Abstract Interpretation IV

Semantics and Application to Program Verification

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Overview

Selected advanced topics (not for the exam):

- Disjunctive abstract domains
- Inter-procedural analyses
- Abstracting arrays

Practical session:

- finish interval and relational analyses
- help with the project

Disjunctive domains

Motivation

Remark: most domains abstract **convex sets** (conjunctions of constraints)
$\Longrightarrow \cup^\sharp$ causes a loss of precision!

The need for non-convex invariants

$X \leftarrow \textbf{rand}(10, 20);$
$Y \leftarrow \textbf{rand}(0, 1);$
$\textbf{if } Y > 0 \textbf{ then } X \leftarrow -X;$
• $Z \leftarrow 100/X$

Concrete semantics:

At •, $X \in [-20, -10] \cup [10, 20]$
$\Longrightarrow$ there is no division by zero

Abstract analysis:

Convex analyses (intervals, polyhedra) will find $X \in [-20, 20]$
(with intervals, $[-20, -10] \cup^\sharp [10, 20] = [-20, 20]$)
$\Longrightarrow$ possible division by zero (false alarm)

Disjunctive domains

Principle:

generic constructions to lift any numeric abstract domain to a domain able to represent disjunctions exactly

Example constructions:

- powerset completion
  unordered "soup" of abstract elements
- state partitioning
  abstract elements keyed to selected subsets of environments
- decision tree abstract domains
  efficient representation of state partitioning
- path-sensitive analyses
  partition with respect to the history of execution

each construction has its strength and weakness
they can be combined during an analysis to exploit the best of each

Powerset completion

Powerset completion

Given: $(\mathcal{E}^\sharp, \sqsubseteq, \gamma, \cup^\sharp, \cap^\sharp, \triangledown, \mathsf{S}^\sharp[\![\, \mathit{stat} \,]\!])$

abstract domain $\mathcal{E}^\sharp$
ordered by $\sqsubseteq$, which also acts as a sound abstraction of $\subseteq$ (i.e., $\subseteq^\sharp = \sqsubseteq$)
with concretization $\gamma : \mathcal{E}^\sharp \to \mathcal{P}(\mathcal{E})$
sound abstractions $\cup^\sharp$, $\cap^\sharp$, $\mathsf{S}^\sharp[\![\, \mathit{stat} \,]\!]$ of $\cup$, $\cap$, $\mathsf{S}[\![\, \mathit{stat} \,]\!]$, and a widening $\triangledown$

Construct: $(\hat{\mathcal{E}}^\sharp, \hat{\sqsubseteq}, \hat{\gamma}, \hat{\cup}^\sharp, \hat{\cap}^\sharp, \hat{\triangledown}, \hat{\mathsf{S}}^\sharp[\![\, \mathit{stat} \,]\!])$

- $\hat{\mathcal{E}}^\sharp \stackrel{\text{def}}{=} \mathcal{P}_{\mathit{finite}}(\mathcal{E}^\sharp)$ (finite sets of abstract elements)
- $\hat{\gamma}(A^\sharp) \stackrel{\text{def}}{=} \cup \{\, \gamma(X^\sharp) \mid X^\sharp \in A^\sharp \,\}$ (join of concretizations)

Example: using the interval domain as $\mathcal{E}^\sharp$

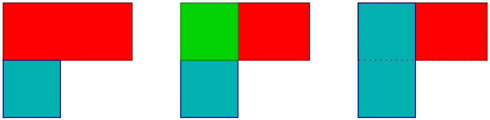
$\hat{\gamma}(\{[-10,-5],[2,4],[0,0],[2,3]\}) = [-10,-5] \cup \{0\} \cup [2,4]$

Ordering

Issue: how can we compare two elements of $\hat{\mathcal{E}}^\sharp$?

- $\hat{\gamma}$ is generally not injective
  there is no canonical representation for $\hat{\gamma}(A^\sharp)$
- testing $\hat{\gamma}(A^\sharp) = \hat{\gamma}(B^\sharp)$ or $\hat{\gamma}(A^\sharp) \subseteq \hat{\gamma}(B^\sharp)$ is difficult

Example: powerset completion of the interval domain



$A^\sharp$ $B^\sharp$ $C^\sharp$

$A^\sharp = \{\{0\} \times \{0\}, [0,1] \times \{1\}\}$
$B^\sharp = \{\{0\} \times \{0\}, \{0\} \times \{1\}, \{1\} \times \{1\}\}$
$C^\sharp = \{\{0\} \times [0,1], [0,1] \times \{1\}\}$

$\hat{\gamma}(A^\sharp) = \hat{\gamma}(B^\sharp) = \hat{\gamma}(C^\sharp)$

$B^\sharp$ is more costly to represent: it requires three abstract elements instead of two
$C^\sharp$ is a covering and not a partition (red $\cap$ blue $= \{0\} \times \{1\} \neq \emptyset$)

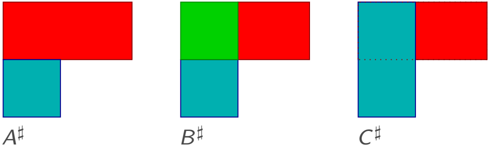
Ordering (cont.)

Solution: sound approximation of $\subseteq$

$A^\sharp \mathrel{\hat{\sqsubseteq}} B^\sharp \overset{\text{def}}{\iff} \forall X^\sharp \in A^\sharp : \exists Y^\sharp \in B^\sharp : X^\sharp \sqsubseteq Y^\sharp$ (Hoare powerdomain order)

- $\hat{\sqsubseteq}$ is a partial order (when $\sqsubseteq$ is)
- $\hat{\sqsubseteq}$ is a sound approximation of $\subseteq$ (when $\sqsubseteq$ is)
  $A^\sharp \mathrel{\hat{\sqsubseteq}} B^\sharp \implies \hat{\gamma}(A^\sharp) \subseteq \hat{\gamma}(B^\sharp)$ but the converse may not hold
- testing $\hat{\sqsubseteq}$ reduces to testing $\sqsubseteq$ finitely many times

Example: powerset completion of the interval domain



$\hat{\gamma}(A^\sharp) = \hat{\gamma}(B^\sharp) = \hat{\gamma}(C^\sharp)$

$B^\sharp \mathrel{\hat{\sqsubseteq}} A^\sharp \mathrel{\hat{\sqsubseteq}} C^\sharp$

Abstract operations

Abstract operators

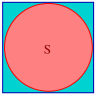
- $\hat{\mathsf{S}}^\sharp[\![\, \mathit{stat}\,]\!]\, A^\sharp \stackrel{\text{def}}{=} \{\, \mathsf{S}^\sharp[\![\, \mathit{stat}\,]\!]\, X^\sharp \mid X^\sharp \in A^\sharp \,\}$
  apply *stat* on each abstract element independently
- $A^\sharp \mathbin{\hat{\cup}^\sharp} B^\sharp \stackrel{\text{def}}{=} A^\sharp \cup B^\sharp$
  keep elements from both arguments without applying any abstract operation
  $\hat{\cup}^\sharp$ is exact
- $A^\sharp \mathbin{\hat{\cap}^\sharp} B^\sharp \stackrel{\text{def}}{=} \{\, X^\sharp \cap^\sharp Y^\sharp \mid X^\sharp \in A^\sharp,\ Y^\sharp \in B^\sharp \,\}$
  $\hat{\cap}^\sharp$ is exact if $\cap^\sharp$ is (as $\cup$ and $\cap$ are distributive)

Galois connection:

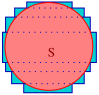
in general, there is no abstraction function $\hat{\alpha}$ corresponding to $\hat{\gamma}$

Example: powerset completion $\hat{\mathcal{E}}^\sharp$ of the interval domain $\mathcal{E}^\sharp$

given the disc $S \stackrel{\text{def}}{=} \{\, (x,y) \mid x^2 + y^2 \leq 1 \,\}$
$\alpha(S) = [-1,1] \times [-1,1]$ (optimal interval abstraction)
but there is no best abstraction in $\hat{\mathcal{E}}^\sharp$



$\alpha(S)$



not $\hat{\alpha}(S)$

Dynamic approximation

Issue: the size $|A^\sharp|$ of elements $A^\sharp \in \hat{\mathcal{E}}^\sharp$ is unbounded

every application of $\hat{\cup}^\sharp$ adds some more elements

$\implies$ efficiency and convergence problems

Solution: to reduce the size of elements

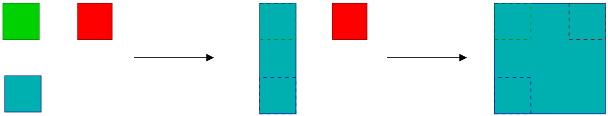
- redundancy removal

  $simplify(A^\sharp) \stackrel{\text{def}}{=} \{\, X^\sharp \in A^\sharp \,|\, \forall Y^\sharp \neq X^\sharp \in A^\sharp : X^\sharp \not\sqsubseteq Y^\sharp \,\}$

  no loss of precision: $\hat{\gamma}(simplify(A^\sharp)) = \hat{\gamma}(A^\sharp)$

- collapse: join elements in $\mathcal{E}^\sharp$

  $collapse(A^\sharp) \stackrel{\text{def}}{=} \{\cup^\sharp \{\, X^\sharp \in A^\sharp \,\}\}$

  large loss of precision, but very effective: $|collapse(A^\sharp)| = 1$

- partial collapse: limit $|A^\sharp|$ to a fixed size k by $\cup^\sharp$

  but how to choose which elements to merge? no easy solution!

Widening

Issue: for loops, abstract iterations $(A^\sharp_n)_{n\in\mathbb{N}}$ may not converge

- the size of $A^\sharp_n$ may grow arbitrarily large
- even if $|A^\sharp_n|$ is stable, some elements in $A^\sharp_n$ may not converge
  if $\mathcal{E}^\sharp$ has infinite increasing sequences

$\Longrightarrow$ we need a widening $\triangledown$

Widenings for powerset domains are difficult to design

Example widening: collapse after a fixed number N of iterations

$$A^\sharp_{n+1} \stackrel{\text{def}}{=} \begin{cases} \mathit{simplify}(A^\sharp_n \mathbin{\hat{\cup}^\sharp} B^\sharp_{n+1}) & \text{if } n < N \\ \mathit{collapse}(A^\sharp_n) \triangledown \mathit{collapse}(B^\sharp_{n+1}) & \text{otherwise} \end{cases}$$

this is very naïve, see Bagnara et al. STTT06 for more interesting widenings

State partitioning

State partitioning

Principle:

- partition *a priori* $\mathcal{E}$ into finitely many sets
- abstract each partition of $\mathcal{E}$ independently using an element of $\mathcal{E}^\sharp$

Abstract domain:

Given an abstract partition $P^\sharp \subseteq \mathcal{E}^\sharp$, i.e., a set such that:

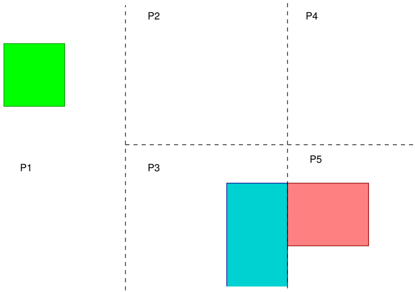
- $P^\sharp$ is finite
- $\cup \{ \gamma(X^\sharp) \mid X^\sharp \in P^\sharp \} = \mathcal{E}$

  for generality, we have in fact a covering, not a partitioning of $\mathcal{E}$
  i.e., we can have $X^\sharp \neq Y^\sharp \in P^\sharp$ with $\gamma(X^\sharp) \cap \gamma(Y^\sharp) \neq \emptyset$

We define $\tilde{\mathcal{E}}^\sharp \stackrel{\text{def}}{=} P^\sharp \to \mathcal{E}^\sharp$

representable in memory, as $P^\sharp$ is finite

Ordering



Example: $\mathcal{E}^\sharp$ is the interval domain

$$
\begin{aligned}
P^\sharp = \{P_1, P_2, P_3, P_4, P_5\} \text{ where}\\
P_1 &= [-\infty, 0] \times [-\infty, +\infty]\\
P_2 &= [0, 10] \times [0, +\infty]\\
P_3 &= [0, 10] \times [-\infty, 0]\\
P_4 &= [10, +\infty] \times [0, +\infty]\\
P_5 &= [10, +\infty] \times [-\infty, 0]
\end{aligned}
$$

$$
\begin{aligned}
X^\sharp = [&P_1 \mapsto [-6, -5] \times [5, 6],\\
&P_2 \mapsto \perp,\\
&P_3 \mapsto [9, 10] \times [-\infty, -1],\\
&P_4 \mapsto \perp,\\
&P_5 \mapsto [10, 12] \times [-3, -1]]
\end{aligned}
$$

- $\tilde{\mathcal{E}}^\sharp \stackrel{\text{def}}{=} P^\sharp \to \mathcal{E}^\sharp$
- $\tilde{\gamma}(A^\sharp) \stackrel{\text{def}}{=} \cup \; \{ \gamma(A^\sharp(X^\sharp)) \cap \gamma(X^\sharp) \,|\, X^\sharp \in P^\sharp \}$
- $A^\sharp \mathrel{\tilde{\sqsubseteq}} B^\sharp \stackrel{\text{def}}{\iff} \forall X^\sharp \in P^\sharp\colon A^\sharp(X^\sharp) \sqsubseteq B^\sharp(X^\sharp)$ (point-wise order)
- $\tilde{\alpha}(S) \stackrel{\text{def}}{=} \lambda X^\sharp \in P^\sharp . \alpha(S \cap \gamma(X^\sharp))$

  if $\mathcal{E}^\sharp$ enjoys a Galois connection, so does $\tilde{\mathcal{E}}^\sharp$

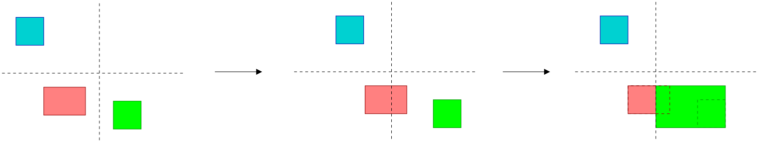
Abstract operators

Abstract operators: point-wise extension from $\mathcal{E}^\sharp$ to $P^\sharp \to \mathcal{E}^\sharp$

- $A \;\tilde{\cup}^\sharp\; B \stackrel{\text{def}}{=} \lambda X^\sharp \in P^\sharp . A(X^\sharp) \cup^\sharp B(X^\sharp)$
- $A \;\tilde{\cap}^\sharp\; B \stackrel{\text{def}}{=} \lambda X^\sharp \in P^\sharp . A(X^\sharp) \cap^\sharp B(X^\sharp)$
- $A \;\tilde{\triangledown}\; B \stackrel{\text{def}}{=} \lambda X^\sharp \in P^\sharp . A(X^\sharp) \triangledown B(X^\sharp)$
- $\tilde{\mathrm{S}}^\sharp[\![\, e \leq 0?\,]\!]\, A^\sharp \stackrel{\text{def}}{=} \lambda X^\sharp \in P^\sharp . \mathrm{S}^\sharp[\![\, e \leq 0?\,]\!]\, A^\sharp(X^\sharp)$
- $\tilde{\mathrm{S}}^\sharp[\![\, V \leftarrow e\,]\!]\, A^\sharp$ is more complex

  any $\mathrm{S}^\sharp[\![\, V \leftarrow e\,]\!]\, A^\sharp(X^\sharp)$ may escape its partition $X^\sharp$; we must cut them at partition borders and glue the pieces falling into the same partition

example: $X \leftarrow X + 2$



$$\tilde{\mathrm{S}}^\sharp[\![\, V \leftarrow e\,]\!]\, A^\sharp \stackrel{\text{def}}{=} \lambda X^\sharp . \cup^\sharp \{\, X^\sharp \cap^\sharp \mathrm{S}^\sharp[\![\, V \leftarrow e\,]\!]\, A(Y^\sharp) \mid Y^\sharp \in P^\sharp \,\}$$

Example analysis

Example

$X \leftarrow \textbf{rand}(10, 20);$
$Y \leftarrow \textbf{rand}(0, 1);$
$\textbf{if } Y > 0 \textbf{ then } X \leftarrow -X;$
• $Z \leftarrow 100/X$

Analysis:

- $\mathcal{E}^\sharp$ is the interval domain
- partition with respect to the sign of X
  $P^\sharp \stackrel{\text{def}}{=} \{\, X^+, X^- \,\}$ where
  $X^+ \stackrel{\text{def}}{=} [0, +\infty] \times \mathbb{Z} \times \mathbb{Z}$ and $X^- \stackrel{\text{def}}{=} [-\infty, 0] \times \mathbb{Z} \times \mathbb{Z}$
- at • we find:
  $X^+ \mapsto [X \in [10, 20], Y \mapsto [0, 0], Z \mapsto [0, 0]]$
  $X^- \mapsto [X \in [-20, -10], Y \mapsto [1, 1], Z \mapsto [0, 0]]$

  $\Longrightarrow$ no division by zero

Binary decision trees

Binary decision trees

Principle: data-structure to compactly represent partitions

Example: boolean partitions

- assume that variables have a type: $\mathbb{V} \stackrel{\text{def}}{=} \mathbb{V}_b \cup \mathbb{V}_n$
  - each $V \in \mathbb{V}_b$ has value in $\{0,1\}$ (boolean variable)
  - each $V \in \mathbb{V}_n$ has value in $\mathbb{Z}$ (numeric variable)
- $\mathcal{E} \simeq \{0,1\}^{|\mathbb{V}_b|} \times \mathbb{Z}^{|\mathbb{V}_n|}$

  $P^\sharp \stackrel{\text{def}}{=} \{\, \langle b_1, \ldots, b_{|\mathbb{V}_b|}\rangle \times \mathbb{Z}^{|\mathbb{V}_n|} \mid b_1, \ldots, b_{|\mathbb{V}_b|} \in \{0,1\}\,\}$

  a partition corresponds to a precise valuation of all the boolean variables and no information on the numeric variables
- assume that $\mathcal{E}_n^\sharp$ abstracts $\mathcal{P}(\mathbb{V}_n \to \mathbb{Z})$ (numeric domain)

  the boolean partitioning domain based on $\mathcal{E}_n^\sharp$ is:

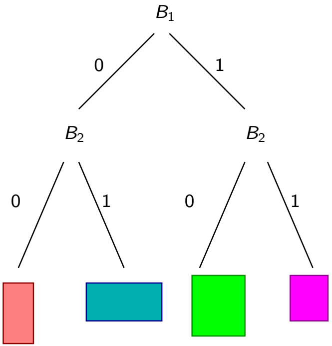
  $\tilde{\mathcal{E}}^\sharp \stackrel{\text{def}}{=} \{0,1\}^{|\mathbb{V}_b|} \to \mathcal{E}_n^\sharp$

Binary decision trees (cont.)

Representation: for $\tilde{\mathcal{E}}^\sharp \stackrel{\text{def}}{=} \{0,1\}^{|\mathbb{V}_b|} \to \mathcal{E}_n^\sharp$

binary trees:

- nodes are labelled with boolean variables $B_i \in \mathbb{V}_b$
- two children: $B_i = 0$ and $B_i = 1$
- leaves are abstract elements in $\mathcal{E}_n^\sharp$ (abstraction of $\mathcal{P}(\mathbb{V}_n \to \mathbb{Z})$)

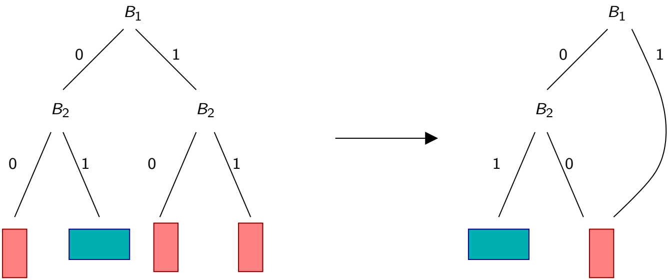


Reduced binary decision trees

Optimization: similar to Reduced Ordered Binary Decision Diagrams

- merge identical sub-trees (memory sharing)
- remove nodes if both children are identical

$\Longrightarrow$ we get a directed acyclic graphs



if $\gamma_n : \mathcal{E}_n^\sharp \to \mathbb{Z}^{|\mathbb{V}_n|}$ is injective and we use memoization
then $\tilde{\gamma}(A^\sharp) = \tilde{\gamma}(B^\sharp) \iff A^\sharp$ and $B^\sharp$ occupy the same address in memory
e.g., == in OCaml, which is faster to test than structural equality =

Abstract operations

- numeric operations: performed independently on each leaf
  e.g., $\tilde{\mathsf{S}}^\sharp[\![\, V \leftarrow e \,]\!]$ reverts to applying $\mathsf{S}^\sharp[\![\, V \leftarrow e \,]\!]$ on each leaf

- boolean operations: manipulate trees
  - $\tilde{\mathsf{S}}^\sharp[\![\, B_i \leftarrow \textbf{rand}(0,1) \,]\!]$: merge B_i's subtrees recursively
  - $\tilde{\mathsf{S}}^\sharp[\![\, B_i = 0? \,]\!]$: set all $B_i = 1$ branches to $\perp$
  - $\cdots$

- binary operations: $\tilde{\cup}^\sharp$, $\tilde{\cap}^\sharp$, $\tilde{\nabla}$, $\tilde{\sqsubseteq}$
  - first, unify tree structures (unshare trees and add missing nodes)
  - then, apply the operation pair-wise on leaves

- optimization needs to be performed again after each operation
  ensures that abstract elements do not grow too large

Example analysis

Example

$X \leftarrow \textbf{rand}(0, 100);$
$\textbf{if } X = 0 \textbf{ then } B \leftarrow 0 \textbf{ else } B \leftarrow 1;$
$\ldots$
$\color{red}\bullet$ $\textbf{if } B = 1 \textbf{ then } {\color{blue}\bullet}\ Y \leftarrow 100/X$

Analysis: using the interval domain for $\mathcal{E}_n^\sharp$

at $\color{red}\bullet$, we can infer the invariant:
$(B = 0 \implies X = 0) \wedge (B = 1 \implies X \in [1, 100])$

at $\color{blue}\bullet$, we deduce that $B = 1 \wedge X \in [1, 100]$
$\implies$ there is no division by zero

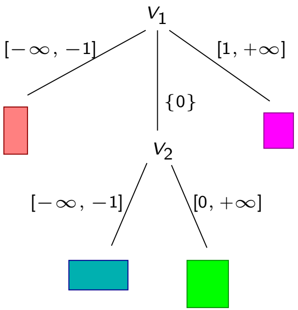
More tree-based partitioning structures

Other tree-based partitioning data-structure

we can extend partition trees in many ways

- allow $n-$array nodes
  partition wrt. abstract values in a non-relational domain

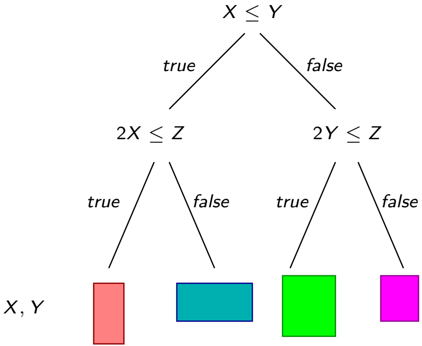
  Example: partitioning integer variables in the interval domain



More tree-based partitioning structures (cont.)

- partitioning with respect to predicates

Example: linear relations over $\mathbb{V} \stackrel{\text{def}}{=} \{X, Y, Z\}$



the same variables may appear in predicates and in the leaves
$\Longrightarrow$ $S^\sharp[\![\, \mathit{stat} \,]\!]$ must generally update both the nodes and the leaves

the set of node predicates may be fixed before the analysis
or chosen dynamically during the analysis

Path partitioning

Path sensitivity

Principle: partition wrt. the history of computation

- keep different abstract elements for different execution paths
  e.g., different branches taken, different loop iterations
- avoid merging with $\cup^\sharp$ elements at control-flow joins
  at the end of **if** $\cdots$ **then** $\cdots$ **else**, or at loop head

Intuition: as a program transformation

$X \leftarrow$ **rand**$(-50, 50)$;
if $X \geq 0$ **then**
  $Y \leftarrow X + 10$
else
  $Y \leftarrow X - 10$;
assert $Y \neq 0$

$\longrightarrow$

$X \leftarrow$ **rand**$(-50, 50)$;
if $X \geq 0$ **then**
  $Y \leftarrow X + 10$;
  assert $Y \neq 0$
else
  $Y \leftarrow X - 10$;
  assert $Y \neq 0$

the **assert** is tested in the context of each branch
instead of after the control-flow join

the interval domain can prove the assertion on the right, but not on the left

Abstract domain

Formalization: we consider here only **if** $\cdots$ **then** $\cdots$ **else**

- $\mathcal{L}$ denote syntactic labels of **if** $\cdots$ **then** $\cdots$ **else** instructions
- history abstraction $\mathbb{H} \stackrel{\text{def}}{=} \mathcal{L} \to \{\text{true}, \text{false}, \bot\}$

  $H \in \mathbb{H}$ indicates the outcome of the last time we executed each test:
  - $H(\ell) = \text{true}$: we took the **then** branch
  - $H(\ell) = \text{false}$: we took the **else** branch
  - $H(\ell) = \bot$: we never executed the test

  Notes:

  – $\mathbb{H}$ can remember the outcome of several successive tests
  ℓ_1 : **if** $\cdots$ **then** $\cdots$ **else**; ℓ_2 : **if** $\cdots$ **then** $\cdots$ **else**

  – for tests in loops, $\mathbb{H}$ remembers only the last outcome
  while $\cdots$ **do** ℓ : **if** $\cdots$ **then** $\cdots$ **else**

  – we could extend $\mathbb{H}$ to longer histories with $\mathbb{H} = (\mathcal{L} \to \{\,\text{true}, \text{false}, \bot\,\})^*$

  – we could extend $\mathbb{H}$ to track loop iterations with $\mathbb{H} = \mathcal{L} \to \mathbb{N}$
- $\breve{\mathcal{E}}^\sharp \stackrel{\text{def}}{=} \mathbb{H} \to \mathcal{E}^\sharp$

  use a different abstract element for each abstract history

Abstract operators

- $\breve{\mathcal{E}}^\sharp \stackrel{\text{def}}{=} \mathbb{H} \to \mathcal{E}^\sharp$
- $\breve{\gamma}(A^\sharp) = \cup \{ \gamma(A^\sharp(H)) \mid H \in \mathbb{H} \}$
- $\breve{\sqsubseteq}$, $\breve{\cup}^\sharp$, $\breve{\cap}^\sharp$, $\breve{\nabla}$ are point-wise
- $\breve{S}^\sharp[\![V \leftarrow e]\!]$ and $\breve{S}^\sharp[\![e \leq 0?]\!]$ are point-wise
- $\breve{S}^\sharp[\![\ell : \textbf{if } c \textbf{ then } s_1 \textbf{ else } s_2]\!] A^\sharp$ is more complex
  - we merge all information about ℓ
    $C^\sharp = \lambda H.A^\sharp(H[\ell \mapsto \text{true}]) \cup^\sharp A^\sharp(H[\ell \mapsto \text{false}]) \cup^\sharp A^\sharp(H[\ell \mapsto \bot])$
  - we compute the then branch, where $H(\ell) = \text{true}$
    $T'^\sharp = \breve{S}^\sharp[\![s_1]\!] (\breve{S}^\sharp[\![c?]\!] T^\sharp)$ where
    $T^\sharp = \lambda H.C^\sharp(H)$ if $H(\ell) = \text{true}$, $\bot$ otherwise
  - we compute the else branch, where $H(\ell) = \text{false}$
    $F'^\sharp = \breve{S}^\sharp[\![s_2]\!] (\breve{S}^\sharp[\![\neg c?]\!] F^\sharp)$ where
    $F^\sharp = \lambda H.C^\sharp(H)$ if $H(\ell) = \text{false}$, $\bot$ otherwise
  - we join both branches: $T'^\sharp \;\breve{\cup}^\sharp\; F'^\sharp$
    the join is exact as $\forall H \in \mathbb{H}$: either $T'^\sharp(H) = \bot$ or $F'^\sharp(H) = \bot$

$\Longrightarrow$ we get a semantic by induction on the syntax of the original program

Complex example

Linear interpolation

$X \leftarrow$ **rand**$(TX[0], TX[N])$;
$I \leftarrow 0$;
while $I < N \wedge X > TX[I+1]$ **do**
    $I \leftarrow I + 1$;
done;
$Y \leftarrow TY[I] + (X - TX[I]) \times TS[I]$

Concrete semantics: table-based interpolation based on the value of X

- look-up index I in the interpolation table: $TX[I] \leq X \leq TX[I+1]$
- interpolate from value $TY[I]$ when $X = TX[I]$ with slope $TS[I]$

Analysis: in the interval domain

- without partitioning:
  $Y \in [\min\ TY, \max\ TY] + (X - [\min\ TX, \max\ TX]) \times [\min\ TS, \max\ TS]$
- partitioning with respect to the number of loop iterations:
  $Y \in \cup_{I \in [0,N]}\ TY[I] + ([0, TX[I+1] - TX[I]]) \times TS[I]$
  more precise as it keeps the relation between table indices

Inter-procedural analyses

Overview

- Analysis on the control-flow graph
  reduce function calls and returns to *gotos*
  useful for the project!
- Inlining
  simple and precise
  but not efficient and may not terminate
- Call-site and call-stack abstraction
  terminates even for recursive programs
  parametric cost-precision trade-off
- Tabulated abstraction
  optimal reuse of analysis partial results
- Summary-based abstraction
  modular bottom-up analysis
  leverage relational domains

in general, these different abstractions give incomparable results
(there is no clear winner)

Analysis on the control-flow graph

Inter-procedural control-flow graphs

Extend control-flow graphs:

- one subgraph for each function
- additional arcs to denote function calls and returns

we get one big graph without procedures nor calls, only gotos

$\Longrightarrow$ reduced to a classic analysis based on equation systems

but difficult to use in a denotational-style analysis by induction on the syntax

Note: to simplify, we assume here no local variables and no function arguments:

- locals and arguments are transformed into locals
- only possible if there are no recursive calls

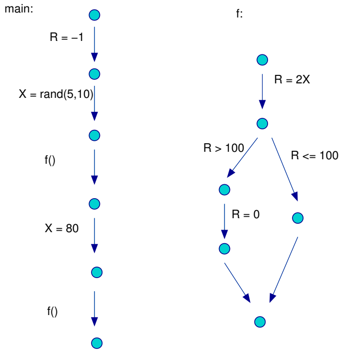
this will be fixed in the following

Example: Control-flow graph

Example

main :
$R \leftarrow -1$;
$X \leftarrow \textbf{rand}(5,10)$; $f()$;
$X \leftarrow 80$; $f()$

f :
$R \leftarrow 2 \times X$;
if $R > 100$ **then** $R \leftarrow 0$



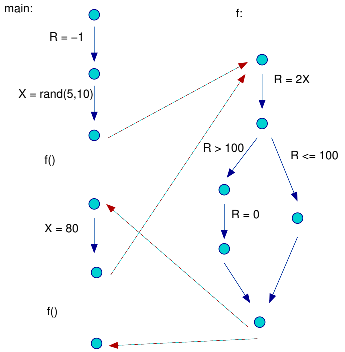
create one control-flow graph for each function

Example: Control-flow graph

Example

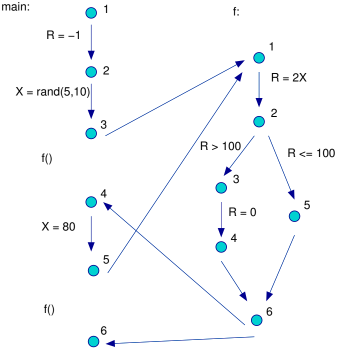
main :
$R \leftarrow -1;$
$X \leftarrow \textbf{rand}(5, 10);\ f();$
$X \leftarrow 80;\ f()$

f :
$R \leftarrow 2 \times X;$
$\textbf{if } R > 100 \textbf{ then } R \leftarrow 0$



replace call instructions with gotos

Example: Equation system

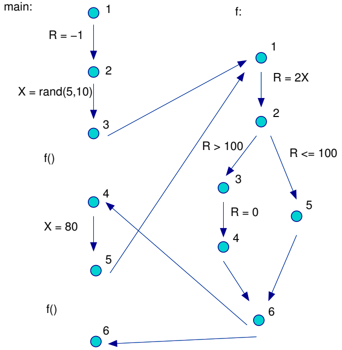


$$\begin{aligned}
\mathcal{S}_{\mathrm{main},1} &= \top \\
\mathcal{S}_{\mathrm{main},2} &= \mathsf{S}[\![\, R \leftarrow 1 \,]\!]\, \mathcal{S}_{\mathrm{main},1} \\
\mathcal{S}_{\mathrm{main},3} &= \mathsf{S}[\![\, X \leftarrow \mathbf{rand}(5,10) \,]\!]\, \mathcal{S}_{\mathrm{main},2} \\
\mathcal{S}_{\mathrm{main},4} &= \mathcal{S}_{\mathrm{f},6} \\
\mathcal{S}_{\mathrm{main},5} &= \mathsf{S}[\![\, X \leftarrow 80 \,]\!]\, \mathcal{S}_{\mathrm{main},4} \\
\mathcal{S}_{\mathrm{main},6} &= \mathcal{S}_{\mathrm{f},6}
\end{aligned}$$

$$\begin{aligned}
\mathcal{S}_{\mathrm{f},1} &= \mathcal{S}_{\mathrm{main},3} \cup \mathcal{S}_{\mathrm{main},5} \\
\mathcal{S}_{\mathrm{f},2} &= \mathsf{S}[\![\, R \leftarrow 2X \,]\!]\, \mathcal{S}_{\mathrm{f},1} \\
\mathcal{S}_{\mathrm{f},3} &= \mathsf{S}[\![\, R > 100 \,]\!]\, \mathcal{S}_{\mathrm{f},2} \\
\mathcal{S}_{\mathrm{f},4} &= \mathsf{S}[\![\, R \leftarrow 0 \,]\!]\, \mathcal{S}_{\mathrm{f},3} \\
\mathcal{S}_{\mathrm{f},5} &= \mathsf{S}[\![\, R \leq 100 \,]\!]\, \mathcal{S}_{\mathrm{f},2} \\
\mathcal{S}_{\mathrm{f},6} &= \mathcal{S}_{\mathrm{f},4} \cup \mathcal{S}_{\mathrm{f},5}
\end{aligned}$$

- each variable $\mathcal{S}_i$ denotes a set of environments at a control location i
- we can derive an abstract version of the system
  e.g.: $\mathcal{S}^\sharp_{\mathrm{f},2} = \mathsf{S}^\sharp[\![\, R \leftarrow 2X \,]\!]\, \mathcal{S}^\sharp_{\mathrm{f},1}$, $\mathcal{S}^\sharp_{\mathrm{f},6} = \mathcal{S}^\sharp_{\mathrm{f},4} \cup^\sharp \mathcal{S}^\sharp_{\mathrm{f},5}$, etc.
- we can solve the abstract system, using widenings to terminate
  c.f. project

Example: Equation system



$$
\begin{aligned}
\mathcal{S}_{\text{main},1} &= \top \\
\mathcal{S}_{\text{main},2} &= \mathsf{S}[\![\, R \leftarrow 1 \,]\!]\, \mathcal{S}_{\text{main},1} \\
\color{blue}{\mathcal{S}_{\text{main},3}} &= \mathsf{S}[\![\, X \leftarrow \textbf{rand}(5,10) \,]\!]\, \mathcal{S}_{\text{main},2} \\
\mathcal{S}_{\text{main},4} &= \color{red}{\mathcal{S}_{\text{f},6}} \\
\color{blue}{\mathcal{S}_{\text{main},5}} &= \mathsf{S}[\![\, X \leftarrow 80 \,]\!]\, \mathcal{S}_{\text{main},4} \\
\mathcal{S}_{\text{main},6} &= \color{red}{\mathcal{S}_{\text{f},6}}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{S}_{\text{f},1} &= \color{blue}{\mathcal{S}_{\text{main},3}} \cup \color{blue}{\mathcal{S}_{\text{main},5}} \\
\mathcal{S}_{\text{f},2} &= \mathsf{S}[\![\, R \leftarrow 2X \,]\!]\, \mathcal{S}_{\text{f},1} \\
\mathcal{S}_{\text{f},3} &= \mathsf{S}[\![\, R > 100 \,]\!]\, \mathcal{S}_{\text{f},2} \\
\mathcal{S}_{\text{f},4} &= \mathsf{S}[\![\, R \leftarrow 0 \,]\!]\, \mathcal{S}_{\text{f},3} \\
\mathcal{S}_{\text{f},5} &= \mathsf{S}[\![\, R \leq 100 \,]\!]\, \mathcal{S}_{\text{f},2} \\
\color{red}{\mathcal{S}_{\text{f},6}} &= \mathcal{S}_{\text{f},4} \cup \mathcal{S}_{\text{f},5}
\end{aligned}
$$

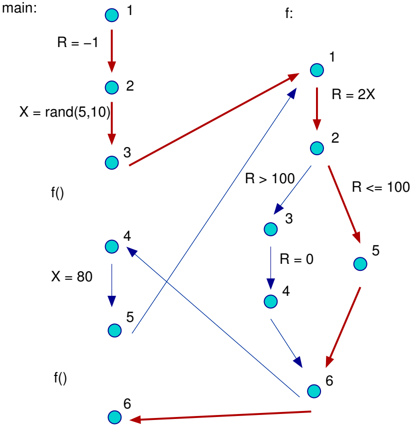
using intervals we get the following solution:

$$
\begin{aligned}
\mathcal{S}^\sharp_{\text{main},1} &: X, R \in \mathbb{Z} \\
\mathcal{S}^\sharp_{\text{main},2} &: X \in \mathbb{Z}, R = -1 \\
\mathcal{S}^\sharp_{\text{main},3} &: \color{blue}{X \in [5,10], R = -1} \\
\mathcal{S}^\sharp_{\text{main},4} &: \color{red}{X \in [5,80], R \in [0,100]} \\
\mathcal{S}^\sharp_{\text{main},5} &: \color{blue}{X = 80, R \in [0,100]} \\
\mathcal{S}^\sharp_{\text{main},6} &: \color{red}{X \in [5,80], R \in [0,100]}
\end{aligned}
\qquad
\begin{aligned}
\mathcal{S}^\sharp_{\text{f},1} &: \color{blue}{X \in [5,80], R \in [-1,100]} \\
\mathcal{S}^\sharp_{\text{f},2} &: X \in [5,80], R \in [10,160] \\
\mathcal{S}^\sharp_{\text{f},3} &: X \in [5,80], R \in [101,160] \\
\mathcal{S}^\sharp_{\text{f},4} &: X \in [5,80], R = 0 \\
\mathcal{S}^\sharp_{\text{f},5} &: X \in [5,80], R \in [10,100] \\
\mathcal{S}^\sharp_{\text{f},6} &: \color{red}{X \in [5,80], R \in [0,100]}
\end{aligned}
$$

Imprecision

In fact, in our example, $R = 0$ holds at the end of the program!
$\Longrightarrow$ the analysis is imprecise

Explanation: the control-flow graph adds impossible executions paths



General case: concrete semantics

Procedures

Syntax:

- $\mathcal{F}$ finite set of procedure names
- $body : \mathcal{F} \to stat$: procedure bodies
- $main \in stat$: entry point body
- $\mathbb{V}_G$: set of global variables
- $\mathbb{V}_f$: set of local variables for procedure $f \in \mathcal{F}$
  procedure f can only access $\mathbb{V}_f \cup \mathbb{V}_G$
  main has no local variable and can only access $\mathbb{V}_G$
- $stat ::= f(expr_1, \ldots, expr_{|\mathbb{V}_f|}) \mid \cdots$

  procedure call, $f \in \mathcal{F}$, setting all its local variables

  local variables double as procedure arguments
  no special mechanism to return a value (a global variable can be used)

Concrete environments

Notes:

- when f calls g, we must remember the value of f's locals $\mathbb{V}_f$ in the semantics of g and restore them when returning
- several copies of each $V \in \mathbb{V}_f$ may exist at a given time
  due to recursive calls, i.e.: cycles in the call graph

$\Longrightarrow$ concrete environments use per-variable stacks

Stacks: $\mathcal{S} \stackrel{\text{def}}{=} \mathbb{Z}^*$ (finite sequences of integers)

- **push**$(v, s) \stackrel{\text{def}}{=} v \cdot s$ $(v, v' \in \mathbb{Z},\ s, s' \in \mathcal{S})$
- **pop**$(s) \stackrel{\text{def}}{=} s'$ when $\exists v: s = v \cdot s'$, undefined otherwise
- **peek**$(s) \stackrel{\text{def}}{=} v$ when $\exists s': s = v \cdot s'$, undefined otherwise
- **set**$(v, s) \stackrel{\text{def}}{=} v \cdot s'$ when $\exists v': s = v' \cdot s'$, undefined otherwise

Environments: $\mathcal{E} \stackrel{\text{def}}{=} (\cup_{f \in \mathcal{F}} \mathbb{V}_f \cup \mathbb{V}_G) \to \mathcal{S}$

for $\mathbb{V}_G$, stacks are not necessary but simplify the presentation

traditionally, there is a single global stack for all local variables
using per-variable stacks instead also makes the presentation simpler

Concrete semantics

Concrete semantics: on $\mathcal{E} \stackrel{\text{def}}{=} (\cup_{f \in \mathcal{F}} \mathbb{V}_f \cup \mathbb{V}_G) \to \mathcal{S}$

variable read and update only consider the top of the stack
procedure calls push and pop local variables

- $\mathsf{E}[\![\, V \,]\!]\, \rho \stackrel{\text{def}}{=} \mathbf{peek}(\rho(V))$
- $\mathsf{S}[\![\, V \leftarrow e \,]\!]\, R \stackrel{\text{def}}{=} \{\, \rho[V \mapsto \mathbf{set}(x, \rho(V))] \mid \rho \in R, x \in \mathsf{E}[\![\, e \,]\!]\, \rho \,\}$
- $\mathsf{S}[\![\, f(e_{V_1}, \ldots, e_{V_n}) \,]\!]\, R = R_3$, where:

  $R_1 \stackrel{\text{def}}{=} \{\, \rho[\forall V \in \mathbb{V}_f \colon V \mapsto \mathbf{push}(x_V, \rho(V))] \mid \rho \in R, \forall V \in \mathbb{V}_f \colon x_V \in \mathsf{E}[\![\, e_V \,]\!]\, \rho \,\}$
  (evaluate each argument e_V and push its value x_V on the stack $\rho(V)$)

  $R_2 \stackrel{\text{def}}{=} \mathsf{S}[\![\, body(f) \,]\!]\, R_1$ (evaluate the procedure body)

  $R_3 \stackrel{\text{def}}{=} \{\, \rho[\forall V \in \mathbb{V}_f \colon V \mapsto \mathbf{pop}(\rho(V))] \mid \rho \in R_2 \,\}$ (pop local variables)

- initial environment: $\rho_0 \stackrel{\text{def}}{=} \lambda V \in \mathbb{V}_G . 0$

other statements are unchanged

Semantic inlining

Semantic inlining

Naïve abstract procedure call: mimic the concrete semantics

- assign abstract variables to stack positions:
  $\mathbb{V}^\sharp \stackrel{\text{def}}{=} \mathbb{V}_G \cup (\cup_{f \in \mathcal{F}} \mathbb{V}_f \times \mathbb{N})$

  $\mathbb{V}^\sharp$ is infinite, but each abstract environment uses finitely many variables

- $\mathcal{E}^\sharp_{\mathbb{V}}$ abstracts $\mathcal{P}(\mathbb{V} \to \mathbb{Z})$, for any finite $\mathbb{V} \subseteq \mathbb{V}^\sharp$

  $V \in \mathbb{V}_f$ denotes $(V, 0)$ in $\mathbb{V}^\sharp$
  push V: shift variables, replacing (V, i) with $(V, i+1)$, then add $(V, 0)$
  pop V: remove $(V, 0)$ and shift each (V, i) to $(V, i-1)$

- $\mathsf{S}^\sharp[\![\, f(e_1, \ldots, e_n) \,]\!]\, X^\sharp$ is then reduced to:

$$
\begin{array}{ll}
X_1^\sharp = \mathsf{S}^\sharp[\![\, \textbf{push } V_1; \ldots; \textbf{push } V_n \,]\!]\, X^\sharp & \text{(add fresh variables for } \mathbb{V}_f\text{)} \\
X_2^\sharp = \mathsf{S}^\sharp[\![\, V_1 \leftarrow e_1; \ldots; V_n \leftarrow e_n \,]\!]\, X_1^\sharp & \text{(bind arguments to locals)} \\
X_3^\sharp = \mathsf{S}^\sharp[\![\, body(f) \,]\!]\, X_2^\sharp & \text{(execute the procedure body)} \\
X_4^\sharp = \mathsf{S}^\sharp[\![\, \textbf{pop } V_1; \ldots; \textbf{pop } V_n \,]\!]\, X_3^\sharp & \text{(delete local variables)}
\end{array}
$$

Limitations:

- does not terminate in case of unbounded recursivity
- requires many abstract variables to represent the stacks
- procedures must be re-analyzed for every call

  full context-sensitivity: precise but costly

Example

Example

main :
  $R \leftarrow -1;$
  $f(\textbf{rand}(5,10));$
  $f(80)$

$f(X)$:
  $R \leftarrow 2 \times X;$
  if $R > 100$ **then** $R \leftarrow 0$

Analysis using intervals

- after the first call to f, we get $R \in [10,20]$
- after the second call to f, we get $R = 0$

Call-site abstraction

Call-site abstraction

Abstracting stacks: into a fixed, bounded set $\mathbb{V}^\sharp$ of variables

- $\mathbb{V}^\sharp \stackrel{\text{def}}{=} \cup_{f\in\mathcal{F}} \{\, V, \hat{V} \mid V \in \mathbb{V}_f \,\} \cup \mathbb{V}_G$
  two copies of each local variable
  V abstracts the value at the top of the stack (current call)
  $\hat{V}$ abstracts the rest of the stack
- $S^\sharp[\![\, \textbf{push } V \,]\!] X^\sharp \stackrel{\text{def}}{=} X^\sharp \cup^\sharp S^\sharp[\![\, \hat{V} \leftarrow V \,]\!] X^\sharp$
  $S^\sharp[\![\, \textbf{pop } V \,]\!] X^\sharp \stackrel{\text{def}}{=} X^\sharp \cup^\sharp S^\sharp[\![\, V \leftarrow \hat{V} \,]\!] X^\sharp$
  weak updates, similar to array manipulation
  no need to create and delete variables dynamically
- assignments and tests always access V, not $\hat{V}$
  $\Longrightarrow$ strong update (precise)

Note: when there is no recursivity, $\hat{V}$, **push** and **pop** can be omitted

Call-site abstraction

Principle: merge all the contexts in which each function is called

- we maintain two global maps $\mathcal{F} \to \mathcal{E}^\sharp$:

  $C^\sharp(f)$: abstracts the environments when calling f
  $R^\sharp(f)$: abstracts the environments when returning from f
  gather environments from all possible calls to f, disregarding the call sites

- during the analysis, when encountering a call $\mathsf{S}^\sharp[\![\, body(f)\,]\!]\, X^\sharp$:
  we return $R^\sharp(f)$
  but we also replace $C^\sharp$ with $C^\sharp[f \mapsto C^\sharp(f) \cup^\sharp X^\sharp]$

- $R^\sharp(f)$ is computed from $C^\sharp(f)$ as

  $R^\sharp(f) = \mathsf{S}^\sharp[\![\, body(f)\,]\!]\,(C^\sharp(f))$

Call-site abstraction

Fixpoint:

there may be circular dependencies between $C^\sharp$ and $R^\sharp$

e.g., in $f(2); f(3)$, the input for $f(3)$ depends on the output from $f(2)$

$\Longrightarrow$ we compute a fixpoint for $C^\sharp$ by iteration:

- initially, $\forall f\colon C^\sharp(f) = R^\sharp(f) = \bot$
- analyze *main*
- while $\exists f\colon C^\sharp(f)$ not stable
  apply widening $\triangledown$ to the iterates of $C^\sharp(f)$
  update $R^\sharp(f) = S^\sharp[\![\, body(f)\,]\!]\, C^\sharp(f)$
  analyze *main* and all the procedures again
  (this may modify some $C^\sharp(g)$)

$\Longrightarrow$ using $\triangledown$, the analysis always terminates in finite time

we can be more efficient and avoid re-analyzing procedures when not needed
e.g., use a workset algorithm, track procedure dependencies, etc.

Example

Example

main :
  $R \leftarrow -1$;
  $f(\textbf{rand}(5,10))$;
  $f(80)$

$f(X)$:
  $R \leftarrow 2 \times X$;
  if $R > 100$ **then** $R \leftarrow 0$

Analysis: using intervals (without widening as there is no dependency)

- first analysis of *main*: we get $\bot$ (as $R^\sharp(f) = \bot$)
  but $C^\sharp(f) = [R \mapsto [-1,-1], X \mapsto [5,10]]$
- first analysis of f: $R^\sharp(f) = [R \mapsto [10,20], X \mapsto [5,10]]$
- second analysis of *main*: we get
  $C^\sharp(f) = [R \mapsto [-1,20], X \mapsto [5,80]]$
- second analysis of f: $R^\sharp(f) = [R \mapsto [0,100], X \mapsto [5,80]]$
- final analysis of *main*, we find $R \in [0,100]$ at the program end
  less precise than $R = 0$ found by semantic inlining

Partial context-sensitivity

Variants: $k-$limiting, k is a constant

- stack:

  assign a distinct variable for the k highest levels of V
  abstract the lower (unbounded) stack part with $\hat{V}$
  more precise than keeping only the top of the stack separately

- context-sensitivity:

  each syntactic call has a unique call-site $\ell \in \mathcal{L}$
  a call stack is a sequence of nested call sites: $c \in \mathcal{L}^*$
  an abstract call stack remembers the last k call sites: $c^\sharp \in \mathcal{L}^k$
  the $C^\sharp$ and $R^\sharp$ maps now distinguish abstract call stacks
  $C^\sharp, R^\sharp : \mathcal{L}^k \to \mathcal{E}^\sharp$
  more precise than a partitioning by function only

larger k give more precision but less efficiency

Example: context-sensitivity

Example

main :
$R \leftarrow -1$;
ℓ_1 : $f(\mathbf{rand}(5,10))$;
ℓ_2 : $f(80)$

$f(X)$:
$R \leftarrow 2 \times X$;
if $R > 100$ **then** $R \leftarrow 0$

Analysis: using intervals and $k = 1$

- $C^\sharp(\ell_1) = [R \mapsto [-1,1], X \mapsto [5,10]]$
  $\implies R^\sharp(\ell_1) = [R \mapsto [10,20], X \mapsto [5,10]]$
- $C^\sharp(\ell_2) = [R \mapsto [10,20], X \mapsto [80,80]]$
  $\implies R^\sharp(\ell_2) = [R \mapsto [0,0], X \mapsto [80,80]]$
- at the end of the analysis, we get $R = 0$
  more precise than $R \in [0,100]$ found without context-sensitivity

Tabulation abstraction

Cardinal power

Principle:

the semantic of a function is $\mathsf{S}[\![\, body(f) \,]\!] : \mathcal{P}(\mathcal{E}) \to \mathcal{P}(\mathcal{E})$

$\Longrightarrow$ abstract it as an abstract function in $\mathcal{E}^\sharp \rightharpoonup \mathcal{E}^\sharp$

we use a partial function as the image of most abstract elements is not useful

Analysis: tabulated analysis

- use a global partial map $F^\sharp : \mathcal{F} \times \mathcal{E}^\sharp \rightharpoonup \mathcal{E}^\sharp$
- $F^\sharp$ is initially empty, and is filled on-demand
- when encountering $\mathsf{S}^\sharp[\![\, body(f) \,]\!]\, X^\sharp$

  return $F^\sharp(f, X^\sharp)$ if defined

  else, compute $\mathsf{S}^\sharp[\![\, body(f) \,]\!]\, X^\sharp$, store it in $F^\sharp(f, X^\sharp)$ and return it

Optimizations: trade precision for efficiency

- if $X^\sharp \sqsubseteq Y^\sharp$ and $F^\sharp(f, X^\sharp)$ is not defined, we can use $F^\sharp(f, Y^\sharp)$ instead
- if the size of $F^\sharp$ grows too large, use $F^\sharp(f, \top)$ instead

  sound, and ensures that the analysis terminates in finite time

Example

Example

main :
$R \leftarrow -1;$
$f(\mathbf{rand}(5,10));$
$f(80)$

$f(X)$:
$R \leftarrow 2 \times X;$
if $R > 100$ **then** $R \leftarrow 0$

Analysis using intervals

- $F^\sharp =$
$$[\ (f,[R \mapsto [-1,-1], X \mapsto [5,10]]) \mapsto [R \mapsto [10,20], X \mapsto [5,10]],$$
$$(f,[R \mapsto [10,20], X \mapsto [80,80]]) \mapsto [R \mapsto [0,0], X \mapsto [80,80]]\ ]$$
- at the end of the analysis, we get again $R = 0$

here, the function partitioning gives the same result as the call-site partitioning

Dynamic partitioning: complex example

Example: McCarthy's 91 function

main :
    $Mc(\textbf{rand}(0,+\infty))$

$Mc(n)$:
    if $n > 100$ **then** $r \leftarrow n - 10$
    else $Mc(n+11)$; $Mc(r)$

- in the concrete, when terminating:
  $r = n - 10$ when $n > 101$, and $r = 91$ wen $n \in [0,101]$

- using a widening $\triangledown$ to choose tabulated abstract values $F^\sharp(f, X^\sharp)$
  we find:

$$\begin{array}{lcl}
n \in [0,72] & \Rightarrow & r = 91 \\
n \in [73,90] & \Rightarrow & r \in [91,101] \\
n \in [91,101] & \Rightarrow & r = 91 \\
n \in [102,111] & \Rightarrow & r \in [91,101] \\
n \in [112,+\infty] & \Rightarrow & r \in [91,+\infty]
\end{array}$$

(source: Bourdoncle, JFP 1992)

Summary-based abstraction

Summary-based analyses

Principle:

- abstract the input-output relation using a relational domain
- analyze each procedure out of context
  no information about its possible arguments
- analyze a procedure given the analysis of the procedures it calls
  bottom-up analysis, from leaf functions to *main*

  $\Longrightarrow$ completely modular analysis
  for recursive calls, we still need to iterate the analysis of call cycles, with $\triangledown$

Analysis:

- analyze f with abstract variables $\mathbb{V}_f^\sharp \stackrel{\text{def}}{=} \{\, V, V' \mid V \in \mathbb{V}_G \cup \mathbb{V}_f \,\}$
  V' denotes the current value of the variable
  V denotes the value of the variable at the function entry
- at the beginning of the procedure, start with $\forall V \in \mathbb{V}_G \cup \mathbb{V}_f\colon V = V'$
  the analysis updates only V', never V
  at the end of the procedure, the invariant gives an input-output relation
  it summarizes the effect of the procedure, store it as $T^\sharp(f)$
- $S^\sharp[\![\, body(f) \,]\!]\, X^\sharp$ can be computed using $T^\sharp(f)$ and variable substitution

$$S^\sharp[\![\, \forall i\colon \textbf{del}\ V_i'' \,]\!]\, (X^\sharp[\forall i\colon V_i''/V_i'] \cap^\sharp T^\sharp(f)[\forall i\colon V_i''/V_i])$$

Example

Example

$max(a, b)$:
  if $a > b$ **then** $r \leftarrow a$;
  else $r \leftarrow b$; $c \leftarrow c + 1$;

main :
  $x \leftarrow [0, 10]; y \leftarrow [0, 10]$;
  $c \leftarrow 0; max(x, y)$;
  $r \leftarrow r - x$

Analysis using polyhedra

- the analysis of *max* gives:
  $r' \geq a \wedge r' \geq b \wedge c' \geq c \wedge c' \leq c + 1 \wedge a = a' \wedge b = b' \wedge x = x' \wedge y = y'$
- at *main*'s call to *max*
  before *max*: $c' = 0 \wedge x' \in [0, 10] \wedge y' \in [0, 10]$
  applying the summary: $c' \in [0, 1] \wedge x' \in [0, 10] \wedge y' \in [0, 10] \wedge r' \geq x' \wedge r' \geq y'$

  at the end of the program, $x \in [0, 10]$, $y \in [0, 10]$, $r \in [0, 10]$, $c \in [0, 1]$

the method requires a relational domain to infer interesting input-output relations
it compensates for the lack of information about the entry point

Abstracting arrays

Example

Example: increasing subsequence

$p[0] \leftarrow 0$; $B[0] \leftarrow A[0]$;
$i \leftarrow 1$; $k \leftarrow 1$;
while $i < N$ **do**
  if $A[i] > B[k-1]$ **then**
    $B[k] \leftarrow A[i]$;
    $p[k] \leftarrow i$;
    $k \leftarrow k+1$;
  $i \leftarrow i+1$

Given an array $A[0], \ldots, A[N-1]$
the program computes an increasing sub-array $B[0], \ldots, B[k-1]$
and the index sequence $p[0], \ldots, p[k-1]$

Invariants:

$1 \leq k \leq i \leq N$ $\quad$ $\forall x < k: B[x] = A[p[x]]$

$\forall x: 0 \leq p[x] < N$ $\quad$ $\forall x < k-1: B[x+1] > B[x]$

Overview

- Syntax and concrete semantics
- Non-relational abstract semantics
  e.g., $\forall i$: $A[i] \leq$ *constant*
  - application to interval analysis
- Relational (uniform) abstract semantics
  e.g., $\forall i$: $A[i] \leq V$
  - expand and fold operations
  - application to polyhedral analysis
- Non-uniform abstraction
  e.g., $\forall i$: $A[i] \leq i$

Syntax extension

Modified expressions and statements

$$
\begin{array}{lcll}
\textit{expr} & ::= & V & (\textit{scalar access, } V \in \mathbb{V}) \\
 & | & A[\textit{expr}] & (\textit{array access, } A \in \mathbb{A}) \\
 & | & \cdots & \\
\textit{stat} & ::= & V \leftarrow \textit{expr} & (\textit{scalar update, } V \in \mathbb{V}) \\
 & | & A[\textit{expr}] \leftarrow \textit{expr} & (\textit{array update, } A \in \mathbb{A}) \\
 & | & \cdots &
\end{array}
$$

Our language now has two ways to access the memory

- $\mathbb{V}$: scalar integer variables (as before)
- $\mathbb{A}$: arrays of integer values (new)
  - arrays are indiced by positive integers
  - arrays are unbounded (to simplify, we ignore overflows)

$\Longrightarrow$ an array A is similar to a map $A : \mathbb{N} \to \mathbb{Z}$

Concrete semantics

Concrete environments: $\mathcal{E} \stackrel{\text{def}}{=} (\mathbb{V} \cup (\mathbb{A} \times \mathbb{N})) \to \mathbb{Z}$

$\rho \in \mathcal{E}$ assigns an integer value to "memory cells" as follows:

- $\rho(V)$ for every scalar variable $V \in \mathbb{V}$
- $\rho(A, i)$ for every array position $A \in \mathbb{A}$, $i \geq 0$

Concrete semantics:

$$
\begin{array}{lcl}
\mathsf{E}[\![\, V \,]\!]\, \rho & \stackrel{\text{def}}{=} & \{\rho(V)\} \\
\mathsf{E}[\![\, A[e] \,]\!]\, \rho & \stackrel{\text{def}}{=} & \{\, \rho(A, i) \mid i \in \mathsf{E}[\![\, e \,]\!]\, \rho \,\} \\
\mathsf{S}[\![\, V \leftarrow e \,]\!]\, R & \stackrel{\text{def}}{=} & \{\, \rho[V \mapsto v] \mid \rho \in R,\ v \in \mathsf{E}[\![\, e \,]\!]\, \rho \,\} \\
\mathsf{S}[\![\, A[f] \leftarrow e \,]\!]\, R & \stackrel{\text{def}}{=} & \{\, \rho[(A, i) \mapsto v] \mid \rho \in R,\ v \in \mathsf{E}[\![\, e \,]\!]\, \rho,\ i \in \mathsf{E}[\![\, f \,]\!]\, \rho,\ i \geq 0 \,\} \\
\dots & &
\end{array}
$$

Non-relational abstractions

Summarization abstraction

Goal: reuse existing numeric abstract domains

issue: numeric domains only abstract subsets of $\mathbb{Z}^n$, for finite n

solution: reduce $\mathcal{E}$ to maps on finite set of abstract variables

Abstract variables: $\mathbb{V}^\sharp \stackrel{\text{def}}{=} \mathbb{V} \cup \mathbb{A}$

- scalar variables in $\mathbb{V}$ are exactly represented in $\mathbb{V}^\sharp$
- the contents of an array $A \in \mathbb{A}$ is abstracted with a single summary variable A (modeling the contents of the whole array)
- $\mathbb{V}^\sharp$ is finite

Summarization Galois Connection:

$$(\mathcal{P}(\mathcal{E}), \subseteq) \underset{\alpha_s}{\overset{\gamma_s}{\leftrightarrows}} (\mathcal{P}(\mathbb{V}^\sharp \to \mathbb{Z}), \subseteq)$$

- $\alpha_s(R) \stackrel{\text{def}}{=} \{\, [V \mapsto \rho(V), A \mapsto \rho(A, \iota(A))] \,|\, \rho \in R,\ \iota \in \mathbb{A} \to \mathbb{N} \,\}$
  (folds all array elements (A, i) into the abstract variable A)
- $\gamma_s(S) \stackrel{\text{def}}{=} \{\, \rho \,|\, \forall \iota \in \mathbb{A} \to \mathbb{N}: [V \mapsto \rho(V), A \mapsto \rho(A, \iota(A))] \in S \,\}$
  (indeed, $\gamma_s(S) = \{\, \rho \,|\, \alpha_s(\{\rho\}) \subseteq S \,\} = \cup \{\, R \,|\, \alpha_s(R) \subseteq S \,\}$)

Non-relational abstraction

Reminder: Interval abstraction

- $\mathcal{P}(\mathbb{V}^\sharp \to \mathbb{Z})$ is abstracted into $\mathbb{V}^\sharp \to \mathcal{P}(\mathbb{Z})$ (Cartesian abstraction)
- $\mathcal{P}(\mathbb{Z})$ is abstracted as an interval in $\mathbb{I}$

(Note: the Cartesian and summarization abstractions commute)

Abstract semantics: in $\mathcal{E}^\sharp \stackrel{\text{def}}{=} \mathbb{V}^\sharp \to \mathbb{I}$

- $\mathsf{E}^\sharp[\![\, V \,]\!]\, X^\sharp \stackrel{\text{def}}{=} X^\sharp(V)$
  $\mathsf{E}^\sharp[\![\, A[e] \,]\!]\, X^\sharp \stackrel{\text{def}}{=} X^\sharp(A)$ (e is ignored)
- $\mathsf{S}^\sharp[\![\, V \leftarrow e \,]\!]\, X^\sharp \stackrel{\text{def}}{=} X^\sharp[V \mapsto \mathsf{E}^\sharp[\![\, e \,]\!]\, X^\sharp]$
  $\mathsf{S}^\sharp[\![\, A[f] \leftarrow e \,]\!]\, X^\sharp \stackrel{\text{def}}{=} X^\sharp[A \mapsto X^\sharp(A) \cup^\sharp \mathsf{E}^\sharp[\![\, e \,]\!]\, X^\sharp]$
  f is ignored, we perform a weak update that accumulates values
- assuming $X^\sharp(V) = X^\sharp(A) = [a, b]$:
  $\mathsf{S}^\sharp[\![\, V \leq c \,]\!]\, X^\sharp \stackrel{\text{def}}{=} X^\sharp[V \mapsto [a, \min(b, c)]]$ if $a \leq c$, $\perp$ otherwise
  $\mathsf{S}^\sharp[\![\, A[e] \leq c \,]\!]\, X^\sharp \stackrel{\text{def}}{=} X^\sharp$ if $a \leq c$, $\perp$ otherwise
  we test for satisfiability but do not refine $X^\sharp(A)$; the case $A[e] \leq A[f]$ is similar
- other operations are unchanged, including $\cap^\sharp$, $\cup^\sharp$, ...

Interval analysis example

Example: increasing subsequence

$p[0] \leftarrow 0; B[0] \leftarrow A[0];$
$i \leftarrow 1; k \leftarrow 1;$
while $i < N$ **do**
  if $A[i] > B[k-1]$ **then**
    $B[k] \leftarrow A[i];$
    $p[k] \leftarrow i;$
    $k \leftarrow k + 1;$
  $i \leftarrow i + 1$

Analysis result:

Assuming that $N \in [N_\ell, N_h]$, $\forall x$: $A[x] \in [A_\ell, A_h]$, we get:

- $\forall x$: $p[x] \in [0, N_h - 1]$
- $\forall x$: $B[x] \in [\min(0, A_\ell), \max(0, A_h)]$

Relational abstractions

Variable addition and removal

Concrete semantics:

The set $\mathbb{V}$ of variables is not always fixed during program execution:
e.g., local variables

now $\mathcal{E} \stackrel{\text{def}}{=} \bigcup_{\mathbb{V} \text{ finite}} \mathbb{V} \to \mathbb{Z}$

- $\mathsf{S}[\![\, \textbf{add } V \,]\!]\, R \stackrel{\text{def}}{=} \{\, \rho[V \mapsto v] \,|\, \rho \in R,\ v \in \mathbb{Z} \,\}$
  add an uninitialized variable
- $\mathsf{S}[\![\, \textbf{del } V \,]\!]\, R \stackrel{\text{def}}{=} \{\, \rho_{|_{\mathrm{dom}(\rho) \setminus \{V\}}} \,|\, \rho \in R \,\}$
  remove a variable

Abstract semantics:

$\mathcal{E}^\sharp \stackrel{\text{def}}{=} \bigcup_{\mathbb{V} \text{ finite}} \mathcal{E}^\sharp_{\mathbb{V}}$
one abstract $|\mathbb{V}|$-dimensional abstract domain for each $\mathbb{V}$, e.g.: $\mathcal{E}^\sharp_{\mathbb{V}}$ = polyhedra of $\mathbb{R}^{|\mathbb{V}|}$

Example, in the interval domain:

- $\mathsf{S}^\sharp[\![\, \textbf{add } V \,]\!]\, X^\sharp \stackrel{\text{def}}{=} X^\sharp[V \mapsto [-\infty, +\infty]]$
- $\mathsf{S}^\sharp[\![\, \textbf{del } V \,]\!]\, X^\sharp \stackrel{\text{def}}{=} X^\sharp_{|_{\mathrm{dom}(X^\sharp) \setminus \{V\}}}$

Variable duplication and fold

Expanding and folding: model dynamic summarization

$$\mathsf{S}[\![\, \textbf{expand } V \to V' \,]\!]\, R \stackrel{\text{def}}{=} \{\, \rho[V' \mapsto v] \mid \rho \in R \wedge \rho[V \mapsto v] \in R \,\}$$

$$\mathsf{S}[\![\, \textbf{fold } V \hookleftarrow V' \,]\!]\, R \stackrel{\text{def}}{=} \{\, \rho \mid \exists v\colon \rho[V' \mapsto v] \in R \vee \rho[V' \mapsto \rho(V), V \mapsto v] \in R \,\}$$

- expand duplicates a variable and its constraints
  ($1 \leq V \leq X \Longrightarrow 1 \leq V \leq X \wedge 1 \leq V' \leq X$; but $V = V'$ does not hold!)
- fold summarizes V and V' into V
  ($1 \leq V \leq X \wedge 2 \leq V' \leq Y \Longrightarrow 1 \leq V \leq X \vee 2 \leq V \leq Y$)
- fold is an abstraction, expand is its associated concretization:

$$\mathcal{P}(\mathbb{V} \to \mathbb{Z}) \underset{\mathsf{S}[\![\, \textbf{fold } V \hookleftarrow V' \,]\!]}{\overset{\mathsf{S}[\![\, \textbf{expand } V \to V' \,]\!]}{\leftrightarrows}} \mathcal{P}((\mathbb{V} \setminus \{V'\}) \to \mathbb{Z})$$

  we have a Galois insertion

Relational expand and join

Polyhedral abstraction:

- expand can be exactly modeled by copying constraints:

  $S^\sharp[\![\, \textbf{expand } V_a \to V_b \,]\!] \{ \sum_i \alpha_{ij} V_i \geq \beta_j \} \stackrel{\text{def}}{=}$

  $\{ \sum_i \alpha_{ij} V_i \geq \beta_j \} \cup \{ \sum_{i \neq a} \alpha_{ij} V_i + \alpha_{aj} V_b \geq \beta_j \}$

- join can be approximated using a weak copy:

  $S^\sharp[\![\, \textbf{fold } V \hookleftarrow V' \,]\!] X^\sharp \stackrel{\text{def}}{=} S^\sharp[\![\, \textbf{del } V' \,]\!] (X^\sharp \cup^\sharp S^\sharp[\![\, V \leftarrow V' \,]\!] X^\sharp)$

  (assignment that keeps new and old values, instead of replacing old by new)

  example: $0 \leq V \leq 3 \wedge 10 \leq V' \leq 13 \Longrightarrow 0 \leq V \leq 13$
  which over-approximates $0 \leq V \leq 3 \vee 10 \leq V \leq 13$

- $S^\sharp[\![\, \textbf{add } V \,]\!]$ keeps the constraint set unchanged

- $S^\sharp[\![\, \textbf{del } V \,]\!]$ projects out V

Relational array abstraction

Goal: abstract $\mathcal{P}(\mathcal{E})$ using polyhedra over $\mathbb{V}^\sharp \stackrel{\text{def}}{=} \mathbb{V} \cup \mathbb{A}$

Principle: use temporary variables, join and expand

Abstract assignment: $\mathsf{S}^\sharp[\![\, A[f] \leftarrow e \,]\!]\, X^\sharp$

- replace each array expression $A[expr]$ in e with a fresh copy of A
  we get a new expression e' and environment $X_1^\sharp$
  e.g., replace $B[expr]$ in $X^\sharp$, with B' in $X_1^\sharp \stackrel{\text{def}}{=} \mathsf{S}^\sharp[\![\, \textbf{expand } B \to B' \,]\!]\, X^\sharp$
- create a new copy A' of A to hold the result
  $X_2^\sharp \stackrel{\text{def}}{=} \mathsf{S}^\sharp[\![\, \textbf{expand } A \to A' \,]\!]\, X_1^\sharp$
- assign e' into A'
  $X_3^\sharp \stackrel{\text{def}}{=} \mathsf{S}^\sharp[\![\, A' \leftarrow e' \,]\!]\, X_2^\sharp$
- fold A' back into A
  $X_4^\sharp \stackrel{\text{def}}{=} \mathsf{S}^\sharp[\![\, \textbf{fold } A \hookleftarrow A' \,]\!]\, X_3^\sharp$
- remove all fresh copies of arrays:
  $\mathsf{S}^\sharp[\![\, \textbf{del } B' \,]\!]\, X_4^\sharp$

The cases for $\mathsf{S}^\sharp[\![\, V \leftarrow e \,]\!]$ and $\mathsf{S}^\sharp[\![\, c? \,]\!]$ are similar, and a bit simpler

Polyhedral analysis example

Example: increasing subsequence

$p[0] \leftarrow 0; B[0] \leftarrow A[0];$
$i \leftarrow 1; k \leftarrow 1;$
while $i < N$ **do**
    if $A[i] > B[k-1]$ **then**
        $B[k] \leftarrow A[i];$
        $p[k] \leftarrow i;$
        $k \leftarrow k+1;$
    $i \leftarrow i+1$

Analysis result:

Assuming that $\forall x\colon A[x] \in [A_\ell, A_h]$, we get:

- $\forall x\colon 0 \leq p[x] < N$
  which is stronger than $\forall k\colon 0 \leq p[k] < N_h$
- $\forall x\colon B[x] \in [\min(0, A_\ell), \max(0, A_h)]$
  $B \leq A$ would mean $\forall i, j\colon B[i] \leq A[j]$, which does not hold

Non-uniform abstractions

Beyond uniform abstractions

The summarization $\alpha_s : \mathcal{P}(\mathcal{E}) \to \mathcal{P}(\mathbb{V}^\sharp \to \mathbb{Z})$ is uniform:
it forgets relations between array element indices and element values

Non-uniform abstraction example: array segmentation

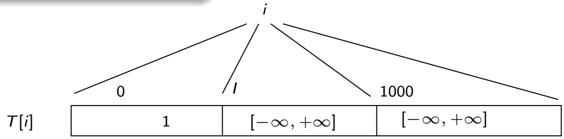
Initialization loop

$I \leftarrow 0$;
while • $I < 1000$ **do**
  $T[I] \leftarrow 1$;
  $I \leftarrow I + 1$

we wish to analyze the loop without unrolling

at • we need to express the loop invariant:
$\forall i < I \colon T[i] = 1$

$\Longrightarrow$ at loop exit, T is initialized until 1000



abstract domain: partition the array contents into uniform segments
segments have constant or symbolic bounds (0, I, 1000,. . .)
segments have a contents in an abstract domain (intervals,. . .)