Abstract Interpretation III

Semantics and Application to Program Verification

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Overview

- Last week: non-relational abstract domains (intervals)

  abstract each variable independently from the others
  can express important properties (e.g., absence of overflow)
  unable to represent relations between variables

- This week: **relational abstract domains**

  more precise, but more costly

  - the need for relational domains
  - linear equality domain ($\sum_i \alpha_i V_i = \beta_i$)
  - polyhedra domain ($\sum_i \alpha_i V_i \geq \beta_i$)
  - practical exercises: relational analysis with the Apron library

- Next week: selected advanced topics on abstract domains

Motivation

Relational assignments and tests

Example

$X \leftarrow \textbf{rand}(0,10);\; Y \leftarrow \textbf{rand}(0,10);$
$\textbf{if } X \geq Y \textbf{ then } X \leftarrow Y \textbf{ else skip};$
$D \leftarrow Y - X;$
$\textbf{assert } D \geq 0$

Interval analysis:

- $S^\sharp[\![\, X \geq Y ?\,]\!]$ is abstracted as the identity

  given $R^\sharp \stackrel{\text{def}}{=} [X \mapsto [0,10], Y \mapsto [0,10]]$

  $S^\sharp[\![\, \textbf{if } X \geq Y \textbf{ then } \cdots]\!]\, R^\sharp = R^\sharp$

- $D \leftarrow Y - X$ gives $D \in [0,10] -^\sharp [0,10] = [-10,10]$
- the assertion $D \geq 0$ fails

Relational assignments and tests

Example

$X \leftarrow \textbf{rand}(0,10);\ \ Y \leftarrow \textbf{rand}(0,10);$
$\textbf{if } X \geq Y \textbf{ then } X \leftarrow Y \textbf{ else skip};$
$D \leftarrow Y - X;$
$\textbf{assert } D \geq 0$

Solution: relational domain

- represent explicitly the information $X \leq Y$
- infer that $X \leq Y$ holds after the **if** $\cdots$ **then** $\cdots$ **else** $\cdots$
  $X \leq Y$ both after $X \leftarrow Y$ when $X \geq Y$, and after **skip** when $X \leq Y$
- use $X \leq Y$ to deduce that $Y - X \in [0, 10]$

Note:
the invariant we seek, $D \geq 0$, can be exactly represented in the interval domain
but inferring $D \geq 0$ requires a more expressive domain locally

Relational loop invariants

Example

$I \leftarrow 1;\ X \leftarrow 0;$
while $I \leq 1000$ **do**
    $I \leftarrow I + 1;\ X \leftarrow X + 1;$
assert $X \leq 1000$

Interval analysis:

- after iterations with widening, we get in 2 iterations:
  as loop invariant: $I \in [1, +\infty]$ and $X \in [0, +\infty]$
  after the loop: $I \in [1001, +\infty]$ and $X \in [0, +\infty] \implies$ **assert** fails
- using a decreasing iteration after widening, we get:
  as loop invariant: $I \in [1, 1001]$ and $X \in [0, +\infty]$
  after the loop: $I = 1001$ and $X \in [0, +\infty] \implies$ **assert** fails
  (the test $I \leq 1000$ only refines I, but gives no information on X)
- without widening, we get $I = 1001$ and $X = 1000 \implies$ **assert** passes
  but we need 1000 iterations! (≃ concrete fixpoint computation)

Relational loop invariants

Example

$I \leftarrow 1;\ X \leftarrow 0;$
while $I \leq 1000$ **do**
    $I \leftarrow I + 1;\ X \leftarrow X + 1;$
assert $X \leq 1000$

Solution: relational domain

- infer a relational loop invariant: $I = X + 1 \wedge 1 \leq I \leq 1001$

  $I = X + 1$ holds before entering the loop as $1 = 0 + 1$
  $I = X + 1$ is invariant by the loop body $I \leftarrow I + 1; X \leftarrow X + 1$

  (can be inferred in 2 iterations with widening in the polyhedra domain)

- propagate the loop exit condition $I > 1000$ to get:

  $I = 1001$
  $X = I - 1 = 1000 \implies$ **assert** passes

Note:
the invariant we seek after the loop exit has an interval form: $X \leq 1000$
but we need to infer a more expressive loop invariant to deduce it

Affine Equalities

The affine equality domain

We look for invariants of the form:

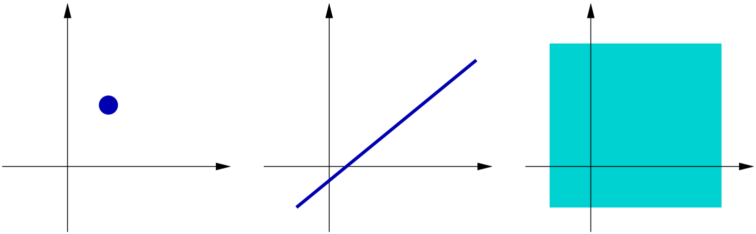
$$\wedge_j \left(\sum_{i=1}^{n} \alpha_{ij} V_i = \beta_j \right),\ \alpha_{ij}, \beta_j \in \mathbb{Q}$$

where all the α_{ij} and β_j are inferred automatically

We use a domain of affine spaces proposed by Karr in 1976

$$\mathcal{E}^\sharp \simeq \{ \text{affine subspaces of } \mathbb{V} \to \mathbb{R} \}$$

(with a suitable machine representation)



Affine equality representation

Machine representation:

$$\mathcal{E}^\sharp \stackrel{\text{def}}{=} \cup_m \{ \langle \mathbf{M}, \vec{C} \rangle \mid \mathbf{M} \in \mathbb{Q}^{m\times n}, \vec{C} \in \mathbb{Q}^m \} \cup \{\bot\}$$

- either the constant $\bot$
- or a pair $\langle \mathbf{M}, \vec{C} \rangle$ where
  - $\mathbf{M} \in \mathbb{Q}^{m\times n}$ is a $m \times n$ matrix, $n = |\mathbb{V}|$ and $m \leq n$,
  - $\vec{C} \in \mathbb{Q}^m$ is a row-vector with m rows

$\langle \mathbf{M}, \vec{C} \rangle$ represents an equation system, with solutions:

$$\gamma(\langle \mathbf{M}, \vec{C} \rangle) \stackrel{\text{def}}{=} \{ \vec{V} \in \mathbb{R}^n \mid \mathbf{M} \times \vec{V} = \vec{C} \}$$

M should be in row echelon form:

- $\forall i \leq m: \exists k_i: M_{ik_i} = 1$ and $\forall c < k_i: M_{ic} = 0$, $\forall l \neq i: M_{lk_i} = 0$,
- if $i < i'$ then $k_i < k_{i'}$ (leading index)

example:

$$\begin{bmatrix} 1 & 0 & 0 & 5 & 0 \\ 0 & 1 & 0 & 6 & 0 \\ 0 & 0 & 1 & 7 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Remarks:

the representation is unique

as $m \leq n = |\mathbb{V}|$, the memory cost is in $\mathcal{O}(n^2)$ at worst

$\top$ is represented as the empty equation system: $m = 0$

Galois connection

Galois connection: (actually, a Galois insertion)

between arbitrary subsets and affine subsets

$$(\mathcal{P}(\mathbb{R}^{|\mathbb{V}|}), \subseteq) \xleftarrow[\alpha]{\gamma} \twoheadrightarrow (\mathit{Aff}(\mathbb{R}^{|\mathbb{V}|}), \subseteq)$$

- $\gamma(X) \stackrel{\text{def}}{=} X$ (identity)
- $\alpha(X) \stackrel{\text{def}}{=}$ smallest affine subset containing X

  $\mathit{Aff}(\mathbb{R}^{|\mathbb{V}|})$ is closed under arbitrary intersections, so we have:

  $\alpha(X) = \cap \{ Y \in \mathit{Aff}(\mathbb{R}^{|\mathbb{V}|}) \mid X \subseteq Y \}$

  $\mathit{Aff}(\mathbb{R}^{|\mathbb{V}|})$ contains every point in $\mathbb{R}^{|\mathbb{V}|}$

  we can also construct $\alpha(X)$ by (abstract) union:

  $\alpha(X) = \cup^\sharp \{ \{x\} \mid x \in X \}$

Notes:

- we have assimilated $\mathbb{V} \to \mathbb{R}$ to $\mathbb{R}^{|\mathbb{V}|}$
- we have used $\mathit{Aff}(\mathbb{R}^{|\mathbb{V}|})$ instead of the matrix representation $\mathcal{E}^\sharp$ for simplicity; a Galois connection also exists between $\mathcal{P}(\mathbb{R}^{|\mathbb{V}|})$ and $\mathcal{E}^\sharp$

Normalisation and emptiness testing

Let $\mathbf{M} \times \vec{V} = \vec{C}$ be a system, not necessarily in normal form

The Gaussian reduction tells in $\mathcal{O}(n^3)$ time:

- whether the system is satisfiable, and in that case
- gives an equivalent system in normal form

i.e.: it returns an element in $\mathcal{E}^\sharp$

Example:

$$\left\{\begin{array}{rcrcrcr} 2X & + & Y & + & Z & = & 19 \\ 2X & + & Y & - & Z & = & 9 \\ & & & & 3Z & = & 15 \end{array}\right.$$

$$\Downarrow$$

$$\left\{\begin{array}{rcrcrcr} X & + & 0.5Y & & & = & 7 \\ & & & & Z & = & 5 \end{array}\right.$$

Normalisation and emptiness testing (cont.)

Gaussian reduction algorithm: $Gauss(\langle \mathbf{M}, \vec{C} \rangle)$

$r \leftarrow 0$ (rank r)
for c from 1 to n (column c)
    if $\exists \ell > r$: $M_{\ell c} \neq 0$ (pivot ℓ)
        $r \leftarrow r + 1$
        swap $\langle \vec{M}_\ell, C_\ell \rangle$ and $\langle \vec{M}_r, C_r \rangle$
        divide $\langle \vec{M}_r, C_r \rangle$ by M_{rc}
        for j from 1 to n, $j \neq r$
            replace $\langle \vec{M}_j, C_j \rangle$ with $\langle \vec{M}_j, C_j \rangle - M_{jc} \langle \vec{M}_r, C_r \rangle$

if $\exists \ell$: $\langle \vec{M}_\ell, C_\ell \rangle = \langle 0, \ldots, 0, c \rangle, c \neq 0$
    then return $\perp$

remove all rows $\langle \vec{M}_\ell, C_\ell \rangle$ that equal $\langle 0, \ldots, 0, 0 \rangle$

Affine equality operators

Abstract operators:

If $X^\sharp, Y^\sharp \neq \bot$, we define:

$$X^\sharp \cap^\sharp Y^\sharp \stackrel{\text{def}}{=} Gauss\left(\left\langle \begin{bmatrix} \mathbf{M}_{X^\sharp} \\ \mathbf{M}_{Y^\sharp} \end{bmatrix}, \begin{bmatrix} \vec{C}_{X^\sharp} \\ \vec{C}_{Y^\sharp} \end{bmatrix} \right\rangle\right) \qquad \textit{(join equations)}$$

$$X^\sharp =^\sharp Y^\sharp \stackrel{\text{def}}{\iff} \mathbf{M}_{X^\sharp} = \mathbf{M}_{Y^\sharp} \text{ and } \vec{C}_{X^\sharp} = \vec{C}_{Y^\sharp} \qquad \textit{(uniqueness)}$$

$$X^\sharp \subseteq^\sharp Y^\sharp \stackrel{\text{def}}{\iff} X^\sharp \cap^\sharp Y^\sharp =^\sharp X^\sharp$$

$$S^\sharp [\![\textstyle\sum_j \alpha_j V_j = \beta ?]\!] X^\sharp \stackrel{\text{def}}{=} Gauss\left(\left\langle \begin{bmatrix} \mathbf{M}_{X^\sharp} \\ \alpha_1 \cdots \alpha_n \end{bmatrix}, \begin{bmatrix} \vec{C}_{X^\sharp} \\ \beta \end{bmatrix} \right\rangle\right) \qquad \textit{(add equation)}$$

$$S^\sharp [\![e \bowtie e' ?]\!] X^\sharp \stackrel{\text{def}}{=} X^\sharp \qquad \text{for other tests}$$

Remark:

$\subseteq^\sharp$, $=^\sharp$, $\cap^\sharp$, $=^\sharp$ and $S^\sharp [\![\sum_j \alpha_j V_j - \beta = 0 ?]\!]$ are exact:

$(X^\sharp \subseteq^\sharp Y^\sharp \iff \gamma(X^\sharp) \subseteq \gamma(Y^\sharp), \quad \gamma(X^\sharp \cap^\sharp Y^\sharp) = \gamma(X^\sharp) \cap \gamma(Y^\sharp), \ldots)$

Affine equality assignment

Non-deterministic assignment: $S^\sharp[\![V_j \leftarrow [-\infty, +\infty]]\!]$

Principle: remove all the occurrences of V_j
but reduce the number of equations by only one
(add a single degree of freedom)

Algorithm: assuming V_j occurs in $\mathbf{M}$

- Pick the row $\langle \vec{M}_i, C_i \rangle$ such that $M_{ij} \neq 0$ and i maximal
- Use it to eliminate all the occurrences of V_j in lines before i

  (i maximal $\implies$ $\mathbf{M}$ stays in row echelon form)
- Remove the row $\langle \vec{M}_i, C_i \rangle$

Example: forgetting Z

$$\begin{cases} X \quad\; + Z = 10 \\ \quad\; Y + Z = 7 \end{cases} \implies \{\; X - Y = 3$$

The operator is exact

Affine equality assignment

Affine assignments: $S^\sharp[\![V_j \leftarrow \sum_i \alpha_i V_i + \beta]\!]$

$S^\sharp[\![V_j \leftarrow \sum_i \alpha_i V_i + \beta]\!] X^\sharp \stackrel{\text{def}}{=}$

if $\alpha_j = 0$, $(S^\sharp[\![V_j = \sum_i \alpha_i V_i + \beta?]\!] \circ S^\sharp[\![V_j \leftarrow [-\infty, +\infty]]\!])X^\sharp$

if $\alpha_j \neq 0$, $\langle \mathbf{M}, \vec{C} \rangle$ where V_j is replaced with $\frac{1}{\alpha_j}(V_j - \sum_{i \neq j} \alpha_i V_i - \beta)$

(variable substitution)

Proof sketch: based on properties in the concrete

non-invertible assignment: $\alpha_j = 0$

$S[\![V_j \leftarrow e]\!] = S[\![V_j \leftarrow e]\!] \circ S[\![V_j \leftarrow [-\infty, +\infty]]\!]$ as the value of V is not used in e

so $S[\![V_j \leftarrow e]\!] = S[\![V_j = e?]\!] \circ S[\![V_j \leftarrow [-\infty, +\infty]]\!]$

invertible assignment: $\alpha_j \neq 0$

$S[\![V_j \leftarrow e]\!] \subsetneq S[\![V_j \leftarrow e]\!] \circ S[\![V_j \leftarrow [-\infty, +\infty]]\!]$ as e depends on V

$$\begin{aligned}
\rho \in S[\![V_j \leftarrow e]\!] R &\iff \exists \rho' \in R: \rho = \rho'[V_j \mapsto \textstyle\sum_i \alpha_i \rho'(V_i) + \beta] \\
&\iff \exists \rho' \in R: \rho[V_j \mapsto (\rho(V_j) - \textstyle\sum_{i \neq j} \alpha_i \rho'(V_i) - \beta)/\alpha_j] = \rho' \\
&\iff \rho[V_j \mapsto (\rho(V_j) - \textstyle\sum_{i \neq j} \alpha_i \rho(V_i) - \beta)/\alpha_j] \in R
\end{aligned}$$

Non-affine assignments: revert to non-deterministic case

$S^\sharp[\![V_j \leftarrow e]\!] X^\sharp \stackrel{\text{def}}{=} S^\sharp[\![V_j \leftarrow [-\infty, +\infty]]\!] X^\sharp$ (imprecise but sound)

Affine equality join

Join: $\langle \mathbf{M}, \vec{C} \rangle \cup^\sharp \langle \mathbf{N}, \vec{D} \rangle$

Idea: unify columns 1 to n of $\langle \mathbf{M}, \vec{C} \rangle$ and $\langle \mathbf{N}, \vec{D} \rangle$ using row operations

Example:

Assume that we have unified columns 1 to k to get $\begin{pmatrix} \mathbf{R} \\ \mathbf{0} \end{pmatrix}$, arguments are in row echelon form, and we have to unify at column $k+1$: ${}^t(\vec{0}\ 1\ \vec{0})$ with ${}^t(\vec{\beta}\ 0\ \vec{0})$

$$\begin{pmatrix} \mathbf{R} & \vec{0} & \mathbf{M_1} \\ \vec{0} & 1 & \vec{M_2} \\ \mathbf{0} & \vec{0} & \mathbf{M_3} \end{pmatrix}, \begin{pmatrix} \mathbf{R} & \vec{\beta} & \mathbf{N_1} \\ \vec{0} & 0 & \vec{N_2} \\ \mathbf{0} & \vec{0} & \mathbf{N_3} \end{pmatrix} \implies \begin{pmatrix} \mathbf{R} & \vec{\beta} & \mathbf{M_1'} \\ \vec{0} & 0 & \vec{0} \\ \mathbf{0} & \vec{0} & \mathbf{M_3} \end{pmatrix}, \begin{pmatrix} \mathbf{R} & \vec{\beta} & \mathbf{N_1} \\ \vec{0} & 0 & \vec{N_2} \\ \mathbf{0} & \vec{0} & \mathbf{N_3} \end{pmatrix}$$

Use the row $(\vec{0}\ 1\ \vec{M_2})$ to create $\vec{\beta}$ in the left argument
Then remove the row $(\vec{0}\ 1\ \vec{M_2})$
The right argument is unchanged
$\implies$ we have now unified columns 1 to $k+1$

Unifying ${}^t(\vec{\alpha}\ 0\ \vec{0})$ and ${}^t(\vec{0}\ 1\ \vec{0})$ is similar
Unifying ${}^t(\vec{\alpha}\ 0\ \vec{0})$ and ${}^t(\vec{\beta}\ 0\ \vec{0})$ is a bit more complicated...
No other case possible as we are in row echelon form

Analysis example

No infinite increasing chain: we can iterate without widening!

Example

$X \leftarrow 10;\ Y \leftarrow 100;$
while $X \neq 0$ **do**
  $X \leftarrow X - 1;$
  $Y \leftarrow Y + 10$

Abstract loop iterations: $\lim\ \lambda X^\sharp . I^\sharp \cup^\sharp \mathsf{S}^\sharp[\![\, body \,]\!]\,(\mathsf{S}^\sharp[\![\, X \neq 0? \,]\!]\, X^\sharp)$

- loop entry: $I^\sharp = (X = 10 \wedge Y = 100)$
- after one loop body iteration: $F^\sharp(I^\sharp) = (X = 9 \wedge Y = 110)$
- $\implies X^\sharp \stackrel{\text{def}}{=} I^\sharp \cup^\sharp F^\sharp(I^\sharp) = (10X + Y = 200)$
- $X^\sharp$ is stable

at loop exit, we get $\mathsf{S}^\sharp[\![\, X = 0? \,]\!]\,(10X + Y = 200) = (X = 0 \wedge Y = 200)$

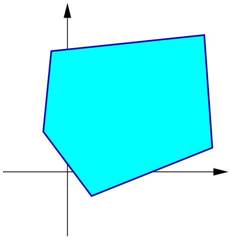
Polyhedra

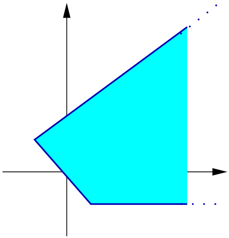
The polyhedron domain

We look for invariants of the form: $\wedge_j \left(\sum_{i=1}^{n} \alpha_{ij} V_i \geq \beta_j\right)$

We use the polyhedron domain by Cousot and Halbwachs (1978)

$\mathcal{E}^\sharp \simeq \{$ closed convex polyhedra of $\mathbb{V} \to \mathbb{R} \}$





Note: polyhedra need not be bounded ($\neq$ polytopes)

Double description of polyhedra

Polyhedra have dual representations (Weyl–Minkowski Theorem)

Constraint representation

$\langle \mathbf{M}, \vec{C} \rangle$ with $\mathbf{M} \in \mathbb{Q}^{m \times n}$ and $\vec{C} \in \mathbb{Q}^m$

represents: $\gamma(\langle \mathbf{M}, \vec{C} \rangle) \stackrel{\text{def}}{=} \{ \vec{V} \mid \mathbf{M} \times \vec{V} \geq \vec{C} \}$

We will also often use a constraint set notation $\{ \sum_i \alpha_{ij} V_i \geq \beta_j \}$

Generator representation

$[\mathbf{P}, \mathbf{R}]$ where

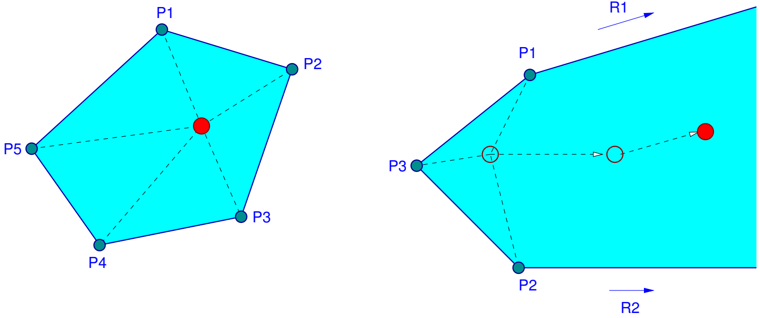
- $\mathbf{P} \in \mathbb{Q}^{n \times p}$ is a set of p points: $\vec{P}_1, \ldots, \vec{P}_p$
- $\mathbf{R} \in \mathbb{Q}^{n \times r}$ is a set of r rays: $\vec{R}_1, \ldots, \vec{R}_r$

$\gamma([\mathbf{P}, \mathbf{R}]) \stackrel{\text{def}}{=} \{ (\sum_{j=1}^{p} \alpha_j \vec{P}_j) + (\sum_{j=1}^{r} \beta_j \vec{R}_j) \mid \forall j, \alpha_j, \beta_j \geq 0\colon \sum_{j=1}^{p} \alpha_j = 1 \}$

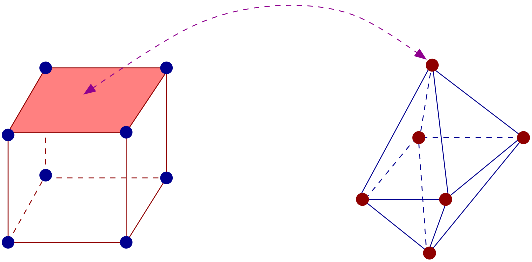
Double description of polyhedra (cont.)

Generator representation examples:

$$\gamma([\mathbf{P},\mathbf{R}]) \stackrel{\text{def}}{=} \{ (\sum_{j=1}^{p} \alpha_j \vec{P}_j) + (\sum_{j=1}^{r} \beta_j \vec{R}_j) \mid \forall j, \alpha_j, \beta_j \geq 0 : \sum_{j=1}^{p} \alpha_j = 1 \}$$



Duality in polyhedra



P $\qquad\qquad$ P^*

$0x + 0y + 1z \leq 1 \quad \Longleftrightarrow \quad (0, 0, 1)$

Duality: P^* is the dual of P, so that:

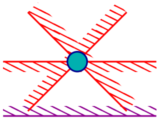
- the generators of P^* are the constraints of P
- the constraints of P^* are the generators of P
- $P^{**} = P$

Polyhedra representations

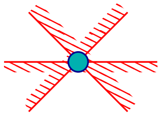
Minimal representations

- A constraint / generator system is minimal if no constraint / generator can be omitted without changing the concretization
- Minimal representations are not unique

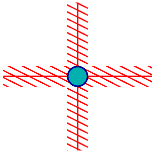
Example: three different constraint representations for a point



(a)



(b)

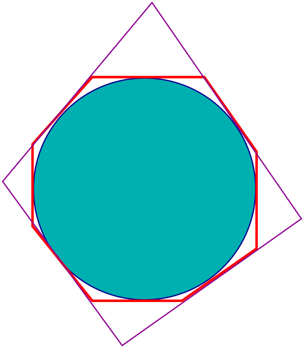


(c)

- (a) $y + x \geq 0, y - x \geq 0, y \leq 0, y \geq -5$ (non mimimal)
- (b) $y + x \geq 0, y - x \geq 0, y \leq 0$ (minimal)
- (c) $x \leq 0, x \geq 0, y \leq 0, y \geq 0$ (minimal)

Polyhedra representations (cont.)

- No bound on the size of representations (even minimal ones)
- No best abstraction α



Example: a disc has infinitely many polyhedral over-approximations, but no best one

Chernikova's algorithm

Algorithm by Chernikova (1968), improved by LeVerge (1992) to switch from a constraint system to an equivalent generator system

Motivation: most operators are easier on one representation

- By duality, we can use the same algorithm to switch from generators to constraints
- The minimal generator system can be exponential in the original constraint system (e.g., hypercube: $2n$ constraints, 2^n vertices)
- Equality constraints and lines (pairs of opposed rays) may be handled separately and more efficiently
- Chernikova's algorithm minimizes the representation on-the-fly (not presented here)

Algorithm: incrementally add constraints one by one

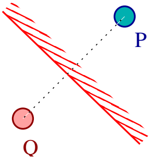
Start with: $\begin{cases} \mathbf{P}_0 = \{ (0, \ldots, 0) \} & \text{(origin)} \\ \mathbf{R}_0 = \{ \vec{x}_i,\ -\vec{x}_i \mid 1 \leq i \leq n \} & \text{(axes)} \end{cases}$

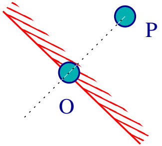
Chernikova's algorithm (cont.)

Update $[\mathbf{P}_{k-1}, \mathbf{R}_{k-1}]$ to $[\mathbf{P}_k, \mathbf{R}_k]$
by adding one constraint $\vec{M}_k \cdot \vec{V} \geq C_k \in \langle \mathbf{M}, \vec{C} \rangle$:
start with $\mathbf{P}_k = \mathbf{R}_k = \emptyset$,

- for any $\vec{P} \in \mathbf{P}_{k-1}$ s.t. $\vec{M}_k \cdot \vec{P} \geq C_k$, add $\vec{P}$ to $\mathbf{P}_k$
- for any $\vec{R} \in \mathbf{R}_{k-1}$ s.t. $\vec{M}_k \cdot \vec{R} \geq 0$, add $\vec{R}$ to $\mathbf{R}_k$

- for any $\vec{P}, \vec{Q} \in \mathbf{P}_{k-1}$ s.t. $\vec{M}_k \cdot \vec{P} > C_k$ and $\vec{M}_k \cdot \vec{Q} < C_k$, add to $\mathbf{P}_k$:
  $\vec{O} \stackrel{\text{def}}{=} \frac{C_k - \vec{M}_k \cdot \vec{Q}}{\vec{M}_k \cdot \vec{P} - \vec{M}_k \cdot \vec{Q}} \vec{P} - \frac{C_k - \vec{M}_k \cdot \vec{P}}{\vec{M}_k \cdot \vec{P} - \vec{M}_k \cdot \vec{Q}} \vec{Q}$





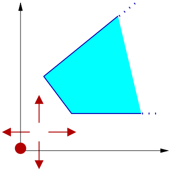
Chernikova's algorithm (cont.)

- for any $\vec{R}, \vec{S} \in \mathbf{R}_{k-1}$ s.t. $\vec{M}_k \cdot \vec{R} > 0$ and $\vec{M}_k \cdot \vec{S} < 0$, add to $\mathbf{R}_k$:
  $\vec{O} \stackrel{\text{def}}{=} (\vec{M}_k \cdot \vec{S})\vec{R} - (\vec{M}_k \cdot \vec{R})\vec{S}$

- for any $\vec{P} \in \mathbf{P}_{k-1}$, $\vec{R} \in \mathbf{R}_{k-1}$ s.t. either $\vec{M}_k \cdot \vec{P} > C_k$ and $\vec{M}_k \cdot \vec{R} < 0$, or $\vec{M}_k \cdot \vec{P} < C_k$ and $\vec{M}_k \cdot \vec{R} > 0$, add to $\mathbf{P}_k$:
  $\vec{O} \stackrel{\text{def}}{=} \vec{P} + \frac{C_k - \vec{M}_k \cdot \vec{P}}{\vec{M}_k \cdot \vec{R}} \vec{R}$

Chernikova's algorithm example

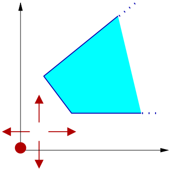
Example:

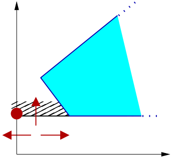


$\mathbf{P}_0 = \{(0,0)\}$ $\mathbf{R}_0 = \{(1,0),\ (-1,0),\ (0,1),\ (0,-1)\}$

Chernikova's algorithm example

Example:


(0)


(1)

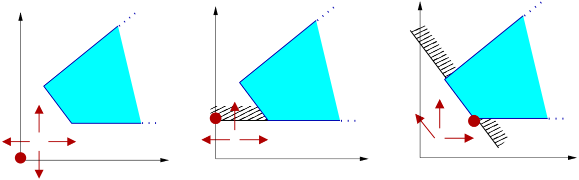
$Y \geq 1$

$\mathbf{P}_0 = \{(0,0)\}$
$\mathbf{P}_1 = \{(0,1)\}$

$\mathbf{R}_0 = \{(1,0),\ (-1,0),\ (0,1),\ (0,-1)\}$
$\mathbf{R}_1 = \{(1,0),\ (-1,0),\ (0,1)\}$

Chernikova's algorithm example

Example:

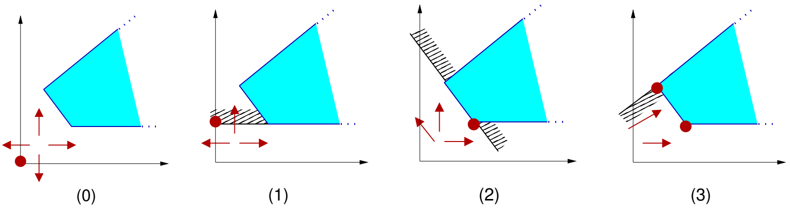


(0) (1) (2)

$$
\begin{array}{lll}
 & \mathbf{P}_0 = \{(0,0)\} & \mathbf{R}_0 = \{(1,0),\ (-1,0),\ (0,1),\ (0,-1)\} \\
Y \geq 1 & \mathbf{P}_1 = \{(0,1)\} & \mathbf{R}_1 = \{(1,0),\ (-1,0),\ (0,1)\} \\
X + Y \geq 3 & \mathbf{P}_2 = \{(2,1)\} & \mathbf{R}_2 = \{(1,0),\ (-1,1),\ (0,1)\}
\end{array}
$$

Chernikova's algorithm example

Example:



	$\mathbf{P}_0 = \{(0,0)\}$	$\mathbf{R}_0 = \{(1,0),\ (-1,0),\ (0,1),\ (0,-1)\}$
$Y \geq 1$	$\mathbf{P}_1 = \{(0,1)\}$	$\mathbf{R}_1 = \{(1,0),\ (-1,0),\ (0,1)\}$
$X + Y \geq 3$	$\mathbf{P}_2 = \{(2,1)\}$	$\mathbf{R}_2 = \{(1,0),\ (-1,1),\ (0,1)\}$
$X - Y \leq 1$	$\mathbf{P}_3 = \{(2,1),\ (1,2)\}$	$\mathbf{R}_3 = \{(0,1),\ (1,1)\}$

Operators on polyhedra

Abstract operators:

Given $X^\sharp, Y^\sharp \neq \bot$, we define:

$$X^\sharp \subseteq^\sharp Y^\sharp \quad \stackrel{\text{def}}{\iff} \quad \begin{cases} \forall \vec{P} \in \mathbf{P}_{X^\sharp} : \mathbf{M}_{Y^\sharp} \times \vec{P} \geq \vec{C}_{Y^\sharp} \\ \forall \vec{R} \in \mathbf{R}_{X^\sharp} : \mathbf{M}_{Y^\sharp} \times \vec{R} \geq \vec{0} \end{cases}$$

$$X^\sharp =^\sharp Y^\sharp \quad \stackrel{\text{def}}{\iff} \quad X^\sharp \subseteq^\sharp Y^\sharp \text{ and } Y^\sharp \subseteq^\sharp X^\sharp$$

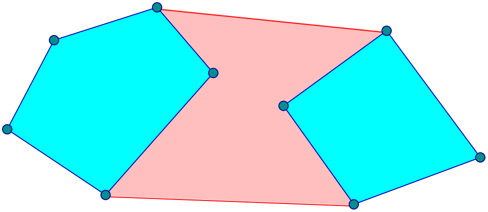
$$X^\sharp \cap^\sharp Y^\sharp \quad \stackrel{\text{def}}{=} \quad \left\langle \begin{bmatrix} \mathbf{M}_{X^\sharp} \\ \mathbf{M}_{Y^\sharp} \end{bmatrix}, \begin{bmatrix} \vec{C}_{X^\sharp} \\ \vec{C}_{Y^\sharp} \end{bmatrix} \right\rangle$$ *(join constraint sets)*

$\subseteq^\sharp$, $=^\sharp$ and $\cap^\sharp$ are exact (in $\mathcal{P}(\mathbb{V} \to \mathbb{R})$)

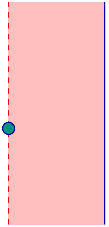
Operators on polyhedra (cont.)

Join: $X^\sharp \cup^\sharp Y^\sharp \stackrel{\text{def}}{=} [\,[\mathbf{P}_{X^\sharp}\ \mathbf{P}_{Y^\sharp}],\ [\mathbf{R}_{X^\sharp}\ \mathbf{R}_{Y^\sharp}]\,]$ *(join generator sets)*

Examples:



two polytopes



a point and a line

$\cup^\sharp$ is optimal (in $\mathcal{P}(\mathbb{V} \to \mathbb{R})$):
we get the topological closure of the convex hull of $\gamma(X^\sharp) \cup \gamma(Y^\sharp)$

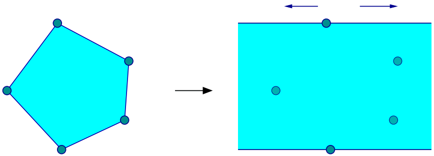
Operators on polyhedra (cont.)

Affine tests:

$$S^\sharp[\![\, \sum_i \alpha_i V_i \geq \beta ?\,]\!]\, X^\sharp \stackrel{\text{def}}{=} \left\langle \begin{bmatrix} \mathbf{M}_{X^\sharp} \\ \alpha_1 \cdots \alpha_n \end{bmatrix}, \begin{bmatrix} \vec{C}_{X^\sharp} \\ \beta \end{bmatrix} \right\rangle$$

Non-deterministic assignment:

$$S^\sharp[\![\, V_j \leftarrow [-\infty, +\infty]\,]\!]\, X^\sharp \stackrel{\text{def}}{=} [\, \mathbf{P}_{X^\sharp},\ [\, \mathbf{R}_{X^\sharp}\ \ \vec{x}_j\ \ (-\vec{x}_j)\,]\,]$$



- these operators are exact (in $\mathcal{P}(\mathbb{V} \to \mathbb{R})$)
- other tests can be abstracted as $S^\sharp[\![\, c?\,]\!]\, X^\sharp \stackrel{\text{def}}{=} X^\sharp$ (sound but not optimal)

Operators on polyhedra (cont.)

Affine assignment:

$\mathsf{S}^\sharp[\![\, V_j \leftarrow \sum_i \alpha_i V_i + \beta \,]\!]\, X^\sharp \stackrel{\text{def}}{=}$

if $\alpha_j = 0, (\mathsf{S}^\sharp[\![\, \sum_i \alpha_i V_i = V_j - \beta ? \,]\!] \circ \mathsf{S}^\sharp[\![\, V_j \leftarrow [-\infty, +\infty] \,]\!])X^\sharp$

if $\alpha_j \neq 0, \langle \mathbf{M}, \vec{C} \rangle$ where V_j is replaced with $\frac{1}{\alpha_j}(V_j - \sum_{i \neq j} \alpha_i V_i - \beta)$

- similar to the assignment in the equality domain
- the assignment is exact (in $\mathcal{P}(\mathbb{V} \to \mathbb{R})$)
- assignments can also be defined on the generator system
- for non-affine assignments: $\mathsf{S}^\sharp[\![\, V \leftarrow e \,]\!] \stackrel{\text{def}}{=} \mathsf{S}^\sharp[\![\, V \leftarrow [-\infty, +\infty] \,]\!]$
  (sound but not optimal)

Polyhedra widening

$\mathcal{E}^\sharp$ has strictly increasing infinite chains $\implies$ we need a widening

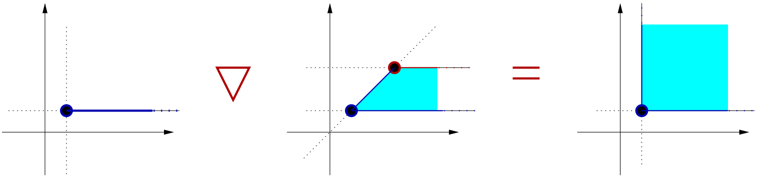
Definition:

Take $X^\sharp$ and $Y^\sharp$ in minimal constraint-set form

$$X^\sharp \triangledown Y^\sharp \stackrel{\text{def}}{=} \{\, c \in X^\sharp \mid Y^\sharp \subseteq^\sharp \{c\} \,\}$$

We suppress any unstable constraint $c \in X^\sharp$, i.e., $Y^\sharp \not\subseteq^\sharp \{c\}$

Example:



Polyhedra widening

$\mathcal{E}^\sharp$ has strictly increasing infinite chains $\implies$ we need a widening

Definition:

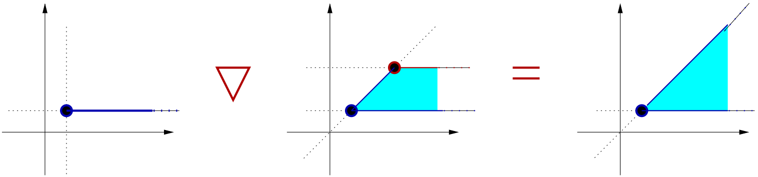
Take $X^\sharp$ and $Y^\sharp$ in minimal constraint-set form

$$X^\sharp \mathbin{\triangledown} Y^\sharp \stackrel{\text{def}}{=} \{\, c \in X^\sharp \mid Y^\sharp \subseteq^\sharp \{c\} \,\} \cup \{\, c \in Y^\sharp \mid \exists c' \in X^\sharp \colon X^\sharp =^\sharp (X^\sharp \setminus c') \cup \{c\} \,\}$$

We suppress any unstable constraint $c \in X^\sharp$, i.e., $Y^\sharp \not\subseteq^\sharp \{c\}$

We also keep constraints $c \in Y^\sharp$ equivalent to those in $X^\sharp$, i.e., when $\exists c' \in X^\sharp \colon X^\sharp =^\sharp (X^\sharp \setminus c') \cup \{c\}$

Example:



Example analysis

Example

$X \leftarrow 2; I \leftarrow 0;$
while $I < 10$ **do**
  if rand$(0,1) = 0$ **then** $X \leftarrow X + 2$ **else** $X \leftarrow X - 3;$
  $I \leftarrow I + 1$

Loop invariant:

increasing iterations with widening:

$$\begin{aligned} X_1^\sharp &= \{X = 2, I = 0\} \\ X_2^\sharp &= \{X = 2, I = 0\} \triangledown (\{X = 2, I = 0\} \cup^\sharp \{X \in [-1,4],\ I = 1\}) \\ &= \{X = 2, I = 0\} \triangledown \{\, I \in [0,1],\ 2 - 3I \leq X \leq 2I + 2 \,\} \\ &= \{I \geq 0,\ 2 - 3I \leq X \leq 2I + 2\} \end{aligned}$$

decreasing iteration: (recover $I \leq 10$)

$$\begin{aligned} X_3^\sharp &= \{X = 2, I = 0\} \cup^\sharp \{\, I \in [1,10],\ 2 - 3I \leq X \leq 2I + 2 \,\} \\ &= \{I \in [0,10],\ 2 - 3I \leq X \leq 2I + 2\} \end{aligned}$$

at the loop exit, we find eventually: $I = 10 \wedge X \in [-28, 22]$

Partial conclusion

Cost vs. precision:

Domain	Invariants	Memory cost	Time cost (per op.)
intervals	$V \in [\ell, h]$	$\mathcal{O}(\lvert\mathbb{V}\rvert)$	$\mathcal{O}(\lvert\mathbb{V}\rvert)$
affine equalities	$\sum_i \alpha_i V_i = \beta_i$	$\mathcal{O}(\lvert\mathbb{V}\rvert^2)$	$\mathcal{O}(\lvert\mathbb{V}\rvert^3)$
polyhedra	$\sum_i \alpha_i V_i \geq \beta_i$	unbounded, exponential in practice	

- domains provide a tradeoff between precision and cost
- relational invariants are sometimes necessary
  even to prove non-relational properties
- an abstract domain is defined by
  - a choice of abstract properties and operators (semantic aspect)
  - data-structures and algorithms (algorithmic aspect)
- an abstract domain mixes two kinds of approximations:
  - static approximations (choice of abstract properties)
  - dynamic approximations (widening)

Weakly relational domains

Principle: restrict the expressiveness of polyhedra
to be more efficient at the cost of precision

Example domains:

- Based on constraint propagation: (closure algorithms)
  - Octagons: $\pm X \pm Y \leq c$
    shortest path closure: $x + y \leq c \wedge -y + z \leq d \implies x + z \leq c + d$
    quadratic memory cost, cubic time cost
  - Two-variables per inequality: $\alpha x + \beta y \leq c$
    slightly more complex closure algorithm, by Nelson
  - Octahedra: $\sum \alpha_i V_i \leq c$, $\alpha_i \in \{ -1, 0, 1 \}$
    incomplete propagation, to avoid exponential cost
  - Pentagons: $X - Y \leq 0$
    restriction of octagons
    incomplete propagation, aims at linear cost
- Based on linear programming:
  - Template polyhedra: $\mathbf{M} \times \vec{V} \geq \vec{C}$ for a fixed $\mathbf{M}$

Integers

Issue:

in relational domains we used implicitly real-valued environments $\mathbb{V} \to \mathbb{R}$
our concrete semantics is based on integer-valued environments $\mathbb{V} \to \mathbb{Z}$

In fact, an abstract element $X^\sharp$ does not represent $\gamma(X^\sharp) \subseteq \mathbb{R}^{|\mathbb{V}|}$, but:

$\gamma_{\mathbb{Z}}(X^\sharp) \stackrel{\text{def}}{=} \gamma(X^\sharp) \cap \mathbb{Z}^{|\mathbb{V}|}$ (keep only integer points)

Soundness and exactness for $\gamma_{\mathbb{Z}}$

- $\subseteq^\sharp$ and $=^\sharp$ are is no longer exact
  e.g., $\gamma(2X = 1) \neq \gamma(\bot)$, but $\gamma_{\mathbb{Z}}(2X = 1) = \gamma(\bot) = \emptyset$
- $\cap^\sharp$ and affine tests are still exact
- affine and non-deterministic assignments are no longer exact
  e.g., $R^\sharp = (Y = 2X)$, $S^\sharp[\![X \leftarrow [-\infty, +\infty]]\!] R^\sharp = \top$,
  but $S[\![X \leftarrow [-\infty, +\infty]]\!] (\gamma_{\mathbb{Z}}(R^\sharp)) = \mathbb{Z} \times (2\mathbb{Z})$
- all the operators are **still sound**
  $\mathbb{Z}^{|\mathbb{V}|} \subseteq \mathbb{R}^{|\mathbb{V}|}$, so $\forall X^\sharp: \gamma_{\mathbb{Z}}(X^\sharp) \subseteq \gamma(X^\sharp)$

(in general, soundness, exactness, optimality depend on the definition of γ)

Integers (cont.)

Possible solutions:

- enrich the domain (add exact representations for operation results)
  - congruence equalities: $\wedge_i \sum_j \alpha_{ij} V_j \equiv \beta_i \; [\gamma_i]$ (Granger 1991)
  - Pressburger arithmetic (first order logic with 0, 1, $+$)
    decidable, but with very costly algorithms
- design optimal (non-exact) operators

  also based on costly algorithms, e.g.:
  - normalization: integer hull
    smallest polyhedra containing $\gamma_Z(X^\sharp)$
  - emptiness testing: integer programming
    NP-hard, while linear programming is P
- pragmatic solution (efficient, non-optimal)

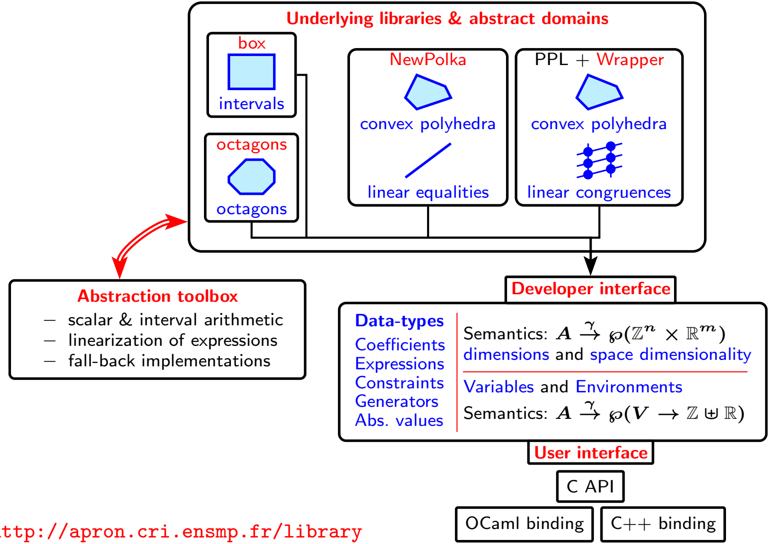
  use regular operators for $\mathbb{R}^{|\mathbb{V}|}$, then tighten each constraint to remove as many non-integer points as possible

  e.g.: $2X + 6Y \geq 3 \to X + 3Y \geq 2$

Note: we abstract integers as reals!

Using the Apron Library

Apron library



http://apron.cri.ensmp.fr/library

Apron modules

The `Apron` module contains sub-modules:

- `Abstract1`
  abstract elements
- `Manager`
  abstract domains (arguments to all `Abstract1` operations)
- `Polka`
  creates a manager for polyhedra abstract elements
- `Var`
  integer or real program variables (denoted as a string)
- `Environment`
  sets of integer and real program variables
- `Texpr1`
  arithmetic expression trees
- `Tcons1`
  arithmetic constraints (based on `Texpr1`)
- `Coeff`
  numeric coefficients (appear in `Texpr1`, `Tcons1`)

Variables and environments

Variables: `type Var.t`

variables are denoted by their name, as a string:

(assumes implicitly that no two program variables have the same name)

- `Var.of_string:  string -> Var.t`

Environments: `type Environment.t`

an abstract element abstracts a set of mappings in $\mathbb{V} \to \mathbb{R}$

$\mathbb{V}$ is the environment; it contains integer-valued and real-valued variables

- `Environment.make:  Var.t array -> Var.t array -> t`
  `make ivars rvars` creates an environment with `ivars` integer variables and `rvars` real variables;
  `make [||] [||]` is the empty environment
- `Environment.add:  Environment.t -> Var.t array -> Var.t array -> t`
  `add env ivars rvars` adds some integer or real variables to env
- `Environment.remove:  t -> Var.t array -> t`

internally, an abstract element abstracts a set of points in $\mathbb{R}^n$;
the environment maintains the mapping from variable names to dimensions in $[1, n]$

Expressions

Concrete expression trees: `type Texpr1.expr`

```
type expr =  | Cst of Coeff.t                                  (constants)
             | Var of Var.t                                    (variables)
             | Unop of unop * expr * typ * round               (unary op.)
             | Binop of binop * expr * expr * typ * round      (binary op.)
```

- unary operators

  `type Texpr1.unop = Neg | ···`

- binary operators

  `type Texpr1.binop = Add | Sub | Mul | Div | ···`

- numeric type:
  (we only use integers, but reals and floats are also possible)

  `type Texpr1.typ = Int | ···`

- rounding direction:
  (only useful for the division on integers; we use rounding to zero, i.e., truncation)

  `type Texpr1.round = Zero | ···`

Expressions (cont.)

Internal expression form: `type Texpr1.t`

concrete expression trees must be converted to an internal form to be used in abstract operations

- `Texpr1.of_expr: Environment.t -> Texpr1.expr -> Texpr1.t`
  (the environment is used to convert variable names to dimensions in $\mathbb{R}^n$)

Coefficients: `type Coeff.t`

can be either a scalar $\{c\}$ or an interval $[a, b]$

we can use the `Mpqf` module to convert from strings to arbitrary precision integers, before converting them into `Coeff.t`:

- for scalars $\{c\}$:
  ```
  Coeff.s_of_mpqf (Mpqf.of_string c)
  ```
- for intervals $[a, b]$:
  ```
  Coeff.i_of_mpqf (Mpqf.of_string a) (Mpqf.of_string b)
  ```

Constraints

Constraints: `type Tcons1.t`

constructor *expr* $\bowtie$ 0:

- `Tcons1.make: Texpr1.t -> TCons1.typ -> Tcons1.t`

  where:

  `type Tcons1.typ = SUPEQ | SUP | EQ | DISEQ | ...`

  $\geq$ (SUPEQ), $>$ (SUP), $=$ (EQ), $\neq$ (DISEQ)

Note: avoid using `DISEQ` directly, which is not very precise;
but use a disjunction of two `SUP` constraints instead

Constraint arrays: `type Tcons1.earray`

abstract operators do not use constraints, but constraint arrays instead

Example: constructing an array `ar` containing a single constraint:

```
let c = Tcons1.make texpr1 typ in
let ar = Tcons1.array_make env 1 in
Tcons1.array_set ar 0 c
```

Abstract operators

Abstract elements: type `Abstract1.t`

- `Abstract1.top: Manager.t -> Environment.t -> t`
  create an abstract element where variables have any value
- `Abstract1.env: t -> Environment.t`
  recover the environment on which the abstract element is defined
- `Abstract1.change_environment: Manager.t -> t -> Environment.t -> bool -> t`
  set the new environment, adding or removing variables if necessary
  the `bool` argument should be set to `false`: variables are not initialized
- `Abstract1.assign_texpr: Manager.t -> t -> Var.t -> Texpr1.t -> t option -> t`
  abstract assignment; the `option` argument should be set to `None`
- `Abstract1.forget_array: Manager.t -> t -> Var.t array -> bool -> t`
  non-deterministic assignment: forget the value of variables (when `bool` is `false`)
- `Abstract1.meet_tcons_array: Manager.t -> t -> Tcons1.earray -> t`
  abstract test: add one or several constraint(s)

Abstract operators (cont.)

- `Abstract1.join: Manager.t -> t -> t -> t`
  abstract union $\cup^\sharp$
- `Abstract1.meet: Manager.t -> t -> t -> t`
  abstract intersection $\cap^\sharp$
- `Abstract1.widen: Manager.t -> t -> t -> t`
  widening ∇
- `Abstract1.is_leq: Manager.t -> t -> t -> bool`
  $\subseteq^\sharp$: return `true` if the first argument is included in the second
- `Abstract1.is_bottom: Manager.t -> t -> t bool`
  whether the abstract element represents $\emptyset$
- `Abstract1.print: Format.formatter -> t -> unit`
  print the abstract element

Contract:

- operators return a new, immutable abstract element (functional style)
- operators return over-approximations
  (not always optimal; e.g.: for non-linear expressions)
- predicates return `true` (definitely true) or `false` (don't know)

Managers

Managers: `type Manager.t`

The manager denotes a choice of abstract domain
To use the polyhedra domain, construct the manager with:

- `let manager = Polka.manager_alloc_loose ()`

the same `manager` variable is passed to all `Abstract1` function

to choose another domain, you only need to change the line defining `manager`

Other libraries:

- `Polka.manager_alloc_equalities` (affine equalities)
- `Polka.manager_alloc_strict` ($\geq$ and $>$ affine inequalities over $\mathbb{R}$)
- `Box.manager_alloc` (intervals)
- `Oct.manager_alloc` (octagons)
- `Ppl.manager_alloc_grid` (affine congruences)
- `PolkaGrid.manager_alloc` (affine inequalities and congruences)

Errors

Argument compatibility: ensure that:

- the same manager is used when creating and using an abstract element

  the type system checks for the compatibility between `'a Manager.t` and `'a Abstract1.t`

- expressions and abstract elements have the same environment
- assigned variables exist in the environment of the abstract element
- both abstract elements of binary operators ($\cup$, $\cap$, $\triangledown$, $\subseteq$) are defined on the same environment

Failure to ensure this results in a `Manager.Error` exception

Abstract domain skeleton using Apron

```
open Apron

module RelationalDomain = (struct
  (* manager *)
  type man = Polka.loose Polka.t
  let manager = Polka.manager_alloc_loose ()

  (* abstract elements *)
  type t = man Abstract1.t

  (* utilities *)
  val expr_to_texpr:  expr -> Texpr1.expr

  (* implementation *)
  ...

end:  ENVIRONMENT_DOMAIN)
```

To compile: add to the `Makefile`:

```
OCAMLINC = ··· -I +zarith -I +apron -I +gmp
CMA = bigarray.cma gmp.cma apron.cma polkaMPQ.cma
```

Fall-back assignments and tests

```
let rec expr_to_texpr = function
| AST_binary (op, e1, e2) ->
  match op with
    | AST_PLUS -> Texpr1.Binop ···
    | ···
    | _ -> raise Top

let assign env var expr =
  try
    let e = expr_to_texpr expr in
    Abstract1.assign_texpr ···
  with Top -> Abstract1.forget_array ···

let compare abs e1 e2 =
  try
    ···
    Abstract1.meet_tcons_array ···
  with Top -> abs
```

Idea:

raise Top to abort a computation
catch it to fall-back to sound coarse assignments and tests