Introduction

Semantics and applications to verification

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Program of this first lecture

Introduction to the course:

1. a study of some **examples** of **software errors**
   - what are the causes ? what kind of properties do we want to verify ?
2. a panel of the main **verification methods**
   with a fundamental limitation: **indecidability**
   - many techniques allow to **compute semantic properties**
   - each comes with **advantages** and **drawbacks**
3. an introduction to the **theory of ordered sets**
   (or, most likely, mostly a refresher...)
   - **order relations** are pervasive in **semantics** and **verification**
   - **fixpoints** of operators are also very common

Outline

Ariane 5 – Flight 501

Ariane 5:

- **a satellite launcher**
- replacement of **Ariane 4**, a lot more powerful
- **first flight**, June, 4th, 1996: **failure!**

Flight story:

- nominal take-off, normal flight for 36 seconds
- **T + 36.7 s** : **angle of attack change**, trajectory lost
- **T + 39 s** : **disintegration** of the launcher

Consequences:

- **loss of satellites** : more than \$ 370 000 000...
- **launcher** unusable for more than a year (delay !)

Full report available online:

`http://esamultimedia.esa.int/docs/esa-x-1819eng.pdf`

Trajectory control system design overview

Sensors: gyroscopes, inertial reference systems...

Calculators (hardware + software) :

- **"Inertial Reference System" (SRI) :**
  integrates data about the trajectory (read on sensors)
- **"On Board Computer" (OBC) :**
  computes the engine actuations that are required to follow the pre-determined theoretical trajectory

Actuators: **engines** of the launcher **follow orders** from the **OBC**

Redundant systems (failure tolerant system):

- **keep running** even in the presence of one or several system failures
- **traditional solution in embedded systems**: **duplication** of systems
  aircraft flight system: 2 or 3 hydraulic circuits
  launcher like Ariane 5 : 2 SRI units (SRI 1 and SRI 2)
- there is also a **control monitor**

The root cause: an unhandled arithmetic error

Processor registers

Each **register** has a size of 16, 32, 64 bits:

- **64-bits floating point**: values in range $[-3.6 \cdot 10^{308}, 3.6 \cdot 10^{308}]$
- **16-bits signed integers**: values in range $[-32768, 32767]$
- upon **copy of data**: conversions are performed such as **rounding**
- when the values are **too large**:
  - **interruption**: run error handling code if any, otherwise crash
  - or **unexpected behavior**: modulo arithmetic or other

Ariane 5:

- the SRI hardware runs in **interruption mode**
- it has **no error handling code** for arithmetic interruptions
- an **unhandled arithmetic conversion overflow crashes the SRI**

From the root cause to the failure

A **not so trivial** sequence of events:

1. a **conversion from 64-bits float to 16-bits signed int** is performed and **causes an overflow**
2. an **interruption** is raised
3. due to the lack of error handling code, the SRI **crashes**
4. the crash causes an **error return** (negative integer value) value be **sent to the OBC** (On-Board Computer)
5. the OBC interprets this illegal value as **flight data**
6. this causes the computation of an **absurd trajectory**
7. hence the **loss of control** of the launcher

Let us discuss **a few specific points**

A crash due to an unaddressed software case

Several solutions would have prevented this mishappening:

1. **Deactivate interruptions on overflows**:
   - then, an overflow may happen, and produce wrong values in the SRI
   - but, these wrong values will not cause the computation to stop!
     and most likely, the flight will not be impacted too much
2. **Fix the SRI code**, so that **no overflow can happen**:
   - all conversions must be **guarded against overflows**:

```
double x = /* ... */;
short  i = /* ... */;
if( -32768. <= x && x <= 32767. )
  i = (short) x;
else
  i = /* default value */;
```

   - this may be costly (many tests), but redundant tests can be removed
3. **Handle** conversion errors (not trivial):
   - the handling code should **identify the problem** and **fix it** at run-time
   - the OBC should **identify illegal input values**

A crash due to a useless task

Piece of code that generated the error:

- part of a gyroscope re-calibration process
- very useful to quickly restart the launch process after a short delay
- can only be done **before lift-off**...
- ... **but not after!**

Re-calibration task shut down:

- normally planned **50 seconds** after lift-off...
- no chance of a need for such a re-calibration after $T_0 + 3$ seconds
- the crash occurred at **36 seconds**

A crash due to legacy software

Software history:

- **already used in Ariane 4** (previous launcher, before Ariane 5)
- the software was tested and ran in real conditions many times yet never failed...
- but Ariane 4 was a **much less powerful** launcher

Software optimization:

- many conversions were **initially protected by a safety guard**
- but these tests were considered **expensive**
  (a test and a branching take processor cycles, interact with the pipeline...)
- thus, conversions were ultimately **removed** for the sake of performance

Yet, Ariane 5 violates the assumptions that were valid with Ariane 4

- **higher values** of *horizontal bias* were generated
- those **were never seen** in Ariane 4, hence the failure

A crash not prevented by redundant systems

Principle of redundant systems: survive the failure of a component by the use of redundant systems

System redundancy in Ariane 5:

- one OBC unit
- **two SRI units**... yet **running the same software**

Obviously, physical redundancy does not address software issues

Other implementation of system redundancy (e.g., Airbus FBW):

- two independent set of controls
- three computing units per set of controls
- each computing unit comprises **two computers** with **distinct softwares** (design and implementation is also performed in **distinct teams**)

Ariane 501, a summary of the issues

A **long series of design errors**, all related to a lack of understanding of what the software does:

1. **Non-guarded** conversion raising an **interruption** due to **overflow**
2. **Removal of pre-existing guards**, too high confidence in the software
3. **Non revised assumptions** on the inputs when moving from Ariane 4 to Ariane 5
4. Redundant systems **running the same software**
5. Useless task **not shutdown** at the right time

Current status: such issues can be found by static analysis tools

Outline

High-speed runway overshoot at landing

Landing at Warsaw airport, Lufthansa A320:

- **bad weather conditions:** rain, high side wind
- **wet runway**
- landing (300 km/h) followed by **aqua-planing**, and **delayed braking**
- **runway overrun** at 132 km/h
- **impact** against a hillside at about 100 km/h

Consequences:

- **2 fatalities**, **56 injured** (among 70 passengers + crew)
- **aircraft completely destroyed** (impact + fire)

Full report available online:

`http://www.rvs.uni-bielefeld.de/publications/Incidents/DOCS/ComAndRep/Warsaw/warsaw-report.html`

Causes of the accident

1. **Root cause:**
   - **bad weather conditions** not well assessed by the crew
   - side wind **exceeding** aircraft certification specification
   - **wrong action from the crew:**
     a "Go Around" (missed landing, acceleration + climb) should have been done
2. **Contributing factor:** **delayed action of the brake system**

time (seconds)	distance (meters) from runway threshold	events
T_0	770 m	main landing gear landed
$T_0 + 3$ s	1030 m	nose landing gear landed **brake command activated**
$T_0 +$ **12 s**	1680 m	spoilers activated
$T_0 + 14$ s	1800 m	thrust reversers activated
$T_0 + 31$ s	2700 m	end of runway

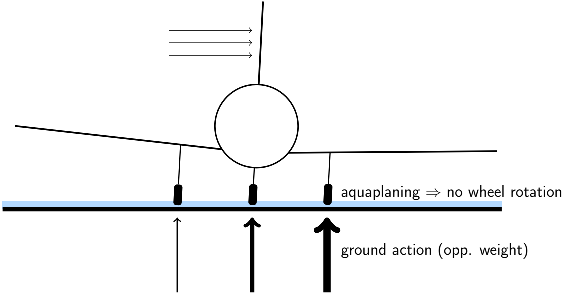
Protection of aircraft brake systems

- **Braking systems inhibition**: **Prevent in-flight activation !**
  - **spoilers:** increase in aerodynamic load (drag)
  - **thrust reversers:** could destroy the plane if activated in-flight ! (ex : crash of a B 767-300 ER Lauda Air, 1991, 223 fatalities; thrust reversers in-flight activation, electronic circuit issue)
- **Braking software specification:**
  DO NOT activate spoilers and thrust reverse **unless the following condition is met:**
  - thrust lever should be set to **minimum** by the flight crew
  - **AND** either of the following conditions:
    - ⋆ **weight** on the main gear should be at least **12 T** i.e., **6 T** for each side
    - ⋆ OR **wheels should be spinning**, with a speed of at least 130 km/h

[Minimum Thrust] **AND** ([Weight] **OR** [Wheels spinning])

Understanding the braking delay

- **Landing configuration:**



- **Braking systems: inhibited**
  - **thrust command** properly set to minimum
  - **no weight** on the left landing gear due to **the wind**
  - **no speed on wheels** due to **aquaplanning**

[Minimum Thrust] **AND** ([Weight] **OR** [Wheels spinning])

Flight 2904, a summary of the issues

Main factor is human (landing in weather conditions the airplane is not certified for), but the specification of the software is a contributing factor:

- **Old condition** that failed to be satisfied:

$$(P_{\text{left}} > 6T)\,\textbf{AND}\,(P_{\text{right}} > 6T)$$

- **Fixed condition** (used in the new version of the software):

$$(P_{\text{left}} + P_{\text{right}}) > 12T$$

- The fix can be understood **only with knowledge of the environment**
  - conditions which the airplane will be used in
  - behavior of the sensors

Outline

The anti-missile "Patriot" system

- **Purpose**: destroy foe missiles before they reach their target
- **Use in wars**:
  - **first Gulf war** (1991)
    protection of towns and military facilities in Israël and Saudi Arabia (against "Scud" missiles launched by Irak)
  - **success rate:**
    - around 50 % of the "Scud" missiles are successfully destroyed
    - almost all launched Patriot missiles destroy their target
    - **failures** are due to **failure to launch a Patriot missile**
- **Constraints on the system:**
  - **hit very quickly moving targets:**
    "Scud" missiles fly at around 1700 m/s ; travel about 1000 km in 10 minutes
  - **not to destroy a friendly target** (it happened at least twice!)
  - very high cost: about **$1 000 000** per launch

System components

Detection / trajectory identification:

- **detection** using radar systems
- **trajectory confirmation** (to make sure a foe missile is tracked):
  1. **trajectory identification** using a sequence of points at various instants
  2. **trajectory confirmation**
     computation of a predictive window (from position and speed vector)
     + confirmation of the predicted trajectory
  3. **identification of the target** (friend / foe)

Guidance system:

- **interception trajectory** computation
- **launch** of a Missile, and control until it hits its target
  high precision required (both missiles travel at more than 1500 m/s)

Very short process: about ten minutes

Dahran failure (1991)

1. **Launch of a "Scud" missile**
2. **Detection** by the radars of the Patriot system
   but failure to confirm the trajectory:
   - **imprecision** in the computation of the **clock** of the detection system
   - computation of **a wrong confirmation window**
   - the "Scud" cannot be found **in the predicted window**
     failure to confirm the trajectory
   - the detection computer concludes it is a **false alert**
3. **The "Scud" missile hits its target:**
   28 fatalities and around 100 people injured

Fixed precision arithmetic

- **Fixed precision numbers** are of the form $\epsilon\, N\, 2^{-p}$ where:
  - p is fixed
  - $\epsilon \in \{-1, 1\}$ is the **sign**
  - $N \in [-2^n, 2^n - 1]_{\mathbb{Z}}$ is an **integer** ($n > p$)
- **In 32 bits fixed precision**, with one sign bit, $n = 31$; thus we may let $p = 20$
- **A few examples:**

decimal value	sign	truncated value	fractional portion
2	0	00000000010	00000000000000000000
-5	1	00000000101	00000000000000000000
0.5	0	00000000000	10000000000000000000
-9.125	1	00000001001	00100000000000000000

- **Range of values that can be represented:**

$$\pm 2^{12}(1 - 2^{-32})$$

Rounding errors in fixed precision computations

- **Not all real numbers in the right range can be represented**
  rounding is unavoidable
  may happen both for **basic operations** and for **program constants**...
- **Example:** fraction 1/10
  - 1/10 **cannot be represented exactly** in fixed precision arithmetic
  - let us decompose 1/10 as a sum of terms of the form $\frac{1}{2^i}$) :

$$\begin{array}{rcl} \frac{1}{10} & = & \frac{1}{2} \cdot \frac{1}{5} \\ \frac{1}{5} & = & \frac{1}{8} + \frac{1}{16} + \frac{1}{16} \cdot \frac{1}{5} = \frac{1}{8} + \frac{1}{16} + \frac{1}{16} \cdot (\frac{1}{8} + \frac{1}{16} + \frac{1}{16} \cdot \frac{1}{5}) = \ldots \end{array}$$

  - **infinite binary representation:** 0.000110011001100110011001100...
  - if $p = 24$:
    representation: "0.000110011001100110011001"
    rounding error is $9.5 \cdot 10^{-8}$
- **Floating precision numbers** (more commonly used today) have the same limitation

The root cause: a clock drift

Trajectory confirmation algorithm (summary):

- **hardware clock** T_d ticks every **tenth of a second**
- time T_c is computed **in seconds**: $T_c = \frac{1}{10} \times T_d$
- in **binary:** $T_c = 0.000110011001100110011001b \times_b T_d$!
- **relative error is** 10^{-6}
- after the computer has been running for **100 h** :
  - the absolute error is **0.34 s**
  - as a "Scud" travels at 1700 m/s : the predicted window is about **580 m** from where it should be
    this explains **the trajectory confirmation failure!**

Remarks:

- the issue was **discovered** by israeli users, who noticed the clock drift
  their solution: **frequently restart the control computer...** (daily)
- this was not done in Dahran... the system had been running for 4 days

Patriot missile failure, a summary of the issues

Precision issues in the fixed precision arithmetic:

- A scalar **constant** used in the code was **invalid**
  i.e., bound to be rounded to an approximate value, incurring a significant approximation the designers were unaware of
- There was **no adequate study of the precision** achieved by the system, although precision is clearly critical here !

Current status: such issues can be found by static analysis tools

Outline

Common issues causing software problems

The examples given so far **are not isolated cases**
See for instance:

`www.cs.tau.ac.il/~nachumd/horror.html`

(not up-to-date)

Typical reasons:

- **Improper specification** or understanding of the environment, conditions of execution...
- **Incorrect implementation of a specification**
  e.g., the code should be free of runtime errors
  e.g., the software should produce a result that meets some property
- **Incorrect understanding of the execution model**
  e.g., generation of too imprecise results

New challenges to ensure embedded systems do not fail

Complex software architecture: e.g. **parallel softwares**

- single processor **multi-threaded**, **distributed** (several computers)
- more and more common: multi-core architectures
- **very hard to reason about**
  - other kinds of issues: **dead-locks**, **races**...
  - very complex execution model: **interleavings**, **memory models**

Complex properties to ensure: e.g., **security**

- the system should resist even in the presence of an **attacker** (agent with malicious intentions)
- attackers may try to **access sensitive data**, to **corrupt** critical data...
- security properties are often even **hard to express**

Techniques to ensure software safety

Software development techniques:

- **software engineering**, with a focus on specification, and software quality (may be more or less formal...)
- **programming rules** for specific areas (e.g., DO 178 c in avionics)
- usually do not guarantee any strong property, but make softwares "cleaner"

Formal methods:

- should have **sound mathematical foundations**
- should allow to **guarantee** softwares meet some complex properties
- should be **trustable** (is a paper proof ok ???)
- **increasingly used in real life applications**, but still a lot of open problems

What is to be verified ?

What do the C programs below do ?

What do these C programs do ?

P0.c

```
int x = 0
int f0( int y ){
  return x * y;
}
int f1( int y ){
  x = y;
  return 0;
}
void main( ){
  int z = f0( 10 ) +
     f1( 100 );
}
```

P1.c

```
void main( ){
  int i;
  int t[100] = { 0, 1, 2,
                ..., 99 };
  while( i < 100 ){
    t[i]++;
    i++;
  }
}
```

P2.c

```
void main( ){
  float f = 0.;
  for( int i = 0;
       i < 1000000;
       i++ )
    f = f + 0.1;
```

Semantic subtleties...

P0.c

```
int x = 0
int f0( int y ){
  return x * y;
}
int f1( int y ){
  x = y;
  return 0;
}
void main( ){
  int z = f0( 10 ) + f1
      ( 100 );
}
```

Execution order:

- **not specified in C**
- specified in Java
- if left to right, $z = 0$
- if right to left, $z = 1000$

Semantic subtleties...

P1.c

```
void main( ){
  int i;
  int t[100] = { 0, 1, 2,
                 ..., 99 };
  while( i < 100 ){
    t[i]++;
    i++;
  }
}
```

P2.c

```
void main( ){
  float f = 0.;
  for( int i = 0;
       i < 1000000;
       i++ )
    f = f + 0.1;
}
```

Initialization:

- runtime error in Java
- read of a random value in C
  (the value that was stored before)

Floating point semantics:

- 0.1 is not representable exactly
  what is it rounded to by the compiler ?
- rounding errors
  what is the rounding mode

The two main parts of this course

1. **Semantics**
   - allow to **describe precisely the behavior of programs**
     should account for execution order, initialization, scope...
   - allow to **express the properties to verify**
     several important families of properties: safety, liveness, security...
   - also important to **transform** and **compile** programs

2. **Verification**
   - aim at **proving** semantic properties of programs
   - a very strong limitation: **indecidability**
   - **several approaches**, that make various compromises around indecidability

Outline

The termination problem

Termination

Program P terminates on input X if and only if
any execution of P, with input X eventually reaches a final state

- **Final state:** final point in the program (i.e., not error)
- **We may want to ensure termination:**
  - processing of a task, such as, e.g., printing a document
  - computation of a mathematical function
- **We may want to ensure *non*-termination:**
  - operating system
  - device drivers

The termination problem

Can we find a program `Pt` that **takes as argument a program P and data X and that returns "`TRUE`" if P terminates on X and "`FALSE`" otherwise ?**

The termination problem is not computable

- **Proof by reductio ad absurdum**, using a *diagonal argument*
  We assume **there exists a program `Pa` such that:**
  - `Pa` always terminates
  - $\mathtt{Pa}(P, X) = 1$ **if** P **terminates** on input X
  - $\mathtt{Pa}(P, X) = 0$ **if** P **does not terminate** on input X
- We consider the following program:

```
void P0( P ){
  if( Pa( P, P ) == 1 ){
    while( 1 ){
      // loop forever
    }
  } else {
    return; // do nothing
  }
}
```

- **What is the return value of $\mathtt{Pa}(\mathtt{P0}, \mathtt{P0})$?**
  i.e., **does `P0` terminate on input `P0` ?**

The termination problem is not computable

- **What is the return value of `Pa(P0, P0)` ?**
  We know `Pa` always terminates and returns either 0 or 1 (assumption).
  Therefore, we need to consider only two cases:
  - if `Pa(P0, P0)` returns 1, then `P0(P0)` **loops forever**,
    thus `Pa(P0, P0)` should return 0, so we have reached a **contradiction**
  - if `Pa(P0, P0)` returns 0, then `P0(P0)` **terminates**,
    thus `Pa(P0, P0)` should 1, so we have reached a **contradiction**

- In both cases, we **reach a contradiction**

- Therefore we conclude **no such a `Pa` exists**

The termination problem is not decidable

There exists no program `Pt` that always terminates and always recognizes whether a program P terminates on input X

Absence of runtime errors

- Can we find a program `Pc` that takes a program P and input X as arguments, always terminates and returns
  - 1 if and only if P runs safely on input X, i.e., without a runtime error
  - 0 if P crashes on input X
- Answer: **No**, the same diagonal argument applies
  if $\mathtt{Pc}(P, X)$ decides whether P will run safely on X, consider

```
void P1( P ){
  if( Pc( P, P ) == 1 ){
    0 / 0; // deliberately crash
         (unsafe)
  } else {
    return; // do nothing
  }
}
```

Non-computability result

The absence of runtime errors is not computable

Rice theorem

- **Semantic specification**: set of *correct* program executions
- **"Trivial" semantic specifications:**
  - empty set
  - set of all possible executions

$\Rightarrow$ intuitively, the non interesting verification problems...

Rice theorem (1953)

Considering a Turing complete language, any non trivial semantic specification is not computable

- **Intuition:** there is no algorithm to decide non trivial specifications, starting with only the program code
- Therefore **all interesting properties are not computable** :
  - **termination**,
  - **absence of runtime errors**,
  - **absence of arithmetic errors**, etc...

Outline

Towards partial solutions

The initial verification problem is **not computable**

Solution: solve a weaker problem

Several compromises can be made:

- **simulation / testing:** observe only **finitely many finite executions** infinite system, but only finite exploration (no proof beyond that)
- **assisted theorem proving:** we **give up on automation** (no proof inference algorithm in general)
- **model checking:** we consider only **finite systems** (with finitely many states)
- **bug-finding:** search for "patterns" indicating "likely errors" (may miss real program errors, and report non existing issues)
- **static analysis with abstraction: attempt at automatic correctness proofs** (yet, may fail to verify some correct programs)

Safety verification method characteristics

Safety verification problem

- **Semantics** $[\![P]\!]$ of program P: set of behaviors of P (e.g., states)
- **Property to verify** $\mathcal{S}$: set of admissible behaviors (e.g., safe states)

Goal: **establish** $[\![P]\!] \subseteq \mathcal{S}$

- **Automation**: **existence of an algorithm**
- **Scalability**: should allow to **handle large softwares**
- **Soundness**: identify **any wrong program**
- **Completeness**: accept **all correct programs**
- **Apply to program source code**, i.e., not require a **modelling phase**

1. Testing by simulation

Principle

Run the program on **finitely many finite inputs**

- maximize **coverage**
- **inspect erroneous traces** to fix bugs

- **Very widely used**:
  - **unit testing**: each function is tested separately
  - **integration testing**: with all surrounding systems, hardware e.g., **iron bird** in avionics
- **Automated**
- **Complete**: will never raise a false alarm
- **Unsound** unless exhaustive: **may miss program defects**
- **Costly**: needs to be re-done when software gets updated

2. Machine assisted proof

Principle

Have a **machine checked** proof, that is partly **human written**

- **tactics / solvers** may help in the inference
- the **hardest invariants** have to be user-supplied

- **Applications**
  - software industry (rare): Line 14 in Paris Subway
  - hardware: ACL 2
  - academia: CompCert compiler, SEL4 verified micro-kernel
  - also for math: four colour theorem, Feith-Thomson theorem
- **Not fully automated**
  often turns out **costly** as complex proof arguments have to be found
- **Sound** and quasi-**complete** (in practice fine...)

3. Model-Checking

Principle

Consider **finite systems** only, using algorithms for

- **exhaustive exploration**,
- **symmetry reduction**...

- **Applications**:
  - **hardware** verification
  - **driver protocols** verification (Microsoft)
- Applies on **a model**: a model extraction phase is needed
  - for infinite systems, this is **necessarily approximate**
  - not always automated
- **Automated**, **sound**, **complete** **with respect to the model**

4. "Bug finding"

Principle

Identify **"likely" issues**, i.e., patterns known to often indicate an error

- use **bounded symbolic execution**, **model exploration**...
- **rank "defect" reports** using heuristics

- Intuition: **model checking made unsound**
- **Example**: Coverity
- **Automated**
- **Not complete**: may report false alarms
- **Not sound**: may accept false programs
  thus **inadequate** for safety-critical systems

5. Static analysis with abstraction (1/4)

Principle

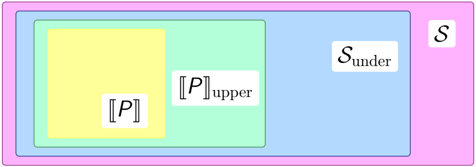
Use some approximation, but **always in a conservative manner**

- **Under-approximation** of the property to verify: $\mathcal{S}_{\mathrm{under}} \subseteq \mathcal{S}$
- **Over-approximation** of the semantics: $[\![P]\!] \subseteq [\![P]\!]_{\mathrm{upper}}$
- We let an automatic static analyzer attempt to prove that:

$$[\![P]\!]_{\mathrm{upper}} \subseteq \mathcal{S}_{\mathrm{under}}$$

  If it succeeds, $[\![P]\!] \subseteq \mathcal{S}$

- In practice, the static analyzer **computes** $[\![P]\!]_{\mathrm{upper}}, \mathcal{S}_{\mathrm{under}}$

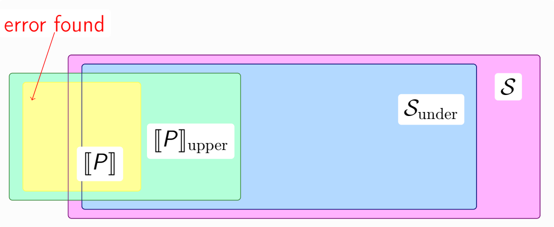


5. Static analysis with abstraction (2/4)

Soundness

The abstraction will catch **any incorrect program**

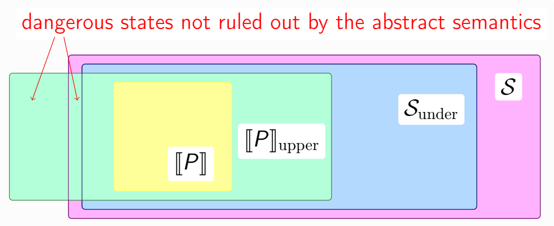
- If $[\![P]\!] \not\subseteq \mathcal{S}$, then $[\![P]\!]_{\rm upper} \not\subseteq \mathcal{S}_{\rm under}$
  since $\begin{cases} \mathcal{S}_{\rm under} & \subseteq & \mathcal{S} \\ [\![P]\!] & \subseteq & [\![P]\!]_{\rm upper} \end{cases}$



5. Static analysis with abstraction (3/4)

Incompleteness

The abstraction may fail to certify **some correct programs**



Case of a false alarm:

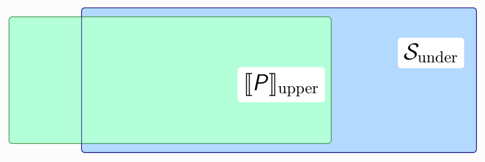
- program P is **correct**
- but the static analysis **fails**

5. Static analysis with abstraction (4/4)

Incompleteness

The abstraction may fail to certify **some correct programs**

- In the following case, the analysis cannot conclude anything



- One goal of the static analyzer designer is to avoid such cases

Static analysis using abstraction

- **Automatic**: $[\![P]\!]_{\mathrm{upper}}$, $\mathcal{S}_{\mathrm{under}}$ computed automatically
- **Sound**: reports any incorrect program
- **Incomplete**: may reject correct programs

A summary of common verification techniques

	Automatic	Sound	Complete	Source level	Scalable
Simulation	Yes	No [1]	Yes	Yes	sometimes [2]
Assisted proving	No	Yes	Almost	Partially	sometimes [3]
Model-checking	Yes	Yes	Partially [4]	No	sometimes
Bug-finding	Yes	No	No	Yes	sometimes
Static analysis	Yes	Yes	No	Yes	sometimes

- Obviously, no approach checks all characteristics
- Scalability is a challenge for all

[1] unless full testing is doable
[2] full testing usually not possible except for small programs with finite state space
[3] quickly requires huge manpower
[4] only with respect to the finite models... but not with respect to infinite semantics

Outline

Order relations

Very useful in semantics and verification:

- **logical ordering**, expresses **implication** of logical facts
- **computational ordering**, useful to establish well-foundedness of fixpoint definitions, and for termination

Definition: partially ordered set (poset)

Let a set $\mathcal{S}$ and a binary relation $\sqsubseteq \subseteq \mathcal{S} \times \mathcal{S}$ over $\mathcal{S}$. Then, $\sqsubseteq$ is an **order relation** (and $(\mathcal{S}, \sqsubseteq)$ is called a **poset**) if and only if it is

- **reflexive:** $\forall x \in \mathcal{S},\ x \sqsubseteq x$
- **transitive:** $\forall x, y, z \in \mathcal{S},\ x \sqsubseteq y \wedge y \sqsubseteq z \implies x \sqsubseteq z$
- **antisymmetric:** $\forall x, y \in \mathcal{S},\ x \sqsubseteq y \wedge y \sqsubseteq x \implies x = y$

- notation: $x \sqsubset y ::= (x \sqsubseteq y \wedge x \neq y)$

Graphical representation

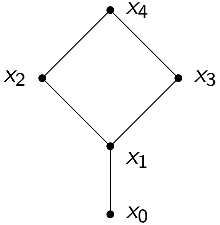
We often use **Hasse diagrams** to represent posets:

Extensive definition:

- $\mathcal{S} = \{x_0, x_1, x_2, x_3, x_4\}$
- $\sqsubset$ defined by:

$$\begin{aligned} x_0 &\sqsubset x_1 \\ x_1 &\sqsubset x_2 \\ x_1 &\sqsubset x_3 \\ x_2 &\sqsubset x_4 \\ x_3 &\sqsubset x_4 \end{aligned}$$

Diagram:



- By reflexivity, we have, e.g., $x_1 \sqsubseteq x_1$
- By transitivity, we have, e.g., $x_1 \sqsubseteq x_4$

Order relations are very useful in semantics...

Example: semantics of automata

In the following, we **illustrate order relations** and their **usefulness in semantics** using **word automata**.

We consider the classical notion of **finite word automata** and let

- L be a finite set of **letters**
- Q be a finite set of **states**
- $q_{\mathrm{i}}, q_{\mathrm{f}} \in Q$ denote the **initial** state and **final** state
- $\rightarrow \subseteq Q \times L \times Q$ be a **transition relation**

Semantics of an automaton

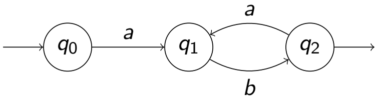
The set of words recognized by $\mathcal{A} = (Q, q_{\mathrm{i}}, q_{\mathrm{f}}, \rightarrow)$ is defined by:

$$\mathcal{L}[\mathcal{A}] = \{a_0 a_1 \ldots a_n \mid \exists q_0 \ldots q_{n-1} \in Q,\ q_i \xrightarrow{a_0} q_0 \xrightarrow{a_1} q_1 \ldots q_{n-1} \xrightarrow{a_n} q_f\}$$

Example: automata and semantic properties

A simple automaton:

$$L = \{a, b\} \qquad Q = \{q_0, q_1, q_2\}$$
$$q_{\mathrm{i}} = q_0 \qquad q_{\mathrm{f}} = q_2$$
$$q_0 \xrightarrow{a} q_1 \qquad q_1 \xrightarrow{b} q_2 \qquad q_2 \xrightarrow{a} q_1$$



A few semantic properties:

- $\mathcal{P}_0$: no recognized word contains two consecutive b

$$\mathcal{L}[\mathcal{A}] \subseteq L^* \setminus L^* bbL^*$$

- $\mathcal{P}_1$: all recognized words contain at least one occurrence of a

$$\mathcal{L}[\mathcal{A}] \subseteq L^* aL^*$$

- we could also consider under-approximation properties (of the form $\mathcal{P}_2 \subseteq \mathcal{L}[\mathcal{A}]$), but do not in this lecture

Total ordering

Definition: total order relation

Order relation $\sqsubseteq$ over $\mathcal{S}$ is a **total** order if and only if

$$\forall x, y \in \mathcal{S},\ x \sqsubseteq y \vee y \sqsubseteq x$$

Examples:

- **real numbers**:
  $(\mathbb{R}, \leq)$ is a total ordering
- **powerset**:
  if set $\mathcal{S}$ has at least two distinct elements x, y then its powerset $(\mathcal{P}(\mathcal{S}), \subseteq)$ is **not** a total order
  indeed $\{x\}, \{y\}$ cannot be compared

Most of the order relations we will use are ***not* be total**
indeed: very often, powerset or similar

Minimum and maximum elements

Definition: extremal elements

Let $(\mathcal{S}, \sqsubseteq)$ be a poset and $\mathcal{S}' \subseteq \mathcal{S}$. Then x is

- **minimum element** of $\mathcal{S}'$ if and only if $x \in \mathcal{S}' \wedge \forall y \in \mathcal{S}',\ x \sqsubseteq y$
- **maximum element** of $\mathcal{S}'$ if and only if $x \in \mathcal{S}' \wedge \forall y \in \mathcal{S}',\ y \sqsubseteq x$

- maximum and minimum elements **may not exist**
  example: $\{\{x\}, \{y\}\}$ in the powerset, where $x \neq y$
- **infimum $\perp$ ("bottom"):** minimum element of $\mathcal{S}$
- **supremum $\top$ ("top"):** maximum element of $\mathcal{S}$

Exercise:

what are the logical interpretations of infimum / supremum elements ?

Upper bounds and least upper bound

Definition: bounds

Given poset $(\mathcal{S}, \sqsubseteq)$ and $\mathcal{S}' \subseteq \mathcal{S}$, then $x \in \mathcal{S}$ is

- an **upper bound** of $\mathcal{S}'$ if
$$\forall y \in \mathcal{S}',\ y \sqsubseteq x$$
- the **least upper bound** (lub) of $\mathcal{S}'$ (noted $\sqcup \mathcal{S}'$) if
$$\forall y \in \mathcal{S}',\ y \sqsubseteq x \wedge \forall z \in \mathcal{S}, (\forall y \in \mathcal{S}',\ y \sqsubseteq z) \implies x \sqsubseteq z$$

- if it exists, the least upper bound is **unique**: if x, y are least upper bounds of $\mathcal{S}$, then $x \sqsubseteq y$ and $y \sqsubseteq x$, thus $x = y$ by antisymmetry
- notation: $x \sqcup y ::= \sqcup\{x, y\}$
- upper bounds and least upper bounds **may not exist**
- **dual notions:** lower bound, greatest lower bound (glb, noted $\sqcap \mathcal{S}'$)

Exercise: logical interpretations ?

Duality principle

So far all definitions admit a symmetric counterpart

- **dual relation**: given an order relation $\sqsubseteq$, $\mathcal{R}$ defined by

$$x\mathcal{R}y \iff y \sqsubseteq x$$

  is also an order relation
- thus all properties that can be proved about $\sqsubseteq$ also have a symmetric property that also holds

This is the **duality principle**:

minimum element	maximum element
infimum	supremum
lower bound	upper bound
greatest lower bound	least upper bound

... more to follow

Complete lattice

Definition: complete lattice

A **complete lattice** is a tuple $(\mathcal{S}, \sqsubseteq, \perp, \top, \sqcup, \sqcap)$ where:

- $(\mathcal{S}, \sqsubseteq)$ is a poset
- $\perp$ is the infimum of $\mathcal{S}$
- $\top$ is the supremum of $\mathcal{S}$
- any subset $\mathcal{S}'$ of $\mathcal{S}$ has a lub $\sqcup \mathcal{S}'$ and a glb $\sqcap \mathcal{S}'$

Properties:

- $\perp = \sqcup \emptyset = \sqcap \mathcal{S}$
- $\top = \sqcap \emptyset = \sqcup \mathcal{S}$

Example:

the **powerset** $(\mathcal{P}(\mathcal{S}), \subseteq, \emptyset, \mathcal{S}, \cup, \cap)$ of set $\mathcal{S}$ is a complete lattice

Lattice

The existence of lubs and glbs for all subsets is often a very strong property, that may not be met:

Definition: lattice

A **lattice** is a tuple $(\mathcal{S}, \sqsubseteq, \perp, \top, \sqcup, \sqcap)$ where:

- $(\mathcal{S}, \sqsubseteq)$ is a poset
- $\perp$ is the infimum of $\mathcal{S}$
- $\top$ is the supremum of $\mathcal{S}$
- any **pair** $\{x, y\}$ of $\mathcal{S}$ has a lub $x \sqcup y$ and a glb $x \sqcap y$

- let $\mathcal{Q} = \{q \in \mathbb{Q} \mid 0 \leq q \leq 1\}$;
  then $(\mathcal{Q}, \leq)$ is a **lattice** but **not a complete lattice**
  indeed, $\{q \in \mathcal{Q} \mid q \leq \frac{\sqrt{2}}{2}\}$ has no lub in $\mathcal{Q}$
- property: a **finite** lattice is also a complete lattice

Chains

Definition: increasing chain

Let $(\mathcal{S}, \sqsubseteq)$ be a poset and $\mathcal{C} \subseteq \mathcal{S}$.
It is an **increasing chain** if and only if

- it has an infimum
- poset $(\mathcal{C}, \sqsubseteq)$ is total (i.e., any two elements can be compared)

Example, in the powerset $(\mathcal{P}(\mathbb{N}), \subseteq)$:

$$\mathcal{C} = \{c_i \mid i \in \mathbb{N}\} \quad \text{where} \quad c_i = \{2^0, 2^2, \ldots, 2^i\}$$

Definition: increasing chain condition

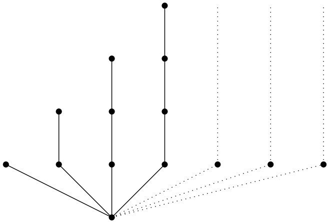
The poset $(\mathcal{S}, \sqsubseteq)$ **satisfies the increasing chain condition** if and only if any increasing chain $\mathcal{C} \subseteq \mathcal{S}$ is finite.

Complete partial orders

Definition: complete partial order

A **complete partial order** (cpo) is a poset $(\mathcal{S}, \sqsubseteq)$ such that any increasing chain $\mathcal{C}$ of $\mathcal{S}$ has a least upper bound. A **pointed cpo** is a cpo with an infimum $\perp$.

- clearly, any complete lattice is a cpo
- the opposite is not true:



Outline

Towards a constructive definition of the automata semantics

We now look for a **constructive version of the automaton semantics** as hinted by the following observations

Observation 1: $\mathcal{L}[\mathcal{A}] = [\![\mathcal{A}]\!](q_f)$ where

$$\begin{array}{rccl} [\![\mathcal{A}]\!]: & Q & \longrightarrow & \mathcal{P}(L^*) \\ & q & \longmapsto & \{w \in L^* \mid \exists n,\ w = a_0 a_1 \dots a_n \\ & & & \quad \exists q_0 \dots q_{n-1} \in Q,\ q_i \xrightarrow{a_0} q_0 \xrightarrow{a_1} \dots q_{n-1} \xrightarrow{a_n} q\} \end{array}$$

Observation 2: $[\![\mathcal{A}]\!] = \dot{\bigcup}_{n \in \mathbb{N}} [\![\mathcal{A}]\!]_n$ where

$$\begin{array}{rccl} [\![\mathcal{A}]\!]_n: & Q & \longrightarrow & \mathcal{P}(L^*) \\ & q & \longmapsto & \{a_0 a_1 \dots a_{n-1} \mid \\ & & & \quad \exists q_0 \dots q_{n-2} \in Q,\ q_i \xrightarrow{a_0} q_0 \xrightarrow{a_1} \dots q_{n-1} \xrightarrow{a_{n-1}} q\} \end{array}$$

Observation 3: $[\![\mathcal{A}]\!]_{n+1}$ can be computed directly from $[\![\mathcal{A}]\!]_n$

$$[\![\mathcal{A}]\!]_{n+1}(q) \quad = \quad \bigcup_{q' \in Q} \{wa \mid w \in [\![\mathcal{A}]\!]_n(q') \wedge q' \xrightarrow{a} q\}$$

Towards a constructive definition of the automata semantics

Alternate approach:

1. Let $[\![\mathcal{A}]\!]_n$ denote **recognized words of length at most** n:

$$[\![\mathcal{A}]\!](q) = \{w \in [\![\mathcal{A}]\!](q) \mid \mathbf{length}(w) \leq n\}$$

2. Compute $[\![\mathcal{A}]\!]_{n+1}$ from $[\![\mathcal{A}]\!]_n$
3. Define the semantics of the automaton as the union of the iterates of this sequence:

$$[\![\mathcal{A}]\!] = \dot{\bigcup_{n \in \mathbb{N}}} [\![\mathcal{A}]\!]_n$$

In the following, we **study such a way of defining semantics**, based on **general mathematical tools**, that we will use throughout the course

Operators over a poset

Definition: operators and orderings

Let $(\mathcal{S}, \sqsubseteq)$ be a poset and $\phi : \mathcal{S} \to \mathcal{S}$ be an operator over $\mathcal{S}$. Then, ϕ is:

- **monotone** if and only if $\forall x, y \in \mathcal{S},\ x \sqsubseteq y \Longrightarrow \phi(x) \sqsubseteq \phi(y)$
- **continuous** if and only if, for any chain $\mathcal{S}' \subseteq \mathcal{S}$ then:
  $$\left\{ \begin{array}{l} \text{if } \sqcup \mathcal{S}' \text{ exists, so does } \sqcup\{\phi(x) \mid x \in \mathcal{S}'\} \\ \text{and } \phi(\sqcup \mathcal{S}') = \sqcup\{\phi(x) \mid x \in \mathcal{S}'\} \end{array} \right.$$
- **$\sqcup$-preserving** if and only if:
  $$\forall \mathcal{S}' \subseteq \mathcal{S}, \left\{ \begin{array}{l} \text{if } \sqcup \mathcal{S}' \text{ exists, then } \sqcup\{\phi(x) \mid x \in \mathcal{S}'\} \text{ exists} \\ \text{and } \phi(\sqcup \mathcal{S}') = \sqcup\{\phi(x) \mid x \in \mathcal{S}'\} \end{array} \right.$$

Notes:

- “monotone” in English means “*croissante*” in French ; “*décroissante*” translates into “anti-monotone” and “monotone” into “*isotone*”
- the dual of “monotone” is “monotone”

Operators over a poset

A few interesting properties:

Continuity implies monotonicity

If ϕ is continuous, then it is also **monotone**

We assume ϕ is continuous, and $x, y \in \mathcal{S}$ are such that $x \sqsubseteq y$:
Then $\{x, y\}$ is a chain with lub y, thus $\phi(x) \sqcup \phi(y)$ exists and is equal to $\phi(\sqcup\{x, y\}) = \phi(y)$. Therefore $\phi(x) \sqsubseteq \phi(y)$.

$\sqcup$-preserving implies monotonicity

If ϕ preserves $\sqcup$, then it is also **monotone**

Same argument.

Fixpoints

Definition: fixpoints

Let $(\mathcal{S}, \sqsubseteq)$ be a poset and $\phi : \mathcal{S} \rightarrow \mathcal{S}$ be an operator over $\mathcal{S}$.

- a **fixpoint** of ϕ is an element x such that $\phi(x) = x$
- a **pre-fixpoint** of ϕ is an element x such that $x \sqsubseteq \phi(x)$
- a **post-fixpoint** of ϕ is an element x such that $\phi(x) \sqsubseteq x$
- the **least fixpoint** $\mathbf{lfp}\,\phi$ of ϕ (if it exists, it is unique) is the smallest fixpoint of ϕ
- the **greatest fixpoint** $\mathbf{gfp}\,\phi$ of ϕ (if it exists, it is unique) is the greatest fixpoint of ϕ

Note: the existence of a least fixpoint, a greatest fixpoint or even a fixpoint is *not guaranteed*; we will see several theorems that establish their existence under specific assumptions...

Tarski's Theorem

Theorem

Let $(\mathcal{S}, \sqsubseteq, \bot, \top, \sqcup, \sqcap)$ be a complete lattice and $\phi : \mathcal{S} \to \mathcal{S}$ be a monotone operator over $\mathcal{S}$. Then:

1. ϕ has a least fixpoint $\mathbf{lfp}\,\phi$ and $\mathbf{lfp}\,\phi = \sqcap\{x \in \mathcal{S} \mid \phi(x) \sqsubseteq x\}$.
2. ϕ has a greatest fixpoint $\mathbf{gfp}\,\phi$ and $\mathbf{gfp}\,\phi = \sqcup\{x \in \mathcal{S} \mid x \sqsubseteq \phi(x)\}$.
3. the set of fixpoints of ϕ is a complete lattice.

Proof of point 1:

We let $X = \{x \in \mathcal{S} \mid \phi(x) \sqsubseteq x\}$ and $x_0 = \sqcap X$.

Let $y \in X$:

- $x_0 \sqsubseteq y$ by definition of the glb;
- thus, since ϕ is monotone, $\phi(x_0) \sqsubseteq \phi(y)$;
- thus, $\phi(x_0) \sqsubseteq y$ since $\phi(y) \sqsubseteq y$, by definition of X.

Therefore $\phi(x_0) \sqsubseteq x_0$, since $x_0 = \sqcap X$ and $\phi(x_0)$ is a lower bound.

Tarski's Theorem

We proved that $\phi(x_0) \sqsubseteq x_0$. We derive from this that:

- $\phi(\phi(x_0)) \sqsubseteq \phi(x_0)$ since ϕ is monotone;
- $\phi(x_0)$ is a post-fixpoint of ϕ, thus $\phi(x_0) \in X$;
- $x_0 \sqsubseteq \phi(x_0)$ by definition of the greatest lower bound

We have established both inclusions so $\phi(x_0) = x_0$.

If x_1 is another fixpoint, then $x_1 \in X$, so $x_0 \sqsubseteq x_1$.

Proof of point 2: similar, by duality.

Proof of point 3:

- if X is a set of fixpoints of ϕ, we need to consider ϕ over $\{y \in \mathcal{S} \mid y \sqsubseteq_{\mathcal{S}} \sqcap X\}$ to establish the existence of a **glb of X in the poset of fixpoints**
- the existence of **least upper bounds in the poset of fixpoints** follows by duality

Tarski's theorem: example (1)

A function over the powerset:

We consider a set $\mathcal{E}$, and a subset $\mathcal{A} \subseteq \mathcal{E}$
We let:

$$\begin{array}{rccc} f: & \mathcal{P}(\mathcal{E}) & \longrightarrow & \mathcal{P}(\mathcal{E}) \\ & X & \longmapsto & X \cup \mathcal{A} \end{array}$$

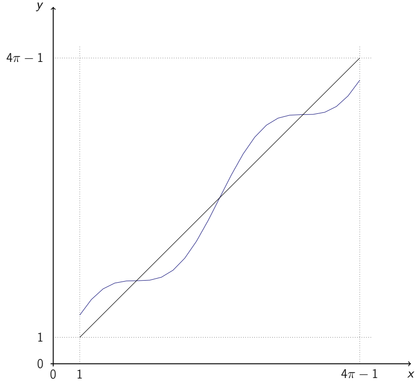
Exercise:

- apply Tarski's theorem, characterize the least and greatest fixpoints

Tarski's theorem: example (2)

Function:

$$\begin{array}{rlcl} f: & [1, 4\pi - 1] & \longrightarrow & [1, 4\pi - 1] \\ & x & \longmapsto & x + \sin x \end{array}$$



Exercise:

- apply Tarski's theorem, and derive the fixpoints of the function

Automata example, fixpoint definition

Lattice:

- $\mathcal{S} = Q \to \mathcal{P}(L^*)$
- the ordering is the pointwise extension $\dot{\sqsubseteq}$ of $\sqsubseteq$

Operator:

- we let $\phi_0 : \mathcal{S} \to \mathcal{S}$ be defined by
  $\phi_0(f) = \lambda(q \in Q) \cdot \bigcup_{q' \in Q} \{wa \mid w \in f(q') \wedge q' \xrightarrow{a} q\}$
- we let $\phi : \mathcal{S} \to \mathcal{S}$ by defined by

$$\phi(f) = \lambda(q \in Q) \cdot \begin{cases} f(q) \cup \phi_0(f)(q_{\mathrm{i}}) \cup \{\epsilon\} & \textbf{if } q = q_{\mathrm{i}} \\ f(q) \cup \phi_0(f)(q) & \textbf{otherwise} \end{cases}$$

Proof steps to complete:

- **the existence of lfp ϕ** follows from Tarski's theorem
- **the equality lfp $\phi = [\![\mathcal{A}]\!]$** can be established by induction and double inclusion... but there is a simpler way

Kleene's Theorem

Tarski's theorem guarantees existence of an lfp, but is not constructive.

Theorem

Let $(\mathcal{S}, \sqsubseteq, \perp)$ be a pointed cpo and $\phi : \mathcal{S} \to \mathcal{S}$ be a continuous operator over $\mathcal{S}$. Then ϕ has a least fixpoint, and

$$\mathbf{lfp}\, \phi = \bigsqcup_{n \in \mathbb{N}} \phi^n(\perp)$$

First, we prove **the existence of the lub**:

Since ϕ is continuous, it is also monotone. We can prove by induction over n that $\{\phi^n(\perp) \mid n \in \mathbb{N}\}$ is a chain:

- $\phi^0(\perp) = \perp \sqsubseteq \phi(\perp)$ by definition of the infimum;
- if $\phi^n(\perp) \sqsubseteq \phi^{n+1}(\perp)$, then
  $\phi^{n+1}(\perp) = \phi(\phi^n(\perp)) \sqsubseteq \phi(\phi^{n+1}(\perp)) = \phi^{n+2}(\perp)$

By definition of the cpo structure, the lub exists. We let x_0 denote it.

Kleene's Theorem

Secondly, we prove that **it is a fixpoint of ϕ**:

Since ϕ is continuous, $\{\phi^{n+1}(\bot) \mid n \in \mathbb{N}\}$ has a lub, and

$$\begin{array}{rcll}
\phi(x_0) & = & \phi(\sqcup\{\phi^n(\bot) \mid n \in \mathbb{N}\}) & \\
& = & \sqcup\{\phi^{n+1}(\bot) \mid n \in \mathbb{N}\} & \text{by continuity of } \phi \\
& = & \bot \sqcup (\sqcup\{\phi^{n+1}(\bot) \mid n \in \mathbb{N}\}) & \text{by definition of } \bot \\
& = & x_0 & \text{by simple rewrite}
\end{array}$$

Last, we show that it is the **least** fixpoint:

Let x_1 denote another fixpoint of ϕ. We show by induction over n that $\phi^n(\bot) \sqsubseteq x_1$:

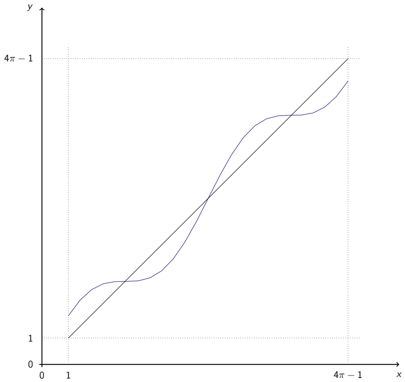
- $\phi^0(\bot) = \bot \sqsubseteq x_1$ by definition of $\bot$;
- if $\phi^n(\bot) \sqsubseteq x_1$, then $\phi^{n+1}(\bot) \sqsubseteq \phi(x_1) = x_1$ by monotony, and since x_1 is a fixpoint.

By definition of the lub, $x_0 \sqsubseteq x_1$

Kleene's theorem: example

Function:

$$\begin{array}{rccl} f: & [1, 4\pi - 1] & \longrightarrow & [1, 4\pi - 1] \\ & x & \longmapsto & x + \sin x \end{array}$$



Exercise:

- apply Kleene's theorem and sketch the iterations

Automata: constructive semantics

We can now state a **constructive definition** of the automaton semantics.
Operator ϕ is defined by

$$\phi(f) = \lambda(q \in Q) \cdot \begin{cases} f(q) \cup \phi_0(f)(q_{\mathrm{i}}) \cup \{\epsilon\} & \text{if } q = q_{\mathrm{i}} \\ f(q) \cup \phi_0(f)(q) & \text{otherwise} \end{cases}$$

Proof steps:

- ϕ is continuous
- thus, Kleene's theorem applies so $\mathbf{lfp}\,\phi$ exists and $\mathbf{lfp}\,\phi = \bigcup_{n \in \mathbb{N}} \phi^n(\bot)$...
  ... this actually saves the double inclusion proof to establish that $[\![\mathcal{A}]\!] = \mathbf{lfp}\,\phi$

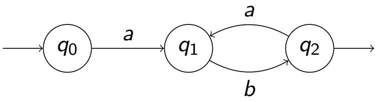
Furthermore, $[\![\mathcal{A}]\!] = \bigcup_{n \in \mathbb{N}} \phi^n(\bot)$.

This fixpoint definition will be very useful to infer or verify semantic properties.

Automata: constructive semantics iterates

A simple automaton:

$$L = \{a, b\} \qquad Q = \{q_0, q_1, q_2\}$$
$$q_{\mathrm{i}} = q_0 \qquad q_{\mathrm{f}} = q_2$$
$$q_0 \xrightarrow{a} q_1 \qquad q_1 \xrightarrow{b} q_2 \qquad q_2 \xrightarrow{a} q_1$$



Iterates of function ϕ from $\perp$:

Iterate	0	1	2	3	4	5
q_0	$\emptyset$	$\{\epsilon\}$	$\{\epsilon\}$	$\{\epsilon\}$	$\{\epsilon\}$	$\{\epsilon\}$
q_1	$\emptyset$	$\emptyset$	$\{a\}$	$\{a\}$	$\{a, aba\}$	$\{a, aba\}$
q_2	$\emptyset$	$\emptyset$	$\emptyset$	$\{ab\}$	$\{ab\}$	$\{ab, abab\}$

Duality principle

We can **extend the duality notion to fixpoints**:

monotone	monotone
anti-monotone	anti-monotone
post-fixpoint	pre-fixpoint
least fixpoint	greatest fixpoint
increasing chain	decreasing chain

Furthermore both Tarski's theorem and Kleene's theorem have a dual version (Tarski's theorem is its own dual).

On the topic of inductive reasoning...

Formalizing inductive definitions:

Definition based on inference rules:

$$\frac{}{x_0 \in \mathcal{X}} \qquad \frac{x \in \mathcal{X}}{f(x) \in \mathcal{X}}$$

Same property based on a least-fixpoint:

$$\mathbf{lfp}(Y \longmapsto \{x_0\} \cup Y \cup \{f(x) \mid x \in Y\})$$

Proving the inclusion of a fixpoint in a given set:

- Let $\phi : \mathcal{S} \longrightarrow \mathcal{S}$ be a continuous operator
- Let $\mathcal{I} \in \mathcal{S}$ such that:

$$\forall x \in \mathcal{S},\ x \sqsubseteq \mathcal{I} \Longrightarrow \phi(x) \sqsubseteq \mathcal{I}$$

- We obviously have $\perp \sqsubseteq \mathcal{I}$
- We can prove that $\mathbf{lfp}\,\phi \sqsubseteq \mathcal{I}$

Exercise: language of a grammar

Language of a grammar as a least-fixpoint

Assumptions:

- Alphabet $\mathcal{A}$, finite set of nodes $\mathcal{N}$
- Finite set of rules $\mathcal{R} \subseteq \mathcal{N} \times (\mathcal{A} \uplus \mathcal{N})^*$
- Starting node $S \in \mathcal{N}$

Questions:

- Define the set of words recognized by the grammar with inductive rules
- Do the same using a least-fixpoint

Hints:

- start with a function that maps each node into the set of words recognized by this node
- compute such a function by induction

Outline

Main points to remember

Foundations:

- **program semantics**: express program behaviors
- **target semantic property**: express proof goal
- **conservative approximation** usually required due to undecidability

Order relations:

- **counterpart for logical implication** (among other)
- will be pervasive in this course

Fixpoints and induction:

- **encode general iteration**
- will also be pervasive in this course

In the next lectures...

- Families of **semantics**, for a general model of programs
- Families of **semantic properties of programs**
- **Verification techniques:**
  - abstract interpretation based static analysis
  - machine assisted theorem proving
  - model checking

Next week: transition systems and operational semantics

Practical information about the course

The course will be taught by:

- **Marc Chevalier** (DIENS, TDs)
- **Sylvain Conchon** (LRI, Paris-Orsay, Model-Checking / SMT)
- **Jérôme Feret** (DIENS, Semantics, Abstract interpretation)
- **Xavier Rival** (DIENS, Semantics, Abstract interpretation, Coq)

Practical organization:

- 1h30 Cours + 2h00 TD or TP depending on week

Evaluation:

$$n = \frac{p + e}{2}$$

- **project** p: several projects will be proposed in a few weeks
- **exam** e: 1st of June, 2018