Abstract Interpretation II

Semantics and Application to Program Verification

Antoine Miné

École normale supérieure, Paris

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Interval lattice

Reminder: Intervals

Idea: abstract program states by the bounds of each variable

- non-relational abstraction

  (aka. attribute-independent, no relation between variables)

- sufficient to express freedom from overflow

  (e.g., computations in machine integers or floats, array accesses)

Intervals: abstraction of sets of integers $\mathcal{P}(\mathbb{Z})$

$$\mathbb{I} \stackrel{\text{def}}{=} \{\, [a,b] \mid a \in \mathbb{Z} \cup \{-\infty\}, b \in \mathbb{Z} \cup \{+\infty\}, a \leq b \,\} \cup \{\perp\}$$

- $-\infty$, $+\infty$ bounds are needed to abstract unbounded sets

  $[-\infty,+\infty] = \top$ represents $\mathbb{Z}$, $[0,+\infty]$ represents $\mathbb{N}$, etc.

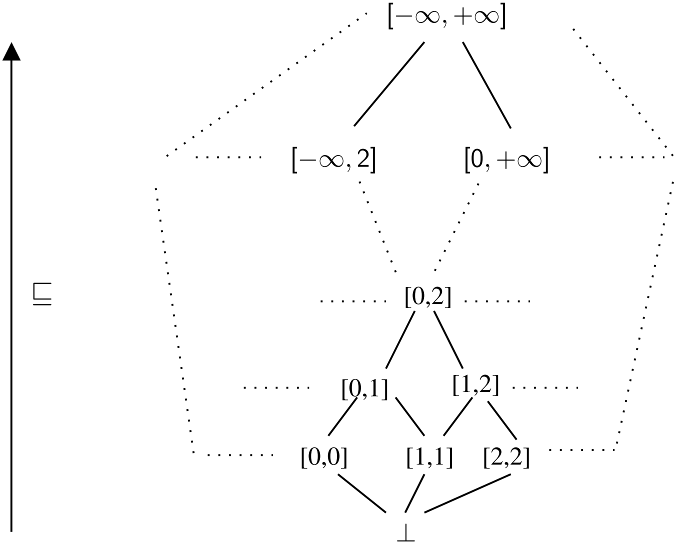
  $\Longrightarrow$ any integer set may be over-approximated in $\mathbb{I}$

  (we can always resort to $\top$)

- $\perp$ (uniquely) represents $\emptyset$

  (in $[a,b]$, we have $a \leq b$ so that non$-\perp$ intervals are never empty)

Interval lattice



Algebraic structure

partial order: $\sqsubseteq$

- $\forall I \in \mathbb{I}: \bot \sqsubseteq I$
- $[a, b] \sqsubseteq [c, d] \iff a \geq c \land b \leq d$

  (where $\leq$ is extended naturally to $\mathbb{Z} \cup \{-\infty, +\infty\}$ as: $\forall c \in \mathbb{Z}: -\infty < c < +\infty$)

lattice structure: $\sqcup, \sqcap$

- least upper bound $\sqcup$ for $\sqsubseteq$
  - $\forall I \in \mathbb{I}: \bot \sqcup I = I \sqcup \bot = I$
  - $[a, b] \sqcup [c, d] = [\min(a, c), \max(b, d)]$
- greatest lower bound $\sqcap$:
  - $\forall I: \bot \sqcap I = I \sqcap \bot = \bot$
  - $[a, b] \sqcap [c, d] =$

    $$\begin{cases} [\max(a, c), \min(b, d)] & \text{if } \max(a, c) \leq \min(b, d) \\ \bot & \text{if } \max(a, c) > \min(b, d) \end{cases}$$

Algebraic structure

Notes:

- the lattice is complete

  $\forall I \subseteq \mathbb{I}$: $\sqcup\ I$ and $\sqcap\ I$ exist

  $\sqcup \{ [a_j, b_j] \mid j \in J \} = [\min_{j\in J} a_j, \max_{j\in J} b_j]$

  $\sqcap \{ [a_j, b_j] \mid j \in J \} = [\max_{j\in J} a_j, \min_{j\in J} b_j]$ if max $\leq$ min, or $\perp$ otherwise

- intervals are closed by $\cap$

  $[a, b] \cap [c, d] = [a, b] \sqcap [c, d]$

$\Longrightarrow$ this will be useful to define best interval approximations

- intervals are not closed by $\cup$

  $[0, 0] \cup [2, 2] = \{0, 2\}$, which is not an interval; $[0, 0] \sqcup [2, 2] = [0, 2]$

- $\sqcup$ and $\sqcap$ are not distributive

  $([0, 0] \sqcup [2, 2]) \sqcap [1, 1] = [0, 2] \sqcap [1, 1] = [1, 1]$

  but $([0, 0] \sqcap [1, 1]) \sqcup ([2, 2] \sqcap [1, 1]) = \emptyset \sqcup \emptyset = \emptyset$

$\Longrightarrow$ this can be a cause of precision loss

Reminder: Interval Galois connection

Interval Galois connection: $(\mathcal{P}(\mathbb{Z}), \subseteq) \xleftarrow[\alpha]{\gamma} (\mathbb{I}, \sqsubseteq)$

- $\begin{cases} \gamma(\perp) \stackrel{\text{def}}{=} \emptyset \\ \gamma([a,b]) \stackrel{\text{def}}{=} \{\, x \in \mathbb{Z} \mid a \leq x \leq b \,\} \end{cases}$
- $\alpha(X) \stackrel{\text{def}}{=} \begin{cases} \perp & \text{if } X = \emptyset \\ [\min X, \max X] & \text{if } X \neq \emptyset \end{cases}$
- Galois connection definition: $\forall I, X \colon \alpha(X) \sqsubseteq I \iff X \subseteq \gamma(I)$
- main property: $\alpha(X)$ is the best abstraction in $\mathbb{I}$ of $X \subseteq \mathbb{Z}$

Proof: that $\alpha(X) \sqsubseteq I \iff X \subseteq \gamma(I)$

$$\begin{array}{lll}
& \alpha(X) \sqsubseteq (a,b) & \\
\iff & \min X \geq a \wedge \max X \leq b & (\textit{def. } \alpha, \sqsubseteq) \\
\iff & \forall x \in X \colon a \leq x \leq b & (\textit{def. } \min, \max) \\
\iff & \forall x \in X \colon x \in \{\, y \mid a \leq y \leq b \,\} & \\
\iff & \forall x \in X \colon x \in \gamma([a,b]) & (\textit{def. } \gamma) \\
\iff & X \subseteq \gamma([a,b]) & (\textit{prop. } \subseteq)
\end{array}$$

Reminder: Sound, optimal, exact abstractions

Given $F : \mathcal{P}(\mathbb{Z}) \to \mathcal{P}(\mathbb{Z})$, how do we construct $F^\sharp : \mathbb{I} \to \mathbb{I}$?

Optimality: define $F^\sharp$ as $F^\sharp = \alpha \circ F \circ \gamma$

then $\forall I \in \mathbb{I}$: $F^\sharp(I) = \min_\sqsubseteq \{\, I' \in \mathbb{I} \mid F(\gamma(I)) \subseteq \gamma(I') \,\}$

(by definition of Galois connections)

this implies: $\forall I \in \mathbb{I}$: $\gamma(F^\sharp(I)) = \min_\subseteq \{\, \gamma(I') \mid F(\gamma(I)) \subseteq \gamma(I') \,\}$

($F^\sharp$ outputs the smallest interval encompassing all the concrete results)

Note: not all domains have a best abstraction α!

we will see abstract interpretation with just γ

in that case, we say that $F^\sharp$ is optimal if
$\gamma(F^\sharp(I)) = \min_\subseteq \{\, \gamma(I') \mid F(\gamma(I)) \subseteq \gamma(I') \,\}$

(such a $F^\sharp$ may not always exist, nor be unique)

Reminder: Sound, optimal, exact abstractions

Soundness: core property of abstract operators

$\alpha \circ F \circ \gamma \sqsubseteq F^\sharp$

this is equivalent to $F \circ \gamma \subseteq \gamma \circ F^\sharp$

(an abstract step $F^\sharp$ over-approximates a concrete one F)

if no α exist, we take $F \circ \gamma \subseteq \gamma \circ F^\sharp$ as definition of soundness

Exactness: $\forall I \in \mathbb{I}$: $F(\gamma(I)) = \gamma(F^\sharp(I))$

- quite rare: $\forall I \in \mathbb{I}$: $F(\gamma(I))$ must be exactly representable in $\mathbb{I}$
- $\alpha \circ F \circ \gamma$ is not always exact
- even if it exists, such a $F^\sharp$ may be difficult to compute

Summary:

- α provides a systematic way to define an optimal $F^\sharp$
- it may not be possible or practicable to use $\alpha \circ F \circ \gamma$
  $\Longrightarrow$ we settle for a sound $F^\sharp$ instead of an optimal one

Concrete integer operations

Goal: design interval versions of core operators

building blocks to design an interval semantics

Concrete arithmetic operators:

$+$, $-$, $\times$, $/$, lifted to sets $(\mathcal{P}(\mathbb{Z}))^n \to \mathcal{P}(\mathbb{Z})$

$$\begin{array}{lcl}
\overline{-}\, X & \stackrel{\text{def}}{=} & \{\, -x \mid x \in X \,\} \\
X \mathbin{\overline{+}} Y & \stackrel{\text{def}}{=} & \{\, x + y \mid x \in X, y \in Y \,\} \\
X \mathbin{\overline{-}} Y & \stackrel{\text{def}}{=} & \{\, x - y \mid x \in X, y \in Y \,\} \\
X \mathbin{\overline{\times}} Y & \stackrel{\text{def}}{=} & \{\, x \times y \mid x \in X, y \in Y \,\} \\
X \mathbin{\overline{/}} Y & \stackrel{\text{def}}{=} & \{\, x / y \mid x \in X, y \in Y, y \neq 0 \,\}
\end{array}$$

where $/$ rounds towards 0 (truncation)

Set operators: $\cup$, $\cap$, $\subseteq$, $=$

Interval set operators

Optimal binary operators: $A_1 \diamond^\sharp A_2 \stackrel{\text{def}}{=} \alpha(\gamma(A_1) \diamond \gamma(A_2))$

- $\cap^\sharp = \sqcap$

  as $\gamma([a,b] \sqcap [c,d]) = \gamma([a,b]) \cap \gamma([c,d])$

- $\cup^\sharp = \sqcup$

  as $$\begin{aligned} &\alpha(\gamma([a,b]) \cup \gamma([c,d])) \\ &= \alpha(\{\, x \mid a \leq x \leq b \vee c \leq x \leq d \,\}) \\ &= [\min \{\, x \mid a \leq x \leq b \vee c \leq x \leq d \,\}, \max \{\, x \mid a \leq x \leq b \vee c \leq x \leq d \,\}] \\ &= [\min(a,c), \max(b,d)] \\ &= [a,b] \sqcup [c,d] \end{aligned}$$

Optimal predicates: $A_1 \bowtie^\sharp A_2 \stackrel{\text{def}}{\iff} \gamma(A_1) \bowtie \gamma(A_2)$

- $\subseteq^\sharp$ is $\sqsubseteq$

  as $\gamma([a,b]) \subseteq \gamma([c,d]) \iff a \geq c \wedge b \leq d \iff [a,b] \sqsubseteq [c,d]$

- $=^\sharp$ is $=$

  as $\gamma([a,b]) = \gamma([c,d]) \iff a = c \wedge b = d$

Note: for soundness, $A_1 \bowtie^\sharp A_2 \implies \gamma(A_1) \bowtie \gamma(A_2)$ is actually sufficient

Interval arithmetic: addition, subtraction

- $-^\sharp\ [a,b] = [-b,-a]$
- $[a,b]\ +^\sharp\ [c,d] = [a+c, b+d]$
- $[a,b]\ -^\sharp\ [c,d] = [a-d, b-c]$
- $\forall I \in \mathbb{I}$: $-^\sharp\ \bot = \bot\ +^\sharp\ I = I\ +^\sharp\ \bot = \cdots = \bot$ (strictness)

where: $+$ and $-$ is extended to $+\infty$, $-\infty$ as:
$\forall x \in \mathbb{Z}$: $(+\infty) + x = +\infty,\ (-\infty) + x = -\infty,\ -(+\infty) = (-\infty), \ldots$

Proof: optimality of $+^\sharp$

$$
\begin{aligned}
&\alpha(\gamma([a,b])\ \overline{+}\ \gamma([c,d])) \\
&= \alpha(\{\, x \mid a \leq x \leq b \,\}\ \overline{+}\ \{\, y \mid c \leq y \leq d \,\}) \\
&= \alpha(\{\, x+y \mid a \leq x \leq b \wedge c \leq y \leq d \,\}) \\
&= [\min \{\, x+y \mid a \leq x \leq b \wedge c \leq y \leq d \,\}, \max \{\, x+y \mid a \leq x \leq b \wedge c \leq y \leq d \,\}] \\
&= [a+c, b+d] \\
&= [a,b]\ +^\sharp\ [c,d]
\end{aligned}
$$

Interval arithmetic: multiplication

- $[a,b] \times^\sharp [c,d] = [\ \min(a \times c,\ a \times d,\ b \times c,\ b \times d),$
  $\max(a \times c,\ a \times d,\ b \times c,\ b \times d)\ ]$

where $\times$ is extended to $+\infty$ and $-\infty$ by the rule of signs:

$c \times (+\infty) = (+\infty)$ if $c > 0$, $(-\infty)$ if $c < 0$

$c \times (-\infty) = (-\infty)$ if $c > 0$, $(+\infty)$ if $c < 0$

we also need the non-standard rule: $0 \times (+\infty) = 0 \times (-\infty) = 0$

Proof sketch: by decomposition into negative and positive intervals

$$\begin{array}{rcrcl}
a \geq 1 & \wedge & c \geq 1 & \Longrightarrow & [a,b] \times^\sharp [c,d] = [a \times c, b \times d] \\
b \leq -1 & \wedge & c \geq 1 & \Longrightarrow & [a,b] \times^\sharp [c,d] = [a \times d, b \times c] \\
a \geq 1 & \wedge & d \leq -1 & \Longrightarrow & [a,b] \times^\sharp [c,d] = [b \times c, a \times d] \\
b \leq -1 & \wedge & d \leq -1 & \Longrightarrow & [a,b] \times^\sharp [c,d] = [b \times d, a \times c] \\
a = b = 0 & \vee & c = d = 0 & \Longrightarrow & [a,b] \times^\sharp [c,d] = [0,0]
\end{array}$$

$$\begin{aligned}
[a,b] \times^\sharp [c,d] = {} & ([a,b] \sqcap [1,+\infty]) \times^\sharp ([c,d] \sqcap [1,+\infty]) \sqcup \\
& ([a,b] \sqcap [1,+\infty]) \times^\sharp ([c,d] \sqcap [0,0]) \sqcup \\
& ([a,b] \sqcap [1,+\infty]) \times^\sharp ([c,d] \sqcap [-\infty,-1]) \sqcup \cdots
\end{aligned}$$

Interval arithmetic: division

- $/^\sharp$ by case split:
  $([a,b] \;/^\sharp\; ([c,d] \sqcap [1,+\infty])) \quad \sqcup \quad ([a,b] \;/^\sharp\; ([c,d] \sqcap [-\infty,-1]))$
  where
  $$[a,b] \;/^\sharp\; [c,d] = \begin{cases} [\min(a/c,a/d), \max(b/c,b/d)] & \text{if } 1 \leq c \\ [\min(b/c,b/d), \max(a/c,a/d)] & \text{if } d \leq -1 \end{cases}$$

where $/$ is extended to $+\infty$ and $-\infty$ by the rule of signs:

$c/(+\infty) = c/(-\infty) = 0$, including $(+\infty)/(+\infty) = 0$
$(+\infty)/c = (+\infty)$ if $c > 0$, $(-\infty)$ if $c < 0$
$(-\infty)/c = (-\infty)$ if $c > 0$, $(+\infty)$ if $c < 0$

Examples:

$[-5,5]/^\sharp[0,0] = \perp$

$[5,10]/^\sharp[-1,1] = ([5,10]/^\sharp[1,1]) \sqcup ([5,10]/^\sharp[-1,-1]) = [5,10] \sqcup [-10,-5] = [-10,10]$

Interval operator exactness

- exact interval operations: $\cap^\sharp$, $+^\sharp$, $-^\sharp$
- non-exact interval operations: $\cup^\sharp$, $\times^\sharp$, $/^\sharp$

$$[0,1] \cup^\sharp [10,11] = [0,11] \quad \text{but} \quad \gamma([0,1]) \cup \gamma([10,11]) = \{0,1,10,11\}$$
$$[0,1] \times^\sharp [2,2] = [0,2] \quad \text{but} \quad \gamma([0,1]) \mathbin{\overline{\times}} \gamma([2,2]) = \{0,2\}$$
$$[10,10] /^\sharp [-1,1] = [-10,10] \quad \text{but} \quad \gamma([10,10]) \mathbin{\overline{/}} \gamma([-1,1]) = \{-10,10\}$$

Note: $F^\sharp$ is exact if it is optimal and $\forall a \in A$: $F(\gamma(a)) \in \{\, \gamma(x) \mid x \in A \,\}$

Operator composition

- if $F^\sharp$ and $G^\sharp$ are sound and F is monotonic, then $F^\sharp \circ G^\sharp$ is sound

  Proof:
  $G(\gamma(I)) \subseteq \gamma(G^\sharp(I))$, so: $F(G(\gamma(I))) \subseteq F(\gamma(G^\sharp(I))) \subseteq \gamma(F^\sharp(G^\sharp(I)))$

- if $F^\sharp$ and $G^\sharp$ are exact, then $F^\sharp \circ G^\sharp$ is exact

  Proof: $F(G(\gamma(I))) = F(\gamma(G^\sharp(I))) = \gamma(F^\sharp(G^\sharp(I)))$

- if $F^\sharp$ and $G^\sharp$ are optimal, then $F^\sharp \circ G^\sharp$ is sound
  but not necessarily optimal!

  Example:
  $F(X) \stackrel{\text{def}}{=} \{\, 2x \mid x \in X \,\}$ and $G(X) \stackrel{\text{def}}{=} \{\, x \in X \mid x \geq 1 \,\}$
  $F^\sharp([a,b]) = [2a, 2b]$ and $G^\sharp([a,b]) = [a,b] \cap^\sharp [1, +\infty]$ are optimal
  but $G^\sharp(F^\sharp([0,1])) = [0,2] \cap^\sharp [1,+\infty] = [1,2]$
  while $\alpha(G(F(\gamma([0,1])))) = [2,2]$

$\Longrightarrow$ decomposing the semantics into more fined-grained operators

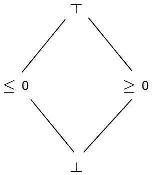
- simplifies analysis design and enhances reusability
- but can degrade the precision

Side-note: Meet closure and optimality

Reminder: $\gamma(a \sqcap a') = \gamma(a) \cap \gamma(a')$

$\Longrightarrow \{\, \gamma(a) \mid a \in A \,\}$ must be closed under $\cap$

Counter-example: invalid sign domain



$A \stackrel{\text{def}}{=} \{\bot, \leq 0, \geq 0, \top\}$

$\gamma(\leq 0) \cap \gamma(\geq 0) = \{0\} \notin \gamma(A)$

no best abstraction for $\{0\}$

$\Longrightarrow$ no Galois connection

possible fixes:

- complete A by $\cap$: $A \stackrel{\text{def}}{=} \{\bot, 0, \leq 0, \geq 0, \top\}$
- hollow A, removing elements: $A \stackrel{\text{def}}{=} \{\bot, \geq 0, \top\}$
- fix elements: $A \stackrel{\text{def}}{=} \{\bot, < 0, \geq 0, \top\}$

Side-note: Complete meet closure and optimality

Reminder: $\gamma(\sqcap X) = \cap \{ \gamma(x) \mid x \in X \}$

$\implies \{ \gamma(a) \mid a \in A \}$ must be closed under arbitrary $\cap$

α can be actually defined as $\alpha(c) = \sqcap \{ a \in A \mid c \subseteq \gamma(a) \}$

Counter-example: rational intervals

▯ $\stackrel{\text{def}}{=} \{ [a, b] \mid a \in \mathbb{Q} \cup \{-\infty\}, b \in \mathbb{Q} \cup \{+\infty\}, a \leq b \} \cup \{\bot\}$

$X = \{ c \mid c^2 \leq 2 \}$ has no best abstraction
because $\max X = \sqrt{2} \notin \mathbb{Q}$
$\implies$ no Galois connection

we can still define optimal $\cup^\sharp, \cap^\sharp, +^\sharp, -^\sharp, \times^\sharp, /^\sharp$
such that $\forall a_1, a_2: \gamma(a_1 \diamond^\sharp a_2) = \min_\subseteq \{ \gamma(a) \mid \gamma(a) \subseteq \gamma(a_1) \diamond \gamma(a_2) \}$

but some operators, such as $F(X) \stackrel{\text{def}}{=} \{ \sqrt{x} \mid x \in X \}$, have no best abstraction

$\implies$ we can study abstract domains wrt. the functions they can abstract precisely

Interval analysis

Reminder: Language

Expressions and conditions

$$
\begin{array}{lcll}
\textit{expr} & ::= & V & V \in \mathbb{V} \\
& | & c & c \in \mathbb{Z} \\
& | & -\textit{expr} & \\
& | & \textit{expr} \diamond \textit{expr} & \diamond \in \{+,-,\times,/\} \\
& | & \textbf{rand}(a,b) & a,b \in \mathbb{Z} \\[1ex]
\textit{cond} & ::= & \textit{expr} \bowtie \textit{expr} & \bowtie \in \{\leq,\geq,=,\neq,<,>\} \\
& | & \neg\textit{cond} & \\
& | & \textit{cond} \diamond \textit{cond} & \diamond \in \{\wedge,\vee\}
\end{array}
$$

Statements

$$
\begin{array}{lcl}
\textit{stat} & ::= & V \leftarrow \textit{expr} \\
& | & \textbf{if } \textit{cond} \textbf{ then } \textit{stat} \textbf{ else } \textit{stat} \\
& | & \textbf{while } \textit{cond} \textbf{ do } \textit{stat} \\
& | & \textit{stat}; \textit{stat} \\
& | & \textbf{skip}
\end{array}
$$

Reminder: Concrete semantics of expressions

Classic non-deterministic concrete semantics, in denotational style:

$\underline{\mathsf{E}[\![\, expr \,]\!] : \mathcal{E} \to \mathcal{P}(\mathbb{Z})}$ *(arithmetic expressions)*

$$
\begin{array}{lcl}
\mathsf{E}[\![\, V \,]\!]\,\rho & \stackrel{\text{def}}{=} & \{\rho(V)\} \\
\mathsf{E}[\![\, c \,]\!]\,\rho & \stackrel{\text{def}}{=} & \{c\} \\
\mathsf{E}[\![\, \mathbf{rand}(a,b) \,]\!]\,\rho & \stackrel{\text{def}}{=} & \{\, x \mid a \leq x \leq b \,\} \\
\mathsf{E}[\![\, -e \,]\!]\,\rho & \stackrel{\text{def}}{=} & \{\, -v \mid v \in \mathsf{E}[\![\, e \,]\!]\,\rho \,\} \\
\mathsf{E}[\![\, e_1 \diamond e_2 \,]\!]\,\rho & \stackrel{\text{def}}{=} & \{\, v_1 \diamond v_2 \mid v_1 \in \mathsf{E}[\![\, e_1 \,]\!]\,\rho,\ v_2 \in \mathsf{E}[\![\, e_2 \,]\!]\,\rho,\ \diamond \neq / \vee v_2 \neq 0 \,\}
\end{array}
$$

$\underline{\mathsf{C}[\![\, cond \,]\!] : \mathcal{E} \to \mathcal{P}(\{\text{true}, \text{false}\})}$ *(boolean conditions)*

$$
\begin{array}{lcl}
\mathsf{C}[\![\, \neg c \,]\!]\,\rho & \stackrel{\text{def}}{=} & \{\, \neg v \mid v \in \mathsf{C}[\![\, c \,]\!]\,\rho \,\} \\
\mathsf{C}[\![\, c_1 \diamond c_2 \,]\!]\,\rho & \stackrel{\text{def}}{=} & \{\, v_1 \diamond v_2 \mid v_1 \in \mathsf{C}[\![\, c_1 \,]\!]\,\rho,\ v_2 \in \mathsf{C}[\![\, c_2 \,]\!]\,\rho \,\} \\
\mathsf{C}[\![\, e_1 \bowtie e_2 \,]\!]\,\rho & \stackrel{\text{def}}{=} & \{\, \text{true} \mid \exists v_1 \in \mathsf{E}[\![\, e_1 \,]\!]\,\rho,\ v_2 \in \mathsf{E}[\![\, e_2 \,]\!]\,\rho \colon v_1 \bowtie v_2 \,\} \cup \\
 & & \{\, \text{false} \mid \exists v_1 \in \mathsf{E}[\![\, e_1 \,]\!]\,\rho,\ v_2 \in \mathsf{E}[\![\, e_2 \,]\!]\,\rho \colon v_1 \not\bowtie v_2 \,\}
\end{array}
$$

where $\mathcal{E} \stackrel{\text{def}}{=} \mathbb{V} \to \mathbb{Z}$

Reminder: Concrete semantics of statements

$\underline{\mathsf{S}[\![\, stat \,]\!] : \mathcal{P}(\mathcal{E}) \to \mathcal{P}(\mathcal{E})}$

$$
\begin{array}{lcl}
\mathsf{S}[\![\, \textbf{skip} \,]\!]\, R & \stackrel{\text{def}}{=} & R \\
\mathsf{S}[\![\, s_1;\; s_2 \,]\!]\, R & \stackrel{\text{def}}{=} & \mathsf{S}[\![\, s_2 \,]\!]\, (\mathsf{S}[\![\, s_1 \,]\!]\, R) \\
\mathsf{S}[\![\, V \leftarrow e \,]\!]\, R & \stackrel{\text{def}}{=} & \{\, \rho[V \mapsto v] \mid \rho \in R,\ v \in \mathsf{E}[\![\, e \,]\!]\, \rho \,\} \\
\mathsf{S}[\![\, \textbf{if}\ c\ \textbf{then}\ s_1\ \textbf{else}\ s_2 \,]\!]\, R & \stackrel{\text{def}}{=} & \mathsf{S}[\![\, s_1 \,]\!]\, (\mathsf{S}[\![\, c? \,]\!]\, R) \cup \mathsf{S}[\![\, s_2 \,]\!]\, (\mathsf{S}[\![\, \neg c? \,]\!]\, R) \\
\mathsf{S}[\![\, \textbf{while}\ c\ \textbf{do}\ s \,]\!]\, R & \stackrel{\text{def}}{=} & \mathsf{S}[\![\, \neg c? \,]\!]\, (\mathrm{lfp}\ \lambda I.R \cup \mathsf{S}[\![\, s \,]\!]\, (\mathsf{S}[\![\, c? \,]\!]\, I))
\end{array}
$$

where

$$
\mathsf{S}[\![\, c? \,]\!]\, R \stackrel{\text{def}}{=} \{\, \rho \in R \mid \text{true} \in \mathsf{C}[\![\, c \,]\!]\, \rho \,\}
$$

$\mathsf{S}[\![\, stat \,]\!]$ is a $\cup-$morphism in the complete lattice $(\mathcal{P}(\mathcal{E}), \subseteq, \cup, \cap, \emptyset, \mathcal{E})$

Reminder: Non-relational abstraction

Reminder: we compose two abstractions:

- $\mathcal{P}(\mathbb{V} \to \mathbb{Z})$ is abstracted as $\mathbb{V} \to \mathcal{P}(\mathbb{Z})$ (forget relationship)
- $\mathcal{P}(\mathbb{Z})$ is abstracted as intervals $\mathbb{I}$ (keep only bounds)

Cartesian lattice:

- $\mathcal{E}^\sharp \stackrel{\text{def}}{=} \mathbb{V} \to \mathbb{I}$
- point-wise order: $X_1^\sharp \mathrel{\dot\sqsubseteq} X_2^\sharp \iff \forall V \in \mathbb{V}: X_1^\sharp(V) \sqsubseteq X_2^\sharp(V)$
- join: $X_1^\sharp \mathbin{\dot\sqcup} X_2^\sharp \stackrel{\text{def}}{=} \lambda V. X_1^\sharp(V) \sqcup X_2^\sharp(V)$
- meet: $X_1^\sharp \mathbin{\dot\sqcap} X_2^\sharp \stackrel{\text{def}}{=} \lambda V. X_1^\sharp(V) \sqcap X_2^\sharp(V)$

$\Longrightarrow$ we still have a complete lattice

Cartesian Galois connection: $(\mathcal{P}(\mathbb{V} \to \mathbb{Z}), \subseteq) \underset{\dot\alpha}{\overset{\dot\gamma}{\leftrightarrows}} (\mathbb{V} \to \mathbb{I}, \dot\sqsubseteq)$

- $\dot\alpha(E) \stackrel{\text{def}}{=} \lambda V. \alpha(\{\, \rho(V) \mid \rho \in R \,\})$
- $\dot\gamma(X^\sharp) \stackrel{\text{def}}{=} \{\, \rho \mid \forall V \in \mathbb{V}: \rho(V) \in \gamma(X^\sharp(V)) \,\}$

Interval expression evaluation

$\mathsf{E}^\sharp[\![\, \mathit{expr} \,]\!] : \mathcal{E}^\sharp \to \mathbb{I}$

interval version of $\mathsf{E}[\![\, \mathit{expr} \,]\!] : \mathcal{E} \to \mathcal{P}(\mathbb{Z})$

Definition by structural induction, very similar to $\mathsf{E}[\![\, \mathit{expr} \,]\!]$

$$
\begin{array}{lcl}
\mathsf{E}^\sharp[\![\, V \,]\!]\, X^\sharp & \stackrel{\text{def}}{=} & X^\sharp(V) \\
\mathsf{E}^\sharp[\![\, c \,]\!]\, X^\sharp & \stackrel{\text{def}}{=} & [c, c] \\
\mathsf{E}^\sharp[\![\, \mathbf{rand}(a, b) \,]\!]\, X^\sharp & \stackrel{\text{def}}{=} & [a, b] \\
\mathsf{E}^\sharp[\![\, -e \,]\!]\, X^\sharp & \stackrel{\text{def}}{=} & -^\sharp\ \mathsf{E}^\sharp[\![\, e \,]\!]\, X^\sharp \\
\mathsf{E}^\sharp[\![\, e_1 \diamond e_2 \,]\!]\, X^\sharp & \stackrel{\text{def}}{=} & \mathsf{E}^\sharp[\![\, e_1 \,]\!]\, X^\sharp\ \diamond^\sharp\ \mathsf{E}^\sharp[\![\, e_2 \,]\!]\, X^\sharp
\end{array}
$$

Soundness of interval expression evaluation

Soundness: $\cup\,\{\,\mathsf{E}[\![\,e\,]\!]\,\rho \mid \rho \in \dot{\gamma}(X^\sharp)\,\} \subseteq \gamma(\mathsf{E}^\sharp[\![\,e\,]\!]\,X^\sharp)$

Proof:

by induction on the expression syntax, using the soundness of abstract operators;
the base cases $\mathsf{E}^\sharp[\![\,V\,]\!]$, $\mathsf{E}^\sharp[\![\,c\,]\!]$, $\mathsf{E}^\sharp[\![\,\mathbf{rand}(a,b)\,]\!]$ are straightforward;
for $+$ (the same holds for $-$, $\times$, $/$):

$$
\begin{array}{ll}
\gamma(\mathsf{E}^\sharp[\![\,e_1+e_2\,]\!]\,X^\sharp) & \\
= \gamma(\mathsf{E}^\sharp[\![\,e_1\,]\!]\,X^\sharp \;+^\sharp\; \mathsf{E}^\sharp[\![\,e_2\,]\!]\,X^\sharp) & (\textit{def. } \mathsf{E}^\sharp[\![\ ]\!]) \\
\supseteq \{\,v_1+v_2 \mid v_1 \in \gamma(\mathsf{E}^\sharp[\![\,e_1\,]\!]\,X^\sharp),\ v_2 \in \gamma(\mathsf{E}^\sharp[\![\,e_2\,]\!]\,X^\sharp)\,\} & (\textit{sound. } +^\sharp) \\
\supseteq \{\,v_1+v_2 \mid \exists \rho_1,\rho_2 \in \dot{\gamma}(X^\sharp)\colon v_1 \in \mathsf{E}[\![\,e_1\,]\!]\,\rho_1,\ v_2 \in \mathsf{E}[\![\,e_2\,]\!]\,\rho_2\,\} & (\textit{induction}) \\
\supseteq \{\,v_1+v_2 \mid \exists \rho \in \dot{\gamma}(X^\sharp)\colon v_1 \in \mathsf{E}[\![\,e_1\,]\!]\,\rho,\ v_2 \in \mathsf{E}[\![\,e_2\,]\!]\,\rho\,\} & (\textit{prop. } \exists) \\
= \cup\,\{\,\mathsf{E}[\![\,e_1+e_2\,]\!]\,\rho \mid \rho \in \dot{\gamma}(X^\sharp)\,\} & (\textit{def. } \mathsf{E}[\![\ ]\!])
\end{array}
$$

Non-optimality except in rare cases because:

- the composition of optimal operators is not always optimal
- of the core non-relational abstraction: $\mathcal{P}(\mathbb{V} \to \mathbb{Z}) \xleftrightarrow[\alpha]{\gamma} \mathbb{V} \to \mathcal{P}(\mathbb{Z})$

  e.g.: $\mathsf{E}^\sharp[\![\,V-V\,]\!]\,[V \mapsto [0,1]] = [0,1] -^\sharp [0,1] = [-1,1]$
  but $\alpha(\cup\,\{\,\mathsf{E}[\![\,V-V\,]\!]\,\rho \mid \rho \in \dot{\gamma}([V \mapsto [0,1]])\,\}) = [0,0]$

Abstract interval statements: first part

$S^\sharp[\![\, stat \,]\!] : \mathcal{E}^\sharp \to \mathcal{E}^\sharp$ interval version of $S[\![\, stat \,]\!] : \mathcal{P}(\mathcal{E}) \to \mathcal{P}(\mathcal{E})$

- $S[\![\, \textbf{skip} \,]\!] R \stackrel{\text{def}}{=} R$
  $S^\sharp[\![\, \textbf{skip} \,]\!] X^\sharp \stackrel{\text{def}}{=} X^\sharp$ (identity)
- $S[\![\, s_1;\ s_2 \,]\!] R \stackrel{\text{def}}{=} S[\![\, s_2 \,]\!] (S[\![\, s_1 \,]\!] R)$
  $S^\sharp[\![\, s_1; s_2 \,]\!] X^\sharp \stackrel{\text{def}}{=} S^\sharp[\![\, s_2 \,]\!] (S^\sharp[\![\, s_1 \,]\!] X^\sharp)$ (composition)
- $S[\![\, V \leftarrow e \,]\!] R \stackrel{\text{def}}{=} \{\, \rho[V \mapsto v] \mid \rho \in R,\ v \in E[\![\, e \,]\!] \rho \,\}$
  $S^\sharp[\![\, V \leftarrow e \,]\!] X^\sharp \stackrel{\text{def}}{=} \begin{cases} X^\sharp[V \mapsto E^\sharp[\![\, e \,]\!] X^\sharp] & \text{if } E^\sharp[\![\, e \,]\!] X^\sharp \neq \bot \\ \dot\bot & \text{if } E^\sharp[\![\, e \,]\!] X^\sharp = \bot \end{cases}$

(tests and loops are more complex, they are presented in the next slides)

Soundness proof: i.e., $S[\![\, s \,]\!] (\dot\gamma(X^\sharp)) \subseteq \dot\gamma(S^\sharp[\![\, s \,]\!] X^\sharp)$

obvious for **skip**; by composition of soundness for $s_1; s_2$;
for $V \leftarrow e$ we derive:

$$\begin{array}{ll}
\dot\gamma(S^\sharp[\![\, V \leftarrow e \,]\!] X^\sharp) & \\
= \{\, \rho[V \mapsto v] \mid \forall W: \rho(W) \in \gamma(X^\sharp(W)),\ v \in \gamma(E^\sharp[\![\, e \,]\!] X^\sharp) \,\} & (\textit{def. } \dot\gamma,\ S^\sharp[\![\ ]\!]) \\
\supseteq \{\, \rho[V \mapsto v] \mid \forall W: \rho(W) \in \gamma(X^\sharp(W)),\ v \in E[\![\, e \,]\!] \rho \,\} & (\textit{sound. } E^\sharp[\![\ ]\!]) \\
= S[\![\, V \to e \,]\!] \dot\gamma(X^\sharp) & (\textit{def. } \dot\gamma,\ S[\![\ ]\!])
\end{array}$$

Tests

Abstract tests

conditionals and loops use the auxiliary "test" statement:

$\mathsf{S}[\![\, c? \,]\!]\, R \stackrel{\text{def}}{=} \{\, \rho \in R \mid \text{true} \in \mathsf{C}[\![\, c \,]\!]\, \rho \,\}$

Abstract tests: $\mathsf{S}^\sharp[\![\, c? \,]\!]$

Preprocessing: remove $\neg$, $=$, $\neq$, $>$, $\geq$, $<$

- $\neg$ can be removed using De Morgan's law:

  $\neg(c_1 \vee c_2) \rightsquigarrow \neg c_1 \wedge \neg c_2$

  $\neg(c_1 \wedge c_2) \rightsquigarrow \neg c_1 \vee \neg c_2$

  $\neg(e_1 \leq e_2) \rightsquigarrow e_1 > e_2 \ \ldots$

- $=$, $\neq$, $>$, $\geq$, $<$ can be expressed using only $\leq$, $\vee$ and $\wedge$:

  $e_1 < e_2 \rightsquigarrow e_1 \leq (e_2 - 1)$

  $e_1 \geq e_2 \rightsquigarrow e_2 \leq e_1$

  $e_1 > e_2 \rightsquigarrow e_2 \leq (e_1 - 1)$

  $e_1 = e_2 \rightsquigarrow (e_1 \leq e_2) \wedge (e_2 \leq e_1)$

  $e_1 \neq e_2 \rightsquigarrow (e_1 \leq (e_2 - 1)) \vee (e_2 \leq (e_1 - 1))$

Interval test (cont.)

Handling boolean operators: by induction

- $\mathsf{S}^\sharp[\![\, c_1 \vee c_2 ? \,]\!]\, X^\sharp \stackrel{\text{def}}{=} (\mathsf{S}^\sharp[\![\, c_1 ? \,]\!]\, X^\sharp) \,\dot{\cup}^\sharp\, (\mathsf{S}^\sharp[\![\, c_2 ? \,]\!]\, X^\sharp)$
- $\mathsf{S}^\sharp[\![\, c_1 \wedge c_2 ? \,]\!]\, X^\sharp \stackrel{\text{def}}{=} (\mathsf{S}^\sharp[\![\, c_1 ? \,]\!]\, X^\sharp) \,\dot{\cap}^\sharp\, (\mathsf{S}^\sharp[\![\, c_2 ? \,]\!]\, X^\sharp)$

Simple tests: comparing a variable to a variable or a constant

assuming $X^\sharp(V) = [a, b]$ and $X^\sharp(W) = [c, d]$

- $\mathsf{S}^\sharp[\![\, V \leq v ? \,]\!]\, X^\sharp \stackrel{\text{def}}{=} \begin{cases} X^\sharp[V \mapsto [a, \min(b, v)]] & \text{if } a \leq v \\ \dot{\perp} & \text{if } a > v \end{cases}$
- $\mathsf{S}^\sharp[\![\, V \leq W ? \,]\!]\, X^\sharp \stackrel{\text{def}}{=} \begin{cases} X^\sharp[\ V \mapsto [a, \min(b, d)], & \text{if } a \leq d \\ \qquad W \mapsto [\max(a, c), d]\ ] & \\ \dot{\perp} & \text{if } a > d \end{cases}$

  (W's upper bound refines V's, V's lower bound refines W's)

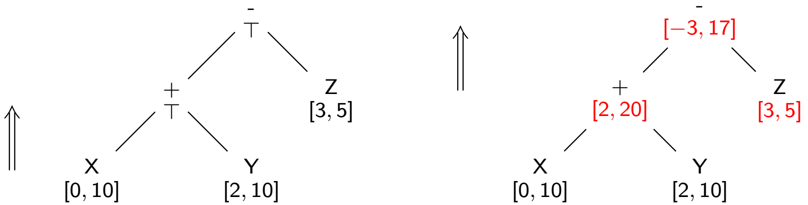
(next slides: how to handle tests with arbitrary expressions)

Example of complex interval test

Example: $\mathsf{S}^\sharp[\![\, X + Y - Z \leq 0 \,]\!]\, X^\sharp$

with $X^\sharp = \{\, X \mapsto [0, 10], Y \mapsto [2, 10], Z \mapsto [3, 5] \,\}$

First step: annotate the expression tree



(1) (2)

bottom-up evaluation using abstract interval operators $+^\sharp$, $-^\sharp$, etc.

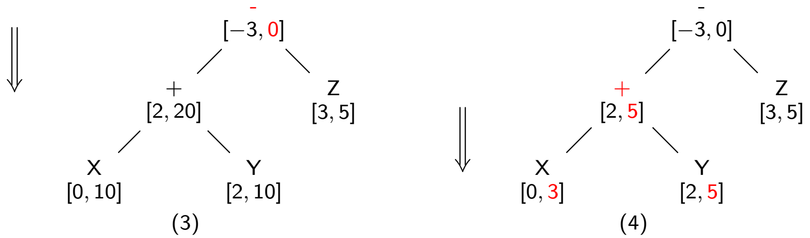
(similar to interval assignment)

Example of complex interval test (cont.)

Example: $\mathsf{S}^\sharp[\![\, X + Y - Z \leq 0 \,]\!]\, X^\sharp$

with $X^\sharp = \{\, X \mapsto [0,10], Y \mapsto [2,10], Z \mapsto [3,5] \,\}$

Second step: refine the expression tree top-down



- refine the root: we know the result is negative
- propagate refined nodes downward toward leaves
- use refined variable values: $\{\, X \mapsto [0,3], Y \mapsto [2,5], Z \mapsto [3,5] \,\}$

$\Longrightarrow$ we need new abstract operators to model refinement $\overleftarrow{\leq 0}^\sharp$, $\overleftarrow{+}^\sharp$, etc.

Backward arithmetic and comparison operators

Sound backward arithmetic and comparison operators
that refine their argument given a result.

- Backward comparison operators, applied at the root:

$$X^{\sharp\prime} = \overleftarrow{\leq 0}^{\sharp}(X^{\sharp})$$
$$\implies \{\, x \in \gamma(X^{\sharp}) \,|\, x \leq 0 \,\} \subseteq \gamma(X^{\sharp\prime}) \subseteq \gamma(X^{\sharp})$$

- Backward arithmetic operators, applied at internal expression nodes:

$$X^{\sharp\prime} = \overleftarrow{-}^{\sharp}(X^{\sharp}, R^{\sharp})$$
$$\implies \{\, x \,|\, x \in \gamma(X^{\sharp}),\ -x \in \gamma(R^{\sharp}) \,\} \subseteq \gamma(X^{\sharp\prime}) \subseteq \gamma(X^{\sharp})$$

$$(X^{\sharp\prime}, Y^{\sharp\prime}) = \overleftarrow{+}^{\sharp}(X^{\sharp}, Y^{\sharp}, R^{\sharp})$$
$$\implies \{\, x \in \gamma(X^{\sharp}) \,|\, \exists y \in \gamma(Y^{\sharp}),\ x + y \in \gamma(R^{\sharp}) \,\} \subseteq \gamma(X^{\sharp\prime}) \subseteq \gamma(X^{\sharp})$$
$$\text{and } \{\, y \in \gamma(Y^{\sharp}) \,|\, \exists x \in \gamma(X^{\sharp}),\ x + y \in \gamma(R^{\sharp}) \,\} \subseteq \gamma(Y^{\sharp\prime}) \subseteq \gamma(Y^{\sharp})$$

$\vdots$

Note: best backward operators can be designed with α

e.g. for $\overleftarrow{+}^{\sharp}$: $X^{\sharp\prime} = \alpha(\{\, x \in \gamma(X^{\sharp}) \,|\, \exists y \in \gamma(Y^{\sharp}),\ x + y \in \gamma(R^{\sharp}) \,\})$

Generic backward operator construction

Synthesizing (non optimal) backward arithmetic operators from forward arithmetic operators

$$\overleftarrow{\leq 0}^\sharp(X^\sharp) \stackrel{\text{def}}{=} X^\sharp \cap^\sharp [-\infty, 0]^\sharp$$

$$\overleftarrow{-}^\sharp(X^\sharp, R^\sharp) \stackrel{\text{def}}{=} X^\sharp \cap^\sharp (-^\sharp\, R^\sharp)$$

$$\overleftarrow{+}^\sharp(X^\sharp, Y^\sharp, R^\sharp) \stackrel{\text{def}}{=} (X^\sharp \cap^\sharp (R^\sharp -^\sharp Y^\sharp),\ Y^\sharp \cap^\sharp (R^\sharp -^\sharp X^\sharp))$$

$$\overleftarrow{-}^\sharp(X^\sharp, Y^\sharp, R^\sharp) \stackrel{\text{def}}{=} (X^\sharp \cap^\sharp (R^\sharp +^\sharp Y^\sharp),\ Y^\sharp \cap^\sharp (X^\sharp -^\sharp R^\sharp))$$

$$\overleftarrow{\times}^\sharp(X^\sharp, Y^\sharp, R^\sharp) \stackrel{\text{def}}{=} (X^\sharp \cap^\sharp (R^\sharp /^\sharp Y^\sharp),\ Y^\sharp \cap^\sharp (R^\sharp /^\sharp X^\sharp))$$

$$\overleftarrow{/}^\sharp(X^\sharp, Y^\sharp, R^\sharp) \stackrel{\text{def}}{=} (X^\sharp \cap^\sharp (S^\sharp \times^\sharp Y^\sharp),\ Y^\sharp \cap^\sharp ((X^\sharp /^\sharp S^\sharp) \cup^\sharp [0,0]^\sharp))$$

$$\text{where } S^\sharp = \begin{cases} R^\sharp & \text{if } \mathbb{I} \neq \mathbb{Z} \\ R^\sharp +^\sharp [-1,1]^\sharp & \text{if } \mathbb{I} = \mathbb{Z} \text{ (as / rounds)} \end{cases}$$

<u>Note:</u> $\overleftarrow{\diamond}^\sharp(X^\sharp, Y^\sharp, R^\sharp) = (X^\sharp, Y^\sharp)$ is always sound (no refinement).

Interval backward operators

Applying the generic construction to the interval domain, we get:

$$\overleftarrow{\leq 0}^\sharp([a,b]) \stackrel{\text{def}}{=} \begin{cases} [a, \min(b,0)] & \text{if } a \geq 0 \\ \perp & \text{otherwise} \end{cases}$$

$$\overleftarrow{-}^\sharp([a,b],[r,s]) \stackrel{\text{def}}{=} [a,b] \cap^\sharp [-s,-r]$$

$$\overleftarrow{+}^\sharp([a,b],[c,d],[r,s]) \stackrel{\text{def}}{=} ([a,b] \cap^\sharp [r-d, s-c], \\ [c,d] \cap^\sharp [r-b, s-a])$$

...

Interval conditionals

Concrete semantics:

$$\mathsf{S}[\![\,\textbf{if } c \textbf{ then } s_1 \textbf{ else } s_2\,]\!]\, R \stackrel{\text{def}}{=} \mathsf{S}[\![\, s_1\,]\!]\,(\mathsf{S}[\![\, c?\,]\!]\, R) \cup \mathsf{S}[\![\, s_2\,]\!]\,(\mathsf{S}[\![\, \neg c?\,]\!]\, R)$$

Abstract semantics: compose existing abstract operators:

$$\mathsf{S}^\sharp[\![\,\textbf{if } c \textbf{ then } s_1 \textbf{ else } s_2\,]\!]\, X^\sharp \stackrel{\text{def}}{=} \mathsf{S}^\sharp[\![\, s_1\,]\!]\,(\mathsf{S}^\sharp[\![\, c?\,]\!]\, X^\sharp) \mathbin{\dot\cup^\sharp} \mathsf{S}^\sharp[\![\, s_2\,]\!]\,(\mathsf{S}^\sharp[\![\, \neg c?\,]\!]\, X^\sharp)$$

Soundness proof:

by soundness of the composition of sound operators

Example: $stat \stackrel{\text{def}}{=} V \leftarrow 2 \times \textbf{rand}(0,1);\ \textbf{if } V > 1 \textbf{ then } V \leftarrow 0 \textbf{ else skip}$

given $E^\sharp \stackrel{\text{def}}{=} [V \mapsto [-\infty, +\infty]]$
we get: $\mathsf{S}^\sharp[\![\, stat\,]\!]\, E^\sharp = [V \mapsto [0,1]]$

note that $\mathsf{S}[\![\, stat\,]\!]\,\mathcal{E} = \{[V \mapsto 0]\}$
$\implies \mathsf{S}^\sharp[\![\, stat\,]\!]$ is sound but not optimal

Loops

Interval loops

Concrete semantics:

$\mathsf{S}[\![\, \textbf{while } c \textbf{ do } s \,]\!]\, R \stackrel{\text{def}}{=} \mathsf{S}[\![\, \neg c? \,]\!]\,(\text{lfp } F)$

where $F(I) \stackrel{\text{def}}{=} R \cup \mathsf{S}[\![\, s \,]\!]\,(\mathsf{S}[\![\, c? \,]\!]\, I))$

Reminder: lfp F exists because F is monotonic
in fact, lfp $F = \cup_{n \in \mathbb{N}} F^n(\emptyset)$ because F is a $\cup-$morphism

Abstract fixpoint computation:

given a sound abstraction $F^\sharp$ of F, how can we abstract lfp F?

- lfp $F^\sharp$ may not exist
  $\Longrightarrow$ we seek only $X^\sharp$ such that $F^\sharp(X^\sharp) \sqsubseteq X^\sharp$ (post fixpoint)
- $F^\sharp$ may be non monotonic (example presented later)
  $\Longrightarrow$ we compute $X^\sharp_{n+1} \stackrel{\text{def}}{=} X^\sharp_n \sqcup F^\sharp(X^\sharp_n)$ (abstract iterations)
- $X^\sharp_n$ may increase infinitely (e.g., $F^\sharp(X^\sharp) = X^\sharp +^\sharp [1,1]$)
  $\Longrightarrow$ we use convergence acceleration

Convergence acceleration

Widening: binary operator $\triangledown : \mathcal{E}^\sharp \times \mathcal{E}^\sharp \to \mathcal{E}^\sharp$ such that:

- $\gamma(X^\sharp) \cup \gamma(Y^\sharp) \subseteq \gamma(X^\sharp \triangledown Y^\sharp)$ (sound abstraction of $\cup$)
- for any sequence $(X^\sharp_n)_{n\in\mathbb{N}}$, the sequence $(Y^\sharp_n)_{n\in\mathbb{N}}$

$$\begin{cases} Y^\sharp_0 & \stackrel{\text{def}}{=} X^\sharp_0 \\ Y^\sharp_{n+1} & \stackrel{\text{def}}{=} Y^\sharp_n \triangledown X^\sharp_{n+1} \end{cases}$$

stabilizes in finite time: $\exists N \in \mathbb{N}$: $Y^\sharp_N = Y^\sharp_{N+1}$

Fixpoint approximation theorem:

- the sequence $X^\sharp_{n+1} \stackrel{\text{def}}{=} X^\sharp_n \triangledown F^\sharp(X^\sharp_n)$ stabilizes in finite time
- when $X^\sharp_{N+1} \sqsubseteq X^\sharp_N$, then $X^\sharp_N$ abstracts $\operatorname{lfp} F$

Soundness proof: assume $X^\sharp_{N+1} \sqsubseteq X^\sharp_N$, then

$\gamma(X^\sharp_N) \supseteq \gamma(X^\sharp_{N+1}) = \gamma(X^\sharp_N \triangledown F^\sharp(X^\sharp_N)) \supseteq \gamma(F^\sharp(X^\sharp_N)) \supseteq F(\gamma(X^\sharp_N))$

$\gamma(X^\sharp_N)$ is a post-fixpoint of F, but $\operatorname{lfp} F$ is F's least post-fixpoint, so, $\gamma(X^\sharp_N) \supseteq \operatorname{lfp} F$

Interval loops (cont.)

Concrete semantics:

$$\mathsf{S}[\![\, \textbf{while } c \textbf{ do } s \,]\!]\, R \stackrel{\text{def}}{=} \mathsf{S}[\![\, \neg c? \,]\!]\, (\mathrm{lfp}\, \lambda I. R \cup \mathsf{S}[\![\, s \,]\!]\, (\mathsf{S}[\![\, c? \,]\!]\, I)))$$

Abstract semantics

compose existing sound abstractions
employ convergence acceleration $\triangledown$

$$\mathsf{S}^\sharp[\![\, \textbf{while } c \textbf{ do } s \,]\!]\, X^\sharp \stackrel{\text{def}}{=} \mathsf{S}^\sharp[\![\, \neg c? \,]\!]\, (\lim \lambda I^\sharp . I^\sharp \triangledown (X^\sharp \mathbin{\dot\cup^\sharp} \mathsf{S}^\sharp[\![\, s \,]\!]\, (\mathsf{S}^\sharp[\![\, c? \,]\!]\, I^\sharp)))$$

(where $\lim F^\sharp$ iterates the function $F^\sharp$ from $\dot\bot$ until $F^\sharp(X^\sharp) \sqsubseteq X^\sharp$)

Interval widening

Interval widening $\quad \triangledown : \mathcal{I} \times \mathcal{I} \to \mathcal{I}$

$$\forall I \in \mathcal{I} \colon \bot \triangledown I = I \triangledown \bot = I$$

$$[a,b] \triangledown [c,d] \stackrel{\text{def}}{=} \left[\begin{cases} a & \text{if } a \leq c \\ -\infty & \text{if } a > c \end{cases}, \begin{cases} b & \text{if } b \geq d \\ +\infty & \text{if } b < d \end{cases} \right]$$

- an unstable lower bound is put to $-\infty$
- an unstable upper bound is put to $+\infty$
- once at $-\infty$ or $+\infty$, the bound becomes stable

Point-wise lifting: $\quad \dot{\triangledown} : \mathcal{E}^\sharp \times \mathcal{E}^\sharp \to \mathcal{E}^\sharp$

$$X^\sharp \,\dot{\triangledown}\, Y^\sharp \stackrel{\text{def}}{=} \lambda V \in \mathbb{V}. X^\sharp(V) \triangledown Y^\sharp(V)$$

extrapolate each variable independently

$\Longrightarrow$ stabilization in at most $2|\mathbb{V}|$ iterations

Analysis example with widening

Example

$V \leftarrow 1;$
while $V \leq 50$ **do** $V \leftarrow V + 2$

We must compute $\mathsf{S}^\sharp[\![\, V > 50 \,]\!]\,(\lim \lambda I^\sharp.I^\sharp \;\dot{\triangledown}\; F^\sharp(I^\sharp))$
where $F^\sharp(I^\sharp) \stackrel{\text{def}}{=} [1,1] \;\dot{\cup}^\sharp\; \mathsf{S}^\sharp[\![\, V \leftarrow V+2 \,]\!]\,(\mathsf{S}^\sharp[\![\, V \leq 50 \,]\!]\, I^\sharp)$

iterates with widening:

$$\begin{array}{lllll}
I_0^\sharp = \bot & & & & \\
I_1^\sharp = I_0^\sharp \triangledown F^\sharp(I_0^\sharp) & = \bot \triangledown [1,1] & = [1,1] & & \\
I_2^\sharp = I_1^\sharp \triangledown F^\sharp(I_1^\sharp) & = [1,1] \triangledown ([1,1] \cup^\sharp [3,3]) & = [1,1] \triangledown [1,3] & = [1,+\infty] & \\
I_3^\sharp = I_2^\sharp \triangledown F^\sharp(I_2^\sharp) & = [1,+\infty] \triangledown ([1,1] \cup^\sharp [3,52]) & = [1,+\infty] \triangledown [1,52] & = [1,+\infty] & = I_2^\sharp
\end{array}$$

$\Longrightarrow \lim \lambda I^\sharp.I^\sharp \triangledown F^\sharp(I^\sharp) = [1,+\infty]$

At the end of the program, we find $\mathsf{S}^\sharp[\![\, V > 50 \,]\!]\, I_3^\sharp = [51,+\infty]$

The concrete semantics would give $\{51\}$

Intuitions behind the widening

Inductive reasoning (philosophical logic)

- induction = generalization from a small set of observations
  e.g., if the upper bound is increasing, it is probably unbounded
  major cognitive process
- $\neq$ induction in mathematics, which is deductive by nature
  (apply an induction axiom)
- in philosophy, induction is unreliable (finite observation)
  but in abstract interpretation, widening is always sound!

Inductive invariants

- lfp F defines the most precise invariant (concrete semantics)
- X such that lfp $F \subseteq X$ is a (possibly less precise) invariant
- X such that $F(X) \subseteq X$ is an inductive invariant
  (X is an invariant, and it can be proved to be invariant without computing lfp F)
- $X^\sharp$ such that $F^\sharp(X^\sharp) \sqsubseteq X^\sharp$ is an abstract inductive invariant
  ($\gamma(X^\sharp)$ can be proved to be invariant in the abstract, without computing lfp F)

Decreasing iterations

Example

$V \leftarrow 1;$
while $V \leq 50$ **do** $V \leftarrow V + 2$

Imprecision

In this example, we found $V \in [1, +\infty]$ as loop invariant
but the most precise interval invariant is $V \in [1, 52]$

Solution: decreasing iterations

after stabilizing an iteration with widening
we can continue iterating without the widening to gain precision

- compute as before $X^\sharp \stackrel{\text{def}}{=} \lim \lambda I^\sharp . I^\sharp \triangledown F^\sharp(I^\sharp)$
  we get an abstract post-fixpoint $X^\sharp \sqsupseteq F^\sharp(X^\sharp)$, so $F(\gamma(X^\sharp)) \subseteq \gamma(X^\sharp)$
- then compute $Y^\sharp_n \stackrel{\text{def}}{=} F^{\sharp n}(X^\sharp)$
  by soundness, $\gamma(Y^\sharp_n)$ is also a post-fixpoint of F for every n
  we stop after a fixed finite n, or when $Y^\sharp_{n+1} = Y^\sharp_n$

Decreasing iterations

Example

$V \leftarrow 1;$
while $V \leq 50$ **do** $V \leftarrow V + 2$

Imprecision

In this example, we found $V \in [1, +\infty]$ as loop invariant
but the most precise interval invariant is $V \in [1, 52]$

Solution: decreasing iterations

here: $F^\sharp(I^\sharp) \stackrel{\text{def}}{=} [1,1] \,\dot{\cup}^\sharp\, \mathsf{S}^\sharp[\![\, V \leftarrow V + 2 \,]\!]\, (\mathsf{S}^\sharp[\![\, V \leq 50 \,]\!]\, I^\sharp)$

$$X^\sharp \stackrel{\text{def}}{=} \lim \lambda I^\sharp . I^\sharp \,\triangledown\, F^\sharp(I^\sharp) = [1, +\infty]$$
$$Y_1^\sharp = F^\sharp(X^\sharp) = [1,1] \cup^\sharp [2,52] = [1,52]$$
$$Y_2^\sharp = F^\sharp(Y_1^\sharp) = [1,52] = Y_1^\sharp$$

we find the most precise loop invariant expressible using intervals!

at the loop exit, we get: $\mathsf{S}^\sharp[\![\, V > 50 \,]\!]\, ([1,52]) = [51, 52]$

Widening with thresholds

Example

$V \leftarrow 40;$
while $V \neq 0$ **do** $V \leftarrow V - 1$

Imprecision

V decreases form 40 (to 0)

$\Longrightarrow$ iterations with widening find the loop invariant: $V \in [-\infty, 40]$

$\mathsf{S}^\sharp[\![\, V \leftarrow V - 1 \,]\!]\,(\mathsf{S}^\sharp[\![\, V \neq 0 \,]\!]\,[-\infty, 40]) = [-\infty, 39]$

$\Longrightarrow$ decreasing iterations are ineffective

Note: this is caused by the $\neq 0$ test instead of ≥ 0
with $\neq 0$, every set $[a, 40] \setminus \{-1\}$ for $a \leq 0$ is a fixpoint
with ≥ 0, we have a single fixpoint: $[0, 40]$

Widening with thresholds

Example

$V \leftarrow 40;$
while $V \neq 0$ **do** $V \leftarrow V - 1$

Solution widening with **thresholds** T

T: fixed finite set of integers containing $-\infty$ and $+\infty$

$\triangledown$ "jumps" to the next value in T

$\implies \triangledown$ tests the stability of values in T

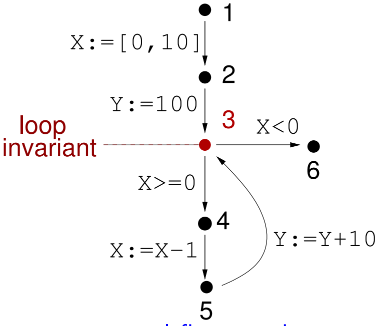
$$[a, b] \triangledown [c, d] \stackrel{\text{def}}{=} \left[\begin{cases} a & \text{if } a \leq c \\ \max\{\, t \in T \mid t \leq c \,\} & \text{if } a > c \end{cases}, \begin{cases} b & \text{if } b \geq d \\ \min\{\, t \in T \mid t \geq d \,\} & \text{if } b < d \end{cases} \right]$$

In our example, we find as loop invariant: $[\max\{\, t \in T \mid t \leq 0 \,\}, 40]$
if $0 \in T$, we find the most precise invariant $[0, 40]$

Solving equation systems with widening

Program semantics as equation system

Alternate view of program semantics:



control flow graph

$$\begin{cases} R_1 = (\{X,Y\} \to \mathbb{Z}) \\ R_2 = \mathsf{S}[\![X \leftarrow [0,10]]\!] R_1 \\ R_3 = \mathsf{S}[\![Y \leftarrow 100]\!] R_2 \cup \\ \qquad \mathsf{S}[\![Y \leftarrow Y + 10]\!] R_5 \\ R_4 = \mathsf{S}[\![X \geq 0]\!] R_3 \\ R_5 = \mathsf{S}[\![X \leftarrow X - 1]\!] R_4 \\ R_6 = \mathsf{S}[\![X < 0]\!] R_3 \end{cases}$$

equation system

- the system has a unique smallest solution
  (it's a least fixpoint in the complete lattice $\mathcal{P}(\{X,Y\} \to \mathbb{Z})$!)
- R_i is the best invariant at program point i
  (e.g. $R_3 = \{\, \rho \mid \rho(X) \in [0,10],\ 10\rho(X) + \rho(Y) \in [100,200] \cap 10\mathbb{Z} \,\}$)
- one big equation system of the form $R_i = F_i(R_1, \ldots, R_n)$

Resolution

Concrete resolution: iterations R^k

$$\begin{cases} R_i^0 \stackrel{\text{def}}{=} \emptyset \\ R_i^{k+1} \stackrel{\text{def}}{=} F_i(R_1^k, \ldots, R_n^k) \end{cases}$$

may not converge in finite time...

Abstract resolution: iterations $X^{\sharp k}$ in the abstract with widening

- choose an abstract domain and sound versions $F_i^\sharp$ of the F_i
- choose a set of widening points W
  every cycle in the CFG should pass through W
  e.g., choose loop heads as W

$$\begin{cases} X_i^{\sharp 0} \stackrel{\text{def}}{=} \bot \\ X_i^{\sharp k+1} \stackrel{\text{def}}{=} F_i^\sharp(X_1^{\sharp k}, \ldots, X_n^{\sharp k}) & \text{if } i \notin W \\ X_i^{\sharp k+1} \stackrel{\text{def}}{=} X_i^{\sharp k} \triangledown F_i^\sharp(X_1^{\sharp k}, \ldots, X_n^{\sharp k}) & \text{if } i \in W \end{cases}$$

$\Longrightarrow$ converges in finite time

(more clever algorithms exist: worklist iterator, see project)

Backward analysis

Forward versus backward analysis

Example

$Y \leftarrow 0;$
while $Y \leq X$ **do** $Y \leftarrow Y + 1$

Forward analysis:

- given $X \in [-10, 10]$ at the beginning of the program
  $Y \in [0, 11]$ at the end of the program

Backward analysis:

- to have $Y \in [10, 20]$ at the end of the program
  we must have $X \in [9, 19]$ at the beginning of the program

Backward-forward combination

Goal: given initial states I and finial states F
consider only executions that start in I and end in F

Application: analysis specialization to remove false alarms

Example

$X \leftarrow \textbf{rand}(-100, 100);$
$\textbf{if } X = 0 \textbf{ then } X \leftarrow 1;$
• $Y \leftarrow 100/X$

Analysis: using the interval domain

- a forward analysis finds $X \in [-100, 100]$ at •
  $\Longrightarrow$ false alarm for division by zero
- backward analysis from • assuming $X = 0$
  we find $\perp$ at the program entry
  $\Longrightarrow$ no execution can trigger the division by zero
  (we have removed the false alarm)

more complex combinations exist, such as iterated forward and backward analyses

Reminder: Forward denotational concrete semantics

$\mathsf{S}[\![\, stat \,]\!] : \mathcal{P}(\mathcal{E}) \to \mathcal{P}(\mathcal{E})$

$$
\begin{array}{lcl}
\mathsf{S}[\![\, \textbf{skip} \,]\!]\, R & \stackrel{\text{def}}{=} & R \\
\mathsf{S}[\![\, s_1;\ s_2 \,]\!]\, R & \stackrel{\text{def}}{=} & \mathsf{S}[\![\, s_2 \,]\!]\, (\mathsf{S}[\![\, s_1 \,]\!]\, R) \\
\mathsf{S}[\![\, V \leftarrow e \,]\!]\, R & \stackrel{\text{def}}{=} & \{\, \rho[V \mapsto v] \mid \rho \in R,\ v \in \mathsf{E}[\![\, e \,]\!]\, \rho \,\} \\
\mathsf{S}[\![\, c? \,]\!]\, R & \stackrel{\text{def}}{=} & \{\, \rho \in R \mid \mathrm{true} \in \mathsf{C}[\![\, c \,]\!]\, \rho \,\} \\
\mathsf{S}[\![\, \textbf{if}\ c\ \textbf{then}\ s_1\ \textbf{else}\ s_2 \,]\!]\, R & \stackrel{\text{def}}{=} & \mathsf{S}[\![\, s_1 \,]\!]\, (\mathsf{S}[\![\, c? \,]\!]\, R) \cup \mathsf{S}[\![\, s_2 \,]\!]\, (\mathsf{S}[\![\, \neg c? \,]\!]\, R) \\
\mathsf{S}[\![\, \textbf{while}\ c\ \textbf{do}\ s \,]\!]\, R & \stackrel{\text{def}}{=} & \mathsf{S}[\![\, \neg c? \,]\!]\, (\mathrm{lfp}\ \lambda I.R \cup \mathsf{S}[\![\, s \,]\!]\, (\mathsf{S}[\![\, c? \,]\!]\, I))
\end{array}
$$

- $\mathsf{S}[\![\, stat \,]\!]\, R$

  set of all possible states at the program end
  when starting in a state in R

Backward denotational concrete semantics

$\overleftarrow{S}[\![\, stat \,]\!] : \mathcal{P}(\mathcal{E}) \to \mathcal{P}(\mathcal{E})$

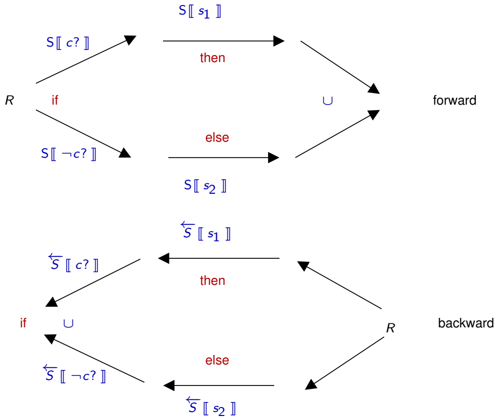
$$
\begin{array}{lcl}
\overleftarrow{S}[\![\, \textbf{skip} \,]\!]\, F & \stackrel{\text{def}}{=} & F \\
\overleftarrow{S}[\![\, s_1;\ s_2 \,]\!]\, F & \stackrel{\text{def}}{=} & \overleftarrow{S}[\![\, s_1 \,]\!]\,(\overleftarrow{S}[\![\, s_2 \,]\!]\, F) \\
\overleftarrow{S}[\![\, V \leftarrow e \,]\!]\, F & \stackrel{\text{def}}{=} & \{\, \rho \mid \exists v \in \mathsf{E}[\![\, e \,]\!]\, \rho \colon \rho[V \mapsto v] \in F \,\} \\
\overleftarrow{S}[\![\, c? \,]\!]\, F & \stackrel{\text{def}}{=} & \{\, \rho \in F \mid \text{true} \in \mathsf{C}[\![\, c \,]\!]\, \rho \,\} \\
\overleftarrow{S}[\![\, \textbf{if}\ c\ \textbf{then}\ s_1\ \textbf{else}\ s_2 \,]\!]\, F & \stackrel{\text{def}}{=} & \overleftarrow{S}[\![\, c? \,]\!]\,(\overleftarrow{S}[\![\, s_1 \,]\!]\, F) \cup \overleftarrow{S}[\![\, \neg c? \,]\!]\,(\overleftarrow{S}[\![\, s_2 \,]\!]\, F) \\
\overleftarrow{S}[\![\, \textbf{while}\ c\ \textbf{do}\ s \,]\!]\, F & \stackrel{\text{def}}{=} & \text{lfp}\ \lambda I.\, \overleftarrow{S}[\![\, \neg c? \,]\!]\, F \cup \overleftarrow{S}[\![\, c? \,]\!]\,(\overleftarrow{S}[\![\, s \,]\!]\, I)
\end{array}
$$

- $\overleftarrow{S}[\![\, stat \,]\!]\, F$
  set of all the states at the program entry
  such that at least one execution ends in a state in F
- $\iota \in \overleftarrow{S}[\![\, stat \,]\!]\, \{\phi\} \iff \phi \in \mathsf{S}[\![\, stat \,]\!]\, \{\iota\}$

Note: – the order of statements reversed (s_2 before s_1, s_1 before $c?$, etc.)
– $\overleftarrow{S}[\![\, c? \,]\!]$ is unchanged

Concrete semantics: flow intuition

Intuition: information propagation for **if** $\cdots$ **then** $\cdots$ **else**



$$\mathsf{S}[\![\, \textbf{if } c \textbf{ then } s_1 \textbf{ else } s_2 \,]\!]\, R = \mathsf{S}[\![\, s_1 \,]\!]\, (\mathsf{S}[\![\, c? \,]\!]\, R) \cup \mathsf{S}[\![\, s_2 \,]\!]\, (\mathsf{S}[\![\, \neg c? \,]\!]\, R)$$

$$\overleftarrow{S}[\![\, \textbf{if } c \textbf{ then } s_1 \textbf{ else } s_2 \,]\!]\, F = \overleftarrow{S}[\![\, c? \,]\!]\, (\overleftarrow{S}[\![\, s_1 \,]\!]\, F) \cup \overleftarrow{S}[\![\, \neg c? \,]\!]\, (\overleftarrow{S}[\![\, s_2 \,]\!]\, F)$$

Backward abstraction denotational semantics

Goal: construct $\overleftarrow{S}^\sharp[\![\, stat \,]\!]$ that soundly approximates $\overleftarrow{S}[\![\, stat \,]\!]$

We can define, by induction:

$\overleftarrow{S}^\sharp[\![\, \textbf{skip} \,]\!]\, F^\sharp \stackrel{\text{def}}{=} F^\sharp$

$\overleftarrow{S}^\sharp[\![\, s_1;\ s_2 \,]\!]\, F^\sharp \stackrel{\text{def}}{=} \overleftarrow{S}^\sharp[\![\, s_1 \,]\!]\,(\overleftarrow{S}^\sharp[\![\, s_2 \,]\!]\, F^\sharp)$

$\overleftarrow{S}^\sharp[\![\, c? \,]\!]\, F^\sharp \stackrel{\text{def}}{=} \mathsf{S}^\sharp[\![\, c? \,]\!]\, F^\sharp$

$\overleftarrow{S}^\sharp[\![\, \textbf{if}\ c\ \textbf{then}\ s_1\ \textbf{else}\ s_2 \,]\!]\, F^\sharp \stackrel{\text{def}}{=} \overleftarrow{S}^\sharp[\![\, c? \,]\!]\,(\overleftarrow{S}^\sharp[\![\, s_1 \,]\!]\, F^\sharp) \cup^\sharp \overleftarrow{S}^\sharp[\![\, \neg c? \,]\!]\,(\overleftarrow{S}^\sharp[\![\, s_2 \,]\!]\, F^\sharp)$

$\overleftarrow{S}^\sharp[\![\, \textbf{while}\ c\ \textbf{do}\ s \,]\!]\, F^\sharp \stackrel{\text{def}}{=} \lim \lambda I^\sharp . I^\sharp \mathbin{\triangledown} (\overleftarrow{S}^\sharp[\![\, \neg c? \,]\!]\, F^\sharp \cup^\sharp \overleftarrow{S}^\sharp[\![\, c? \,]\!]\,(\overleftarrow{S}^\sharp[\![\, s \,]\!]\, I^\sharp))$

Abstract operators:

- we can reuse $\cup^\sharp$, $\triangledown$ and $\mathsf{S}^\sharp[\![\, c? \,]\!]$
- only $\overleftarrow{S}^\sharp[\![\, V \leftarrow e \,]\!]$ needs to be defined on a per-domain basis
- assuming forward-backward combination,
  we can use the pre-condition $X^\sharp$ discovered in the forward phase:

  $$\overleftarrow{S}^\sharp[\![\, X \leftarrow e \,]\!]\,(X^\sharp, F^\sharp) \text{ approximates } \gamma(X^\sharp) \cap \overleftarrow{S}[\![\, X \leftarrow e \,]\!]\, \gamma(F^\sharp)$$

  (makes $\overleftarrow{S}^\sharp[\![\, X \leftarrow e \,]\!]$ easier to implement and more precise: see next slide)

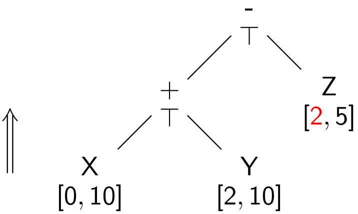
Backward interval assignment

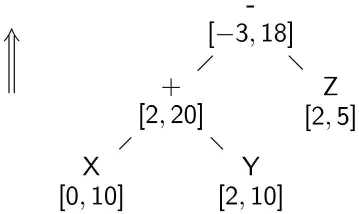
Example: $\overleftarrow{S}^\sharp[\![\, X \leftarrow X + Y - Z \,]\!](X^\sharp, F^\sharp)$

- before the assignment $X^\sharp = \{\, X \mapsto [0, 10], Y \mapsto [2, 10], Z \mapsto [1, 5] \,\}$
- after the assignment $F^\sharp = \{\, X \mapsto [-6, 6], Y \mapsto [2, 10], Z \mapsto [2, 6] \,\}$
- returns: subset $X^{\sharp\prime}$ of $X^\sharp$ that result in $F^\sharp$ after assignment

Similar to test.

Firstly: bottom-up evaluation





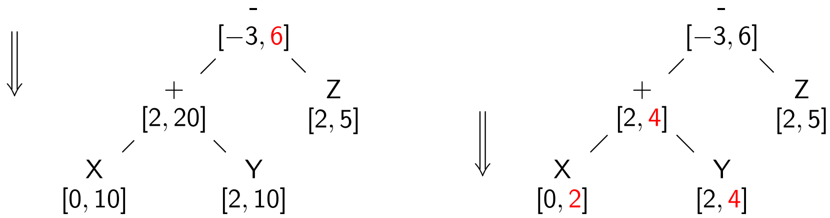
Backward interval assignment (cont.)

Example: $\overleftarrow{S}^\sharp[\![\, X \leftarrow X + Y - Z \,]\!](X^\sharp, F^\sharp)$

- before the assignment $X^\sharp = \{\, X \mapsto [0,10], Y \mapsto [2,10], Z \mapsto [1,5] \,\}$
- after the assignment $F^\sharp = \{\, X \mapsto [-6,6], Y \mapsto [2,10], Z \mapsto [2,6] \,\}$
- returns: subset $X^{\sharp\prime}$ of $X^\sharp$ that result in $F^\sharp$ after assignment

Similar to test.

Secondly: top-down refinement



returns $X^{\sharp\prime} = \{\, X \mapsto [0,2], Y \mapsto [2,4], Z \mapsto [2,5] \,\}$

Conclusion

Conclusion

Summary:

- systematic design of abstract operators (Galois connection)
- optimal and non-optimal (practical) abstractions
- abstract tests through abstract refinement operators
- backward assignment
- fixpoint approximation by iteration with widening
  $\Longrightarrow$ ensure termination even for infinite-height domains!
- application to interval analysis
  (but can be used on any non-relational analysis, e.g., constants)

Next lecture: relational domains (polyhedra)

Practical session: implement the interval domain

(also useful for the project)

TP implementation suggestions

Summary of the (forward) abstract semantics

$$\mathsf{S}^\sharp[\![\, \textbf{skip} \,]\!]\, X^\sharp \stackrel{\text{def}}{=} X^\sharp$$

$$\mathsf{S}^\sharp[\![\, s_1; s_2 \,]\!]\, X^\sharp \stackrel{\text{def}}{=} \mathsf{S}^\sharp[\![\, s_2 \,]\!]\, (\mathsf{S}^\sharp[\![\, s_1 \,]\!]\, X^\sharp)$$

$$\mathsf{S}^\sharp[\![\, V \leftarrow e \,]\!]\, X^\sharp \stackrel{\text{def}}{=} \begin{cases} X^\sharp[V \mapsto \mathsf{E}^\sharp[\![\, e \,]\!]\, X^\sharp] & \text{if } \mathsf{E}^\sharp[\![\, e \,]\!]\, X^\sharp \neq \bot \\ \dot\bot & \text{if } \mathsf{E}^\sharp[\![\, e \,]\!]\, X^\sharp = \bot \end{cases}$$

$$\mathsf{S}^\sharp[\![\, \textbf{if } c \textbf{ then } s_1 \textbf{ else } s_2 \,]\!]\, X^\sharp \stackrel{\text{def}}{=} \mathsf{S}^\sharp[\![\, s_1 \,]\!]\, (\mathsf{S}^\sharp[\![\, c? \,]\!]\, X^\sharp)\ \dot\cup^\sharp\ \mathsf{S}^\sharp[\![\, s_2 \,]\!]\, (\mathsf{S}^\sharp[\![\, \neg c? \,]\!]\, X^\sharp)$$

$$\mathsf{S}^\sharp[\![\, \textbf{while } c \textbf{ do } s \,]\!]\, X^\sharp \stackrel{\text{def}}{=} \mathsf{S}^\sharp[\![\, \neg c? \,]\!]\, (\lim \lambda I^\sharp . I^\sharp\ \dot\triangledown\ (X^\sharp\ \dot\cup^\sharp\ \mathsf{S}^\sharp[\![\, s \,]\!]\, (\mathsf{S}^\sharp[\![\, c? \,]\!]\, I^\sharp)))$$

$\mathsf{E}^\sharp[\![\, e \,]\!]$ by induction on the syntax of expressions

$\mathsf{S}^\sharp[\![\, c? \,]\!]$ by bottom-up evaluation followed by top-down refinement
(for the project only, not required in the practical session)

Value domain signature

```
module type VALUE_DOMAIN = sig
  type t                             // { [a,b] | a ∈ ℤ ∪ {−∞}, b ∈ ℤ ∪ {+∞}, a ≤ b } ∪ {⊥}
  (* constructors *)
  val top:   t                       // [−∞,+∞]
  val bottom:  t                     // ⊥
  val cost:  int -> t                // c ↦ [c,c]
  val rand:  int -> int -> t         // l ↦ h ↦ [l,h]
  (* order *)
  val subset:  t -> t -> bool        // ⊑
  (* set-theoretic operations *)
  val join:  t -> t -> t             // ∪♯
  val meet:  t -> t -> t             // ∩♯
  val widen:  t -> t -> t            // ▽
  (* arithmetic operations *)
  val neg:  t -> t                   // unary −♯
  val add:  t -> t -> t              // +♯
  val sub:  t -> t -> t              // −♯
  val mul:  t -> t -> t              // ×♯
  val div:  t -> t -> t              // /♯
  (* boolean test *)
  val leq:  t -> t -> t * t          // [a,b] ↦ [c,d] ↦ ([a,min(b,d)],[max(a,c),d])
end
```

Interval domain implementation details

```
module Intervals = (struct
  type bound = Int of Z.t | PINF | MINF                  // ℤ ∪ {+∞, −∞}
  type t = Itv of bound * bound | BOT                    // { [a, b] | a ≤ b } ∪ {⊥}

  (* utilities *)
  val bound_cmp:  bound -> bound -> int                  // as OCaml's compare
  val bound_neg:  bound -> bound                         // unary −
  val bound_add:  bound -> bound -> bound                // +
  ...
  val strict:  (bound -> bound -> t) -> t -> t           // maps ⊥ to ⊥

  (* domain implementation *)
  let neg = strict                                       // unary −♯
    (fun a b -> Itv (bound_neg b, bound_neg a))

  let subset a b = match a,b with                        // ⊑
  | BOT,_ -> true | _,BOT -> false
  | Itv (a,b), Itv (c,d) ->
      bound_cmp a c >= 0 && bound_cmp b d >= 0

  ...
end:  VALUE_DOMAIN)
```

Environment domain signature

```
module type ENVIRONMENT_DOMAIN = sig
  type t                                    // ℰ♯
  (* constructors *)
  val init:  id list -> t                   // ∀V ∈ 𝕍: ρ(V) = 0
  (* abstract operators *)
  val assign:  t -> id -> expr -> t         // S♯⟦ id ← expr ⟧
  val compare:  t -> expr -> expr -> t      // S♯⟦ expr ≤ expr? ⟧
  (* set-theoretic operations *)
  val join:  t -> t -> t                    // ∪̇♯
  val meet:  t -> t -> t                    // ∩̇♯
  val widen:  t -> t -> t                   // ∇̇
  (* order *)
  val subset:  t -> t -> bool               // ⊑̇
end
```

Environment domain implementation details

```
module NonRelational(V : VALUE_DOMAIN) = (struct
  module Map = Mapext.Make                          // maps
    (struct type t = id let compare = compare end)
  type env = V.t Map.t                              // 𝕍 → (𝔻 \ {⊥})
  type t = Env of env | BOT                         // ℰ♯ ≝ (𝕍 → (𝔻 \ {⊥})) ∪ {⊥̇}
  (* utilities *)
  val eval:   env -> expr -> V.t                    // E♯⟦ expr ⟧
  val is_bot:   V.t -> bool                         // whether γ(v♯) = ∅
  val strict:  (env -> t) -> t -> t                 // maps ⊥̇ to ⊥̇
  (* operators *)
  let join a b = match a,b with                     // ∪̇♯
  | BOT,x | x,BOT -> x
  | Env m,Env n -> Env (Map.map2z (fun _ x y -> V.join x y) m n)
  ...
end: ENVIRONMENT_DOMAIN)
```

Generic functor to lift a VALUE_DOMAIN to an ENVIRONMENT_DOMAIN

Uses a Map as data-structure for environment (functional array)

and a binary map iterator map2z f
(optimized for idempotent functions: $\forall x$: f $k\ x\ x = x$)