}, a_{i+1}') \geq m$ (and similarly for a_{i+1} and a_{i+1}'').
Suppose that there is a path of length $\leq m-1$ in $\mathcal{R}_{i+1}^-$ from a_{i+1} to a_{i+1}'. This path is valid in $\mathcal{R}_i^-$. Hence, there would be a "vertical loop" in $\mathcal{N}_{m+1}^{\mathcal{R}_i}(a_{i+1})$, contradicting $\mathcal{R}_i \in$ quasi-$\mathbb{H}_\sigma^{m+1}$.

We set $\mathcal{R}' := \mathcal{R}_s$, which has the desired properties. ◀

▶ **Proposition 15.** *For every $\alpha, m \in \mathbb{N}$, there exist $N, d, D \in \mathbb{N}$ such that, for every $\mathcal{P} \in \mathbb{H}_\sigma$, $\mathcal{Q} \in$ quasi-$\mathbb{H}_\sigma^{N+1}$ such that $[\![\mathcal{E}_{N+1}(\mathcal{P})]\!] \leq_d^D [\![\mathcal{E}_{N+1}(\mathcal{Q})]\!]$ and $\mathcal{P}$ is $(N+1)$-pseudo-included in $\mathcal{Q}$ through some h, there are some h' and $\mathcal{Q}' \in$ quasi-$\mathbb{H}_\sigma^{m+1}$ such that $\mathcal{Q}' \equiv_\alpha^{<\text{-inv FO}} \mathcal{Q}$, $\mathcal{S}upp_{m+1}(\mathcal{Q}') \simeq \mathcal{S}upp_{m+1}(\mathcal{Q})$, h' is a reduced $(m+1)$-pseudo-inclusion of $\mathcal{P}$ in $\mathcal{Q}'$ and $Q' \setminus \text{Im}(h')$ is S-stable in $\mathcal{Q}'$.*

Proof. We can assume that $m \geq s(\alpha)$. We will first provide a non-necessarily reduced $(m'+1)$-pseudo-inclusion verifying those conditions, with $m' := 2m+3$, and then modify it as well as the underlying structure to get a fitting reduced $(m+1)$-pseudo-inclusion.

For every $n \in \mathbb{N}$ (we will assign a value to n later on), there is a N such that, under the hypothesis, Lemma 14 yields $\mathcal{R} \equiv_\alpha^{<\text{-inv FO}} \mathcal{Q}$ such that $\mathcal{S}\text{upp}_{n+2}(\mathcal{R}) \simeq \mathcal{S}\text{upp}_{n+2}(\mathcal{Q})$ and g which $(n+2)$-pseudo-includes $\mathcal{P}$ in $\mathcal{R}$ and respects the S-siblings relation; we denote by V the complement of $\text{Im}(g)$ in Q. This implies that two nodes having the same S-parent are both either in $\text{Im}(g)$ (and are the two endpoints of the same thread) or in V.

We say that $z \in V$ is **misassociated** if its S-children are in $\text{Im}(g)$. Likewise, we say that $g(x)$ is misassociated if its S-children are in V. The $(n+2)$-type of this element is called the type of the misassociation. Note that the number of misassociations in V and in $\text{Im}(g)$ is the same.

First, we deal with all but a bounded number of misassociations. There exists a M (which depends only on n) such that, if there are more than $2M$ misassociations, then we can find a misassociated element of V and one of $\text{Im}(g)$ that have the same type, and are at distance $\geq 2(n+1)+5$ from one another: this is because a hollow quasitree has degree at most 4. We can solve these misassociations by a $(n+1)$-guarded crossing-S-swap, according to Lemma 8 and Note 7, which preserves $\mathcal{S}\text{upp}_{n+2}(\mathcal{R})$.

Once we've done that, we're left with at most M misassociations in V, and the same number in $\text{Im}(g)$. Let $(z_1, g(x_1'), g(x_1'')), \cdots, (z_r, g(x_r'), g(x_r''))$ be an arbitrary enumeration of the misassociated elements of V, together with their S-children (recall that x_i' and x_i'' are S-siblings in $\mathcal{P}$, and let x_i be their S-parent).

Fix i between 1 and r. There exists a sequence $x_i = x_i^1, \cdots, x_i^{s_i}$ of elements of P, such that, if we name $x_i'^j$ and $x_i''^j$ the S-children of x_i^j in $\mathcal{P}$, for every j, $g(x_i^j)$ is the S-parent of $g(x_i'^{j+1})$ and $g(x_i''^{j+1})$, and $g(x_i^{s_i})$ is misassociated ; let $z_i', z_i'' \in V$ be its S-children, and rename $y_i := x_i^{s_i}$ for ease. This sequence is represented in Figure 27.

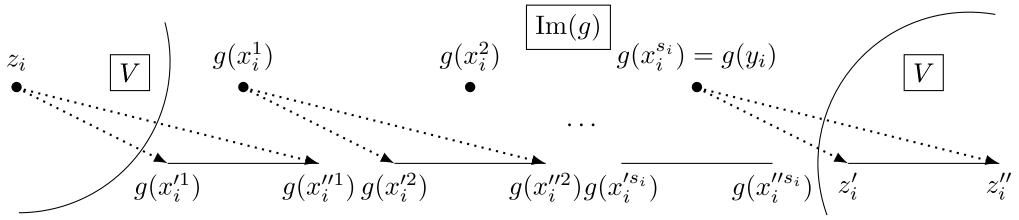

Figure 26 From z_i to (z_i', z_i'')

For every j, $\mathrm{tp}_{\mathcal{P}}^{n+2}(x_i'^{j+1}) = \mathrm{tp}_{\mathcal{R}}^{n+2}(g(x_i'^{j+1}))$, hence $\mathrm{tp}_{\mathcal{P}}^{n+1}(x_i^{j+1}) = \mathrm{tp}_{\mathcal{R}}^{n+1}(g(x_i^j))$, which in turn implies $\mathrm{tp}_{\mathcal{R}}^{n+1}(g(x_i^{j+1})) = \mathrm{tp}_{\mathcal{R}}^{n+1}(g(x_i^j))$

For the same reason, we have that $\mathrm{tp}_{\mathcal{R}}^{n+1}(z_i) = \mathrm{tp}_{\mathcal{R}}^{n+1}(g(x_i^1))$. In the end, we get $\tau_i := \mathrm{tp}_{\mathcal{R}}^{n+1}(z_i) = \mathrm{tp}_{\mathcal{R}}^{n+1}(g(y_i))$

Let B be the set containing the z_i, the $g(y_i)$, for $1 \leq i \leq r$, and their S-children.

Since we've bounded r by M and $\mathcal{R}$ is of degree 4, we can choose d and D large enough so that we are able to find $t_1, \cdots, t_r \in V$ and $u_1, \cdots, u_r \in \mathrm{Im}(g)$, with respective S-children t_i', t_i'' and u_i', u_i'', such that t_i and u_i are of type τ_i (since $z_i \in V$ is of type τ_i, there must be at least d elements of this type in $\mathrm{Im}(g)$ and D in V), and such that $\{t_1\}, \cdots, \{t_r\}, \{u_1\}, \cdots, \{u_r\}, B$ are pairwise $(2n+5)$-distant in $\mathcal{R}$.

We can apply Lemma 24 with $s = 2r$, $(a_1, \cdots, a_s) = (t_1, \cdots, t_r, u_1, \cdots, u_r)$ and $(b_1, \cdots, b_s) = (z_1, \cdots, z_r, g(y_1), \cdots, g(y_r))$.

This ensures that $\mathcal{R}'$ (which is the simultaneous crossing-S-swaps between z_i and t_i and crossing-S-swaps between $g(y_i)$ and u_i) is such that $\mathcal{R}' \equiv_\alpha^{\text{<-inv FO}} \mathcal{R}$ and $\mathcal{S}\mathrm{upp}_{n+1}(\mathcal{R}') = \mathcal{S}\mathrm{upp}_{n+1}(\mathcal{R})$. Furthermore, $\{t_1\}, \{t_1', t_2'\}, \{u_1\}, \{u_1', u_1''\}, \cdots, \{t_r\}, \{t_r', t_r''\}, \{u_r\}, \{u_r', u_r''\}$ are n-scattered wrt. B in $\mathcal{R}'$.

Note that we haven't added any new misassociated element in the process: the only misassociated elements in $\mathcal{R}'$ are now the t_i and the u_i.

Choose retrospectively $n := 2m' + 5$

Let's show that $\{t_1\}, \cdots, \{t_r\}, \{u_1\}, \cdots, \{u_r\}$ are pairwise $(2m'+5)$-distant in $\mathcal{R}'$.

Let x, y be distinct elements among them, and let's prove that $\mathrm{dist}_{\mathcal{R}'}(x, y) \geq 2m' + 5$. Suppose that's false, and consider a shortest path from x to y. It cannot be valid in $\mathcal{R}$, hence it must go though at least one new S-edge, and the first one must be $S(x, x')$, with x' being either z_i' or z_i'' (if $x = u_i$) or $g(x_i')$ or $g(x_i'')$ (if $x = t_i$) for some i.

The only way to reach y from x' in less than $2m' + 4$ is through a S-children y' of y.

Now, $x \neq y$, hence $\mathrm{dist}_{\mathcal{S}\mathrm{upp}_0(\mathcal{R}')}(x', y') \geq 2m' + 3$ (either they are endpoints of two different threads, either the thread they're both in doesn't have the matching endpoint property, which ensures that it is of length $> 2n + 1$). We can thus apply Lemma 21, which states that the path of length $\leq 2m' + 3$ from x' to y' must go through either x or y. This contradicts the minimality hypothesis.

We can proceed to the sequence of m'-guarded crossing-S-swap between u_i and t_i in $\mathcal{R}'$ for every i.

After the r swaps, we end up with $\mathcal{R}'' \equiv_{\alpha}^{<\text{inv FO}} \mathcal{Q}$, such that $\mathcal{S}\text{upp}_{m'+1}(\mathcal{R}'') \simeq \mathcal{S}\text{upp}_{m'+1}(\mathcal{Q})$, and with no misassociation left wrt g.

If the pseudo-inclusion g isn't reduced, we reduce it by eliminating one by one its redundant jumping pairs.

Seeing g as a m-pseudo-inclusion, we get that nodes involved in a jumping pair are at distance $\geq 2m+3$ from the endpoints in $\mathcal{S}\text{upp}_0(\mathcal{R}'')$.

Let $\{x, x'\}$ and $\{y, y'\}$ be two jumping pairs with the same type, and u', u (resp v', v) be the E-neighbors of $g(x), g(x')$ (resp. $g(y), g(y')$) in $[g(x), g(x')]$ (resp. $[g(y), g(y')]$). We have that $\text{tp}^m_{\mathcal{R}''}(g(x), u') = \text{tp}^m_{\mathcal{R}''}(u, g(x')) = \text{tp}^m_{\mathcal{R}''}(g(y), v') = \text{tp}^m_{\mathcal{R}''}(v, g(y'))$

If their images are on two different threads (c.f. Figure 28), we can perform two m-guarded crossing-E-swaps: first, the m-guarded crossing-E-swap between $g(x)u'$ and $vg(y')$ in $\mathcal{R}''$, and then the m-guarded crossing-E-swap between $g(x)g(y')$ and $ug(x')$ in the previous swap, after which $\{x, x'\}$ is no longer a jumping pair.

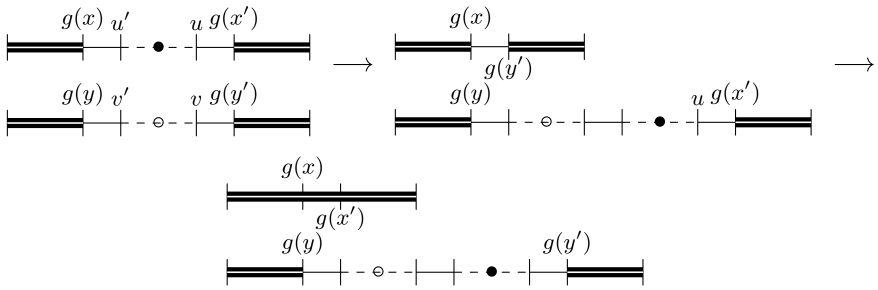

Figure 27 Elimination of a jumping pair among two, in different threads

If their images appear on the same thread in the order $g(y), g(y'), g(x), g(x')$ (c.f. Figure 29), we can perform the m-guarded contiguous-segment-E-swap between $[v', g(x)]$ and $[u', u]$ in $\mathcal{R}''$, after which $\{x, x'\}$ is no longer a jumping pair.

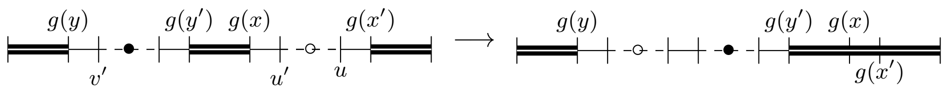

Figure 28 Elimination of a jumping pair among two, with a contiguous-segment-E-swap

Otherwise, we can assume that the images appear in the order $g(y), g(y'), g(x'), g(x)$ (c.f. Figure 30). We can perform two consecutive m-guarded mirror-E-swaps: first the m-guarded mirror-E-swap at $[v', g(x')]$ in $\mathcal{R}''$, and then (in order to reverse again the segment $[g(y'), g(x')]$ into the initial direction) the m-guarded mirror-E-swap at $[g(x'), u']$ in the previous swap, after which $\{x, x'\}$ is no longer a jumping pair.

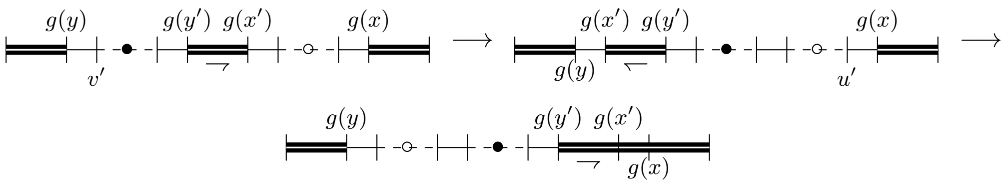

Figure 29 Elimination of a jumping pair among two, with two mirror-E-swaps

In the end, we get $\mathcal{Q}' \equiv_{\alpha}^{<\text{-inv FO}} \mathcal{Q}$, such that $\mathcal{S}\text{upp}_{m+1}(\mathcal{Q}') \simeq \mathcal{S}\text{upp}_{m+1}(\mathcal{Q})$, and a reduced h' that $(m+1)$-pseudo-includes $\mathcal{P}$ in $\mathcal{Q}'$. Notice that during the transformation from g (which was misassociation-free) to h, we never created any misassociation. Hence, $Q' \setminus \text{Im}(h')$ is S-stable.

◀

E Proofs for Section 6 (Removing unnecessary material)

Let $\mathcal{W}_+ \in \mathbb{L}_\sigma^n$, and g be a n-inclusion from $\mathcal{W}$ to some $\mathcal{R} \in$ quasi-$\mathbb{H}_\sigma^n$.

Let $\mathcal{R}_+$ be an extension of $\mathcal{R}$ to Σ_n obtained in the following way. For every $\tau \in \text{Type}_\sigma^n[2]$, and $i \in \{1,2\}$, such that there exists (a unique) $x_\tau^i \in W$ such that $\mathcal{W}_+ \models J_\tau^i(x_\tau^i)$, J_τ^i is interpreted in $\mathcal{R}_+$ as $\{g(x_\tau^i), y_\tau^i\}$, where $y_\tau^i \notin \text{Im}(g)$ and $E(g(x_\tau^i), y_\tau^i)$. The existence and unicity of such y_τ^i is guaranteed. This process is depicted in Figure 31. Every other J_τ^i is interpreted as the empty set. We say that $\mathcal{R}_+$ is the **g-border-extension of $\mathcal{R}$**.

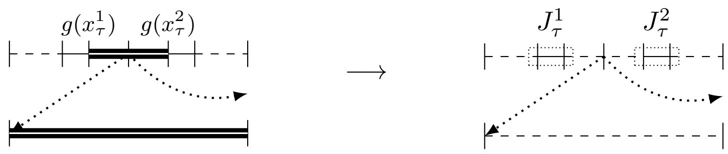

Figure 30 From an inclusion g (double line) of the previous $\mathcal{V}$ in $\mathcal{R}$ to the g-border-extension $\mathcal{R}_+$

Let $\mathcal{I}_n$ be the FO-interpretation from the vocabulary Σ_n to $P_\sigma \cup \{E, S\}$, which adds an E-edge between a and b if $a \neq b$, $J_\tau^i(a)$ and $J_\tau^i(b)$ for some $i \in \{1,2\}$ and τ, and then forgets about the $(J_\tau^i)_{(i,\tau)}$. Every $\mathcal{I}_n$ has arity 1 and depth 0. Hence for every $k \in \mathbb{N}$ and Σ_n-structures $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{A} \equiv_k^{<\text{-inv FO}} \mathcal{B}$ entails $\mathcal{I}_n(\mathcal{A}) \equiv_k^{<\text{-inv FO}} \mathcal{I}_n(\mathcal{B})$.

▶ **Proposition 16.** *For every $\alpha, n \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that for every $\mathcal{P} \in \mathbb{H}_\sigma$, $\mathcal{Q} \in$ quasi-$\mathbb{H}_\sigma^n$ and reduced n-pseudo-inclusion $h : P \to Q$, if $V := Q \setminus \text{Im}(h)$ is S-stable then there exists some $\mathcal{Q}' \in$ quasi-$\mathbb{H}_\sigma^n$ and a reduced n-pseudo-inclusion $h' : P \to Q'$ such that $\mathcal{Q}' \equiv_\alpha^{<\text{-inv FO}} \mathcal{Q}$, $U := Q' \setminus \text{Im}(h')$ is S-stable and $|U| \leq N$.*

Proof. For every equivalence class $\mathcal{C}$ of $\equiv_\alpha^{<\text{-inv FO}}$ on $\mathbb{L}_\sigma^n$, pick a representative $\mathcal{U}_+^{\mathcal{C}}$. Now, set $N := \max\{|U_+^{\mathcal{C}}| : \mathcal{C}$ equivalence class for $\equiv_\alpha^{<\text{-inv FO}}\}$. N is well defined since $\equiv_\alpha^{<\text{-inv FO}}$ is of finite index.

Let $\mathcal{Q}_+$ be a h-jump-extension of $\mathcal{Q}$.

Let $\mathcal{U}_+$ be the representative of the class of $\mathcal{V}_+ := \text{Ctxt}_n(\mathcal{Q}_+|_V)$.

Since V is S-stable in $\mathcal{Q}$, $\mathcal{Q}^+ \setminus \{E(h(x), u'), E(u, h(x')) : \{x, x'\}$ jumping pair$\}$ can be decomposed as $\mathcal{V}_+ \uplus \mathcal{R}_+$ for some Σ_n-structure $\mathcal{R}_+$.

Note that $\mathcal{Q} = \mathcal{I}_n(\mathcal{V}_+ \uplus \mathcal{R}_+)$. We set $\mathcal{Q}' := \mathcal{I}_n(\mathcal{U}_+ \uplus \mathcal{R}_+)$ and $h' := h$ (this makes sense since $R = \text{Im}(h)$).

By definition of $\mathcal{U}_+$, $\mathcal{U}_+ \uplus \mathcal{R}_+ \equiv_\alpha^{<\text{-inv FO}} \mathcal{V}_+ \uplus \mathcal{R}_+$. Applying $\mathcal{I}_n$ yields $\mathcal{Q}' \equiv_\alpha^{<\text{-inv FO}} \mathcal{Q}$,

It remains to show that h' is a n-pseudo-inclusion. Every thread of $\mathcal{P}$ is still sent on a single thread: indeed, for every jumping pair $\{x, x'\}$ for h, $h'(x)$ and $h'(x')$ lie on the same thread. This is because in $\mathcal{U}_+ \in \mathbb{L}_\sigma^n$.

All that's left to prove is that for every $a \in R, \text{tp}_{\mathcal{Q}'}^n(a) = \text{tp}_{\mathcal{Q}}^n(a)$. This follows from the fact that for every τ and $i \in \{1,2\}$, the element of $\mathcal{U}_+$ colored with J_τ^i and the element of $\mathcal{V}_+$ coloured with J_τ^i (if they exist) have the same n-type, once again because $\mathcal{U}_+ \in \mathbb{L}_\sigma^n$. ◀

We now turn to loop elimination.

Our goal is to get rid of the extra material found outside of the image of the pseudo-inclusion. For that, we make sure it is S-stable (Proposition 15), we minimize it (Proposition 16), then we include (Proposition 13) a great number a times this loop in $\mathcal{Q}$. However, to be able to remove a copy while staying in the same $\equiv_\alpha^{<\text{-inv FO}}$-class, we need to recreate every of these loops to the original cape we included: recall indeed that the inclusion preserves the E-edges, but not necessarily the S-edges.

The following lemma gives a method to modify the including structure so that the pseudo-inclusion respects S-edges.

▶ **Lemma 25.** *$\forall \alpha, n, \exists N, \forall M, \exists D \in \mathbb{N}, \forall \mathcal{Q} \in$ quasi-$\mathbb{H}_\sigma^{N+1}, \forall \mathcal{W} \in$ Ctxt$_\sigma^{N+1}$ such that $|W| \leq M$, for all $(N+1)$-inclusion $h : W \to Q$ such that for every $(N+1)$-type τ that occurs in $\mathcal{W}$, there are at least D elements of type τ in $Q \setminus \mathrm{Im}(h)$, there exist some $\mathcal{Q}' \in$ quasi-$\mathbb{H}_\sigma^n$ and some g that n-includes $\mathcal{W}$ into $\mathcal{Q}'$ such that $\mathcal{Q}' \equiv_\alpha^{<\text{-inv FO}} \mathcal{Q}$, $\mathcal{S}upp_n(\mathcal{Q}') \simeq \mathcal{S}upp_n(\mathcal{Q})$ and $\mathcal{W} \models S(x,y) \to \mathcal{Q}' \models S(g(x), g(y))$.*

Proof. We can assume that $n \geq s(\alpha)$. We'll assign values to m and N later on, in that order.

Keep in mind from Note 7 that a crossing-S-swap is guarded as long as it happens between elements of same type that are distant enough. First, we re-associate the S-edges going in/out of the images of every S-parent and S-child. The hypothesis on the number of excess occurrences of every type allows us to scatter their S-neighbors across the including structure. Recall that we introduced the notion of scattering in Definition 23.

Let's enumerate arbitrarily as $(x_1, x_1', x_1''), \cdots, (x_r, x_r', x_r'')$ the elements of $\mathcal{W}$ such that $S(x_i, x_i') \wedge S(x_i, x_i'') \wedge x_i' \neq x_i''$.

First, we use Lemma 14 to find $\mathcal{R} \equiv_\alpha^{<\text{-inv FO}} \mathcal{Q}$ such that $\mathcal{S}\mathrm{upp}_{m+1}(\mathcal{R}) \simeq \mathcal{S}\mathrm{upp}_{m+1}(\mathcal{Q})$ and g such that $g : W \to R$ is a $(m+1)$-pseudo-inclusion that respects the S-siblings relation, for some m to be specified later on. This sets the value for N.

Let $B := \{b_1, b_1', b_1'', \cdots, b_s, b_s', b_s''\}$ be such that b_i', b_i'' are the S-children of b_i and $\forall i \leq r, \exists j, k \leq s,\ g(x_i) = b_j, g(x_i') = b_k'$ and $g(x_i'') = b_k''$ (the existence of a k comes from the fact that g respects the S-siblings relation). Note that the minimal such B is $\mathrm{Im}(g)$ plus the S-children of every $g(x_i)$ (if they are not already in $\mathrm{Im}(g)$), plus the S-parent of every $g(x_i'), g(x_i'')$ (if it's not already in $\mathrm{Im}(g)$). This guarantees that $s \leq 2r$.

Every hollow 1-quasitree has degree at most 4. In $\mathcal{R}$, $|\mathrm{Im}(g)| \leq M$; hence as long as D is large enough, there must exist elements $(a_i)_{1 \leq i \leq s} \in R$, such that for every i, a_i', a_i'' being the S-children of a_i in $\mathcal{R}$, $\mathrm{tp}^m_{\mathcal{R}_0}(a_i, a_i', a_i'') = \mathrm{tp}^m_{\mathcal{R}}(b_i, b_i', b_i'')$, and $\{a_1\}, \cdots, \{a_s\}, B$ are pairwise $(2m+5)$-distant in $\mathcal{R}$, where $m := 2n+5$.

We are in the right conditions to apply Lemma 24, and get $\mathcal{R}' \equiv_\alpha^{<\text{-inv FO}} \mathcal{R}$, with $\mathcal{S}\mathrm{upp}_{2n+6}(\mathcal{R}') = \mathcal{S}\mathrm{upp}_{2n+6}(\mathcal{R})$ and $\{a_1\}, \{a_1', a_1''\}, \cdots, \{a_s\}, \{a_s', a_s''\}$ are $(2n+5)$-scattered wrt. B in $\mathcal{R}'$. Note that $g : W \to R'$ still preserves the S-siblings relation.

Not all of the a_i, a_i', a_i'' are of interest. We re-index them, and focus on $u_1, u_1', u_1'', \cdots, u_r, u_r', u_r''$, where u_i is the S-parent of $g(x_i'), g(x_i'')$ and u_i', u_i'' are the S-children of $g(x_i)$.

The scattering of the a_i, a_i', a_i'' entails that $\{u_1\}, \{u_1', u_1''\}, \cdots, \{u_r\}, \{u_r', u_r''\}$ are $(2n+5)$-scattered wrt. $\mathrm{Im}(g)$ in $\mathcal{R}'$.

Set $\mathcal{W}_i := \mathcal{W} \uplus \{\bar{x}_{i+1}, \cdots, \bar{x}_r\}$ where, for every $j > i$, $S(x_j, x_j')$ and $S(x_j, x_j'')$ have been replaced by $S(\bar{x}_j, x_j')$ and $S(\bar{x}_j, x_j')$. There cannot be a path of length $\leq 2n+5$ from x_j' (or x_j'') to x_j, as long as $N+1 \geq 2n+5$, for otherwise there would be a vertical loop in $\mathrm{tp}_{\mathcal{W}}^{N+1}(x_j)$.

Now, let's re-associate the S-edges back so that g respects S. We construct a sequence of structures $\mathcal{T}_0 \equiv_\alpha^{<\text{-inv FO}} \cdots \equiv_\alpha^{<\text{-inv FO}} \mathcal{T}_r$ having the same $(n+1)$-enriched support, where $\mathcal{T}_i$ is the simultaneous crossing-S-swap between $g(x_j)$ and u_j in $\mathcal{R}'$ for $j \leq i$.

Let's prove that for every i, $\mathcal{T}_i$ verifies (Q_i):

1. $\{u_{i+1}\}, \{u'_{i+1}, u''_{i+1}\}, \cdots, \{u_r\}, \{u'_r, u''_r\}$ are $(2n+5)$-scattered wrt. $\mathrm{Im}(g)$
2. $\forall j, k > i$, let $a_j \in \{x'_j, x''_j\}$. Then

$$\mathrm{dist}_{\mathcal{T}_i}(g(a_j), g(x_k)) \geq \min(\mathrm{dist}_{\mathcal{W}_i}(a_j, x_k), 2n+6)$$

3. $\forall j \neq k > i, a_j \in \{x'_j, x''_j\}$ and $a_k \in \{x'_k, x''_k\}$, $\mathrm{dist}_{\mathcal{T}_i}(g(a_j), g(a_k)) > 2n+5$

Set $\mathcal{T}_0 := \mathcal{R}'$.

We check that (Q_0).2 holds, for x'_j and x_k (it is similar for x''_j). Let them be such that $\mathrm{dist}_{\mathcal{T}_0}(g(x'_j), g(x_k)) \leq 2n+5$, and let's prove that $\mathrm{dist}_{\mathcal{W}_0}(g(x'_j), g(x_k)) \leq \mathrm{dist}_{\mathcal{T}_0}(g(x'_j), g(x_k))$. Consider a shortest path from $g(x'_j)$ to $g(x_k)$ in $\mathcal{T}_0$.

Suppose it goes through at least one S-edge: the first time it does, it must be one that goes out of the thread containing $g(x'_j)$, which is contained (because g is an inclusion) in $\mathrm{Im}(g)$. (Q_0).1 rules out the possibility for this S-edge to be of the form $S(g(x_l), u'_l)$ (or $S(g(x_l), u''_l)$): from u'_l, the only way to reach $g(x_k)$ in $\leq 2n+4$ is through $S(g(x_l), u''_l)$, which contradicts the minimality of this path.

Moreover, it cannot be the S-edge landing on the other endpoint of the thread, since this would mean that the thread is of length $\leq 2n+4$, and since $\mathcal{R}' \in$ quasi-$\mathbb{H}_\sigma^{2n+6}$, the other endpoint is guaranteed to be $g(x''_j)$. In this case, there would be a shortest path from $g(x'_j)$ to u_j, which would directly borrow $S(u_j, g(x'_j))$.

Hence, the first S-edge can only be $S(u_j, g(x'_j))$, and (Q_0).1 ensures that the only way this would result in a path of length $\leq 2n+5$ is if the second edge it goes through is $S(u_j, g(x''_j))$, from which we can repeat the same reasoning to prove that from there, the path doesn't go through any S-edge.

The other possibility is that the path doesn't go through any S-edge. In either case, it means that $g(x'_j)$ or $g(x''_j)$ and $g(x_k)$ are on the same thread, and the shortest path follows the E-edges of this thread. Hence, a path as short exists in $\mathcal{W}_0$ between x'_j and x_k.

We now check that (Q_0).3 holds: let x'_j and x'_k (and similarly for x''_j and for x''_k) be such that $\mathrm{dist}_{\mathcal{T}_0}(g(x'_j), g(x'_k)) \leq 2n+5$, and consider a shortest path from $g(x'_j)$ to $g(x'_k)$.

The same reasoning as before ensures that $g(x'_k)$ is on the same thread as $g(x'_j)$ or $g(x''_j)$, and that the shortest path follows the E-edges of that thread, which must then be of length $\leq 2n+5$ which in turn implies that $j = k$.

Now suppose that we have constructed $\mathcal{T}_i$ and let $\mathcal{T}_{i+1}$ be the n-guarded crossing-S-swap between $g(x_{i+1})$ and u_{i+1} in $\mathcal{T}_i$. Suppose that $\mathrm{dist}_{\mathcal{T}_i}(g(x_{i+1}), u_{i+1}) < 2n+5$: then $\mathrm{dist}_{\mathcal{T}_i}(g(x_{i+1}), g(x'_{i+1})) \leq 2n+5$, and (Q_i).2 ensures that $\mathrm{dist}_{\mathcal{W}_i}(x_{i+1}, x'_{i+1}) \leq 2n+5$, which, as seen above, is absurd.

Lemma 8 ensures that $\mathcal{T}_{i+1} \equiv_\alpha^{<\text{-inv FO}} \mathcal{T}_i$ and $\mathcal{S}\mathrm{upp}_{n+1}(\mathcal{T}_{i+1}) = \mathcal{S}\mathrm{upp}_{n+1}(\mathcal{T}_i)$. Let's show that $\mathcal{T}_{i+1}$ verifies (Q_{i+1}):

(Q_{i+1}).1 is straightforward: we only need to note that the new S-edges appeared at distance $\geq 2n+4$ from every $A \in \{\{u_{i+2}\}, \{u'_{i+2}, u''_{i+2}\}, \cdots, \{u_r\}, \{u'_r, u''_r\}\}$, in $\mathcal{T}_i \setminus S(A)$.

(Q_{i+1}).2 : let $j, k > i+1$ and suppose that there is a path of length $l \leq 2n+5$ between $g(x'_j)$ (and similarly for $g(x''_j)$) and $g(x_k)$ in $\mathcal{T}_{i+1}$, and consider a shortest such path. Let's show that $\mathrm{dist}_{\mathcal{W}_{i+1}}(x'_j, x_k) \leq l$. If this path doesn't go through any of the new S-edges,

(Q_i).2 allows us to conclude (any path going through $\bar{x}_{i+1}$ in $\mathcal{W}_i$ can now through x_{i+1} instead).

Otherwise, (Q_i).1 ensures that it doesn't go through $S(u_{i+1}, u'_{i+1})$ or $S(u_{i+1}, u''_{i+1})$. Thus we can decompose this path in a sequence of two (since it's a shortest path) paths valid in $\mathcal{T}_i$, joined by either $S(g(x_{i+1}), g(x'_{i+1}))$, or $S(g(x_{i+1}), g(x''_{i+1}))$, or one then the other.

It is not possible for the path to be decomposable as $g(x'_j) \overset{p_1}{\rightsquigarrow} g(a_{i+1})Sg(x_{i+1}) \overset{p_2}{\rightsquigarrow} g(x_k)$ (for $a_{i+1} \in \{x'_{i+1}, x''_{i+1}\}$), because p_1 would be a path of length $\leq 2n + 5$ in $\mathcal{T}_i$ from $g(x'_j)$ to $g(a_{i+1})$, which contradicts (Q_i).3

Hence the path can be decomposed as $g(x'_j) \overset{p_1}{\rightsquigarrow} g(x_{i+1})Sg(a_{i+1}) \overset{p_2}{\rightsquigarrow} g(x_k)$, with p_1 and p_2, of respective length l_1 and l_2 (with $l = l_1 + l_2 + 1$) being valid in $\mathcal{T}_i$.

(Q_i).2 allows us to reflect p_1 as a path from x'_j to x_{i+1} in $\mathcal{W}_i$ of length $\leq l_1$, and p_2 as a path from a_{i+1} to x_k in $\mathcal{W}_i$ of length $\leq l_2$. Replacing $\bar{x}_{i+1}$ by x_{i+1} in those paths gives us paths at least as short valid in $\mathcal{W}_{i+1}$. We then link them with $S(x_{i+1}, a_{i+1}) \in \mathcal{W}_{i+1}$, and get that $\text{dist}_{\mathcal{W}_{i+1}}(x'_j, x_k) \leq l_1 + l_2 + 1 = l$.

(Q_{i+1}).3 : let $j \neq k > i + 1$, $a_j \in \{x'_j, x''_j\}$ and $a_k \in \{x'_k, x''_k\}$.

Suppose that there is a path (take a shortest witness) p of length $\leq 2n + 5$ between $g(a_j)$ and $g(a_k)$ in $\mathcal{T}_{i+1}$. Because of (Q_i).3, p cannot be valid in $\mathcal{T}_i$. Because of (Q_i).1, it cannot go through $S(u_{i+1}, u'_{i+1})$ or $S(u_{i+1}, u''_{i+1})$. Hence, it must go through $S(g(x_{i+1}), g(x'_{i+1}))$ or $S(g(x_{i+1}), g(x''_{i+1}))$. It cannot go through both, for otherwise we could replace $g(x'_{i+1})Sg(x_{i+1})Sg(x''_{i+1})$ in p by $g(x'_{i+1})Su_{i+1}Sg(x''_{i+1})$ and get a path as short in $\mathcal{T}_i$.

We can decompose p either as $g(a_j) \overset{p_1}{\rightsquigarrow} g(a_{i+1})Sg(x_{i+1}) \overset{p_2}{\rightsquigarrow} g(a_k)$ or, if it goes through the S-edge in the other direction, as $g(a_j) \overset{p_1}{\rightsquigarrow} g(x_{i+1})Sg(a_{i+1}) \overset{p_2}{\rightsquigarrow} g(a_k)$, with $a_{i+1} \in \{x'_{i+1}, x''_{i+1}\}$ and p_1, p_2 valid in $\mathcal{T}_i$, and of length $\leq 2n + 5$.

This is absurd since either p_1 or p_2 breaks (Q_i).3

We set $\mathcal{Q}' := \mathcal{T}_r$ together with g, which have the desired properties. ◀

▶ **Proposition 17.** *$\forall \alpha \in \mathbb{N}, \exists l \in \mathbb{N}, \forall m \in \mathbb{N}, \exists n \in \mathbb{N}, \forall M \in \mathbb{N}, \exists K \in \mathbb{N}$ such that for every abstract loop $\mathcal{U}_+ \in \mathbb{L}^{n+1}_\sigma$ and every $\mathcal{Q} \in$ quasi-$\mathbb{H}^{n+1}_\sigma$ such that $|U| \leq M$, $(l+1) \cdot [\![\mathcal{E}_{n+1}(\mathcal{U})]\!] < [\![\mathcal{E}_{n+1}(\mathcal{Q})]\!]$ and such that for every $(n+1)$-type χ that occurs in $\mathcal{U}$, $|\mathcal{Q}|_\chi \geq K$, there exists $\mathcal{Q}' \in$ quasi-$\mathbb{H}^m_\sigma$ such that $\mathcal{Q}' \equiv^{<\text{-inv FO}}_\alpha \mathcal{Q}$ and $[\![\mathcal{E}_m(\mathcal{Q})]\!] = [\![\mathcal{E}_m(\mathcal{Q}')]\!] + [\![\mathcal{E}_m(\mathcal{U})]\!]$*

Proof. We can assume that $m \geq s(\alpha)$. Let m_1 be given by Lemma 25 from m, D be given by Lemma 25 from m and M and n be given by Lemma 13 from m_1. Set $K := D + (l+1)M$

We construct $\mathcal{U}^l_+, \mathcal{U}^{l+1}_+ \in \mathbb{L}^{n+1}_\sigma$, such that

- $[\![\mathcal{E}_{n+1}(\mathcal{U}^l)]\!] = l \cdot [\![\mathcal{E}_{n+1}(\mathcal{U})]\!]$
- $[\![\mathcal{E}_{n+1}(\mathcal{U}^{l+1})]\!] = (l+1) \cdot [\![\mathcal{E}_{n+1}(\mathcal{U})]\!]$
- $\mathcal{U}^l_+ \equiv^{<\text{-inv FO}}_\alpha \mathcal{U}^{l+1}_+$

Consider the FO-interpretation $\mathcal{J}$ (of arity 2 and depth d, independent of n) from the vocabulary $\Sigma_{n+1} \cup \{N, <\}$ (where N is a unary relational symbol) to $\Sigma_{n+1} \cup \{<\}$, which, given a structure $\mathcal{V}_+$, returns $\mathcal{J}(\mathcal{V}_+)$ as follows. For the sake of simplicity, we will name $1, \cdots, r$ the elements of $N^{\mathcal{V}_+}$ accordingly to $<^{\mathcal{V}_+}$

- its universe is $\{1, \cdots, r\} \times (V \setminus N^{\mathcal{V}_+})$
- $\mathcal{J}(\mathcal{V}_+) \models S((i, x), (j, y))$ iff $i = j$ and $\mathcal{V}_+ \models S(x, y)$
- $\mathcal{J}(\mathcal{V}_+) \models E((i, x), (j, y)) \wedge E((j, y), (i, x))$ iff $i = j$ and $\mathcal{V} \models E(x, y)$, or $j = i + 1$ and $\mathcal{V}_+ \models J^2_\tau(x)$ and $\mathcal{V}_+ \models J^1_\tau(y)$ for some τ
- for every τ, $\mathcal{J}(\mathcal{V}_+) \models J^1_\tau(i, x)$ iff $i = 1$ and $\mathcal{V}_+ \models J^1_\tau(x)$

- for every τ, $\mathcal{J}(\mathcal{V}_+) \models J^2_\tau(i,x)$ iff $i = r$ and $\mathcal{V}_+ \models J^2_\tau(x)$
- $<^{\mathcal{J}(\mathcal{V}_+)}$ is the lexicographical order

In other words, if we add r elements to the abstract loop $\mathcal{U}_+$, color them with N and add an order, its image by $\mathcal{J}$ is the r-fold concatenation of $\mathcal{U}_+$ to itself (in the same direction each time), with an order.

Fix an arbitrary order $<^U$ on U. For $r \in \mathbb{N}$, let $\mathcal{U}_+^{[r]}$ be the $\Sigma_{n+1} \cup \{N, <\}$-structure obtained by adding $\{1, \cdots, r\}$ to the universe of $\mathcal{U}_+$, interpreting N as $\{1, \cdots, r\}$ and ordering the elements as $1, \cdots, r$ and then accordingly to $<^U$

Now let $(\mathcal{U}_+^r, <^r) := \mathcal{J}(\mathcal{U}_+^{[r]})$. See Figure 32 for an example.

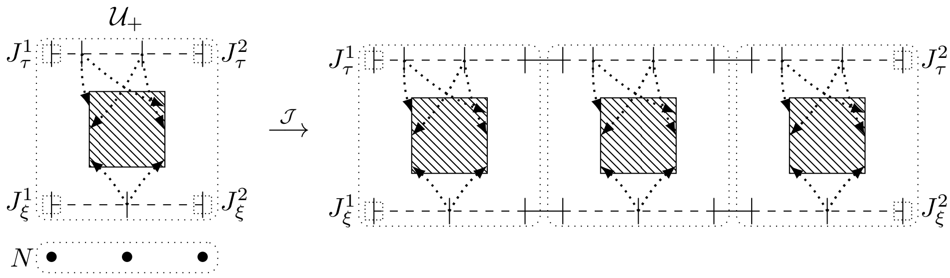

Figure 31 Application of $\mathcal{J}$ to $\mathcal{U}_+^{[3]}$. In this illustration, two jumping pair types τ and ξ are relevant in $\mathcal{U}_+$. The new order $<^3$ is the concatenation of the old ones

If we choose $l := 2^{2\alpha+d}$, we have $\mathcal{U}_+^{[l]} \equiv^{\text{FO}}_{2\alpha+d} \mathcal{U}_+^{[l+1]}$, hence $(\mathcal{U}_+^l, <^l) \equiv^{\text{FO}}_\alpha (\mathcal{U}_+^{l+1}, <^{l+1})$, and $\mathcal{U}_+^l \equiv^{<\text{-inv FO}}_\alpha \mathcal{U}_+^{l+1}$.

By construction, $[\![\mathcal{E}_{n+1}(\mathcal{U}^l)]\!] = l \cdot [\![\mathcal{E}_{n+1}(\mathcal{U})]\!]$ and $[\![\mathcal{E}_{n+1}(\mathcal{U}^{l+1})]\!] = (l+1) \cdot [\![\mathcal{E}_{n+1}(\mathcal{U})]\!]$

By hypothesis, $[\![\mathcal{E}_{n+1}(\mathcal{U}^{l+1})]\!] < [\![\mathcal{E}_{n+1}(\mathcal{Q})]\!]$, thus we can apply Proposition 13 to get $\mathcal{R} \equiv^{<\text{-inv FO}}_\alpha \mathcal{Q}$ such that $[\![\mathcal{E}_{m_1+1}(\mathcal{R})]\!] = [\![\mathcal{E}_{m_1+1}(\mathcal{Q})]\!]$ and a (m_1+1)-inclusion h from $\mathcal{U}^{l+1}$ to $\mathcal{R}$.

Now, for every (m_1+1)-type ξ occurring in $\mathcal{U}^{l+1}$, $|\mathcal{R}|_\xi = |\mathcal{Q}|_\xi \geq K$, hence $|\mathcal{R}|_{R\setminus \text{Im}(h)}|_\xi \geq D$ by choice of K.

We can apply Lemma 25, which yields some $\mathcal{R}^{l+1} \equiv^{<\text{-inv FO}}_\alpha \mathcal{R}$ such that $\mathcal{S}\text{upp}_m(\mathcal{R}^{l+1}) \simeq \mathcal{S}\text{upp}_m(\mathcal{R})$, and g m-includes $\mathcal{U}^{l+1}$ in $\mathcal{R}^{l+1}$, and respects S.

Let $\mathcal{R}_+^{l+1}$ be the g-border-extension of $\mathcal{R}^{l+1}$.

Since $g(\mathcal{U}^{l+1})$ is S-stable in $\mathcal{R}^{l+1}$, we can decompose $\mathcal{R}_+^{l+1} \setminus \{E(x,y) : x, y \in R^{l+1}, i \in \{1,2\}, J^i_\tau(x) \wedge J^i_\tau(y)\}$ as $g(\mathcal{U}_+^{l+1}) \uplus \mathcal{R}'_+$ for some Σ_n structure $\mathcal{R}'_+$, where $g(\mathcal{U}_+^{l+1})$ is the abstract loop based upon $g(\mathcal{U}^{l+1})$ such that g respects every J^i_τ.

Note that $\mathcal{R}^{l+1} = \mathcal{I}_m(g(\mathcal{U}_+^{l+1}) \uplus \mathcal{R}'_+)$, and let $\mathcal{R}^l := \mathcal{I}_m(g(\mathcal{U}_+^l) \uplus \mathcal{R}'_+)$.

$g(\mathcal{U}_+^l) \uplus \mathcal{R}'_+ \equiv^{<\text{-inv FO}}_\alpha g(\mathcal{U}_+^{l+1}) \uplus \mathcal{R}'_+$, hence $\mathcal{R}^l \equiv^{<\text{-inv FO}}_\alpha \mathcal{R}^{l+1}$

Now, set $\mathcal{Q}' := \mathcal{R}^l$. We have that $\mathcal{Q}' \equiv^{<\text{-inv FO}}_\alpha \mathcal{Q}$, and, by construction, $[\![\mathcal{E}_m(\mathcal{Q})]\!] = [\![\mathcal{E}_m(\mathcal{U}^{l+1})]\!] + [\![\mathcal{E}_m(\mathcal{R}')]\!] = [\![\mathcal{E}_m(\mathcal{U})]\!] + [\![\mathcal{E}_m(\mathcal{U}^l)]\!] + [\![\mathcal{E}_m(\mathcal{R}')]\!] = [\![\mathcal{E}_m(\mathcal{U})]\!] + [\![\mathcal{E}_m(\mathcal{Q}')]\!]$ ◀

The notion of *vertical swap* in a tree has been introduced in [5] and is a crucial operation in their proof. We need here a version of these vertical swaps adapted to hollow trees. Unlike the other swaps, vertical-S-swap preserve hollow trees. In the following, $\mathcal{T}$ is a hollow tree on σ.

We start by defining classical notions making use of the tree structure of $\mathcal{T}$.

The (strict) ancestor relation within a hollow tree is inherited from the original tree and is denoted by $x \preceq y$ (resp. $x \prec y$). Note that this relation is not part of the schema and not expressible in FO from E and S.

Let x, y be two nodes of T such that $x \prec y$. We define the **context** $\mathcal{C}_{\mathcal{T}}(x, y)$ **at** x **and** y **in** $\mathcal{T}$ (referred using the simplified notation $\mathcal{C}$ in the following) as the substructure of $\mathcal{T}$ induced by the set $\{z \in T : x \prec z \ \wedge \ y \not\prec z\}$, with three distinguished nodes colored by two new unary predicates $\top$ and $\bot$: the S-children x' and x'' of x are $\mathcal{C}$'s **top-anchors** ($\top^{\mathcal{C}} = \{x', x''\}$), and y its **bottom-anchor** ($\bot^{\mathcal{C}} = \{y\}$). The set $V(\mathcal{C}) := \{z \in C : z \preceq y\}$ is the set of **vertebræ** of $\mathcal{C}$. The **height** height$(\mathcal{C})$ is $|V(\mathcal{C})|$ and correspond to the difference of depth between y and x. Given $n \in \mathbb{N}$, $\mathcal{C}$'s n**-skeleton**, denoted $\mathrm{Sk}_n(\mathcal{C})$, is the substructure of $\mathcal{C}$ induced by the nodes at distance at most n of $V(\mathcal{C})$, of S-children of nodes of $V(\mathcal{C})$, or of $\mathcal{C}$'s top-anchors. Additionally, $\mathrm{Sk}_n(\mathcal{C})$ inherits the restriction of $\prec$ to $V(\mathcal{C})$. Two contexts are said to be n**-similar** if their n-skeletons are isomorphic. Given two contexts $\mathcal{C}$ and $\mathcal{D}$, we denote by $\mathcal{C} \cdot \mathcal{D}$ the context obtained as the disjoint union of $\mathcal{C}$ and $\mathcal{D}$, with a S-edge from $\mathcal{C}$'s bottom-anchor to each of $\mathcal{D}$'s top-anchor, and where the anchors are redefined in the natural way: $\top^{\mathcal{C}\cdot\mathcal{D}} := \top^{\mathcal{C}}$ and $\bot^{\mathcal{C}\cdot\mathcal{D}} := \bot^{\mathcal{D}}$. Similarly, we define the **prefix** $\mathcal{P}_{\mathcal{T}}(y)$ **at** y **in** $\mathcal{T}$ as the substructure of $\mathcal{T}$ induced by $\{z \in T : y \not\prec z\}$ (the only additional relation being $\bot$), and the **suffix** $\mathcal{S}_{\mathcal{T}}(x)$ **at** x **in** $\mathcal{T}$ as the substructure of $\mathcal{T}$ induced by $\{z \in T : x \prec z\}$ (here, the only additional relation is $\top$). The concatenation between a prefix and a context, a prefix and a suffix, and a context and a suffix are defined in the natural way (and results respectively in a prefix, a hollow tree, and a suffix). Concatenation is associative.

Let $x \prec x_A \prec x_B \prec x_C \in T$, and x', x'', x'_A, x''_A, x'_B, x''_B, x'_C, x''_C be their respective S-children. Suppose that $\mathrm{tp}^k_{\mathcal{T}}(x, x', x'') = \mathrm{tp}^k_{\mathcal{T}}(x_B, x'_B, x''_B)$ and $\mathrm{tp}^k_{\mathcal{T}}(x_A, x'_A, x''_A) = \mathrm{tp}^k_{\mathcal{T}}(x_C, x'_C, x''_C)$. Let us define $\mathcal{P} := \mathcal{P}_{\mathcal{T}}(x)$, $\mathcal{A} := \mathcal{C}_{\mathcal{T}}(x, x_A)$, $\mathcal{B} := \mathcal{C}_{\mathcal{T}}(x_A, x_B)$, $\mathcal{C} := \mathcal{C}_{\mathcal{T}}(x_B, x_C)$ and $\mathcal{S} := \mathcal{S}_{\mathcal{T}}(x_C)$. With these definitions, $\mathcal{T} = \mathcal{P} \cdot \mathcal{A} \cdot \mathcal{B} \cdot \mathcal{C} \cdot \mathcal{S}$.

In this case, $\mathcal{T}' := \mathcal{P} \cdot \mathcal{C} \cdot \mathcal{B} \cdot \mathcal{A} \cdot \mathcal{S}$ is called the k**-guarded vertical-**S**-swap between** $[x, x_A]$ **and** $[x_B, x_c]$ **in** $\mathcal{T}$, c.f. Figure 33.

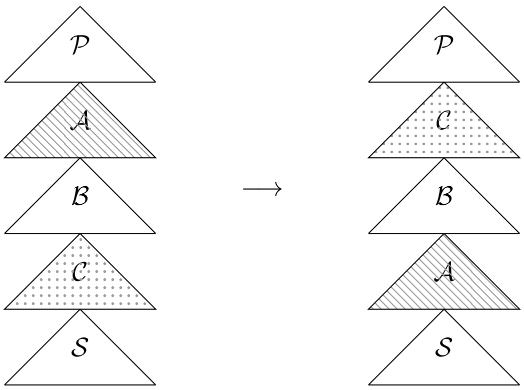

Figure 32 A vertical-S-swap from $\mathcal{T}$ to $\mathcal{T}'$

We wish to show:

▶ **Lemma 26.** *For all $\alpha \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that for every hollow tree $\mathcal{T}$ on σ, if $\mathcal{T}'$ is the N-guarded vertical-S-swap between $[x, x_A]$ and $[x_B, x_C]$ in $\mathcal{T}$, then every node in T has the same $(N+1)$-type in $\mathcal{T}$ and in $\mathcal{T}'$, and $\mathcal{T}' \equiv_\alpha^{<\text{-inv FO}} \mathcal{T}$.*

It is immediate to check that a N-guarded vertical-S-swap preserves $(N+1)$-types. The following is devoted to the proof that it also preserves $\equiv_\alpha^{<\text{-inv FO}}$, concluding the proof of

Lemma 26.

We start by proving a special case of Lemma 26. We will reduce the general case to it. This case is illustrated in Figure 34.

▶ **Lemma 27.** *For all $\alpha \in \mathbb{N}$, there exists a $M \in \mathbb{N}$ such that the following holds.*

Let $\mathcal{T} \in \mathbb{H}_\sigma$ and $x \prec x_A \prec x_B \in T$ having for respective S-children (x', x''), (x'_A, x''_A) and (x'_B, x''_B). Suppose that $tp^M_{\mathcal{T}}(x, x', x'') = tp^M_{\mathcal{T}}(x_A, x'_A, x''_A) = tp^M_{\mathcal{T}}(x_B, x'_B, x''_B)$.

Let $\mathcal{P} := \mathcal{P}_{\mathcal{T}}(x)$, $\mathcal{A} := \mathcal{C}_{\mathcal{T}}(x, x_A)$, $\mathcal{B} := \mathcal{C}_{\mathcal{T}}(x_A, x_B)$ and $\mathcal{S} := \mathcal{S}_{\mathcal{T}}(x_B)$.

Then $\mathcal{P} \cdot \mathcal{A} \cdot \mathcal{B} \cdot \mathcal{S} \equiv^{<\text{-inv FO}}_\alpha \mathcal{P} \cdot \mathcal{B} \cdot \mathcal{A} \cdot \mathcal{S}$

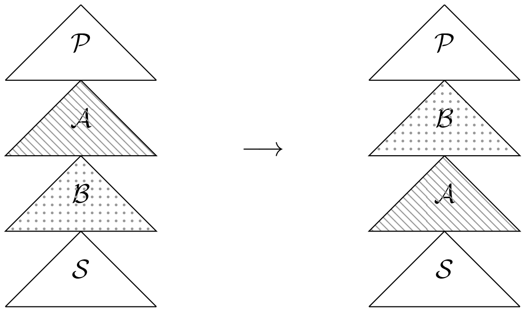

Figure 33 vertical-S-swap (special case) from $\mathcal{P} \cdot \mathcal{A} \cdot \mathcal{B} \cdot \mathcal{S}$ to $\mathcal{P} \cdot \mathcal{B} \cdot \mathcal{A} \cdot \mathcal{S}$

Proof. We will first set $N \in \mathbb{N}$ instead of M, that will be sufficient for most cases. Then, we will define $M \geq N$ which will work for all cases.

Recall the function o^Σ_p introduced in Lemma 6, and consider $n := o^\Sigma_3(\alpha + c)$ where c is to be chosen later on, and $\Sigma := P_\sigma \cup \{E, S, P_{1/2}, P_6\}$ where $P_{1/2}$ and P_6 are new unary symbols. We distinguish between several cases depending on whether x, x_A and x_B are close or not, where "close" is relative to n:

1. Assume first that $\text{tp}^n_{\mathcal{A}}(x', x'', x_A) = \text{tp}^n_{\mathcal{B}}(x'_A, x''_A, x_B)$.
   This case covers the instances where $\mathcal{A}$ and $\mathcal{B}$ are n-similar, as well as those where $\text{dist}_{\mathcal{T}}(x, x_A)$ and $\text{dist}_{\mathcal{T}}(x_A, x_B)$ are $> 2n + 2$.
   Consider the extension $\mathcal{T}^-$ of $\mathcal{P} \uplus \mathcal{A} \uplus \mathcal{B} \uplus \mathcal{S}$ to Σ where the interpretation of $P_{1/2}$ only contains the bottom-anchor of $\mathcal{P}$, and that of P_6 contains the top-anchors of $\mathcal{S}$. Since $P^{\mathcal{T}^-}_{1/2}$ and $P^{\mathcal{T}^-}_6$ are at distance $+\infty$ from $\mathcal{A}$ and $\mathcal{B}$, $\text{tp}^n_{\mathcal{T}^-}(x', x'', x_A) = \text{tp}^n_{\mathcal{T}^-}(x'_A, x''_A, x_B)$.
   Hence, we can apply Lemma 6, and get two orders $<_{AB}$ (whose first elements are $x', x'', x_A, x'_A, x''_A, x_B$) and $<_{BA}$ (whose first elements are $x'_A, x''_A, x_B, x', x'', x_A$) such that $(\mathcal{T}^-, <_{AB}) \equiv^{\text{FO}}_{\alpha+c} (\mathcal{T}^-, <_{BA})$.
   Now, consider the FO-interpretation that adds a S-edge between u and v if either:

   - $P_{1/2}(u)$ and v is either the first or the second element of $<$
   - u is the third element of $<$ and v is either its fourth or fifth one
   - u is the sixth element of $<$ and $P_6(v)$

   and then forgets about $P_{1/2}$ and P_6.
   Take c to be the depth of this FO-interpretation (which has arity 1).
   Note that the result of this FO-interpretation of $(\mathcal{T}^-, <_{AB})$ is an ordered extension of $\mathcal{P} \cdot \mathcal{A} \cdot \mathcal{B} \cdot \mathcal{S}$ and that its result on $(\mathcal{T}^-, <_{BA})$ is an ordered extension of $\mathcal{P} \cdot \mathcal{B} \cdot \mathcal{A} \cdot \mathcal{S}$
   This entails $\mathcal{P} \cdot \mathcal{A} \cdot \mathcal{B} \cdot \mathcal{S} \equiv^{<\text{-inv FO}}_\alpha \mathcal{P} \cdot \mathcal{B} \cdot \mathcal{A} \cdot \mathcal{S}$

2. Assume next that $\mathcal{B}$ can be decomposed as $\mathcal{B}_1 \cdots \mathcal{B}_k$, where each $\mathcal{B}_i$ is n-similar to $\mathcal{A}$. We can then apply k times Case 1 and obtain $\mathcal{P} \cdot \mathcal{A} \cdot \mathcal{B} \cdot \mathcal{S} \equiv_\alpha^{<\text{-inv FO}} \mathcal{P} \cdot \mathcal{B} \cdot \mathcal{A} \cdot \mathcal{S}$ as desired.
3. From now on, $N \geq l(2n+2)+n$ for a large enough l to be chosen later on.
   As we are not in Case 1, we can restrict our study to the cases where $\text{dist}_\mathcal{T}(x, x_A) \leq 2n+2$ (the cases where $\text{dist}_\mathcal{T}(x_A, x_B) \leq 2n+2$ can be treated similarly).
   We will need the following claims. The first one is just a simple observation.
   ▶ **Claim 28.** *Let $\mathcal{U}, \mathcal{V}$ be two n-similar contexts of $\mathcal{T}$.*
   For every decomposition $\mathcal{U} = \mathcal{U}_1 \cdots \mathcal{U}_p$, there exists a decomposition $\mathcal{V} = \mathcal{V}_1 \cdots \mathcal{V}_p$ such that for every i, $\mathcal{U}_i$ and $\mathcal{V}_i$ are n-similar.

   Proof. Let φ be an isomorphism from $\text{Sk}_n(\mathcal{U})$ to $\text{Sk}_n(\mathcal{V})$, and $x_0 \prec x_1 \prec \cdots \prec x_p \in T$ be such that $\mathcal{U}_i = \mathcal{C}_\mathcal{T}(x_{i-1}, x_i)$. Since φ is $\prec$-monotonous on $V(\mathcal{U})$, the $\mathcal{V}_i = \mathcal{C}_\mathcal{T}(\varphi(x_{i-1}), \varphi(x_i))$ are well-defined.
   We have that $\mathcal{V} = \mathcal{V}_1 \cdots \mathcal{V}_p$; it remains to show that for every i, $\mathcal{U}_i$ and $\mathcal{V}_i$ are n-similar. Again, the $\prec$-monotonicity of φ entails that $\varphi(\text{Sk}_n(\mathcal{U}_i)) \subseteq \text{Sk}_n(\mathcal{V}_i)$, which allows us to conclude. ◁

   The next one is a variant of Lyndon-Schützenberger Theorem stated for contexts of hollow trees instead of words.
   ▶ **Claim 29.** *Let $n \in \mathbb{N}$, let $\mathcal{T} \in \mathbb{H}_\sigma$, let $x \prec y \prec z \prec t$ be nodes of T, and let $\mathcal{U} := \mathcal{C}_\mathcal{T}(x,y)$, $\mathcal{V} := \mathcal{C}_\mathcal{T}(y,z)$ and $\mathcal{W} := \mathcal{C}_\mathcal{T}(z,t)$ such that $\mathcal{U}$ and $\mathcal{W}$ are n-similar, $\mathcal{U} \cdot \mathcal{V}$ and $\mathcal{V} \cdot \mathcal{W}$ are n-similar.*
   Then there exist decompositions $\mathcal{U} = \mathcal{U}_1 \cdots \mathcal{U}_p$, $\mathcal{V} = \mathcal{V}_1 \cdots \mathcal{V}_q$, and $\mathcal{W} = \mathcal{W}_1 \cdots \mathcal{W}_p$, where all the $\mathcal{U}_i, \mathcal{V}_i$ and $\mathcal{W}_i$ are n-similar.

   Proof. We define θ_n which maps two successive vertebræ $x_i \prec x_{i+1}$ of a context $\mathcal{U}$ to the type $\text{tp}^n_{\mathcal{C}_\mathcal{U}(x_i, x_{i+1})}(x_i', x_i'', x_{i+1})$, where x_i' and x_i'' are the S-children of x_i.
   Now, let Θ_n be the monoïd morphism from contexts to words extending θ_n; that is, if $x_0 \prec \cdots \prec x_d$ are all the vertebræ of $\mathcal{U}$, then $|\Theta_n(\mathcal{U})| = \text{height}(\mathcal{U}) = d$, and the ith letter of $\Theta_n(\mathcal{U})$ is $\theta_n(x_i, x_{i+1})$.
   Let $u := \Theta_n(\mathcal{U})$, $v := \Theta_n(\mathcal{V})$ and $w := \Theta_n(\mathcal{W})$.
   By n-similarity of $\mathcal{U}$ and $\mathcal{W}$, $u = w$, and by n-similarity of $\mathcal{U} \cdot \mathcal{V}$ and $\mathcal{V} \cdot \mathcal{W}$, $uv = vw$. Hence $uv = vu$.
   By Lyndon-Schützenberger Theorem, there must exist a word a and integers p, q such that $u = w = a^p$ and $v = a^q$ [12].
   We can decompose $\mathcal{U}, \mathcal{V}$ and $\mathcal{W}$ alongside those decompositions of u, v and w, to get $\mathcal{U} = \mathcal{U}_1 \cdots \mathcal{U}_p$, $\mathcal{V} = \mathcal{V}_1 \cdots \mathcal{V}_q$, and $\mathcal{W} = \mathcal{W}_1 \cdots \mathcal{W}_p$, where all the $\mathcal{U}_i, \mathcal{V}_i$ and $\mathcal{W}_i$ are mapped to a by Θ_n, hence are n-similar. ◁

   Let ϕ be an isomorphism between the N-neighborhood of x and that of x_A.
   As $\text{dist}_\mathcal{T}(x, x_A) \leq 2n+2$, x_A is in the N-neighborhood of x and set $x_0 := x_A$ and $x_1 := \phi(x_A)$. Construct by induction $x_{i+1} := \phi(x_i)$ until $i > l$. Our choice of N ensures that x_i is well defined as x_{i-1} remains in the N-neighborhood of x. We claim that for all $j \leq l$, $X_j := \mathcal{C}_\mathcal{T}(x_{j-1}, x_j)$ is n-similar to $\mathcal{A}$. This is a simple consequence of the fact that the n-skeleton of of X_j is included into the N-neighborhood of x.

Likewise, starting from x_B instead of x_A, we show that there exist $y_1, \cdots, y_l \in T$ such that for $j \in [1, l]$ (and with the convention that $y_0 := x_B$), $Y_j := \mathcal{C}_{\mathcal{T}}(y_j, y_{j-1})$ is n-similar to $\mathcal{A}$.
We distinguish several cases:

a. Suppose that $\text{dist}_{\mathcal{T}}(x_A, x_B) \geq 2N$. This ensures that all the $(x_i)_{i\geq 1}$ and $(y_i)_{i\geq 0}$ belong to B.

- Suppose $x_{l-1} \prec y_l$.
If we let $\mathcal{C} := \mathcal{C}_{\mathcal{T}}(x_{l-1}, y_l)$, we can decompose $\mathcal{B}$ as $\mathcal{X}_1 \cdots \mathcal{X}_{l-1} \cdot \mathcal{C} \cdot \mathcal{Y}_l \cdots \mathcal{Y}_1$.
If l is chosen large enough, namely $l \geq 2n+4$, the following decomposition of $\mathcal{T}$

$$\mathcal{P} \cdot \underbrace{\mathcal{A} \cdot \mathcal{X}_1 \cdots \mathcal{X}_{l-1} \cdot \mathcal{C} \cdot \mathcal{Y}_l}_{} \cdot \underbrace{\mathcal{Y}_{l-1} \cdots \mathcal{Y}_1}_{} \cdot \mathcal{S}$$

falls in Case 1 of this Lemma and therefore the following equation holds

$$\mathcal{P} \cdot \underbrace{\mathcal{A} \cdot \mathcal{X}_1 \cdots \mathcal{X}_{l-1} \cdot \mathcal{C} \cdot \mathcal{Y}_l}_{} \cdot \underbrace{\mathcal{Y}_{l-1} \cdots \mathcal{Y}_1}_{} \cdot \mathcal{S} \equiv_{\alpha}^{<\text{-inv FO}} \mathcal{P} \cdot \underbrace{\mathcal{Y}_{l-1} \cdots \mathcal{Y}_1}_{} \cdot \underbrace{\mathcal{A} \cdot \mathcal{X}_1 \cdots \mathcal{X}_{l-1} \cdot \mathcal{C} \cdot \mathcal{Y}_l}_{} \cdot \mathcal{S}$$

Likewise, we get

$$\mathcal{P} \cdot \underbrace{\mathcal{Y}_{l-1} \cdots \mathcal{Y}_1 \cdot \mathcal{A}}_{} \cdot \underbrace{\mathcal{X}_1 \cdots \mathcal{X}_{l-1} \cdot \mathcal{C} \cdot \mathcal{Y}_l}_{} \cdot \mathcal{S} \equiv_{\alpha}^{<\text{-inv FO}} \mathcal{P} \cdot \underbrace{\mathcal{X}_1 \cdots \mathcal{X}_{l-1} \cdot \mathcal{C} \cdot \mathcal{Y}_l}_{} \cdot \underbrace{\mathcal{Y}_{l-1} \cdots \mathcal{Y}_1 \cdot \mathcal{A}}_{} \cdot \mathcal{S}$$

Hence $\mathcal{P} \cdot \mathcal{A} \cdot \mathcal{B} \cdot \mathcal{S} \equiv_{\alpha}^{<\text{-inv FO}} \mathcal{P} \cdot \mathcal{B} \cdot \mathcal{A} \cdot \mathcal{S}$ as desired.
- Suppose now that $x_{l-1} \not\prec y_l$.
Because $\text{dist}_{\mathcal{T}}(x_A, x_B) \geq 2N$, we know that $y_l \not\prec x_{l-1}$: let $\mathcal{T}_1$ be the n-guarded crossing-S-swap between x_{l-1} and y_l in $\mathcal{T}$. Lemma 8 (we can always assume that $n \geq s(\alpha)$) ensures that $\mathcal{T}_1 \equiv_{\alpha}^{<\text{-inv FO}} \mathcal{T}$ and $\forall z \in T$, $\text{tp}_{\mathcal{T}_1}^{n+1}(z) = \text{tp}_{\mathcal{T}}^{n+1}(z)$. We are now in the situation to apply Case 2. It only remains to do again the n-guarded crossing-S-swap between x_{l-1} and y_l afterwards to derive the desired $\mathcal{T}' \equiv_{\alpha}^{<\text{-inv FO}} \mathcal{T}$.

b. Suppose now that $\text{dist}_{\mathcal{T}}(x_A, x_B) < 2N$. Set $M := (2N+1)(2n+2)+n$. Just as before (by replacing l with $2N+1$), we define $x_0, \cdots, x_{2N+1}$ and $y_0, \cdots, y_{2N+1}$, and accordingly, $\mathcal{X}_1, \cdots, \mathcal{X}_{2N+1}$ and $\mathcal{Y}_1, \cdots, \mathcal{Y}_{2N+1}$ that all are n-similar to $\mathcal{A}$.
There are at most $2N$ vertebræ in $\mathcal{B}$, hence not all of the $(y_i)_{0\leq i\leq 2N}$ can be in $\mathcal{B}$. Let k be the smallest index such that y_k is not a vertebrate of $\mathcal{B}$ (we know that $1 \leq k \leq 2N$).
Since $x_A \prec y_{k-1}$ and $y_k \prec y_{k-1}$, x_A and y_k must be related by $\preceq$; by definition of k, we must have $y_k \preceq x_A$. If $y_k = x_A$, we can conclude using Case 2. Otherwise, $y_k \prec x_A \prec y_{k-1}$.
Likewise, either $x \prec y_k$ or $y_k \preceq x$. By n-similarity of $\mathcal{A}$ and $\mathcal{Y}_k$, and because $\text{dist}_{\mathcal{T}}(x, x_A) \leq 2n+2$, we know that $\text{height}(\mathcal{A}) = \text{height}(\mathcal{Y}_k)$. Hence, it cannot be the case that $y_k \preceq x$.
We now have $x \prec y_k \prec x_A \prec y_{k-1}$. Let $\mathcal{U} := C_{\mathcal{T}}(x, y_k), \mathcal{V} := C_{\mathcal{T}}(y_k, x_A)$ and $\mathcal{W} := C_{\mathcal{T}}(x_A, y_{k-1})$. We have that $\mathcal{U} \cdot \mathcal{V} = \mathcal{A}$ and $\mathcal{V} \cdot \mathcal{W} = \mathcal{Y}_k$ are n-similar.
To see that $\mathcal{U}$ and $\mathcal{W}$ are n-similar, look at $\mathcal{Y}_{k+1}$: there is a isomorphism φ from $\text{Sk}_n(\mathcal{Y}_k)$ to $\text{Sk}_n(\mathcal{Y}_{k+1})$, which is by definition $\prec$-monotonous on $V(\mathcal{Y}_k)$. Hence φ sends any vertebrate of $\mathcal{Y}_k$ to the vertebrate of $\mathcal{Y}_{k-1}$ whose depth is $\text{height}(\mathcal{A})$ smaller. This entails $\varphi(x_A) = x$, and by restricting φ, $\mathcal{W}$ and $\mathcal{U}$ are n-similar.
We can now apply Claim 29, and get decompositions $\mathcal{U} = \mathcal{U}_1 \cdots \mathcal{U}_p$, $\mathcal{V} = \mathcal{V}_1 \cdots \mathcal{V}_q$, and $\mathcal{W} = \mathcal{W}_1 \cdots \mathcal{W}_p$, where all the $\mathcal{U}_i, \mathcal{V}_i$ and $\mathcal{W}_i$ are n-similar.

Hence, $\mathcal{A}$ can be decomposed as $\mathcal{U}_1 \cdots \mathcal{U}_p \cdot \mathcal{V}_1 \cdots \mathcal{V}_q$, and such a decomposition can be transposed as in Claim 28 onto each $\mathcal{Y}_i, 0 < i < k$, as $\mathcal{Y}_i = \mathcal{Y}_1^i \cdots \mathcal{Y}_{p+q}^i$, where all the $\mathcal{Y}_j^i$, the $\mathcal{U}_i$, the $\mathcal{V}_i$ and the $\mathcal{W}_i$ are n-similar.

$$\mathcal{T} = \mathcal{P} \cdot \underbrace{\mathcal{U}_1 \cdots \mathcal{U}_p \cdot \mathcal{V}_1 \cdots \mathcal{V}_q}_{\mathcal{A}} \cdot \underbrace{\underbrace{\mathcal{W}_1 \cdots \mathcal{W}_p}_{\mathcal{W}} \cdot \underbrace{\mathcal{Y}_1^{k-1} \cdots \mathcal{Y}_{p+q}^{k-1}}_{\mathcal{Y}_{k-1}} \cdots \underbrace{\mathcal{Y}_1^{1} \cdots \mathcal{Y}_{p+q}^{1}}_{\mathcal{Y}_1}}_{\mathcal{B}} \cdot \mathcal{S}$$

Now, we can use Case 2 with $\mathcal{A} := \mathcal{V}_q$ and derive that $\mathcal{T}$ is $\equiv_\alpha^{<\text{-inv FO}}$ to

$$\mathcal{P} \cdot \mathcal{U}_1 \cdots \mathcal{U}_p \cdot \mathcal{V}_1 \cdots \mathcal{V}_{q-1} \cdot \mathcal{W}_1 \cdots \mathcal{W}_p \cdot \mathcal{Y}_1^{k-1} \cdots \mathcal{Y}_{p+q}^{k-1} \cdots \mathcal{Y}_1^{1} \cdots \mathcal{Y}_{p+q}^{1} \cdot \mathcal{V}_q \cdot \mathcal{S}$$

Repeating this operation $p+q-1$ times allows us to conclude that $\mathcal{P} \cdot \mathcal{A} \cdot \mathcal{B} \cdot \mathcal{S} \equiv_\alpha^{<\text{-inv FO}} \mathcal{P} \cdot \mathcal{B} \cdot \mathcal{A} \cdot \mathcal{S}$.

◀

We are now ready to conclude the proof of Lemma 26. As in the proof of Lemma 27, we distinguish between two cases. Let $n := o_3^\Sigma(\alpha + c)$ where c is the depth of some FO-interpretation to be specified later on, and $\Sigma := P_\sigma \cup \{E, S, P_{1/2}, P_3, P_{4/5}, P_6\}$ where $P_{1/2}$, P_3, $P_{4/5}$ and P_6 are new unary symbols.

1. Assume first that $\text{tp}_\mathcal{A}^n(x', x'', x_A) = \text{tp}_\mathcal{C}^n(x'_B, x''_B, x_C)$.
 This case covers the instances where $\mathcal{A}$ and $\mathcal{C}$ are n-similar, as well as those where $\text{dist}_\mathcal{T}(x, x_A)$ and $\text{dist}_\mathcal{T}(x_B, x_C)$ are $> 2n + 2$.
 Consider the extension $\mathcal{T}^-$ of $\mathcal{P} \uplus \mathcal{A} \uplus \mathcal{B} \uplus \mathcal{C} \uplus \mathcal{S}$ to Σ where $P_{1/2}^{\mathcal{T}^-} := \{x\}$, $P_3^{\mathcal{T}^-} := \{x'_A, x''_A\}$, $P_{4/5}^{\mathcal{T}^-} := \{x_B\}$ and $P_6^{\mathcal{T}^-} := \{x'_C, x''_C\}$
 Since $P_{1/2}^{\mathcal{T}^-}$, $P_3^{\mathcal{T}^-}$, $P_{4/5}^{\mathcal{T}^-}$ and $P_6^{\mathcal{T}^-}$ are at distance $+\infty$ from $\mathcal{A}$ and $\mathcal{C}$, $\text{tp}_{\mathcal{T}^-}^n(x', x'', x_A) = \text{tp}_{\mathcal{T}^-}^n(x'_B, x''_B, x_C)$.
 Hence, we can apply Lemma 6, and get two orders $<_{AC}$ (whose first elements are $x', x'', x_A, x'_B, x''_B, x_C$) and $<_{CA}$ (whose first elements are $x'_B, x''_B, x_C, x', x'', x_A$) such that $(\mathcal{T}^-, <_{AC}) \equiv_{\alpha+c}^{\text{FO}} (\mathcal{T}^-, <_{CA})$.
 Now, consider the FO-interpretation that adds a S-edge between u and v if either:

 - $P_{1/2}(u)$ and v is either the first or the second element of $<$
 - u is the third element of $<$ and $P_3(v)$
 - $P_{4/5}(u)$ and v is either the fourth or the fifth element of $<$
 - u is the sixth element of $<$ and $P_6(v)$

 and then forgets about $P_{1/2}$, P_3, $P_{4/5}$ and P_6.
 Take c to be the depth of this FO-interpretation (which has arity 1).
 Note that the result of this FO-interpretation on $(\mathcal{T}^-, <_{AC})$ is an ordered extension of $\mathcal{P} \cdot \mathcal{A} \cdot \mathcal{B} \cdot \mathcal{C} \cdot \mathcal{S}$ and that its result on $(\mathcal{T}^-, <_{CA})$ is an ordered extension of $\mathcal{P} \cdot \mathcal{C} \cdot \mathcal{B} \cdot \mathcal{A} \cdot \mathcal{S}$
 This entails $\mathcal{P} \cdot \mathcal{A} \cdot \mathcal{B} \cdot \mathcal{C} \cdot \mathcal{S} \equiv_\alpha^{<\text{-inv FO}} \mathcal{P} \cdot \mathcal{C} \cdot \mathcal{B} \cdot \mathcal{A} \cdot \mathcal{S}$

2. We can now, without loss of generality, assume that $\text{dist}_{\mathcal{T}}(x, x_A) \leq 2n+2$.
Set $N := (2n+2) + m$ with $m := \max(M, n, s(\alpha))$, where M is given in Lemma 27. Consider a isomorphism φ from $\mathcal{N}_{\mathcal{T}}^N(x, x', x'')$ to $\mathcal{N}_{\mathcal{T}}^N(x_B, x'_B, x''_B)$.
By choice of N, $\text{tp}_{\mathcal{T}}^m(x_A, x'_A, x''_A) = \text{tp}_{\mathcal{T}}^m(\varphi(x_A), y', y'')$, where y' and y'' are the S-children of $\varphi(x_A)$.
Since $x_B \prec \varphi(x_A)$ and $\varphi(x_A) \neq x_C$ (for otherwise we would be in Case 1), there are only three subcases to consider:

- if $x_B \prec \varphi(x_A) \prec x_C$, set $\mathcal{C}' := \mathcal{C}_{\mathcal{T}}(x_B, \varphi(x_A))$ and $\mathcal{X} := \mathcal{C}_{\mathcal{T}}(\varphi(x_A), x_C)$. We then have $\mathcal{T} = \mathcal{P} \cdot \mathcal{A} \cdot \mathcal{B} \cdot \mathcal{C}' \cdot \mathcal{X} \cdot \mathcal{S}$. Let $\mathcal{T}_1 = \mathcal{P} \cdot \mathcal{C}' \cdot \mathcal{B} \cdot \mathcal{A} \cdot \mathcal{X} \cdot \mathcal{S}$ be the m-guarded vertical-S-swap between $[x, x_A]$ and $[x_B, \varphi(x_A)]$ in $\mathcal{T}$. This swap falls under the scope of Case 1 since $m \geq n$, hence $\mathcal{T}_1 \equiv_\alpha^{<\text{-inv FO}} \mathcal{T}$ and $\forall z \in T$, $\text{tp}_{\mathcal{T}_1}^{m+1}(z) = \text{tp}_{\mathcal{T}}^{m+1}(z)$.
  Hence we are in the conditions (since $m \geq M$) to apply Lemma 27 on $\varphi(x_A) \prec x_A \prec x_C$ in $\mathcal{T}_1$ and get $\mathcal{T}_2 := \mathcal{P} \cdot \mathcal{C}' \cdot \mathcal{X} \cdot \mathcal{B} \cdot \mathcal{A} \cdot \mathcal{S} \equiv_\alpha^{<\text{-inv FO}} \mathcal{T}_1$. Notice that $\mathcal{T}' = \mathcal{T}_2$, which implies that $\mathcal{T}' \equiv_\alpha^{<\text{-inv FO}} \mathcal{T}$
  These sequence of operations is depicted in Figure 35.

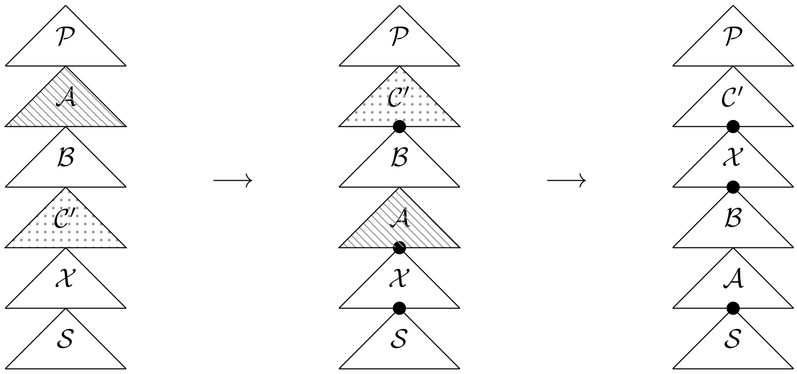

Figure 34 The swaps solving the case $x_B \prec \varphi(x_A) \prec x_C$. The second operation swaps the segments between the dark nodes using Lemma 27.

- if $x_B \prec x_C \prec \varphi(x_A)$, set $\mathcal{C}' := \mathcal{C}_{\mathcal{T}}(x_B, \varphi(x_A))$ and $\mathcal{X} := \mathcal{C}_{\mathcal{T}}(\varphi(x_A), x_C)$. We then have $\mathcal{T} = \mathcal{P} \cdot \mathcal{A} \cdot \mathcal{B} \cdot \mathcal{C} \cdot \mathcal{X} \cdot \mathcal{S}'$. Let $\mathcal{T}_1 = \mathcal{P} \cdot \mathcal{C} \cdot \mathcal{X} \cdot \mathcal{B} \cdot \mathcal{A} \cdot \mathcal{S}'$ be the m-guarded vertical-S-swap between $[x, x_A]$ and $[x_B, \varphi(x_A)]$ in $\mathcal{T}$. This swap falls under the scope of Case 1 since $m \geq n$, hence $\mathcal{T}_1 \equiv_\alpha^{<\text{-inv FO}} \mathcal{T}$ and $\forall z \in T$, $\text{tp}_{\mathcal{T}_1}^{m+1}(z) = \text{tp}_{\mathcal{T}}^{m+1}(z)$.
  Hence we are in the conditions (since $m \geq M$) to apply Lemma 27 on $x_C \prec \varphi(x_A) \prec x_A$ in $\mathcal{T}_1$ and get $\mathcal{T}_2 := \mathcal{P} \cdot \mathcal{C} \cdot \mathcal{B} \cdot \mathcal{A} \cdot \mathcal{X} \cdot \mathcal{S}' \equiv_\alpha^{<\text{-inv FO}} \mathcal{T}_1$. Notice that $\mathcal{T}' = \mathcal{T}_2$, which implies that $\mathcal{T}' \equiv_\alpha^{<\text{-inv FO}} \mathcal{T}$.
  These operations are depicted in Figure 36.
- otherwise, we have $x_B \prec x_C$ and $x_B \prec \varphi(x_A)$ but x_C and $\varphi(x_A)$ are $\prec$-incomparable. Let us decompose $\mathcal{C} \cdot \mathcal{S}$ as $\mathcal{C}'[\mathcal{S}, \mathcal{S}']$ (this notation extends in the natural way that of context, with two bottom anchors), where $\mathcal{S}' := \mathcal{S}_{\mathcal{T}}(\varphi(x_A))$, that is $\mathcal{T} = \mathcal{P} \cdot \mathcal{A} \cdot \mathcal{B} \cdot \mathcal{C}'[\mathcal{S}, \mathcal{S}']$.
  First, let $\mathcal{T}_1 = \mathcal{P} \cdot \mathcal{A} \cdot \mathcal{B} \cdot \mathcal{C}'[\mathcal{S}', \mathcal{S}]$ be the m-guarded crossing-S-swap between x_C and $\varphi(x_A)$ in $\mathcal{T}$. Lemma 8 ensures (since $m \geq s(\alpha)$) that $\mathcal{T}_1 \equiv_\alpha^{<\text{-inv FO}} \mathcal{T}$ and $\forall z \in T$, $\text{tp}_{\mathcal{T}_1}^{m+1}(z) = \text{tp}_{\mathcal{T}}^{m+1}(z)$. The distance precondition in Lemma 8 holds because $\mathcal{T}$ is a hollow tree.

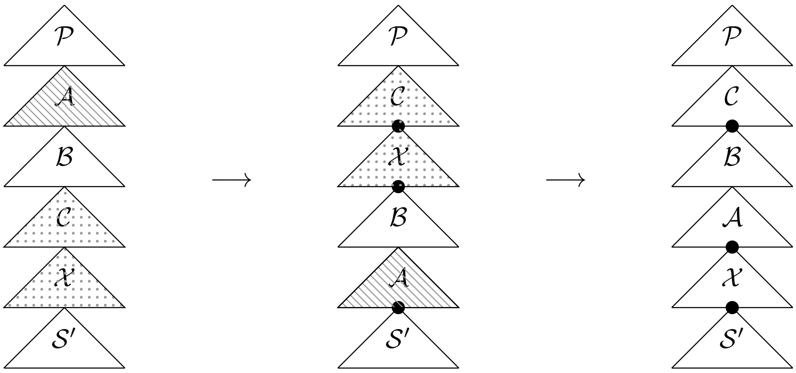

Figure 35 The case where $x_B \prec x_C \prec \varphi(x_A)$. The second operation swaps the segments between the dark nodes using Lemma 27.

Let $\mathcal{T}_2 = \mathcal{P} \cdot \mathcal{C}'[\mathcal{S}', \mathcal{B} \cdot \mathcal{A} \cdot \mathcal{S}]$ be the m-guarded vertical-S-swap between $[x, x_A]$ and $[x_B, \varphi(x_A)]$ in $\mathcal{T}_1$. This swap falls under Case 1, hence ($m \geq n+1$) we get $\mathcal{T}_2 \equiv_\alpha^{<\text{-inv FO}} \mathcal{T}_1$ and $\forall z \in T$, $\text{tp}_{\mathcal{T}_2}^{m+1}(z) = \text{tp}_{\mathcal{T}_1}^{m+1}(z)$.
Now, let $\mathcal{T}_3 = \mathcal{P} \cdot \mathcal{C}'[\mathcal{B} \cdot \mathcal{A} \cdot \mathcal{S}, S']$ be the m-guarded crossing-S-swap between x_A and x_C in $\mathcal{T}_2$. Lemma 8 ensures (since $m \geq s(\alpha)$) that $\mathcal{T}_3 \equiv_\alpha^{<\text{-inv FO}} \mathcal{T}_2$. Notice that $\mathcal{T}_3 = \mathcal{P} \cdot \mathcal{C} \cdot \mathcal{B} \cdot \mathcal{A} \cdot \mathcal{S}$ is nothing but $\mathcal{T}'$. Hence $\mathcal{T}' \equiv_\alpha^{<\text{-inv FO}} \mathcal{T}$.
This process is illustrated in Figure 37.

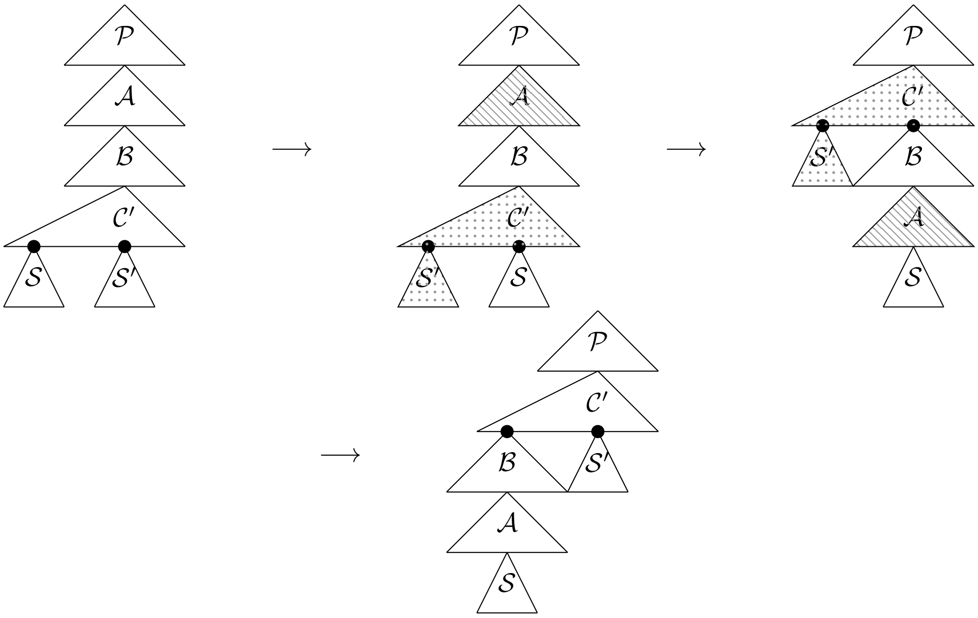

Figure 36 The case where x_C and $\varphi(x_A)$ are $\prec$-unrelated. The first operation is a m-guarded crossing-S-swap between $\varphi(x_A)$ and x_C. The second operation uses Case 1. The last operation is the dual of the first one.

▶ **Proposition 18.** *$\forall \alpha \in \mathbb{N}$, there exists $n_1 \in \mathbb{N}$ such that $\forall \mathcal{P} \in \mathbb{H}_\sigma, \forall \mathcal{Q} \in$ quasi-$\mathbb{H}_\sigma^{n_1}$, if $\mathcal{S}upp_{n_1}(\mathcal{P}) \simeq \mathcal{S}upp_{n_1}(\mathcal{Q})$ then $\mathcal{P} \equiv_\alpha^{<\text{-inv FO}} \mathcal{Q}$.*

Proof. Let n_0 be the maximum between the integers given by Lemma 26 and Lemma 27, and $s(\alpha)$.

Let n_1 be the integer given by Lemma 14 for n_0.

Because of the isomorphism between the n_1-enriched supports, there is a trivial n_1-pseudo-inclusion of $\mathcal{P}$ in $\mathcal{Q}$. Thus, Lemma 14 yields some $\mathcal{Q}' \in$ quasi-$\mathbb{H}_\sigma^{n_0+1}$ such that $\mathcal{Q}' \equiv_\alpha^{<\text{-inv FO}} \mathcal{Q}$, $\mathcal{S}\text{upp}_{n_0+1}(\mathcal{Q}') \simeq \mathcal{S}\text{upp}_{n_0+1}(\mathcal{Q})$ and some h' which (n_0+1)-pseudo-includes $\mathcal{P}$ in $\mathcal{Q}'$ and which respects S-siblings relation.

Now, $\mathcal{P}$ is a hollow tree, hence has the matching endpoints property, and h' must be surjective: this entails that $\mathcal{Q}'$ has the matching endpoints property.

For the remainder of this proof, we will need to apply vertical-S-swaps to $\mathcal{Q}'$ (and subsequent hollow quasitrees), even though it is not necessarily a hollow tree. However, the matching endpoints property ensures that the connected component $\mathcal{R}$ containing its root is a hollow tree.

We will only apply vertical-S-swaps in $\mathcal{R}$; when we talk of the vertical-S-swap in $\mathcal{Q}'$, we mean the disjoint union of the vertical-S-swap in $\mathcal{R}$ and of the other connected components of $\mathcal{Q}'$.

A **tree-prefix** of $\mathcal{P}$ or $\mathcal{Q}'$ is a substructure $\mathcal{T}$ which contains the root, is E-stable and such that if $S(x,y)$ and $y \in T$, then $x \in T$.

Let t be a thread with matching endpoints whose parent is y and an element x. We say that $x \prec t$ if $x \preceq y$. If u is a thread we write $t \prec u$ if $y \prec z$, where z is the S-parent of both of u's endpoints.

Let $\mathcal{T}_0, \cdots, \mathcal{T}_r$ be a sequence of tree-prefixes of $\mathcal{P}$ such that T_0 contains only the root of $\mathcal{P}$, $T_r = P$, and we go from T_i to T_{i+1} by adding a single thread.

We construct a sequence of structures $\mathcal{Q}' = \mathcal{Q}_0, \cdots, \mathcal{Q}_r$ with the following properties:

- $\mathcal{Q}_{i+1} \equiv_\alpha^{<\text{-inv FO}} \mathcal{Q}_i$
- $\mathcal{S}\text{upp}_{n_0+1}(\mathcal{Q}_{i+1}) = \mathcal{S}\text{upp}_{n_0+1}(\mathcal{Q}_i)$
- $\mathcal{T}_i$ is **vertically-pseudo-included** in $\mathcal{Q}_i$, that is for every node x and thread t of $\mathcal{T}_i$, if x is the parent of t in $\mathcal{T}_i$ then $x \prec t$ in $\mathcal{Q}_i$. The smallest tree-prefix of $\mathcal{Q}_i$ containing all the threads of $\mathcal{T}_i$ is called the **$\mathcal{T}_i$-pseudo-tree**.

For $i = 0$, there is nothing to do: the root of $\mathcal{P}$ is vertically-pseudo-included in $\mathcal{Q}_0 = \mathcal{Q}'$. From $\mathcal{T}_i$ to $\mathcal{T}_{i+1}$: we let t be the thread in $T_{i+1} \setminus T_i$ and let x be the parent of t.

- If t is in the $\mathcal{T}_i$-pseudo-tree of $\mathcal{Q}_i$, then there exists some element y and some thread u in $\mathcal{T}_i$ such that in $\mathcal{T}_i$, y is the parent of u, and in $\mathcal{Q}_i$, $y \prec t \prec u$. In $\mathcal{Q}_i$, let's call y' the parent of u, u' the thread whose parent is y, x' the parent of t and t' the thread whose parent is x.

  There are two cases to consider:

  - If $y \prec x$ in $\mathcal{P}$ (c.f. Figure 38). Then in $\mathcal{Q}_i$, we must have $y' \prec x$, and we can apply Lemma 26. Let $\mathcal{Q}_{i+1}$ be the n_0-guarded vertical-S-swap between $[y, x']$ and $[y', x]$ in $\mathcal{Q}_i$. Note that in the limit case where y is the parent of t (that is, $x' = y$), we apply Lemma 27 instead of Lemma 26.
  - Otherwise, x and y must be $\prec$-unrelated in $\mathcal{Q}_i$ (c.f. Figure 39). Note that because we work in a hollow tree, the conditions to apply Lemma 8 are met. Set $\mathcal{Q}'$ to be the n_0-guarded crossing-S-swap between x and x' in $\mathcal{Q}_i$. Then, set $\mathcal{Q}_{i+1}$ to be the n_0-guarded crossing-S-swap between y and y' in $\mathcal{Q}'$.

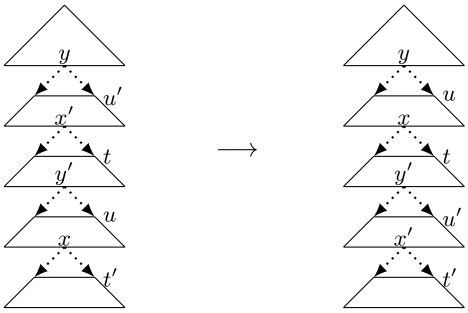

Figure 37 We re-associate y to u and x to t with a vertical-S-swap

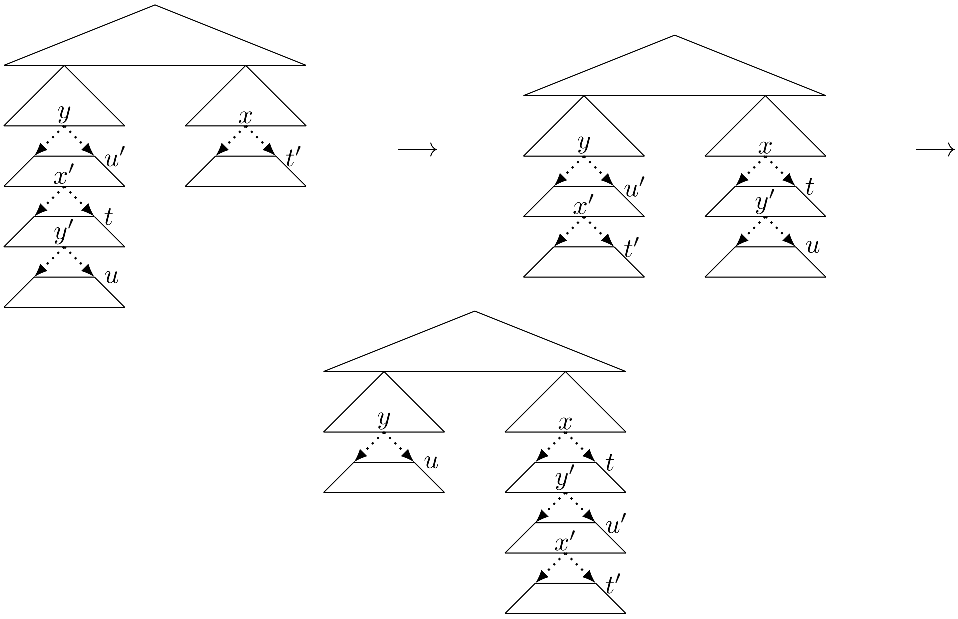

Figure 38 We re-associate y to u and x to t with two crossing-S-swaps

- Otherwise, if $x \prec t$ in $\mathcal{Q}_i$, we set $\mathcal{Q}_{i+1} := \mathcal{Q}_i$
- Otherwise, x and x' are $\prec$-unrelated in $\mathcal{Q}_i$. Once again, we are in the right setting to apply Lemma 8 because we work in a hollow tree. Let $\mathcal{Q}_{i+1}$ be the n_0-guarded crossing-S-swap between x and x' in $\mathcal{Q}_i$.

In the end, we have vertically-pseudo-included $\mathcal{P}$ into $\mathcal{Q}_r$. Since they have the same support, the vertical-pseudo-inclusion is an isomorphism. Hence, $\mathcal{P} \equiv_\alpha^{<\text{-inv FO}} \mathcal{Q}$. ◀