

# Introduction

## Semantics and applications to verification

Xavier RIVAL

École Normale Supérieure

February 7th, 2025

# Program of this first lecture

## Introduction to the course:

- ① a study of some **examples** of **software errors**
  - ▶ what are the causes ? what kind of properties do we want to verify ?
- ② a panel of the main **verification methods** with a fundamental limitation: **undecidability**
  - ▶ many techniques allow to **compute semantic properties**
  - ▶ each comes with **advantages** and **drawbacks**
- ③ an introduction to the **theory of ordered sets** (or, most likely, mostly a refresher...)
  - ▶ **order relations** are pervasive in **semantics** and **verification**
  - ▶ **fixpoints** of operators are also very common

# Outline

## 1 Introduction

## 2 Case studies

- Ariane 5, Flight 501 (1996)
- Lufthansa Flight 2904, Warsaw (1993)
- Patriot missile (anti-missile system), Dahrhan (1991)
- General remarks

## 3 Approaches to verification

## 4 Orderings, lattices, fixpoints

## 5 Conclusion

# Ariane 5 – Flight 501

## Ariane 5:

- a satellite launcher
- replacement of **Ariane 4**, a lot more powerful
- **first flight**, June, 4th, 1996: **failure!**

## Flight story:

- nominal take-off, normal flight for 36 seconds
- **T + 36.7 s** : **angle of attack change**, trajectory lost
- **T + 39 s** : **disintegration** of the launcher



## Consequences:

- **loss of satellites** : more than \$ 370 000 000...
- **launcher** unusable for more than a year (delay !)

## Full report available online:

<http://esamultimedia.esa.int/docs/esa-x-1819eng.pdf>

# Trajectory control system design overview

**Sensors:** gyroscopes, inertial reference systems...

**Calculators** (hardware + software) :

- **“Inertial Reference System” (SRI) :**  
integrates data about the trajectory (read on sensors)
- **“On Board Computer” (OBC) :**  
computes the engine actuations that are required to follow the pre-determined theoretical trajectory

**Actuators:** **engines** of the launcher **follow orders** from the **OBC**

**Redundant systems** (failure tolerant system):

- **keep running** even in the presence of one or several system failures
- **traditional solution in embedded systems: duplication** of systems  
aircraft flight system: 2 or 3 hydraulic circuits  
launcher like Ariane 5 : 2 SRI units (SRI 1 and SRI 2)
- there is also a **control monitor**

# The root cause: an unhandled arithmetic error

## Processor registers

Each **register** has a size of 16, 32, 64 bits:

- **64-bits floating point**: values in range  $[-3.6 \cdot 10^{308}, 3.6 \cdot 10^{308}]$
- **16-bits signed integers**: values in range  $[-32768, 32767]$
- upon **copy of data**: conversions are performed such as **rounding**
- when the values are **too large**:
  - ▶ **interruption**: run error handling code if any, otherwise crash
  - ▶ or **unexpected behavior**: modulo arithmetic or other

## Ariane 5:

- the SRI hardware runs in **interruption mode**
- it has **no error handling code** for arithmetic interruptions
- an **unhandled arithmetic conversion overflow crashes the SRI**

# From the root cause to the failure

A **not so trivial** sequence of events:

- 1 a **conversion from 64-bits float to 16-bits signed int** is performed and **causes an overflow**
- 2 an **interruption** is raised
- 3 due to the lack of error handling code, the SRI **crashes**
- 4 the crash causes an **error return** (negative integer value) value be **sent to the OBC** (On-Board Computer)
- 5 the OBC interprets this illegal value as **flight data**
- 6 this causes the computation of an **absurd trajectory**
- 7 hence the **loss of control** of the launcher

Let us discuss **a few specific points**

# A crash due to an unaddressed software case

**Several solutions** would have prevented this mishappening:

❶ **Deactivate interruptions on overflows:**

- ▶ then, an overflow may happen, and produce wrong values in the SRI
- ▶ but, these wrong values will not cause the computation to stop!  
and most likely, the flight will not be impacted too much

❷ **Fix the SRI code**, so that **no overflow can happen**:

- ▶ all conversions must be **guarded against overflows**:

```
double x = /* ... */;  
short i = /* ... */;  
if( -32768. <= x && x <= 32767. )  
    i = (short) x;  
else  
    i = /* default value */;
```

- ▶ this may be costly (many tests), but redundant tests can be removed

❸ **Handle conversion errors** (not trivial):

- ▶ the handling code should **identify the problem** and **fix it** at run-time
- ▶ the OBC should **identify illegal input values**



# A crash due to a useless task

## Piece of code that generated the error:

- part of a gyroscope re-calibration process
- very useful to quickly restart the launch process after a short delay
- can only be done **before lift-off**...
- ... **but not after!**

## Re-calibration task shut down:

- normally planned **50 seconds** after lift-off...
- no chance of a need for such a re-calibration after  $T_0 + 3$  seconds
- the crash occurred at **36 seconds**

# A crash due to legacy software

## Software history:

- already used in Ariane 4 (previous launcher, before Ariane 5)
- the software was tested and ran in real conditions many times yet never failed...
- but Ariane 4 was a much less powerful launcher

## Software optimization:

- many conversions were initially protected by a safety guard
- but these tests were considered expensive  
(a test and a branching take processor cycles, interact with the pipeline...)
- thus, conversions were ultimately removed for the sake of performance

Yet, Ariane 5 violates the assumptions that were valid with Ariane 4

- higher values of *horizontal bias* were generated
- those were never seen in Ariane 4, hence the failure

# A crash not prevented by redundant systems

**Principle of redundant systems:** survive the failure of a component by the use of redundant systems

## System redundancy in Ariane 5:

- one OBC unit
- **two SRI units**... yet **running the same software**

**Obviously, physical redundancy does not address software issues**

## Other implementation of system redundancy (e.g., Airbus FBW):

- two independent set of controls
- three computing units per set of controls
- each computing unit comprises **two computers** with **distinct softwares** (design and implementation is also performed in **distinct teams**)

# Ariane 501, a summary of the issues

A **long series of design errors**, all related to a lack of understanding of what the software does:

- ① **Non-guarded** conversion raising an **interruption** due to **overflow**
- ② **Removal of pre-existing guards**, too high confidence in the software
- ③ **Non revised assumptions** on the inputs when moving from Ariane 4 to Ariane 5
- ④ Redundant systems **running the same software**
- ⑤ Useless task **not shutdown** at the right time

**Current status:** such issues can be found by static analysis tools

# Outline

## 1 Introduction

## 2 Case studies

- Ariane 5, Flight 501 (1996)
- Lufthansa Flight 2904, Warsaw (1993)
- Patriot missile (anti-missile system), Dahrhan (1991)
- General remarks

## 3 Approaches to verification

## 4 Orderings, lattices, fixpoints

## 5 Conclusion

# High-speed runway overshoot at landing

**Landing** at Warsaw airport, Lufthansa A320:

- **bad weather conditions:** rain, high side wind
- **wet runway**
- landing (300 km/h) followed by **aqua-planing**, and **delayed braking**
- **runway overrun** at 132 km/h
- **impact** against a hillside at about 100 km/h

**Consequences:**

- **2 fatalities, 56 injured** (among 70 passengers + crew)
- **aircraft completely destroyed** (impact + fire)

**Full report available online:**

<http://www.rvs.uni-bielefeld.de/publications/Incidents/DOCS/ComAndRep/Warsaw/warsaw-report.html>

# Causes of the accident

## ① Root cause:

- ▶ **bad weather conditions** not well assessed by the crew
- ▶ side wind **exceeding** aircraft certification specification
- ▶ **wrong action from the crew:**
  - a “Go Around” (missed landing, acceleration + climb) should have been done

## ② Contributing factor: **delayed action of the brake system**

| time (seconds)                | distance (meters)<br>from runway threshold | events   |
|-------------------------------|--|--|
| $T_0$                         | 770 m                                      | main landing gear landed                                   |
| $T_0 + 3 \text{ s}$           | 1030 m                                     | nose landing gear landed<br><b>brake command activated</b> |
| $T_0 + \mathbf{12 \text{ s}}$ | 1680 m                                     | spoilers activated   |
| $T_0 + 14 \text{ s}$          | 1800 m                                     | thrust reversers activated                                 |
| $T_0 + 31 \text{ s}$          | 2700 m                                     | end of runway  |

# Protection of aircraft brake systems

- **Braking systems inhibition: Prevent in-flight activation !**

- ▶ **spoilers:** increase in aerodynamic load (drag)
- ▶ **thrust reversers:** could destroy the plane if activated in-flight !  
(ex : crash of a B 767-300 ER Lauda Air, 1991, 223 fatalities; thrust reversers in-flight activation, electronic circuit issue)

- **Braking software specification:**

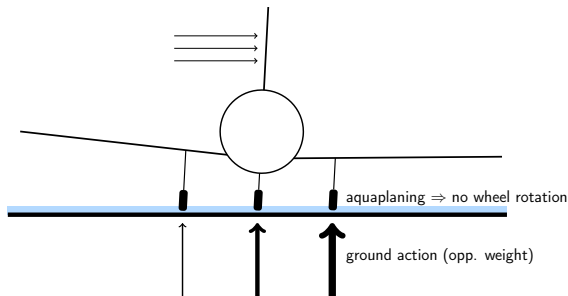
DO NOT activate spoilers and thrust reverse **unless the following condition is met:**

- ▶ thrust lever should be set to **minimum** by the flight crew
  - ▶ **AND** either of the following conditions:
    - ★ **weight** on the main gear should be at least **12 T**  
i.e., **6 T** for each side
    - ★ **OR wheels should be spinning**, with a speed of at least 130 km/h
- [Minimum Thrust] **AND** ([Weight] **OR** [Wheels spinning])



# Understanding the braking delay

- **Landing configuration:**



- **Braking systems: inhibited**

- ▶ **thrust command** properly set to minimum
- ▶ **no weight** on the left landing gear due to **the wind**
- ▶ **no speed on wheels** due to **aquaplaning**

[Minimum Thrust] AND ([Weight] OR [Wheels spinning])

# Flight 2904, a summary of the issues

Main factor is human (landing in weather conditions the airplane is not certified for), but the specification of the software is a contributing factor:

- **Old condition** that failed to be satisfied:

$$(P_{\text{left}} > 6T) \textbf{ AND } (P_{\text{right}} > 6T)$$

- **Fixed condition** (used in the new version of the software):

$$(P_{\text{left}} + P_{\text{right}}) > 12T$$

- The fix can be understood **only with knowledge of the environment**
  - ▶ conditions which the airplane will be used in
  - ▶ behavior of the sensors

# Outline

## 1 Introduction

## 2 Case studies

- Ariane 5, Flight 501 (1996)
- Lufthansa Flight 2904, Warsaw (1993)
- Patriot missile (anti-missile system), Dahran (1991)
- General remarks

## 3 Approaches to verification

## 4 Orderings, lattices, fixpoints

## 5 Conclusion

# The anti-missile “Patriot” system

- **Purpose:** destroy foe missiles before they reach their target
- **Use in wars:**
  - ▶ **first Gulf war** (1991)  
protection of towns and military facilities in Israël and Saudi Arabia (against “Scud” missiles launched by Irak)
  - ▶ **success rate:**
    - ★ around 50 % of the “Scud” missiles are successfully destroyed
    - ★ almost all launched Patriot missiles destroy their target
    - ★ **failures** are due to **failure to launch a Patriot missile**
- **Constraints on the system:**
  - ▶ **hit very quickly moving targets:**  
“Scud” missiles fly at around 1700 m/s ; travel about 1000 km in 10 minutes
  - ▶ **not to destroy a friendly target** (it happened at least twice!)
  - ▶ very high cost: about **\$1 000 000** per launch

# System components

## Detection / trajectory identification:

- **detection** using radar systems
- **trajectory confirmation** (to make sure a foe missile is tracked):
  - ① **trajectory identification** using a sequence of points at various instants
  - ② **trajectory confirmation**  
computation of a predictive window (from position and speed vector)  
+ confirmation of the predicted trajectory
  - ③ **identification of the target** (friend / foe)

## Guidance system:

- **interception trajectory** computation
- **launch** of a Missile, and control until it hits its target  
high precision required (both missiles travel at more than 1500 m/s)

**Very short process:** about ten minutes

# Dahran failure (1991)

- ① **Launch of a “Scud” missile**
- ② **Detection** by the radars of the Patriot system  
**but failure to confirm the trajectory:**
  - ▶ **imprecision** in the computation of the **clock** of the detection system
  - ▶ computation of **a wrong confirmation window**
  - ▶ the “Scud” cannot be found **in the predicted window**  
**failure to confirm the trajectory**
  - ▶ the detection computer concludes it is a **false alert**
- ③ **The “Scud” missile hits its target:**  
**28 fatalities** and around 100 people injured

# Fixed precision arithmetic

- **Fixed precision numbers** are of the form  $\epsilon N 2^{-p}$  where:
  - ▶  $p$  is fixed
  - ▶  $\epsilon \in \{-1, 1\}$  is the **sign**
  - ▶  $N \in [-2^n, 2^n - 1]_{\mathbb{Z}}$  is an **integer** ( $n > p$ )
- In 32 bits fixed precision, with one sign bit,  $n = 31$ ;  
thus we may let  $p = 20$
- A few examples:

| decimal value | sign | truncated value | fractional portion   |
|---------------|------|-----------------|----------------------|
| 2             | 0    | 00000000010     | 00000000000000000000 |
| -5            | 1    | 00000000101     | 00000000000000000000 |
| 0.5           | 0    | 00000000000     | 10000000000000000000 |
| -9.125        | 1    | 00000001001     | 00100000000000000000 |

- Range of values that can be represented:

$$\pm 2^{12}(1 - 2^{-32})$$

# Rounding errors in fixed precision computations

- Not all real numbers in the right range can be represented  
rounding is unavoidable  
may happen both for **basic operations** and for **program constants**...

- **Example:** fraction  $1/10$

- ▶  $1/10$  **cannot be represented exactly** in fixed precision arithmetic
- ▶ let us decompose  $1/10$  as a sum of terms of the form  $\frac{1}{2^i}$  :

$$\begin{aligned}\frac{1}{10} &= \frac{1}{2} \cdot \frac{1}{5} \\ \frac{1}{5} &= \frac{1}{8} + \frac{1}{16} + \frac{1}{16} \cdot \frac{1}{5} = \frac{1}{8} + \frac{1}{16} + \frac{1}{16} \cdot \left( \frac{1}{8} + \frac{1}{16} + \frac{1}{16} \cdot \frac{1}{5} \right) = \dots\end{aligned}$$

- ▶ **infinite binary representation:**  $0.00011001100110011001100\dots$
- ▶ if  $p = 24$ :  
**representation:** "0.000110011001100110011001"  
**rounding error is**  $9.5 \cdot 10^{-8}$

- **Floating precision numbers** (more commonly used today) have the same limitation



# The root cause: a clock drift

## Trajectory confirmation algorithm (summary):

- **hardware clock**  $T_d$  ticks every **tenth of a second**
- time  $T_c$  is computed **in seconds**:  $T_c = \frac{1}{10} \times T_d$
- in **binary**:  $T_c = 0.000110011001100110011001b \times_b T_d$  !
- **relative error is  $10^{-6}$**
- after the computer has been running for **100 h** :
  - ▶ the absolute error is **0.34 s**
  - ▶ as a “Scud” travels at 1700 m/s : the predicted window is about **580 m** from where it should be  
this explains **the trajectory confirmation failure!**

## Remarks:

- the issue was **discovered** by israeli users, who noticed the clock drift  
their solution: **frequently restart the control computer...** (daily)
- this was not done in Dahran... the system had been running for 4 days

# Patriot missile failure, a summary of the issues

**Precision issues** in the fixed precision arithmetic:

- A scalar **constant** used in the code was **invalid** i.e., bound to be rounded to an approximate value, incurring a significant approximation the designers were unaware of
- There was **no adequate study of the precision** achieved by the system, although precision is clearly critical here !

**Current status:** such issues can be found by static analysis tools

# Outline

## 1 Introduction

## 2 Case studies

- Ariane 5, Flight 501 (1996)
- Lufthansa Flight 2904, Warsaw (1993)
- Patriot missile (anti-missile system), Dahrhan (1991)
- General remarks

## 3 Approaches to verification

## 4 Orderings, lattices, fixpoints

## 5 Conclusion

# Common issues causing software problems

The examples given so far **are not isolated cases**

See for instance:

`www.cs.tau.ac.il/~nachumd/horror.html`

(not up-to-date)

## Typical reasons:

- **Improper specification** or understanding of the environment, conditions of execution...
- **Incorrect implementation of a specification**  
e.g., the code should be free of runtime errors  
e.g., the software should produce a result that meets some property
- **Incorrect understanding of the execution model**  
e.g., generation of too imprecise results

# New challenges to ensure embedded systems do not fail

## Complex software architecture: e.g. **parallel softwares**

- single processor **multi-threaded**, **distributed** (several computers)
- more and more common: multi-core architectures
- **very hard to reason about**
  - ▶ other kinds of issues: **dead-locks**, **races**...
  - ▶ very complex execution model: **interleavings**, **memory models**

## Complex properties to ensure: e.g., **security**

- the system should resist even in the presence of an **attacker**  
(agent with malicious intentions)
- attackers may try to **access sensitive data**, to **corrupt** critical data...
- security properties are often even **hard to express**

# Techniques to ensure software safety

## Software development techniques:

- **software engineering**, with a focus on specification, and software quality (may be more or less formal...)
- **programming rules** for specific areas (e.g., DO 178 c in avionics)
- usually do not guarantee any strong property, but make softwares “cleaner”

## Formal methods:

- should have **sound mathematical foundations**
- should allow to **guarantee** softwares meet some complex properties
- should be **trustable** (is a paper proof ok ???)
- **increasingly used in real life applications**, but still a lot of open problems

# What is to be verified ?

What do the C programs below do ?

What do these C programs do ?

## P0.c

```
int x = 0
int f0( int y ){
    return x * y;
}
int f1( int y ){
    x = y;
    return 0;
}
void main( ){
    int z = f0( 10 ) +
           f1( 100 );
}
```

## P1.c

```
void main( ){
    int i;
    int t[100] = { 0, 1, 2,
                  ..., 99 };
    while( i < 100 ){
        t[i]++;
        i++;
    }
}
```

## P2.c

```
void main( ){
    float f = 0.;
    for( int i = 0;
         i < 1000000;
         i++ )
        f = f + 0.1;
}
```

# Semantic subtleties...

P0.c

```
int x = 0
int f0( int y ){
    return x * y;
}
int f1( int y ){
    x = y;
    return 0;
}
void main( ){
    int z = f0( 10 ) + f1
        ( 100 );
}
```

## Execution order:

- **not specified in C**
- **specified in Java**
- if left to right,  $z = 0$
- if right to left,  $z = 1000$



# Semantic subtleties...

## P1.c

```
void main( ){  
    int i;  
    int t[100] = { 0, 1, 2,  
                  ..., 99 };  
    while( i < 100 ){  
        t[i]++;  
        i++;  
    }  
}
```

## P2.c

```
void main( ){  
    float f = 0.;  
    for( int i = 0;  
         i < 1000000;  
         i++ )  
        f = f + 0.1;  
}
```

### Initialization:

- **runtime error in Java**
- **read of a random value in C** (the value that was stored before)

### Floating point semantics:

- **0.1 is not representable exactly**; what is it rounded to by the compiler ?
- **rounding errors**; what is the rounding mode at runtime ?

# The two main parts of this course

## ① Semantics

- ▶ allow to **describe precisely the behavior of programs**  
should account for execution order, initialization, scope...
- ▶ allow to **express the properties to verify**  
several important families of properties: safety, liveness, security...
- ▶ also important to **transform** and **compile** programs

## ② Verification

- ▶ aim at **proving** semantic properties of programs
- ▶ a very strong limitation: **undecidability**
- ▶ **several approaches**, that make various compromises around undecidability

# Outline

- 1 Introduction
- 2 Case studies
- 3 Approaches to verification
  - Indecidability and fundamental limitations
  - Approaches to verification
- 4 Orderings, lattices, fixpoints
- 5 Conclusion

# The termination problem

## Termination

**Program  $P$  terminates on input  $X$  if and only if any execution of  $P$ , with input  $X$  eventually reaches a final state**

- **Final state:** final point in the program (i.e., not error)
- **We may want to ensure termination:**
  - ▶ processing of a task, such as, e.g., printing a document
  - ▶ computation of a mathematical function
- **We may want to ensure *non*-termination:**
  - ▶ operating system
  - ▶ device drivers

## The termination problem

Can we find a program  $P_t$  that **takes as argument a program  $P$  and data  $X$  and that returns “TRUE” if  $P$  terminates on  $X$  and “FALSE” otherwise ?**

# The termination problem is not computable

- **Proof by reductio ad absurdum**, using a *diagonal argument*

We assume **there exists a program  $P_a$  such that:**

- ▶  $P_a$  always terminates
- ▶  $P_a(P, X) = 1$  **if  $P$  terminates** on input  $X$
- ▶  $P_a(P, X) = 0$  **if  $P$  does not terminate** on input  $X$

- We consider the following program:

```
void P0( P ){  
    if( P_a( P, P ) == 1 ){  
        while( 1 ){  
            // loop forever  
        }  
    } else {  
        return; // do nothing  
    }  
}
```

- What is the return value of  $P_a(P_0, P_0)$  ?

i.e., **does  $P_0$  terminate on input  $P_0$  ?**

# The termination problem is not computable

- What is the return value of  $P_a(P_0, P_0)$  ?

We know  $P_a$  always terminates and returns either 0 or 1 (assumption).

Therefore, we need to consider only two cases:

- ▶ if  $P_a(P_0, P_0)$  returns 1, then  $P_0(P_0)$  loops forever, thus  $P_a(P_0, P_0)$  should return 0, so we have reached a **contradiction**
- ▶ if  $P_a(P_0, P_0)$  returns 0, then  $P_0(P_0)$  terminates, thus  $P_a(P_0, P_0)$  should return 1, so we have reached a **contradiction**

- In both cases, we reach a contradiction

- Therefore we conclude **no such a  $P_a$  exists**

The termination problem is not decidable

There exists no program  $P_t$  that always terminates and always recognizes whether a program  $P$  terminates on input  $X$

# Absence of runtime errors

- Can we find a program  $P_c$  that takes a program  $P$  and input  $X$  as arguments, always terminates and returns
  - ▶ 1 if and only  $P$  runs safely on input  $X$ , i.e., without a runtime error
  - ▶ 0 if  $P$  crashes on input  $X$
- Answer: **No**, the same diagonal argument applies  
if  $P_c(P, X)$  decides whether  $P$  will run safely on  $X$ , consider

```
void P1( P ){  
    if( P_c( P, P ) == 1 ){  
        0 / 0; // deliberately crash  
            (unsafe)  
    } else {  
        return; // do nothing  
    }  
}
```

Non-computability result

The absence of runtime errors is not computable

# Rice theorem

- **Semantic specification:** set of *correct* program executions
  - **“Trivial” semantic specifications:**
    - ▶ empty set
    - ▶ set of all possible executions
- ⇒ intuitively, the non interesting verification problems...

## Rice theorem (1953)

**Considering a Turing complete language,  
any non trivial semantic specification is not computable**

- **Intuition:** there is no algorithm to decide non trivial specifications, starting with only the program code
- Therefore **all interesting properties are not computable** :
  - ▶ **termination**,
  - ▶ **absence of runtime errors**,
  - ▶ **absence of arithmetic errors**, etc...



# Outline

- 1 Introduction
- 2 Case studies
- 3 **Approaches to verification**
  - Indecidability and fundamental limitations
  - Approaches to verification
- 4 Orderings, lattices, fixpoints
- 5 Conclusion

# Towards partial solutions

The initial verification problem is **not computable**

**Solution:** solve a weaker problem

Several compromises can be made:

- **simulation / testing:** observe only **finitely many finite executions** infinite system, but only finite exploration (no proof beyond that)
- **assisted theorem proving:** we **give up on automation** (no proof inference algorithm in general)
- **model checking:** we consider only **finite systems** (with finitely many states)
- **bug-finding:** search for “patterns” indicating “likely errors” (may miss real program errors, and report non existing issues)
- **static analysis with abstraction: attempt at automatic correctness proofs** (yet, may fail to verify some correct programs)

# Safety verification method characteristics

## Safety verification problem

- **Semantics**  $\llbracket P \rrbracket$  of program  $P$ : set of behaviors of  $P$  (e.g., states)
- **Property to verify**  $\mathcal{S}$ : set of admissible behaviors (e.g., safe states)

**Goal:** establish  $\llbracket P \rrbracket \subseteq \mathcal{S}$

- **Automation:** **existence of an algorithm**
- **Scalability:** should allow to **handle large softwares**
- **Soundness:** identify **any wrong program**
- **Completeness:** accept **all correct programs**
- **Apply to program source code**, i.e., not require a **modelling phase**

# 1. Testing by simulation

## Principle

Run the program on **finitely many finite inputs**

- maximize **coverage**
- **inspect erroneous traces** to fix bugs
- **Very widely used:**
  - ▶ **unit testing**: each function is tested separately
  - ▶ **integration testing**: with all surrounding systems, hardware  
e.g., **iron bird** in avionics
- **Automated**
- **Complete**: will never raise a false alarm
- **Unsound** unless exhaustive: **may miss program defects**
- **Costly**: needs to be re-done when software gets updated

## 2. Machine assisted proof

### Principle

Have a **machine checked** proof, that is partly **human written**

- **tactics / solvers** may help in the inference
- the **hardest invariants** have to be user-supplied

- **Applications**

- ▶ software industry (rare): Line 14 in Paris Subway
- ▶ hardware: ACL 2
- ▶ academia: CompCert compiler, SEL4 verified micro-kernel
- ▶ also for math: four colour theorem, Feith-Thomson theorem

- **Not fully automated**

often turns out **costly** as complex proof arguments have to be found

- **Sound** and quasi-**complete** (in practice fine...)

### 3. Model-Checking

#### Principle

Consider **finite systems** only, using algorithms for

- **exhaustive exploration**,
- **symmetry reduction**...

- **Applications:**

- ▶ **hardware** verification
- ▶ **driver protocols** verification (Microsoft)

- Applies on **a model**: a model extraction phase is needed

- ▶ for infinite systems, this is **necessarily approximate**
- ▶ not always automated

- **Automated, sound, complete with respect to the model**

## 4. “Bug finding”

### Principle

Identify **“likely” issues**, i.e., patterns known to often indicate an error

- use **bounded symbolic execution, model exploration...**
- **rank "defect" reports** using heuristics

- Intuition: **model checking made unsound**
- **Example:** Coverity
- **Automated**
- **Not complete:** may report false alarms
- **Not sound:** may accept false programs  
thus **inadequate** for safety-critical systems

## 5. Static analysis with abstraction (1/4)

### Principle

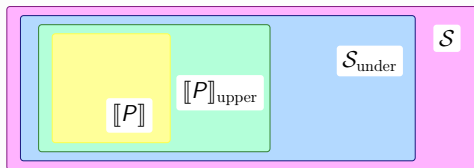
Use some approximation, but **always in a conservative manner**

- **Under-approximation** of the property to verify:  $\mathcal{S}_{\text{under}} \subseteq \mathcal{S}$
- **Over-approximation** of the semantics:  $\llbracket P \rrbracket \subseteq \llbracket P \rrbracket_{\text{upper}}$
- We let an automatic static analyzer attempt to prove that:

$$\llbracket P \rrbracket_{\text{upper}} \subseteq \mathcal{S}_{\text{under}}$$

If it succeeds,  $\llbracket P \rrbracket \subseteq \mathcal{S}$

- In practice, the static analyzer **computes**  $\llbracket P \rrbracket_{\text{upper}}, \mathcal{S}_{\text{under}}$



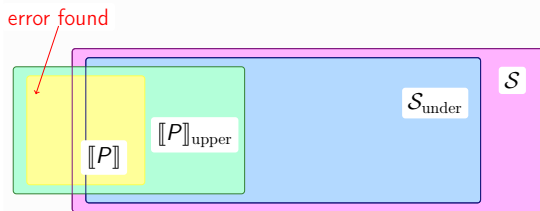


## 5. Static analysis with abstraction (2/4)

### Soundness

The abstraction will catch **any incorrect program**

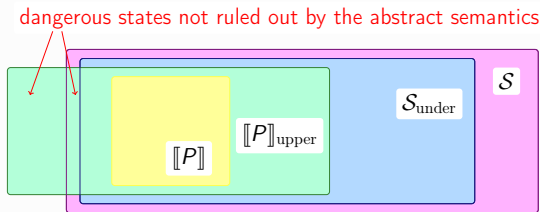
- If  $\llbracket P \rrbracket \not\subseteq \mathcal{S}$ , then  $\llbracket P \rrbracket_{\text{upper}} \not\subseteq \mathcal{S}_{\text{under}}$   
 since  $\begin{cases} \mathcal{S}_{\text{under}} \subseteq \mathcal{S} \\ \llbracket P \rrbracket \subseteq \llbracket P \rrbracket_{\text{upper}} \end{cases}$



## 5. Static analysis with abstraction (3/4)

### Incompleteness

The abstraction may fail to certify **some correct programs**



### Case of a false alarm:

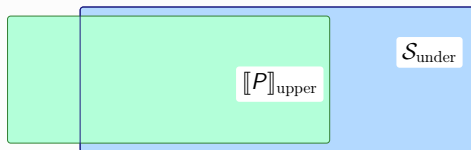
- program  $P$  is **correct**
- but the static analysis **fails**

## 5. Static analysis with abstraction (4/4)

### Incompleteness

The abstraction may fail to certify **some correct programs**

- In the following case, the analysis cannot conclude anything



- One goal of the static analyzer designer is to avoid such cases

### Static analysis using abstraction

- **Automatic**:  $\llbracket P \rrbracket_{upper}$ ,  $S_{under}$  computed automatically
- **Sound**: reports any incorrect program
- **Incomplete**: may reject correct programs

# A summary of common verification techniques

|                  | Automatic | Sound           | Complete               | Source level | Scalable               |
|------------------|-----------|-----------------|------------------------|--------------|------------------------|
| Simulation       | Yes       | No <sup>1</sup> | Yes                    | Yes          | sometimes <sup>2</sup> |
| Assisted proving | No        | Yes             | Almost                 | Partially    | sometimes <sup>3</sup> |
| Model-checking   | Yes       | Yes             | Partially <sup>4</sup> | No           | sometimes              |
| Bug-finding      | Yes       | No              | No                     | Yes          | sometimes              |
| Static analysis  | Yes       | Yes             | No                     | Yes          | sometimes              |

- Obviously, no approach checks all characteristics
- Scalability is a challenge for all

---

<sup>1</sup>unless full testing is doable

<sup>2</sup>full testing usually not possible except for small programs with finite state space

<sup>3</sup>quickly requires huge manpower

<sup>4</sup>only with respect to the finite models... but not with respect to infinite semantics

# Outline

- 1 Introduction
- 2 Case studies
- 3 Approaches to verification
- 4 Orderings, lattices, fixpoints
  - Basic definitions on orderings
  - Operators over a poset and fixpoints
- 5 Conclusion

# Order relations

Very useful in semantics and verification:

- **logical ordering**, expresses **implication** of logical facts
- **computational ordering**, useful to establish well-foundedness of fixpoint definitions and for proving termination

## Definition: partially ordered set (poset)

Let a set  $\mathcal{S}$  and a binary relation  $(\sqsubseteq) \subseteq \mathcal{S} \times \mathcal{S}$  over  $\mathcal{S}$ .

Then,  $\sqsubseteq$  is an **order relation** (and  $(\mathcal{S}, \sqsubseteq)$  is called a **poset**) if and only if it is

- **reflexive**:  $\forall x \in \mathcal{S}, x \sqsubseteq x$
- **transitive**:  $\forall x, y, z \in \mathcal{S}, x \sqsubseteq y \wedge y \sqsubseteq z \implies x \sqsubseteq z$
- **antisymmetric**:  $\forall x, y \in \mathcal{S}, x \sqsubseteq y \wedge y \sqsubseteq x \implies x = y$
- notation:  $x \sqsubset y ::= (x \sqsubseteq y \wedge x \neq y)$

# Graphical representation

We often use **Hasse diagrams** to represent posets:

## Extensive definition:

- $\mathcal{S} = \{x_0, x_1, x_2, x_3, x_4\}$
- $\sqsubseteq$  defined by:

$$x_0 \sqsubseteq x_1$$

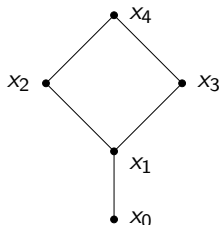
$$x_1 \sqsubseteq x_2$$

$$x_1 \sqsubseteq x_3$$

$$x_2 \sqsubseteq x_4$$

$$x_3 \sqsubseteq x_4$$

## Diagram:



- By reflexivity, we have, e.g.,  $x_1 \sqsubseteq x_1$
- By transitivity, we have, e.g.,  $x_1 \sqsubseteq x_4$

Order relations are very useful in semantics...

## Example: semantics of automata

In the following, we **illustrate order relations** and their **usefulness in semantics** using **word automata**.

We consider the classical notion of **finite word automata** and let

- $L$  be a finite set of **letters**
- $Q$  be a finite set of **states**
- $q_i, q_f \in Q$  denote the **initial** state and **final** state
- $\rightarrow \subseteq Q \times L \times Q$  be a **transition relation**

### Semantics of an automaton

The set of words recognized by  $\mathcal{A} = (Q, q_i, q_f, \rightarrow)$  is defined by:

$$\mathcal{L}[\mathcal{A}] = \{a_0 a_1 \dots a_n \mid \exists q_0 \dots q_{n-1} \in Q, q_i \xrightarrow{a_0} q_0 \xrightarrow{a_1} q_1 \dots q_{n-1} \xrightarrow{a_n} q_f\}$$



# Example: automata and semantic properties

## A simple automaton:

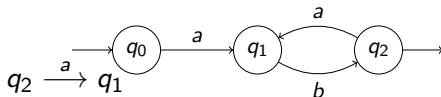
$$L = \{a, b\} \quad Q = \{q_0, q_1, q_2\}$$

$$q_i = q_0$$

$$q_f = q_2$$

$$q_0 \xrightarrow{a} q_1$$

$$q_1 \xrightarrow{b} q_2$$



## A few semantic properties:

- $\mathcal{P}_0$ : no recognized word contains two consecutive  $b$

$$\mathcal{L}[\mathcal{A}] \subseteq L^* \setminus L^*bbL^*$$

- $\mathcal{P}_1$ : all recognized words contain at least one occurrence of  $a$

$$\mathcal{L}[\mathcal{A}] \subseteq L^*aL^*$$

- $\mathcal{P}_2$ : recognized words do not contain  $b$

$$\mathcal{L}[\mathcal{A}] \subseteq (L \setminus \{b\})^*$$

- we could also consider under-approximation properties (of the form  $\mathcal{P}_3 \subseteq \mathcal{L}[\mathcal{A}]$ ), but do not in this lecture

# Total ordering

## Definition: total order relation

Order relation  $\sqsubseteq$  over  $\mathcal{S}$  is a **total** order if and only if

$$\forall x, y \in \mathcal{S}, x \sqsubseteq y \vee y \sqsubseteq x$$

## Examples:

- **real numbers:**

$(\mathbb{R}, \leq)$  is a total ordering

- **powerset:**

if set  $\mathcal{S}$  has at least two distinct elements  $x, y$  then its powerset

$(\mathcal{P}(\mathcal{S}), \subseteq)$  is **not** a total order

indeed  $\{x\}, \{y\}$  cannot be compared

Most of the order relations we will use are **not be total**

indeed: very often, powerset or similar

# Minimum and maximum elements

## Definition: extremal elements

Let  $(\mathcal{S}, \sqsubseteq)$  be a poset and  $\mathcal{S}' \subseteq \mathcal{S}$ . Then  $x$  is

- **minimum element** of  $\mathcal{S}'$  if and only if  $x \in \mathcal{S}' \wedge \forall y \in \mathcal{S}', x \sqsubseteq y$
- **maximum element** of  $\mathcal{S}'$  if and only if  $x \in \mathcal{S}' \wedge \forall y \in \mathcal{S}', y \sqsubseteq x$

- maximum and minimum elements **may not exist**  
example:  $\{\{x\}, \{y\}\}$  in the powerset, where  $x \neq y$
- **infimum**  $\perp$  (“**bottom**”): minimum element of  $\mathcal{S}$
- **supremum**  $\top$  (“**top**”): maximum element of  $\mathcal{S}$

## Exercise:

what are the logical interpretations of infimum / supremum elements ?

# Upper bounds and least upper bound

## Definition: bounds

Given poset  $(\mathcal{S}, \sqsubseteq)$  and  $\mathcal{S}' \subseteq \mathcal{S}$ , then  $x \in \mathcal{S}$  is

- an **upper bound** of  $\mathcal{S}'$  if

$$\forall y \in \mathcal{S}', y \sqsubseteq x$$

- the **least upper bound** (lub) of  $\mathcal{S}'$  (noted  $\sqcup \mathcal{S}'$ ) if

$$\forall y \in \mathcal{S}', y \sqsubseteq x \wedge \forall z \in \mathcal{S}, (\forall y \in \mathcal{S}', y \sqsubseteq z) \implies x \sqsubseteq z$$

- if it exists, the least upper bound is **unique**: if  $x, y$  are least upper bounds of  $\mathcal{S}$ , then  $x \sqsubseteq y$  and  $y \sqsubseteq x$ , thus  $x = y$  by antisymmetry
- notation:  $x \sqcup y ::= \sqcup \{x, y\}$
- upper bounds and least upper bounds **may not exist**
- **dual notions**: lower bound, greatest lower bound (glb, noted  $\sqcap \mathcal{S}'$ )

**Exercise:** logical interpretations ?

# Duality principle

**So far all definitions admit a symmetric counterpart**

- **dual relation**: given an order relation  $\sqsubseteq$ ,  $\mathcal{R}$  defined by

$$x\mathcal{R}y \iff y \sqsubseteq x$$

is also an order relation

- thus all properties that can be proved about  $\sqsubseteq$  also have a symmetric property that also holds

This is the **duality principle**:

|                      |                   |
|----------------------|-------------------|
| minimum element      | maximum element   |
| infimum              | supremum          |
| lower bound          | upper bound       |
| greatest lower bound | least upper bound |

... more to follow

# Complete lattice

## Definition: complete lattice

A **complete lattice** is a tuple  $(\mathcal{S}, \sqsubseteq, \perp, \top, \sqcup, \sqcap)$  where:

- $(\mathcal{S}, \sqsubseteq)$  is a poset
- $\perp$  is the infimum of  $\mathcal{S}$
- $\top$  is the supremum of  $\mathcal{S}$
- any subset  $\mathcal{S}'$  of  $\mathcal{S}$  has a lub  $\sqcup \mathcal{S}'$  and a glb  $\sqcap \mathcal{S}'$

## Properties:

- $\perp = \sqcup \emptyset = \sqcap \mathcal{S}$
- $\top = \sqcap \emptyset = \sqcup \mathcal{S}$

## Example:

the **powerset**  $(\mathcal{P}(\mathcal{S}), \subseteq, \emptyset, \mathcal{S}, \cup, \cap)$  of set  $\mathcal{S}$  is a complete lattice

# Lattice

The existence of lubs and glbs for all subsets is often a very strong property, that may not be met:

## Definition: lattice

A **lattice** is a tuple  $(\mathcal{S}, \sqsubseteq, \perp, \top, \sqcup, \sqcap)$  where:

- $(\mathcal{S}, \sqsubseteq)$  is a poset
  - $\perp$  is the infimum of  $\mathcal{S}$
  - $\top$  is the supremum of  $\mathcal{S}$
  - any **pair**  $\{x, y\}$  of  $\mathcal{S}$  has a lub  $x \sqcup y$  and a glb  $x \sqcap y$
- 
- let  $\mathcal{Q} = \{q \in \mathbb{Q} \mid 0 \leq q \leq 1\}$ ;  
then  $(\mathcal{Q}, \leq)$  is a **lattice** but **not a complete lattice**  
indeed,  $\{q \in \mathcal{Q} \mid q \leq \frac{\sqrt{2}}{2}\}$  has no lub in  $\mathcal{Q}$
  - property: a **finite** lattice is also a complete lattice

# Chains

## Definition: increasing chain

Let  $(\mathcal{S}, \sqsubseteq)$  be a poset and  $\mathcal{C} \subseteq \mathcal{S}$ .

It is an **increasing chain** if and only if

- it has an infimum (thus it is not empty)
- poset  $(\mathcal{C}, \sqsubseteq)$  is total (i.e., any two elements can be compared)

**Example**, in the powerset  $(\mathcal{P}(\mathbb{N}), \subseteq)$ :

$$\mathcal{C} = \{c_i \mid i \in \mathbb{N}\} \quad \text{where} \quad c_i = \{2^0, 2^2, \dots, 2^i\}$$

## Definition: increasing chain condition

The poset  $(\mathcal{S}, \sqsubseteq)$  **satisfies the increasing chain condition** if and only if any increasing chain  $\mathcal{C} \subseteq \mathcal{S}$  is finite.

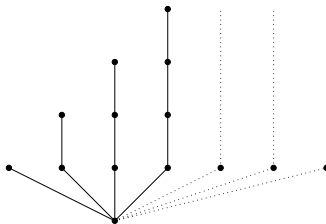


# Complete partial orders

## Definition: complete partial order

A **complete partial order** (cpo) is a poset  $(S, \sqsubseteq)$  such that any increasing chain  $\mathcal{C}$  of  $S$  has a least upper bound. A **pointed cpo** is a cpo with an infimum  $\perp$ .

- clearly, any complete lattice is a cpo
- the opposite is not true:

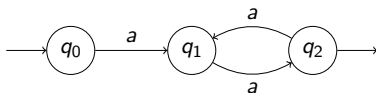


# Outline

- 1 Introduction
- 2 Case studies
- 3 Approaches to verification
- 4 Orderings, lattices, fixpoints**
  - Basic definitions on orderings
  - Operators over a poset and fixpoints
- 5 Conclusion

# How to (informally) prove semantic properties

## Automaton:



## Target property:

recognized words do not contain  $b$

$$\mathcal{L}[\mathcal{A}] \subseteq (L \setminus \{b\})^*$$

## Informal proof:

- ① processing of a word starts at  $q_0$ , with  $\epsilon$
- ② then, processing may continue at  $q_1$ , with an  $a$
- ③ then, processing may continue at  $q_2$ , with an  $a$  (may terminate)
- ④ then, processing may return to  $q_1$ , with an  $a$
- ⑤ ... repeat the previous steps

**we want to do a proof by induction**

## Induction

- it is natural to **reason by induction** over executions
- so we would like a **more suitable way to express the semantics**

# Towards a constructive definition of the automata semantics

We now look for a **constructive version of the automaton semantics** as hinted by the following observations

**Observation 1:**  $\mathcal{L}[\mathcal{A}] = \llbracket \mathcal{A} \rrbracket(q_f)$  where

$$\begin{aligned} \llbracket \mathcal{A} \rrbracket : Q &\longrightarrow \mathcal{P}(L^*) \\ q &\longmapsto \{w \in L^* \mid \exists n, w = a_0 a_1 \dots a_n \\ &\quad \exists q_0 \dots q_{n-1} \in Q, q_i \xrightarrow{a_0} q_0 \xrightarrow{a_1} \dots q_{n-1} \xrightarrow{a_n} q\} \end{aligned}$$

**Observation 2:**  $\llbracket \mathcal{A} \rrbracket = \dot{\bigcup}_{n \in \mathbb{N}} \llbracket \mathcal{A} \rrbracket_n$  where

$$\begin{aligned} \llbracket \mathcal{A} \rrbracket_n : Q &\longrightarrow \mathcal{P}(L^*) \\ q &\longmapsto \{a_0 a_1 \dots a_{n-1} \mid \\ &\quad \exists q_0 \dots q_{n-2} \in Q, q_i \xrightarrow{a_0} q_0 \xrightarrow{a_1} \dots q_{n-1} \xrightarrow{a_{n-1}} q\} \end{aligned}$$

**Observation 3:**  $\llbracket \mathcal{A} \rrbracket_{n+1}$  can be computed directly from  $\llbracket \mathcal{A} \rrbracket_n$

$$\llbracket \mathcal{A} \rrbracket_{n+1}(q) = \bigcup_{q' \in Q} \{wa \mid w \in \llbracket \mathcal{A} \rrbracket_n(q') \wedge q' \xrightarrow{a} q\}$$

# Towards a constructive definition of the automata semantics

## Alternate approach:

- 1 Let  $\llbracket \mathcal{A} \rrbracket_n$  denote **recognized words of length at most  $n$** :

$$\llbracket \mathcal{A} \rrbracket_n(q) ::= \{w \in \llbracket \mathcal{A} \rrbracket(q) \mid \text{length}(w) \leq n\}$$

- 2 Compute  $\llbracket \mathcal{A} \rrbracket_{n+1}$  from  $\llbracket \mathcal{A} \rrbracket_n$
- 3 Define the semantics of the automaton as the union of the iterates of this sequence:

$$\llbracket \mathcal{A} \rrbracket = \bigcup_{n \in \mathbb{N}} \llbracket \mathcal{A} \rrbracket_n$$

In the following, we **study such a way of defining semantics**, based on **general mathematical tools**, that we will use throughout the course

# Operators over a poset

## Definition: operators and orderings

Let  $(\mathcal{S}, \sqsubseteq)$  be a poset and  $\phi : \mathcal{S} \rightarrow \mathcal{S}$  be an operator over  $\mathcal{S}$ . Then,  $\phi$  is:

- **monotone** if and only if  $\forall x, y \in \mathcal{S}, x \sqsubseteq y \implies \phi(x) \sqsubseteq \phi(y)$
- **continuous** if and only if, for any chain  $\mathcal{S}' \subseteq \mathcal{S}$  then:
 
$$\begin{cases} \text{if } \sqcup \mathcal{S}' \text{ exists, so does } \sqcup \{\phi(x) \mid x \in \mathcal{S}'\} \\ \text{and } \phi(\sqcup \mathcal{S}') = \sqcup \{\phi(x) \mid x \in \mathcal{S}'\} \end{cases}$$
- **$\sqcup$ -preserving** if and only if:
 
$$\forall \mathcal{S}' \subseteq \mathcal{S}, \begin{cases} \text{if } \sqcup \mathcal{S}' \text{ exists, then } \sqcup \{\phi(x) \mid x \in \mathcal{S}'\} \text{ exists} \\ \text{and } \phi(\sqcup \mathcal{S}') = \sqcup \{\phi(x) \mid x \in \mathcal{S}'\} \end{cases}$$

## Notes:

- “monotone” in English means “*croissante*” in French ; “*décroissante*” translates into “anti-monotone” and “monotone” into “*isotone*”
- the dual of “monotone” is “monotone”

# Operators over a poset

## A few interesting properties:

### Continuity implies monotonicity

If  $\phi$  is continuous, then it is also **monotone**

We assume  $\phi$  is continuous, and  $x, y \in \mathcal{S}$  are such that  $x \sqsubseteq y$ :  
Then  $\{x, y\}$  is a chain with lub  $y$ , thus  $\phi(x) \sqcup \phi(y)$  exists and is equal to  $\phi(\sqcup\{x, y\}) = \phi(y)$ . Therefore  $\phi(x) \sqsubseteq \phi(y)$ .

### $\sqcup$ -preserving implies monotonicity

If  $\phi$  preserves  $\sqcup$ , then it is also **monotone**

Same argument.

# Fixpoints

## Definition: fixpoints

Let  $(\mathcal{S}, \sqsubseteq)$  be a poset and  $\phi : \mathcal{S} \rightarrow \mathcal{S}$  be an operator over  $\mathcal{S}$ .

- a **fixpoint** of  $\phi$  is an element  $x$  such that  $\phi(x) = x$
- a **pre-fixpoint** of  $\phi$  is an element  $x$  such that  $x \sqsubseteq \phi(x)$
- a **post-fixpoint** of  $\phi$  is an element  $x$  such that  $\phi(x) \sqsubseteq x$
- the **least fixpoint**  $\text{lfp } \phi$  of  $\phi$  (if it exists, it is unique) is the smallest fixpoint of  $\phi$
- the **greatest fixpoint**  $\text{gfp } \phi$  of  $\phi$  (if it exists, it is unique) is the greatest fixpoint of  $\phi$

**Note:** the existence of a least fixpoint, a greatest fixpoint or even a fixpoint is *not guaranteed*; we will see several theorems that establish their existence under specific assumptions...



# Tarski's Theorem

## Theorem

Let  $(\mathcal{S}, \sqsubseteq, \perp, \top, \sqcup, \sqcap)$  be a complete lattice and  $\phi : \mathcal{S} \rightarrow \mathcal{S}$  be a monotone operator over  $\mathcal{S}$ . Then:

- ❶  $\phi$  has a least fixpoint  $\text{lfp } \phi$  and  $\text{lfp } \phi = \sqcap \{x \in \mathcal{S} \mid \phi(x) \sqsubseteq x\}$ .
- ❷  $\phi$  has a greatest fixpoint  $\text{gfp } \phi$  and  $\text{gfp } \phi = \sqcup \{x \in \mathcal{S} \mid x \sqsubseteq \phi(x)\}$ .
- ❸ the set of fixpoints of  $\phi$  is a complete lattice.

## Proof of point 1:

We let  $X = \{x \in \mathcal{S} \mid \phi(x) \sqsubseteq x\}$  and  $x_0 = \sqcap X$ .

For all  $y \in X$ , we remark that:

- $x_0 \sqsubseteq y$  by definition of the glb;
- thus, since  $\phi$  is monotone,  $\phi(x_0) \sqsubseteq \phi(y)$ ;
- thus,  $\phi(x_0) \sqsubseteq y$  since  $\phi(y) \sqsubseteq y$ , by definition of  $X$ .

Therefore  $\phi(x_0) \sqsubseteq x_0$ , since  $x_0 = \sqcap X$  and  $\phi(x_0)$  is a lower bound.

# Tarski's Theorem

We proved that  $\phi(x_0) \sqsubseteq x_0$ . We derive from this that:

- $\phi(\phi(x_0)) \sqsubseteq \phi(x_0)$  since  $\phi$  is monotone;
- $\phi(x_0)$  is a post-fixpoint of  $\phi$ , thus  $\phi(x_0) \in X$ ;
- $x_0 \sqsubseteq \phi(x_0)$  by definition of the greatest lower bound

We have established both inclusions so  $\phi(x_0) = x_0$ .

If  $x_1$  is another fixpoint, then  $x_1 \in X$ , so  $x_0 \sqsubseteq x_1$ .

**Proof of point 2:** similar, by duality.

**Proof of point 3:**

- if  $X$  is a set of fixpoints of  $\phi$ , we need to consider  $\phi$  over  $\{y \in \mathcal{S} \mid y \sqsubseteq_{\mathcal{S}} \sqcap X\}$  to establish the existence of a **glb of  $X$  in the poset of fixpoints**
- the existence of **least upper bounds in the poset of fixpoints** follows by duality

# Tarski's theorem: example (1)

## A function over the powerset:

We consider a set  $\mathcal{E}$ , and a subset  $\mathcal{A} \subseteq \mathcal{E}$

We let:

$$\begin{array}{ccc} f : \mathcal{P}(\mathcal{E}) & \longrightarrow & \mathcal{P}(\mathcal{E}) \\ X & \longmapsto & X \cup \mathcal{A} \end{array}$$

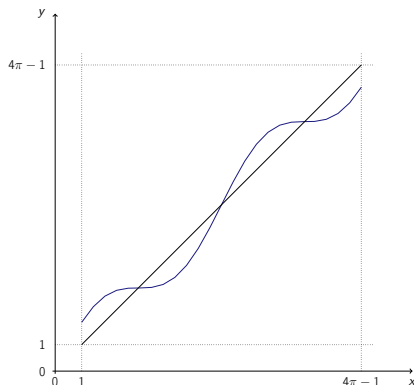
## Exercise:

- apply Tarski's theorem, characterize the least and greatest fixpoints

# Tarski's theorem: example (2)

Function:

$$\begin{array}{ccc} f : [1, 4\pi - 1] & \longrightarrow & [1, 4\pi - 1] \\ x & \longmapsto & x + \sin x \end{array}$$



Exercise:

- apply Tarski's theorem, and derive the fixpoints of the function

# Automata example, fixpoint definition

## Lattice:

- $\mathcal{S} = Q \rightarrow \mathcal{P}(L^*)$
- the ordering is the pointwise extension  $\dot{\sqsubseteq}$  of  $\sqsubseteq$

## Operator:

- we let  $\phi_0 : \mathcal{S} \rightarrow \mathcal{S}$  be defined by  

$$\phi_0(f) = \lambda(q \in Q) \cdot \bigcup_{q' \in Q} \{wa \mid w \in f(q') \wedge q' \xrightarrow{a} q\}$$
- we let  $\phi : \mathcal{S} \rightarrow \mathcal{S}$  be defined by

$$\phi(f) = \lambda(q \in Q) \cdot \begin{cases} f(q_i) \cup \phi_0(f)(q_i) \cup \{\epsilon\} & \text{if } q = q_i \\ f(q) \cup \phi_0(f)(q) & \text{otherwise} \end{cases}$$

## Proof steps to complete:

- the existence of  $\text{lfp } \phi$  follows from Tarski's theorem
- the equality  $\text{lfp } \phi = \llbracket \mathcal{A} \rrbracket$  can be established by induction and double inclusion... but there is a simpler way

# Kleene's Theorem

Tarski's theorem guarantees existence of an lfp, but is not constructive.

## Theorem

Let  $(\mathcal{S}, \sqsubseteq, \perp)$  be a pointed cpo and  $\phi : \mathcal{S} \rightarrow \mathcal{S}$  be a continuous operator over  $\mathcal{S}$ . Then  $\phi$  has a least fixpoint, and

$$\text{lfp } \phi = \bigsqcup_{n \in \mathbb{N}} \phi^n(\perp)$$

First, we prove **the existence of the lub**:

Since  $\phi$  is continuous, it is also monotone. We can prove by induction over  $n$  that  $\{\phi^n(\perp) \mid n \in \mathbb{N}\}$  is a chain:

- $\phi^0(\perp) = \perp \sqsubseteq \phi(\perp)$  by definition of the infimum;
- if  $\phi^n(\perp) \sqsubseteq \phi^{n+1}(\perp)$ , then
$$\phi^{n+1}(\perp) = \phi(\phi^n(\perp)) \sqsubseteq \phi(\phi^{n+1}(\perp)) = \phi^{n+2}(\perp)$$

By definition of the cpo structure, the lub exists. We let  $x_0$  denote it.

# Kleene's Theorem

Secondly, we prove that **it is a fixpoint of  $\phi$** :

Since  $\phi$  is continuous,  $\{\phi^{n+1}(\perp) \mid n \in \mathbb{N}\}$  has a lub, and

$$\begin{aligned}
 \phi(x_0) &= \phi(\sqcup\{\phi^n(\perp) \mid n \in \mathbb{N}\}) \\
 &= \sqcup\{\phi^{n+1}(\perp) \mid n \in \mathbb{N}\} && \text{by continuity of } \phi \\
 &= \perp \sqcup (\sqcup\{\phi^{n+1}(\perp) \mid n \in \mathbb{N}\}) && \text{by definition of } \perp \\
 &= x_0 && \text{by simple rewrite}
 \end{aligned}$$

Last, we show that it is the **least** fixpoint:

Let  $x_1$  denote another fixpoint of  $\phi$ . We show by induction over  $n$  that  $\phi^n(\perp) \sqsubseteq x_1$ :

- $\phi^0(\perp) = \perp \sqsubseteq x_1$  by definition of  $\perp$ ;
- if  $\phi^n(\perp) \sqsubseteq x_1$ , then  $\phi^{n+1}(\perp) \sqsubseteq \phi(x_1) = x_1$  by monotony, and since  $x_1$  is a fixpoint.

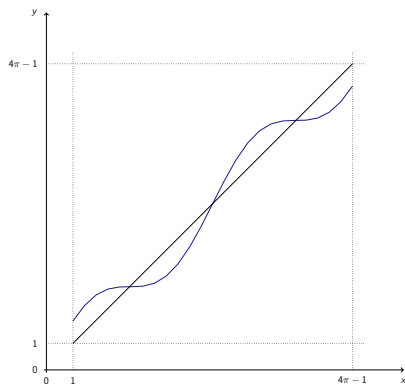
By definition of the lub,  $x_0 \sqsubseteq x_1$

# Kleene's theorem: example

## Function:

$$f : [1, 4\pi - 1] \longrightarrow [1, 4\pi - 1]$$

$$x \longmapsto x + \sin x$$



## Exercise:

- apply Kleene's theorem and sketch the iterations



# Automata: constructive semantics

We can now state a **constructive definition** of the automaton semantics.  
Operator  $\phi$  is defined by

$$\phi(f) = \lambda(q \in Q) \cdot \begin{cases} f(q) \cup \phi_0(f)(q_i) \cup \{\epsilon\} & \text{if } q = q_i \\ f(q) \cup \phi_0(f)(q) & \text{otherwise} \end{cases}$$

## Proof steps:

- $\phi$  is continuous
- thus, Kleene's theorem applies so  $\text{lfp } \phi$  exists and  
 $\text{lfp } \phi = \bigcup_{n \in \mathbb{N}} \phi^n(\perp) \dots$   
 ... this actually saves the double inclusion proof to establish that  
 $\llbracket \mathcal{A} \rrbracket = \text{lfp } \phi$

Furthermore,  $\llbracket \mathcal{A} \rrbracket = \bigcup_{n \in \mathbb{N}} \phi^n(\perp)$ .

This fixpoint definition will be very useful to infer or verify semantic properties.

# Automata: constructive semantics iterates

## A simple automaton:

$$L = \{a, b\} \quad Q = \{q_0, q_1, q_2\}$$

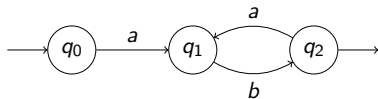
$$q_i = q_0$$

$$q_f = q_2$$

$$q_0 \xrightarrow{a} q_1$$

$$q_1 \xrightarrow{b} q_2$$

$$q_2 \xrightarrow{a} q_1$$



## Iterates of function $\phi$ from $\perp$ :

| Iterate | 0           | 1              | 2              | 3              | 4              | 5              |
|---------|-------------|----------------|----------------|----------------|----------------|----------------|
| $q_0$   | $\emptyset$ | $\{\epsilon\}$ | $\{\epsilon\}$ | $\{\epsilon\}$ | $\{\epsilon\}$ | $\{\epsilon\}$ |
| $q_1$   | $\emptyset$ | $\emptyset$    | $\{a\}$        | $\{a\}$        | $\{a, aba\}$   | $\{a, aba\}$   |
| $q_2$   | $\emptyset$ | $\emptyset$    | $\emptyset$    | $\{ab\}$       | $\{ab\}$       | $\{ab, abab\}$ |

# Duality principle

We can **extend the duality notion to fixpoints**:

|                  |                   |
|------------------|-------------------|
| monotone         | monotone          |
| anti-monotone    | anti-monotone     |
| post-fixpoint    | pre-fixpoint      |
| least fixpoint   | greatest fixpoint |
| increasing chain | decreasing chain  |

Furthermore both Tarski's theorem and Kleene's theorem have a dual version (Tarski's theorem is its own dual).

# On the topic of inductive reasoning...

## Formalizing inductive definitions:

Definition based on  
inference rules:

$$\frac{}{x_0 \in \mathcal{X}} \quad \frac{x \in \mathcal{X}}{f(x) \in \mathcal{X}}$$

Same property based on a  
least-fixpoint:

$$\text{lfp}(Y \mapsto \{x_0\} \cup Y \cup \{f(x) \mid x \in Y\})$$

## Proving the inclusion of a fixpoint in a given set:

- Let  $\phi : \mathcal{S} \longrightarrow \mathcal{S}$  be a continuous operator
- Let  $\mathcal{I} \in \mathcal{S}$  such that:

$$\forall x \in \mathcal{S}, x \sqsubseteq \mathcal{I} \implies \phi(x) \sqsubseteq \mathcal{I}$$

- We obviously have  $\perp \sqsubseteq \mathcal{I}$
- We can prove that  $\text{lfp } \phi \sqsubseteq \mathcal{I}$

## Exercise: language of a grammar

### Language of a grammar as a least-fixpoint

#### Assumptions:

- Alphabet  $\mathcal{A}$ , finite set of nodes  $\mathcal{N}$
- Finite set of rules  $\mathcal{R} \subseteq \mathcal{N} \times (\mathcal{A} \uplus \mathcal{N})^*$
- Starting node  $S \in \mathcal{N}$

#### Questions:

- Define the set of words recognized by the grammar with inductive rules
- Do the same using a least-fixpoint

#### Hints:

- start with a function that maps each node into the set of words recognized by this node
- compute such a function by induction

# Outline

- 1 Introduction
- 2 Case studies
- 3 Approaches to verification
- 4 Orderings, lattices, fixpoints
- 5 Conclusion**

# Main points to remember

## Foundations:

- **program semantics**: express program behaviors
- **target semantic property**: express proof goal
- **conservative approximation** usually required due to undecidability

## Order relations:

- **counterpart for logical implication** (among other)
- will be pervasive in this course

## Fixpoints and induction:

- **encode general iteration**
- will also be pervasive in this course

# In the next lectures...

- Families of **semantics**, for a general model of programs
- Families of **semantic properties of programs**
- **Verification techniques:**
  - ▶ abstract interpretation based static analysis
  - ▶ machine assisted theorem proving
  - ▶ model checking

**Next week: transition systems and operational semantics**



# Practical information about the course

**Course** (1h30) + **TD or TP** (2h00)

**Schedule:** **Friday morning** 8h30–12h15

**Location:** Room É Noether (also known as “U ou V”)

**Course teachers:**

- Sylvain Conchon: SMT
- Charles De Haro: lab sessions
- Jérôme Feret: semantics, typing, abstract interpretation
- Xavier Rival: semantics, program properties, abstract interpretation

**Webpage** with class material:

<https://www.di.ens.fr/~rival/semverif-2024>

material (e.g., video if a class is online) also on Moodle

**Evaluation:** 50 % project + 50 % final exam (or homework, TBC)