

Traces Properties

Semantics and applications to verification

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Program of this lecture

Goal of verification

Prove that $\llbracket P \rrbracket \subseteq \mathcal{S}$

(i.e., all behaviors of P satisfy specification \mathcal{S})

where $\llbracket P \rrbracket$ is the **program semantics** and \mathcal{S} the **desired specification**

Last week, we studied a form of $\llbracket P \rrbracket$...

Today's lecture: we look back at program's properties

- **families of properties:**
what properties can be considered “similar” ? in what sense ?
- **proof techniques:**
how can those kinds of properties be established ?
- **specification of properties:**
are there languages to describe properties ?

A high level overview

- In this lecture we look at **trace properties**
- A property is **a set of traces**, defining the **admissible** executions

Safety properties:

- **something (e.g., bad) will never happen**
- proof by invariance

Liveness properties:

- **something (e.g., good) will eventually happen**
- proof by variance

Beyond safety and liveness: **hyperproperties** (e.g., security...)

State properties

As usual, we consider $\mathcal{S} = (\mathbb{S}, \rightarrow, \mathbb{S}_{\mathcal{I}})$

First approach: properties as sets of states

- A property \mathcal{P} is a set of states $\mathcal{P} \subseteq \mathbb{S}$
- \mathcal{P} is satisfied if and only if all reachable states belong to \mathcal{P} , i.e., $\llbracket \mathcal{S} \rrbracket_{\mathcal{R}} \subseteq \mathcal{P}$ where $\llbracket \mathcal{S} \rrbracket_{\mathcal{R}} = \{s_n \in \mathbb{S} \mid \exists \langle s_0, \dots, s_n \rangle \in \llbracket \mathcal{S} \rrbracket^*, s_0 \in \mathbb{S}_{\mathcal{I}}\}$

Examples:

- Absence of runtime errors:

$$\mathcal{P} = \mathbb{S} \setminus \{\Omega\} \quad \text{where } \Omega \text{ is the error state}$$

- Non termination (e.g., for an operating system):

$$\mathcal{P} = \{s \in \mathbb{S} \mid \exists s' \in \mathbb{S}, s \rightarrow s'\}$$

Trace properties

Second approach: properties as sets of traces

- A property \mathcal{T} is a set of traces $\mathcal{T} \subseteq \mathbb{S}^\infty$
- \mathcal{T} is satisfied if and only if all traces belong to \mathcal{T} , i.e., $[\mathcal{S}]^\infty \subseteq \mathcal{T}$

Examples:

- Obviously, **state properties** are trace properties
- **Functional properties**:
e.g., “program P takes one integer input x and returns its absolute value”
- **Termination**: $\mathcal{T} = \mathbb{S}^*$ (i.e., the system should have no infinite execution)

Monotonicity

Property 1

Let $\mathcal{P}_0, \mathcal{P}_1 \subseteq \mathbb{S}$ be two state properties, such that $\mathcal{P}_0 \subseteq \mathcal{P}_1$.

Then \mathcal{P}_0 is stronger than \mathcal{P}_1 , i.e. if program \mathcal{S} satisfies \mathcal{P}_0 , then it also satisfies \mathcal{P}_1 .

Property 2

Let $\mathcal{T}_0, \mathcal{T}_1 \subseteq \mathbb{S}$ be two trace properties, such that $\mathcal{T}_0 \subseteq \mathcal{T}_1$.

Then \mathcal{T}_0 is stronger than \mathcal{T}_1 , i.e. if program \mathcal{S} satisfies \mathcal{T}_0 , then it also satisfies \mathcal{T}_1 .

Property 3

Let $\mathcal{S}_0, \mathcal{S}_1$ two transition systems, such that \mathcal{S}_1 has more behaviors than \mathcal{S}_0 (i.e., $\llbracket \mathcal{S}_0 \rrbracket \subseteq \llbracket \mathcal{S}_1 \rrbracket$), and \mathcal{P} be a (trace or state) property. Then, if \mathcal{S}_1 satisfies \mathcal{P} , so does \mathcal{S}_0 .

Proofs: straightforward application of the definitions

Outline

1 Safety properties

- Informal and formal definitions
- Proof method

2 Liveness properties

3 Decomposition of trace properties

4 A specification language: temporal logic

5 Beyond safety and liveness

6 Conclusion

Safety properties

Informal definition: safety properties

A safety property is a property which specifies that some (bad) **behavior defined by a finite, irrecoverable observation will never occur**, at any time

- **Absence of runtime errors** is a safety property (“bad thing”: error)
- **State properties** is a safety property (“bad thing”: reaching $\mathbb{S} \setminus \mathcal{P}$)
- **Non termination** is a safety property (“bad thing”: reaching a blocking state)
- “**Not reaching state b after visiting state a** ” is a safety property (and **not** a state property)
- **Termination** is **not** a safety property

We now intend to provide a **formal definition** of safety.

Towards a formal definition

How to refute a safety property ?

- We assume \mathcal{S} does **not** satisfy safety property \mathcal{P}
- Thus, there exists a **counter-example trace**
 $\sigma = \langle s_0, \dots, s_n, \dots \rangle \in \llbracket \mathcal{S} \rrbracket \setminus \mathcal{P}$;
at this point of our study, the trace may be finite or infinite...
- The intuitive definition says this trace **eventually exhibits some bad behavior**, that is **irrecoverable** at **observed** at some given time, thus the observation corresponds to some index i
- Therefore, trace $\sigma' = \langle s_0, \dots, s_i \rangle$ violates \mathcal{P} , i.e. $\sigma' \notin \mathcal{P}$
- Due to the irrecoverability of the observation, the same goes for any trace with the same prefix
- We remark σ' is **finite**

A safety property that does not hold
can always be refuted with a finite, irrecoverable counter-example

A Few Operators on Traces

Prefix: We write $\sigma \upharpoonright_i$ for the prefix of length i of trace σ :

$$\begin{aligned}\langle s_0, \dots, s_n \rangle \upharpoonright_0 &= \epsilon \\ \langle s_0, \dots, s_n \rangle \upharpoonright_{i+1} &= \begin{cases} \langle s_0, \dots, s_i \rangle & \text{if } i < n \\ \langle s_0, \dots, s_n \rangle & \text{otherwise} \end{cases} \\ \langle s_0, \dots \rangle \upharpoonright_{i+1} &= \langle s_0, \dots, s_i \rangle\end{aligned}$$

Suffix (or tail):

$$\begin{aligned}\sigma \upharpoonright_i &= \epsilon \quad \text{if } |\sigma| < i \\ (\langle s_0, \dots, s_i \rangle \cdot \sigma) \upharpoonright_{i+1} &::= \sigma \quad \text{otherwise}\end{aligned}$$

Upper closure operators

Definition: upper closure operator (uco)

We consider a preorder $(\mathcal{S}, \sqsubseteq)$. Function $\phi : \mathcal{S} \rightarrow \mathcal{S}$ is an **upper closure operator** iff:

- **monotone**
- **extensive**: $\forall x \in \mathcal{S}, x \sqsubseteq \phi(x)$
- **idempotent**: $\forall x \in \mathcal{S}, \phi(\phi(x)) = \phi(x)$

Dual: **lower closure operator**, monotone, reductive, idempotent

Examples:

- on real/decimal numbers, or on fraction:
the **ceiling** operator, that returns the next integer is an upper-closure operator

Prefix closure

Definition: prefix closure

The prefix closure operator is defined by:

$$\begin{aligned} \text{PCI} : \mathcal{P}(\mathbb{S}^\infty) &\longrightarrow \mathcal{P}(\mathbb{S}^*) \\ X &\longmapsto \{\sigma \upharpoonright_i \mid \sigma \in X, i \in \mathbb{N}\} \end{aligned}$$

Example: assuming $\mathcal{S} = \{\langle a, b, c \rangle, \langle a, c \rangle\}$ then,

$$\text{PCI}(\mathcal{S}) = \{\epsilon, \langle a \rangle, \langle a, b \rangle, \langle a, b, c \rangle, \langle a, c \rangle\}$$

Properties:

- PCI is monotone
- PCI is idempotent, i.e., $\text{PCI} \circ \text{PCI}(X) = \text{PCI}(X)$
- PCI is not extensive on $\mathcal{P}(\mathbb{S}^\infty)$ (infinite traces do not appear anymore)
its restriction to $\mathcal{P}(\mathbb{S}^*)$

Limit

Definition: limit

The **limit operator** is defined by:

$$\begin{array}{rcl} \text{Lim} : \mathcal{P}(\mathbb{S}^\infty) & \longrightarrow & \mathcal{P}(\mathbb{S}^\infty) \\ X & \longmapsto & X \cup \{\sigma \in \mathbb{S}^\infty \mid \forall i \in \mathbb{N}, \sigma \upharpoonright_i \in X\} \end{array}$$

Operator Lim **is an upper-closure operator**

Proof: exercise!

Example: assuming

$$\mathcal{S} = \{ \epsilon, \langle a \rangle, \langle a, b \rangle, \langle a, b, a \rangle, \langle a, b, a, b \rangle, \langle a, b, a, b, a \rangle, \dots \}$$

then,

$$\text{Lim}(\mathcal{S}) = \mathcal{S} \uplus \{\langle a, b, a, b, a, b, \dots \rangle\}$$

Towards a formal definition for safety

Operator Safe

Operator Safe is defined by $\text{Safe} = \text{Lim} \circ \text{PCI}$.

Operator **Safe saturates** a set of traces S with

- prefixes
- infinite traces all finite prefixes of which can be observed in S

Thus, if $\text{Safe}(S) = S$ and σ is a trace, to establish that σ is not in S , it is sufficient to discover a **finite prefix of σ** that cannot be observed in S .

- if σ is finite the result is clear (consider σ)
- otherwise, if all finite prefixes of σ are in S , then σ is in the limit, thus in S .

Safety: definition

A trace property \mathcal{T} is a **safety** property if and only if $\text{Safe}(\mathcal{T}) = \mathcal{T}$

Safety properties: formal definition

An upper closure operator

Operator **Safe** is an upper closure operator over $\mathcal{P}(\mathbb{S}^\infty)$

Proof:

Safe is monotone since Lim and PCI are monotone

Safe is extensive:

indeed if $X \subseteq \mathbb{S}^\infty$ and $\sigma \in X$, we can show that $\sigma \in \text{Safe}(X)$:

- if σ is a finite trace, it is one of its prefixes, so
 $\sigma \in \text{PCI}(X) \subseteq \text{Lim}(\text{PCI}(X))$
- if σ is an infinite trace, all its prefixes belong to $\text{PCI}(X)$, so
 $\sigma \in \text{Lim}(\text{PCI}(X))$

Safety properties: formal definition

Proof (continued):

Safe is idempotent:

- as Safe is extensive and monotone $\text{Safe} \subseteq \text{Safe} \circ \text{Safe}$, so we simply need to show that $\text{Safe} \circ \text{Safe} \subseteq \text{Safe}$
- let $X \subseteq \mathbb{S}^\infty$, $\sigma \in \text{Safe}(\text{Safe}(X))$; then:

$$\begin{aligned} & \sigma \in \text{Safe}(\text{Safe}(X)) \\ \Rightarrow & \forall i, \sigma \upharpoonright_i \in \text{PCI} \circ \text{Safe}(X) && \text{by def. of Lim} \\ \Rightarrow & \forall i, \exists \sigma', j, \sigma \upharpoonright_i = \sigma' \upharpoonright_j \wedge \sigma' \in \text{Safe}(X) && \text{by def. of PCI} \\ \Rightarrow & \forall i, \exists \sigma', j, \sigma \upharpoonright_i = \sigma' \upharpoonright_j \wedge \forall k, \sigma' \upharpoonright_k \in \text{PCI}(X) \\ & && \text{by def. of Lim and case analysis over finiteness of } \sigma' \\ \Rightarrow & \forall i, \exists \sigma', j, \sigma \upharpoonright_i = \sigma' \upharpoonright_j \wedge \sigma' \upharpoonright_j \in \text{PCI}(X) && \text{if we take } k = j \\ \Rightarrow & \forall i, \sigma \upharpoonright_i \in \text{PCI}(X) && \text{by simplification} \\ \Rightarrow & \sigma \in \text{Lim} \circ \text{PCI}(X) && \text{by def. of Lim} \\ \Rightarrow & \sigma \in \text{Safe}(X) \end{aligned}$$

Safety properties: formal definition

Safety: definition

A trace property \mathcal{T} is a **safety** property if and only if $\text{Safe}(\mathcal{T}) = \mathcal{T}$

Theorem

If \mathcal{T} is a trace property, then $\text{Safe}(\mathcal{T})$ is a **safety property**

Proof:

Straightforward, by idempotence of Safe

Intuition:

- if \mathcal{T} is a trace property (not necessarily a safety property), $\text{Safe}(\mathcal{T})$ is **the strongest safety property, that is weaker than \mathcal{T}**
- at this point, this observation is not so useful...
but it will be soon!

Example

We assume that:

- $\mathbb{S} = \{a, b\}$
- \mathcal{T} states that *a should not be visited after state b is visited*; elements of \mathcal{T} are of the general form

$$\langle a, a, a, \dots, a, b, b, b, b, \dots \rangle \text{ or } \langle a, a, a, \dots, a, a, \dots \rangle$$

Then:

- $\text{PCI}(\mathcal{T})$ elements are all finite traces which are of the above form (i.e., made of n occurrences of a followed by m occurrences of b , where n, m are positive integers)
- $\text{Lim}(\text{PCI}(\mathcal{T}))$ adds to this set the trace made of infinitely many occurrences of a and the infinite traces made of n occurrences of a followed by infinitely many occurrences of b
- thus, $\text{Safe}(\mathcal{T}) = \text{Lim}(\text{PCI}(\mathcal{T})) = \mathcal{T}$

Therefore \mathcal{T} is indeed formally a safety property.

State properties are safety properties

Theorem

Any **state property** is also a **safety property**.

Proof:

Let us consider **state property** \mathcal{P} .

It is equivalent to **trace property** $\mathcal{T} = \mathcal{P}^\alpha$:

$$\begin{aligned}\text{Safe}(\mathcal{T}) &= \text{Lim}(\text{PCI}(\mathcal{P}^\alpha)) \\ &= \text{Lim}(\mathcal{P}^*) \\ &= \mathcal{P}^* \cup \mathcal{P}^\omega \\ &= \mathcal{P}^\alpha \\ &= \mathcal{T}\end{aligned}$$

Therefore \mathcal{T} is indeed a safety property.

Intuition of the formal definition

Operator **Safe saturates** a set of traces S with

- prefixes
- infinite traces all finite prefixes of which can be observed in S

Thus, if $\text{Safe}(S) = S$ and σ is a trace, to establish that σ is not in S , it is sufficient to discover a **finite prefix of σ** that cannot be observed in S .

Alternatively, if all finite prefixes of σ belong to S or can be observed as a prefix of another trace in S , by definition of the limit operator, σ belongs to S (even if it is infinite).

Thus, our definition **indeed captures properties that can be disproved with a finite counter-example**.

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Proof by invariance

- We consider transition system $\mathcal{S} = (\mathbb{S}, \rightarrow, \mathbb{S}_{\mathcal{I}})$, and safety property \mathcal{T} . Finite traces semantics is the least fixpoint of F_* .
- We seek a way of **verifying that \mathcal{S} satisfies \mathcal{T}** , i.e., that $\llbracket \mathcal{S} \rrbracket^\infty \subseteq \mathcal{T}$

Principle of invariance proofs

Let \mathbb{I} be a set of finite traces; it is said to be an **invariant** if and only if:

- $\forall s \in \mathbb{S}_{\mathcal{I}}, \langle s \rangle \in \mathbb{I}$
- $F_*(\mathbb{I}) \subseteq \mathbb{I}$

It is stronger than \mathcal{T} if and only if $\mathbb{I} \subseteq \mathcal{T}$.

The “**by invariance**” proof method is based on finding an invariant that is stronger than \mathcal{T} .

Soundness

Theorem: soundness

The invariance proof method is **sound**: if we can find an invariant for \mathcal{S} , that is stronger than safety property \mathcal{T} , then \mathcal{S} satisfies \mathcal{T} .

Proof:

We assume that \mathbb{I} is an invariant of \mathcal{S} and that it is stronger than \mathcal{T} , and we show that \mathcal{S} satisfies \mathcal{T} :

- by induction over n , we can prove that $F_*^n(\{\langle s \rangle \mid s \in \mathbb{S}_{\mathcal{I}}\}) \subseteq F_*^n(\mathbb{I}) \subseteq \mathbb{I}$
- therefore $\llbracket \mathcal{S} \rrbracket^* \subseteq \mathbb{I}$
- thus, $\text{Safe}(\llbracket \mathcal{S} \rrbracket^*) \subseteq \text{Safe}(\mathbb{I}) \subseteq \text{Safe}(\mathcal{T})$ since Safe is monotone
- we remark that $\llbracket \mathcal{S} \rrbracket^\infty = \text{Safe}(\llbracket \mathcal{S} \rrbracket^*)$
- \mathcal{T} is a safety property so $\text{Safe}(\mathcal{T}) = \mathcal{T}$
- we conclude $\llbracket \mathcal{S} \rrbracket^\infty \subseteq \mathcal{T}$, i.e., \mathcal{S} satisfies property \mathcal{T}

Completeness

Theorem: completeness

The invariance proof method is **complete**: if \mathcal{S} satisfies safety property \mathcal{T} , then we can find an invariant \mathbb{I} for \mathcal{S} , that is stronger than \mathcal{T} .

Proof:

We assume that $\llbracket \mathcal{S} \rrbracket^*$ satisfies \mathcal{T} , and show that we can exhibit an invariant.

Then, $\mathbb{I} = \llbracket \mathcal{S} \rrbracket^*$ is an invariant of \mathcal{S} by definition of $\llbracket . \rrbracket^*$, and it is stronger than \mathcal{T} .

Caveat:

- $\llbracket \mathcal{S} \rrbracket^\infty$ is most likely **not** a very easy to express invariant
- it is just a convenient completeness argument
- so, completeness does not mean the proof is easy !

Example

We consider the proof that the program below **computes the sum of the elements of an array**, i.e., when the exit is reached, $s = \sum_{k=0}^{n-1} t[k]$:

i, s integer variables
 t integer array of length n

ℓ_0 : $\langle \text{true} \rangle$
 $s = 0;$
 ℓ_1 : $\langle s = 0 \rangle$
 $i = 0;$
 ℓ_2 : $\langle i = 0 \wedge s = 0 \rangle$
 $\text{while}(i < n) \{$
 ℓ_3 : $\langle 0 \leq i < n \wedge s = \sum_{k=0}^{i-1} t[k] \rangle$
 $s = s + t[i];$
 ℓ_4 : $\langle 0 \leq i < n \wedge s = \sum_{k=0}^i t[k] \rangle$
 $i = i + 1;$
 ℓ_5 : $\langle 1 \leq i \leq n \wedge s = \sum_{k=0}^{i-1} t[k] \rangle$
 $\}$
 ℓ_6 : $\langle i = n \wedge s = \sum_{k=0}^{n-1} t[k] \rangle$

Principle of the proof:

- for each program point ℓ , we have a **local invariant** \mathbb{I}_ℓ (denoted by a logical formula instead of a set of states in the figure)
- the global **invariant** \mathbb{I} is defined by:

$$\mathbb{I} = \{ \langle (\ell_0, m_0), \dots, (\ell_n, m_n) \rangle \mid \forall n, m_n \in \mathbb{I}_{\ell_n} \}$$

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Liveness properties

Informal definition: liveness properties

A liveness property is a property which specifies that some (good) behavior **will eventually occur**, and that this behavior **may still occur after any finite observation**.

- **Termination** is a liveness property
“good behavior”: reaching a blocking state (no more transition available)
- “**State a will eventually be reached by any execution**” is a liveness property
“good behavior”: reaching state a
- The **absence of runtime errors** is *not* a liveness property

As for safety properties, we intend to provide a **formal definition** of liveness.

Intuition towards a formal definition

How to refute a liveness property ?

- We consider liveness property \mathcal{T} (think \mathcal{T} is termination)
- We assume \mathcal{S} does **not** satisfy liveness property \mathcal{T}
- Thus, there exists a **counter-example trace** $\sigma \in \llbracket \mathcal{S} \rrbracket \setminus \mathcal{T}$;
- The informal definition says:
 ... **may still occur after any finite observation**
 thus, each finite trace σ' can be extended into a good trace
- **Conclusion of this discussion:** σ is necessarily **infinite**

To prove that a liveness property does not hold
we need to look for an infinite counter-example
i.e., no finite trace is a counter-example

Intuition towards a formal definition

To refute a liveness property, we need to look at infinite traces.

Example: if we run a program, and do not see it return...

- should we do Ctrl+C and conclude it does not terminate ?
- should we just wait a few more seconds minutes, hours, years ?

Towards a formal definition:

we expect any finite trace be the prefix of a trace in \mathcal{T}

... since finite executions cannot be used to disprove \mathcal{T}

Formal definition (incomplete)

$$\text{PCI}(\mathcal{T}) = \mathbb{S}^*$$

Definition

Formal definition

Operator Live is defined by $\text{Live}(\mathcal{T}) = \mathcal{T} \cup (\mathbb{S}^\infty \setminus \text{Safe}(\mathcal{T}))$. Given property \mathcal{T} , the following three statements are equivalent:

- (i) $\text{Live}(\mathcal{T}) = \mathcal{T}$
- (ii) $\text{PCI}(\mathcal{T}) = \mathbb{S}^*$
- (iii) $\text{Lim} \circ \text{PCI}(\mathcal{T}) = \mathbb{S}^\infty$

When they are satisfied, \mathcal{T} is said to be a **liveness property**

Example: termination

- The property is $\mathcal{T} = \mathbb{S}^*$
(i.e., there should be no infinite execution)
- Clearly, it satisfies (ii): $\text{PCI}(\mathcal{T}) = \mathbb{S}^*$
thus termination indeed satisfies this definition

Proof of equivalence

Proof of equivalence:

(i) implies (ii):

We assume that $\text{Live}(\mathcal{T}) = \mathcal{T}$, i.e., $\mathcal{T} \cup (\mathbb{S}^\infty \setminus \text{Safe}(\mathcal{T})) = \mathcal{T}$
therefore, $\mathbb{S}^\infty \setminus \text{Safe}(\mathcal{T}) \subseteq \mathcal{T}$.

Let $\sigma \in \mathbb{S}^*$, and let us show that $\sigma \in \text{PCI}(\mathcal{T})$; clearly, $\sigma \in \mathbb{S}^\infty$, thus:

- either $\sigma \in \text{Safe}(\mathcal{T}) = \text{Lim}(\text{PCI}(\mathcal{T}))$, so all its prefixes are in $\text{PCI}(\mathcal{T})$ and $\sigma \in \text{PCI}(\mathcal{T})$
- or $\sigma \in \mathcal{T}$, which implies that $\sigma \in \text{PCI}(\mathcal{T})$

(ii) implies (iii):

If $\text{PCI}(\mathcal{T}) = \mathbb{S}^*$, then $\text{Lim} \circ \text{PCI}(\mathcal{T}) = \mathbb{S}^\infty$

(iii) implies (i):

If $\text{Lim} \circ \text{PCI}(\mathcal{T}) = \mathbb{S}^\infty$, then

$\text{Live}(\mathcal{T}) = \mathcal{T} \cup (\mathbb{S}^\infty \setminus (\text{Lim} \circ \text{PCI}(\mathcal{T}))) = \mathcal{T} \cup (\mathbb{S}^\infty \setminus \mathbb{S}^\infty) = \mathcal{T}$

Example

We assume that:

- $\mathbb{S} = \{a, b, c\}$
- \mathcal{T} states that *b should eventually be visited, after a has been visited*; elements of \mathcal{T} can be described by

$$\mathcal{T} = \text{PCI}(\mathbb{S}^* \cdot a \cdot \mathbb{S}^* \cdot b \cdot \mathbb{S}^\omega)$$

Then \mathcal{T} is a liveness property:

- let $\sigma \in \mathbb{S}^*$; then $\sigma \cdot a \cdot b \in \mathcal{T}$, so $\sigma \in \text{PCI}(\mathcal{T})$
- thus, $\text{PCI}(\mathcal{T}) = \mathbb{S}^*$

A property of Live

Theorem

If \mathcal{T} is a trace property, then $\text{Live}(\mathcal{T})$ is a liveness property (i.e., operator Live is **idempotent**).

Proof: we show that $\text{PCI} \circ \text{Live}(\mathcal{T}) = \mathbb{S}^*$, by considering $\sigma \in \mathbb{S}^*$ and proving that $\sigma \in \text{PCI} \circ \text{Live}(\mathcal{T})$; we first note that:

$$\begin{aligned}\text{PCI} \circ \text{Live}(\mathcal{T}) &= \text{PCI}(\mathcal{T}) \cup \text{PCI}(\mathbb{S}^\infty \setminus \text{Safe}(\mathcal{T})) \\ &= \text{PCI}(\mathcal{T}) \cup \text{PCI}(\mathbb{S}^\infty \setminus \text{Lim} \circ \text{PCI}(\mathcal{T}))\end{aligned}$$

- if $\sigma \in \text{PCI}(\mathcal{T})$, this is obvious.
- if $\sigma \notin \text{PCI}(\mathcal{T})$, then:
 - ▶ $\sigma \notin \text{Lim} \circ \text{PCI}(\mathcal{T})$ by definition of the limit
 - ▶ thus, $\sigma \in \mathbb{S}^\infty \setminus \text{Lim} \circ \text{PCI}(\mathcal{T})$
 - ▶ $\sigma \in \text{PCI}(\mathbb{S}^\infty \setminus \text{Lim} \circ \text{PCI}(\mathcal{T}))$ as PCI is extensive when applied to sets of finite traces, which proves the above result

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Termination proof with ranking function

- We consider only **termination**
- We consider transition system $\mathcal{S} = (\mathbb{S}, \rightarrow, \mathbb{S}_{\mathcal{I}})$, and liveness property \mathcal{T}
- We seek a way of **verifying that \mathcal{S} satisfies termination**, i.e., that $[\mathcal{S}]^\infty \subseteq \mathbb{S}^*$

Definition: ranking function

A **ranking function** is a function $\phi : \mathbb{S} \rightarrow E$ where:

- (E, \sqsubseteq) is a **well-founded ordering**
- $\forall s_0, s_1 \in \mathbb{S}, s_0 \rightarrow s_1 \implies \phi(s_1) \sqsubset \phi(s_0)$

Theorem

If \mathcal{S} has a **ranking function ϕ** , it satisfies termination.

Example

We consider the termination of the array sum program:

i, s integer variables
 t integer array of length n

l_0 : $s = 0;$
 l_1 : $i = 0;$
 l_2 : while($i < n$) {
 l_3 : $s = s + t[i];$
 l_4 : $i = i + 1;$
 l_5 : }
 l_6 : ...

Ranking function:

$$\begin{aligned}
 \phi : \mathbb{S} &\longrightarrow \mathbb{N} \\
 (l_0, m) &\longmapsto 3 \cdot n + 6 \\
 (l_1, m) &\longmapsto 3 \cdot n + 5 \\
 (l_2, m) &\longmapsto 3 \cdot n + 4 \\
 (l_3, m) &\longmapsto 3 \cdot (n - m(i)) + 3 \\
 (l_4, m) &\longmapsto 3 \cdot (n - m(i)) + 2 \\
 (l_5, m) &\longmapsto 3 \cdot (n - m(i)) + 4 \\
 (l_6, m) &\longmapsto 0
 \end{aligned}$$

Proof by variance

- We consider transition system $\mathcal{S} = (\mathbb{S}, \rightarrow, \mathbb{S}_{\mathcal{I}})$, and liveness property \mathcal{T} ; infinite traces semantics is the greatest fixpoint of F_{ω} .
- We seek a way of **verifying that \mathcal{S} satisfies \mathcal{T}** , i.e., that $\llbracket \mathcal{S} \rrbracket^{\omega} \subseteq \mathcal{T}$

Principle of variance proofs

Let $(\mathbb{I}_n)_{n \in \mathbb{N}}, \mathbb{I}_{\omega}$ be elements of \mathbb{S}^{ω} ; these are said to form a variance proof of \mathcal{T} if and only if:

- $\mathbb{S}^{\omega} \subseteq \mathbb{I}_0$
- for all $k \in \{1, 2, \dots, \omega\}$, $\forall s \in \mathbb{S}, \langle s \rangle \in \mathbb{I}_k$
- for all $k \in \{1, 2, \dots, \omega\}$, there exists $l < k$ such that $F_{\omega}(\mathbb{I}_l) \subseteq \mathbb{I}_k$
- $\mathbb{I}_{\omega} \subseteq \mathcal{T}$

Proofs of soundness and completeness: exercise, similar to the previous proof but using the definition of $\llbracket \mathcal{S} \rrbracket^{\omega}$ instead

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The decomposition theorem

Theorem

Let $\mathcal{T} \subseteq \mathbb{S}^\infty$; it can be decomposed into the **conjunction** of **safety property** $\text{Safe}(\mathcal{T})$ and **liveness property** $\text{Live}(\mathcal{T})$:

$$\mathcal{T} = \text{Safe}(\mathcal{T}) \cap \text{Live}(\mathcal{T})$$

- **Reading: Recognizing Safety and Liveness.**
Bowen Alpern and **Fred B. Schneider**.
In Distributed Computing, Springer, 1987.
- **Consequence of this result:**
the proof of any trace property can be decomposed into
 - ▶ a proof of safety
 - ▶ a proof of liveness

Proof

- **Safety part:**

Safe is idempotent, so $\text{Safe}(\mathcal{T})$ is a safety property.

- **Liveness part:**

Live is idempotent, so $\text{Live}(\mathcal{T})$ is a liveness property.

- **Decomposition:**

$$\begin{aligned}\text{Safe}(\mathcal{T}) \cap \text{Live}(\mathcal{T}) &= \text{Safe}(\mathcal{T}) \cap (\mathcal{T} \cup \mathbb{S}^\infty \setminus \text{Safe}(\mathcal{T})) \\ &= \text{Safe}(\mathcal{T}) \cap \mathcal{T} \\ &\quad \cup \text{Safe}(\mathcal{T}) \cap (\mathbb{S}^\infty \setminus \text{Safe}(\mathcal{T})) \\ &= \mathcal{T} \cup \emptyset \\ &= \mathcal{T}\end{aligned}$$

Example: verification of total correctness

```

i, s integer variables
t integer array of length n
l0 : s = 0;
l1 : i = 0;
l2 : while(i < n){
l3 :     s = s + t[i];
l4 :     i = i + 1;
l5 : }
l6 : ...

```

Property to prove:
total correctness

- ➊ the program **terminates**
- ➋ and **does not crash**
- ➌ and **computes the sum of the elements in the array**

Application of the decomposition principle

Conjunction of two proofs:

- ➊ Proved with a **ranking function**
- ➋ Proved with **local invariants**
- ➌ Also proved with **local invariants**

Safety and Liveness Decomposition Example

We consider a very simple **greatest common divider** code function:

```
l0 : int f(int a, int b){  
l1 :     while(a > 0){  
l2 :         int d = b/a;  
l3 :         int r = b - a * d;  
l4 :         b = a;  
l5 :         a = r;  
l6 :     }  
l7 :     return b;  
l8 : }
```

Specification

When applied to positive integers, function `f` should always return their GCD.

Safety and Liveness Decomposition Example

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l7 :     return b;
l8 : }
```

Specification

When applied to positive integers, function f should always return their GCD.

Safety part

For all trace starting with positive inputs, a conjunction of two properties:

- no runtime errors
- the value of b is the GCD

Liveness part

Termination, on all traces starting with positive inputs

The Zoo of semantic properties: current status

Trace properties
total correctness

Safety properties
never reach s_0 before s_1

State properties
absence or runtime errors
partial correctness

Liveness properties
termination

- **Safety:** if wrong, can be refuted with a **finite trace**
proof done by **invariance**
- **Liveness:** if wrong, has to be refuted with an **infinite trace**
proof done by **variance**

Outline

- 1 Safety properties
- 2 Liveness properties
- 3 Decomposition of trace properties
- 4 A specification language: temporal logic
- 5 Beyond safety and liveness
- 6 Conclusion

Notion of specification language

- Ultimately, we would like to **verify or compute** properties
- So far, we simply describe properties with **sets of executions** or worse, with English / French / ... statements
- Ideally, we would prefer to use a **mathematical language** for that
 - ▶ to **gain in concision, avoid ambiguity**
 - ▶ to **define sets of properties to consider, fix the form of inputs for verification tools...**

Definition: specification language

A **specification language** is a set of terms \mathbb{L} with an **interpretation function** (or **semantics**)

$$[\![\cdot]\!] : \mathbb{L} \longrightarrow \mathcal{P}(\mathbb{S}^\infty) \quad (\text{resp., } \mathcal{P}(\mathbb{S}))$$

- We are now going to consider specification languages **for states, for traces...**

A State specification language

A first **example** of a (simple) specification language:

A state specification language

- **Syntax:** we let terms of \mathbb{L}_S be defined by:

$$p \in \mathbb{L}_S ::= @\ell \mid x < x' \mid x < n \mid \neg p' \mid p' \wedge p'' \mid \Omega$$

- **Semantics:** $\llbracket p \rrbracket_s \subseteq \mathbb{S}_\Omega$ is defined by

$$\begin{aligned}
 \llbracket @\ell \rrbracket_s &= \{\ell\} \times \mathbb{M} \\
 \llbracket x \leq x' \rrbracket_s &= \{(\ell, m) \in \mathbb{S} \mid m(x) \leq m(x')\} \\
 \llbracket x \leq n \rrbracket_s &= \{(\ell, m) \in \mathbb{S} \mid m(x) \leq n\} \\
 \llbracket \neg p \rrbracket_s &= \mathbb{S}_\Omega \setminus \llbracket p \rrbracket_s \\
 \llbracket p \wedge p' \rrbracket_s &= \llbracket p \rrbracket_s \cap \llbracket p' \rrbracket_s \\
 \llbracket \Omega \rrbracket_s &= \{\Omega\}
 \end{aligned}$$

Exercise: add $=$, \vee , $\implies \dots$

State properties: examples

Unreachability of control state l_0 :

- **specification:** $\vee \neg @l_0$
- **property:** $\llbracket \neg @l_0 \rrbracket_s = \mathbb{S}_\Omega \setminus \{(l_0, m) \mid m \in \mathbb{M}\}$

Absence of runtime errors:

- **specification:** $\neg \Omega$
- **property:** $\llbracket \neg \Omega \rrbracket_s = \mathbb{S}_\Omega \setminus \{\Omega\} = \mathbb{S}$

Intermittent invariant:

- **principle:** attach a local invariant to each control state
- **example:**

l_0 :	$\text{if}(x \geq 0) \{$		
l_1 :	$y = x;$	$\quad \quad \quad @l_1 \implies x \geq 0$	
l_2 :	$\} \text{else} \{$	$\wedge \quad @l_2 \implies x \geq 0 \wedge y \geq 0$	
l_3 :	$y = -x;$	$\wedge \quad @l_3 \implies x < 0$	
l_4 :	$\}$	$\wedge \quad @l_4 \implies x < 0 \wedge y > 0$	
l_5 :	\dots	$\wedge \quad @l_5 \implies y \geq 0$	

Propositional temporal logic: syntax

We now consider the **specification of trace properties**

- **Temporal logic:** specification of properties in terms of events that occur at distinct times in the execution (hence, the name “temporal”)
- There are **many** instances of temporal logic
- We study a simple one: **Pnueli’s Propositional Temporal Logic**

Definition: syntax of PTL (Propositional Temporal Logic)

Properties over traces are defined as terms of the form

$$\begin{array}{lcl}
 t (\in \mathbb{L}_{\text{PTL}}) & ::= & p \quad \text{state property, i.e., } p \in \mathbb{L}_{\text{S}} \\
 & | & t' \vee t'' \quad \text{disjunction} \\
 & | & \neg t' \quad \text{negation} \\
 & | & \bigcirc t' \quad \text{"next"} \\
 & | & t' \mathfrak{U} t'' \quad \text{"until", i.e., } t' \text{ until } t''
 \end{array}$$

Propositional temporal logic: semantics

The semantics of a temporal property is a set of traces, and it is defined by induction over the syntax:

Semantics of Propositional Temporal Logic formulae

$$\begin{aligned}\llbracket p \rrbracket_t &= \{s \cdot \sigma \mid s \in \llbracket p \rrbracket_s \wedge \sigma \in \mathbb{S}^\infty\} \\ \llbracket t_0 \vee t_1 \rrbracket_t &= \llbracket t_0 \rrbracket_t \cup \llbracket t_1 \rrbracket_t \\ \llbracket \neg t_0 \rrbracket_t &= \mathbb{S}^\infty \setminus \llbracket t_0 \rrbracket_t \\ \llbracket \bigcirc t_0 \rrbracket_t &= \{s \cdot \sigma \mid s \in \mathbb{S} \wedge \sigma \in \llbracket t_0 \rrbracket_t\} \\ \llbracket t_0 \mathfrak{U} t_1 \rrbracket_t &= \{\sigma \in \mathbb{S}^\infty \mid \exists n \in \mathbb{N}, \forall i < n, \sigma_{i\lceil} \in \llbracket t_0 \rrbracket_t \wedge \sigma_{n\lceil} \in \llbracket t_1 \rrbracket_t\}\end{aligned}$$

Temporal logic operators as syntactic sugar

Many useful operators can be added:

- **Boolean constants:**

$$\mathbf{true} ::= (x < 0) \vee \neg(x < 0)$$

$$\mathbf{false} ::= \neg \mathbf{true}$$

- **Sometime:**

$$\Diamond t ::= \mathbf{true} \sqcup t$$

intuition: there exists a rank n at which t holds

- **Always:**

$$\Box t ::= \neg(\Diamond(\neg t))$$

intuition: there is no rank at which the negation of t holds

Exercise: what do $\Diamond \Box t$ and $\Box \Diamond t$ mean ?

Propositional temporal logic: examples

We consider the program below:

```

 $l_0 : x = \text{input}();$ 
 $l_1 : \text{if}(x < 8) \{$ 
 $l_2 : \quad x = 0;$ 
 $l_3 : \} \text{ else } \{$ 
 $l_4 : \quad x = 1;$ 
 $l_5 : \}$ 
 $l_6 : \dots$ 

```

Examples of properties:

- “when l_4 is reached, x is positive”

$$\square(\text{@}l_4 \implies x \geq 0)$$

- “if the value read at point l_0 is negative, and when l_6 is reached, x is equal to 0”

$$\square((\text{@}l_1 \wedge x < 0) \implies \square(\text{@}l_6 \implies x = 0))$$

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Security properties

We now consider other interesting properties of programs, and show that they do not all reduce to trace properties

Security

- Collects many kinds of properties
- So we consider just one:
an unauthorized observer should not be able to guess anything about private information by looking at public information
- **Example:** another user should not be able to guess the content of an email sent to you
- We need to **formalize this property**

A few definitions

Assumptions:

- We let $\mathcal{S} = (\mathbb{S}, \rightarrow, \mathbb{S}_{\mathcal{I}})$ be a transition system
- States are of the form $(\ell, m) \in \mathbb{L} \times \mathbb{M}$
- Memory states are of the form $\mathbb{X} \rightarrow \mathbb{V}$
- We let $\ell, \ell' \in \mathbb{L}$ (program entry and exit)
and $x, x' \in \mathbb{X}$ (private and public variables)

Security property we are looking at

Observing the value of x' at ℓ'
gives no information on the value of x at ℓ

A few examples

A secure program (no information flow, no way to guess x):

$$\begin{array}{ll}\ell : & x' = 84; \\ \ell' : & \dots\end{array}$$

An insecure program (explicit information flow, x' gives a lot of information about x , so that we can simply recompute it):

$$\begin{array}{ll}\ell : & x' = x - 2; \\ \ell' : & \dots\end{array}$$

An insecure program (implicit information flow, through a test):

$$\begin{array}{ll}\ell : & \text{if}(x < 0)\{x' = 0; \} \\ \ell' : & \dots\end{array}$$

How to characterize information flow in the semantic level ?

Non-interference

We consider the **transformer** Φ defined by:

$$\begin{aligned}\Phi : \mathbb{M} &\longrightarrow \mathcal{P}(\mathbb{M}) \\ m &\longmapsto \{m' \in \mathbb{M} \mid \exists \sigma = \langle (\ell, m), \dots, (\ell', m') \rangle \in \llbracket \mathcal{S} \rrbracket\}\end{aligned}$$

Definition: non-interference

There is **no interference** between (ℓ, x) and (ℓ', x') and we write $(\ell', x') \not\rightsquigarrow (\ell, x)$ if and only if the following property holds:

$$\forall m \in \mathbb{M}, \forall v_0, v_1 \in \mathbb{V}, \\ \{m'(x') \mid m' \in \Phi(m[x \leftarrow v_0])\} = \{m'(x') \mid m' \in \Phi(m[x \leftarrow v_1])\}$$

Intuition:

- if two observations at point ℓ differ only in the value of x , there is no difference in observation of x' at ℓ'
- in other words, observing x' at ℓ' (even on many executions) gives no information about the value of x at point ℓ ...

Non-interference is not a trace property

- We assume $\mathbb{V} = \{0, 1\}$ and $\mathbb{X} = \{x, x'\}$ (store m is defined by the pair $(m(x), m(x'))$, and denoted by it)
- We assume $\mathbb{L} = \{\ell, \ell'\}$ and consider two systems such that all transitions are of the form $(\ell, m) \rightarrow (\ell', m')$
(i.e., system \mathcal{S} is isomorphic to its transformer $\Phi[\mathcal{S}]$)

$$\begin{array}{ll} \Phi[\mathcal{S}_0] : & \begin{array}{ll} (0, 0) & \mapsto \mathbb{M} \\ (0, 1) & \mapsto \mathbb{M} \\ (1, 0) & \mapsto \mathbb{M} \\ (1, 1) & \mapsto \mathbb{M} \end{array} & \Phi[\mathcal{S}_1] : & \begin{array}{ll} (0, 0) & \mapsto \mathbb{M} \\ (0, 1) & \mapsto \mathbb{M} \\ (1, 0) & \mapsto \{(1, 0)\} \\ (1, 1) & \mapsto \{(1, 1)\} \end{array} \end{array}$$

- \mathcal{S}_1 has fewer behaviors than \mathcal{S}_0 : $[\mathcal{S}_1]^* \subset [\mathcal{S}_0]^*$
- \mathcal{S}_0 **has the non-interference property**, but \mathcal{S}_1 **does not**
- If non interference was a trace property, \mathcal{S}_1 should have it (monotony)

Thus, the non interference property is not a trace property

Dependence properties

Dependence property

- Many notions of dependences
- So we consider just one:
what inputs may influence on the observation of a given output
- **Applications:**
 - ▶ **reverse engineering:** understand how an input gets computed
 - ▶ **slicing:** extract the fragment of a program that is relevant to a result
- This corresponds to the **negation** of non-interference

Interference

Definition: interference

There is **interference** between (ℓ, x) and (ℓ', x') and we write $(\ell', x') \rightsquigarrow (\ell, x)$ if and only if the following property holds:

$$\exists m \in \mathbb{M}, \exists v_0, v_1 \in \mathbb{V}, \\ \{m'(x') \mid m' \in \Phi(m[x \leftarrow v_0])\} \neq \{m'(x') \mid m' \in \Phi(m[x \leftarrow v_1])\}$$

- This expresses that there is at least one case, where the value of x at ℓ has an impact on that of x' at ℓ'
- It may not hold even if the computation of x' reads x :

$$\begin{aligned} \ell : \quad x' &= 0 \star x; \\ \ell' : \quad \dots \end{aligned}$$

Interference is not a trace property

- We assume $\mathbb{V} = \{0, 1\}$ and $\mathbb{X} = \{x, x'\}$ (store m is defined by the pair $(m(x), m(x'))$, and denoted by it)
- We assume $\mathbb{L} = \{\ell, \ell'\}$ and consider two systems such that all transitions are of the form $(\ell, m) \rightarrow (\ell', m')$
(i.e., system \mathcal{S} is isomorphic to its transformer $\Phi[\mathcal{S}]$)

$$\begin{array}{ll} \Phi[\mathcal{S}_0] : \begin{array}{ll} (0, 0) \mapsto \mathbb{M} & \Phi[\mathcal{S}_1] : \begin{array}{ll} (0, 0) \mapsto \{(1, 1)\} \\ (0, 1) \mapsto \mathbb{M} & (0, 1) \mapsto \{(1, 1)\} \\ (1, 0) \mapsto \{(1, 1)\} & (1, 0) \mapsto \{(1, 1)\} \\ (1, 1) \mapsto \{(1, 1)\} & (1, 1) \mapsto \{(1, 1)\} \end{array} \end{array} \end{array}$$

- \mathcal{S}_1 has fewer behavior than \mathcal{S}_0 : $\llbracket \mathcal{S}_1 \rrbracket^* \subset \llbracket \mathcal{S}_0 \rrbracket^*$
- \mathcal{S}_0 **has the interference property**, but \mathcal{S}_1 **does not**
- If interference was a trace property, \mathcal{S}_1 should have it (monotony)

Thus, the interference property is not a trace property

Hyperproperties

Conclusion:

- The absence of interference between (ℓ, x) and (ℓ', x') **is not a trace property**:
we cannot describe as the set of programs the semantics of which is included into a given set of traces
- It can however **be described by a set of sets of traces**:
we simply collect the set of program semantics that satisfy the property

This is what we call a **hyperproperty**:

Hyperproperties

- **Trace hyperproperties** are described by sets of sets of executions
- **Trace properties** are described by sets of executions

2-safety: to disprove the absence of interference (i.e., to show there exists an interference), we simply need to exhibit **two finite traces**

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The Zoo of semantic properties

Sets of sets of executions
non-interference, dependency

Trace properties
total correctness

Safety properties
never reach s_0 before s_1

State properties
absence or runtime errors
partial correctness

Liveness properties
termination

Summary

To sum-up:

- Trace properties allow to express a large range of program properties
- Safety = absence of bad behaviors
- Liveness = existence of good behaviors
- Trace properties can be decomposed as conjunctions of safety and liveness properties, with dedicated proof methods
- Some interesting properties are not trace properties
security properties are *sets of sets of executions*
- Notion of specification languages to describe program properties