ABSTRACT

Explicit hybrid systems modelers like Simulink/Stateflow allow for programming both discrete- and continuous-time behaviors with complex interactions between them. A key issue in their compilation is the static detection of algebraic or causality loops. Such loops can cause simulations to deadlock and prevent the generation of statically scheduled code.

This paper addresses this issue for a hybrid modeling language that combines synchronous data-flow equations with Ordinary Differential Equations (ODEs). We introduce the operator \texttt{last}($x$) for the left-limit of a signal $x$. This operator is used to break causality loops and permits a uniform treatment of discrete and continuous state variables. The semantics relies on non-standard analysis, defining an execution as a sequence of infinitesimally small steps. A signal is deemed \emph{causally correct} when it can be computed sequentially and only changes infinitesimally outside of announced discrete events like zero-crossings. The causality analysis takes the form of a type system that expresses dependences between signals. In well-typed programs, signals are provably continuous during integration provided that imported external functions are also continuous.

The effectiveness of this system is illustrated with several examples written in ZÉLUS, a Lustre-like synchronous language extended with hierarchical automata and ODEs.

Categories and Subject Descriptors
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Keywords
Hybrid systems; Synchronous programming languages; Type systems; Block diagrams; Static analysis.

1. CAUSALITY AND SCHEDULING

Tools for modeling hybrid systems \cite{7} such as Modelica, LabVIEW, and Simulink/Stateflow are now rightly understood and studied as programming languages. Indeed, models are used not only for simulation, but also for test-case generation, formal verification and translation to embedded code. This explains the need for formal operational semantics for specifying their implementations and proving them correct \cite{15, 9}.

The underlying mathematical model is the synchronous parallel composition of stream equations, Ordinary Differential Equations (ODEs), hierarchical automata, and imperative features. While each of these features taken separately is precisely understood, real languages allow them to be combined in sophisticated ways. One major difficulty in such languages is the treatment of causality loops.

Causality or \emph{algebraic} loops \cite{20, 2-34} pose problems of well-definedness and compilation. They can lead to mathematically unsound models. They can prevent simulators from statically ensuring the existence of a fixed point, and compilers from generating statically scheduled code. The static detection of such loops, termed \emph{causality analysis}, has been studied and implemented since the mid-1980s in synchronous data-flow language compilers \cite{12, 13, 1}. The classical and simplest solution is to reject cycles (feed-back loops) which do not cross a unit delay. For instance, the Lustre-like equations:

\[
x = 0 \rightarrow \text{pre} \ y \quad \text{and} \quad y = \text{if } c \text{ then } x + 1 \text{ else } x
\]

define the two sequences $(x_n)_{n \in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$ such that:

\[
x(0) = 0 \quad y(n) = \text{if } c(n) \text{ then } x(n) + 1 \text{ else } x(n)
\]

\[
x(n) = y(n - 1)
\]

They are causally correct since the feedback loop for $x$ contains a unit delay \texttt{pre} $y$ (‘previous’). Replacing \texttt{pre} $y$ with $y$ gives two non-causal equations. Causally correct equations can be statically scheduled to produce a sequential, loop-free \emph{step} function. Below is an excerpt of the C code generated by the Heptagon compiler \cite{11} of Lustre:

\[
\text{if (self->v_1) (} x = 0; \} \text{ else (} x = \text{self->v}_2;); \\
\text{if (c) (} y = x+1; \} \text{ else (} y = x;); \\
\text{self->v}_2 = y; \text{self->v}_1 = \text{false;}
\]

1. \url{http://www.modelica.org}
2. \url{http://www.ni.com/labview}
3. \url{http://www.mathworks.com/products/simulink}
4. The unit delay $0 \rightarrow \text{pre}()$, initialized to 0, is sometimes written as $0 \cdot \text{byp}$ (‘0 followed by’), or in SIMULINK: $\frac{1}{z}$.
It computes current values of $x$ and $y$ from that of $c$. The internal memory of function $\text{step}$ is in $\text{self}$, with $\text{self->v}_1$ initialized to true and set to false (to encode the LUSTRE operator $\rightarrow$) and $\text{self->v}_2$ storing the value of $\text{pre y}$.

ODEs with resets: Consider now the situation of a program defining continuous-time signals only, made of ODEs and equations. For example:

$$\text{der y} = z \text{ init 4.0 and z} = 10.0 - 0.1 \cdot y \text{ and k} = y + 1.0$$

defines signals $y$, $z$ and $k$, where for all $t \in \mathbb{R}^+$, $\frac{dy}{dt}(t) = z(t)$, $y(0) = 4$, $z(t) = 10 - 0.1 \cdot y(t)$, and $k(t) = y(t) + 1$. This program is causal simply because it is possible to generate a sequential function $\text{derivative}(y) = \text{let} \ z = 10 - 0.1 \cdot y \in \text{z} \text{ returning the current derivative of } y$ and initial value $4$ for $y$ so that a numeric solver $[8]$ can compute a sequence of approximations $y(t_n)$ for increasing values of time $t_n \in \mathbb{R}^+$ and $n \in \mathbb{N}$. Thus, for continuous-time signals, integrators break algebraic loops just as delays do for discrete-time signals.

Can we reuse the simple justification we used for data-flow equations to justify that the above program is causal? Consider the value that $y$ would have if computed by an ideal solver $\text{[8]}$ where $y(t_n)$ and $y(n + 1)$ are causally defined, for all $n \in \mathbb{N}$, yielding a unique solution for $y$ and $k$. The equations are thus causally correct.

Troubles arise when ODEs interact with discrete-time constructs, for example when a reset occurs at every occurrence of an event. E.g., the sawtooth signal $y : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $\frac{dy}{dt}(t) = 1$ and $y(t) = 0$ if $t \in \mathbb{N}$ can be defined by an ODE with reset:

$$\text{der y} = 1.0 \text{ init 0.0 reset up(y - 1.0)} \rightarrow 0.0$$

where $y$ is initialized with $0.0$, has derivative $1.0$, and is reset to $0.0$ every time the zero-crossing $\text{up}(y - 1.0)$ is true, that is, whenever $y - 1.0$ crosses $0.0$ from negative to positive.

Is this program causal? Again, consider the value $y$ would have were it calculated by an ideal solver taking infinitesimal steps of length $\partial$. The value of $\text{y}(n)$ at instant $n\partial$, for all $n \in \mathbb{N}$ would be:

- $y(0) = 0$
- $y(n) = \text{if } \text{z}(n) \text{ then } 0.0 \text{ else } \text{ly}(n)$
- $\text{ly}(n) = \text{y}(n - 1) + \partial$
- $\text{c}(n) = \text{c}(n) \land \neg \text{c}(n - 1)$

This set of equations is clearly not causal: the value of $\text{y}(n)$ depends instanstantaneously on $\text{z}(n)$ which itself depends on $\text{y}(n)$. There are two ways to break this cycle: (a) consider that the effect of the zero-crossing is delayed by one cycle, that is, the test is made on $\text{z}(n - 1)$ instead of on $\text{z}(n)$, or, (b) distinguish the zero-crossing value of $\text{y}(n)$ from the value it would have had were there no reset, namely $\text{ly}(n)$. Testing a zero-crossing of $\text{ly}$ (instead of $y$),

$$\text{c}(n) = (\text{ly}(n) - 1) \geq 0,$$

gives a program that is causal since $\text{y}(n)$ no longer depends instantanenously on itself. We propose writing this: $^5$

$$\text{der y} = \text{c init v}_0 \text{ for y = } \frac{1}{\partial}(c) \text{ initialized to } v_0 \text{ in } \text{SIMULINK}.$$
Figure 1: A miscompiled Simulink model (R2009b) and that Integrator1 is reset to 24 (= −6·−4). The latter result is surprising since, at this instant, the state port of Integrator0 should also be equal to 2, and we would thus expect Integrator1 to be reset to −8 (= 2·−4)!

The Simulink implementation does not satisfy its documented behavior [19, 2-685]. Inspecting the C function which computes the current outputs, mdlOutputs in Figure 2, the code of the two integrators appears in an incorrectly scheduled sequence.\(^5\) At the instant of the zero-crossing (conditions ssIsMajorTimeStep(S) and zcEvent are true), the state port of Integrator0 (stored in sGetContStates(S) -> Integrator0_CSTATE) is reset using the state port value of Integrator1. Thus, Integrator1 does not read the value of Integrator0’s state port (that is \(x(n−1)\)) but rather the current value \(\hat{x}(n)\) leading to an incorrect output. The Simulink model is not correctly compiled—it needs another variable to store the value of \(\hat{x}(n−1)\), just as a third variable is normally needed to swap the values of two others.

We argue that such a program should either be scheduled correctly or give rise to a warning or error message. Providing a means to statically detect and explain causality issues in a hybrid model is a key motivation of this paper.

Any loop in Simulink, whether of discrete- or continuous-time signals, can be broken by inserting the so-called memory block [19, 2-831].\(^6\) If \(x\) is a signal, \(\text{mem}(x)\) is a piecewise constant signal which refers to the value of \(x\) at the previous integration step (or major step). If those steps are taken at increasing instants \(t_i \in \mathbb{R}\), \(\text{mem}(x)(t_0) = x_0\) where \(t_0 = 0\) and \(x_0\) is an explicitly defined initial value, \(\text{mem}(x)(t_i) = x(t_{i−1})\) for \(i > 0\) and \(\text{mem}(x)(t_i + \delta) = x(t_{i−1})\) for \(0 \leq \delta < t_{i+1} – t_i\). As integration is performed globally, \(\text{mem}(y)\) may cause strange behaviors as the previous value of a continuously changing signal \(x\) depends precisely on when the solver decides to stop! Writing \(\text{mem}(y)\) is thus unsafe in general [3].\(^{10}\) There is nonetheless a situation where the use of the memory block is mandatory and still safe: The program only refers to the previous integration step during a discrete step. This situation is very common: it is typically that of a system with continuous modes \(M_1\) and \(M_2\) producing a signal \(x\), with each of them being started with the value computed previously by the solver, and \(\text{mem}(x)\) being used to pass a value from one mode to the other. Instead of using the unsafe operator \(\text{mem}(x)\), we could better refer to the left limit of \(x\), and write it again \(\text{last}(x)\). Yet, the unrestricted use of this operation may cause a new kind of causality loop which has to be statically rejected. Consider the following equation activated on a continuous time base:

\[
y = -1.0 \cdot (\text{last}(y)) + \text{init}(y) = 1.0
\]

which defines, for all \(n \in \mathbb{N}\), the sequence \(y(n)\) such that:

\[
y(n) = - y(n-1) + y(0) = 1
\]

Indeed, this differs little from the equation \(y = -1.0 \cdot y\). Even though \(y(n)\) can be computed sequentially, its value does not increase infinitesimally at each step, that is, \(y\) is not left continuous even though no zero-crossing occurs. For any time \(t \in \mathbb{R}\), the set \(\{n \theta | n \in \mathbb{N} \land n \theta \approx t \land y(n) \neq y(n+1)\}\) is infinite. Thus, the value of \(y(t)\) at any standard instant \(t \in \mathbb{R}\) is undefined.

Contribution and organization of the paper: This paper presents the causality problem for a core language

\(^5\)The same issue exists in release R2014a.

\(^6\)In contrast, the application of a unit delay \(\frac{1}{2}\) to a continuous-time signal is statically detected and results in a warning.

\(^{10}\)Quoting the Simulink manual (http://www.mathworks.com/help/simulink/mdlref/memory.html), “Avoid using the Memory block when both these conditions are true: - Your model uses the variable-step solver ode15s or ode113. - The input to the block changes during simulation.”
that combines Lustre-like stream equations, ODEs with reset and a basic control structure. The operator \texttt{last}(x) stands for the previous value of \(x\) in non-standard semantics and coincides with its left-limit when \(x\) is left-continuous. This operation plays the role of a delay but is safer than \sto{x} as its semantics does not depend on when a particular solver decides to stop. When \(x\) is a continuous-state variable, it coincides with the so-called Simulink state port. We develop a non-standard semantics following [4] and a compile-time causality analysis in order to detect possible instantaneous loops. The static analysis takes the form of a type system, reminiscent of the simple Hindley-Milner type system for core ML [21]. A type signature for a function expresses the instantaneous dependencies between its inputs and outputs. We prove that well typed programs only progress by infinitely small steps outside of zero-crossing events, that is, signals are continuous during integration provided imported operations applied point-wise are continuous. We are not aware of such a correctness theorem based on a static typing discipline for hybrid models.

The presented material is implemented in Zélus [6], a synchronous Lustre-like language extended with ODEs. Moreover, all examples in the paper are written in Zélus.

The paper is organized as follows. Section 2 introduces a core synchronous language with ODEs. Section 3 presents its semantics based on non-standard analysis. Section 4 presents a type system for causality and Section 5 a major property: any well-typed program is proved not to have any discontinuities during integration. Section 6 discusses related work and we conclude in Section 7.

2. A SYNCHRONOUS LANGUAGE + ODES

We now introduce a kernel language. It is not intended to be a full language but a minimal one for the purpose of the present paper. It includes data-flow equations, ordinary differential equations and control structures. Its syntax is:

\[
\begin{align*}
    d & ::= \text{let } x = e | \text{let } k \ f(p) = e \ w h e r e \ E | d ; d \\
    e & ::= x | v | \text{op}(e) | e \ f b y \ e | \text{last}(x) | f(e) | (e, e) | \text{up}(e) \\
    p & ::= (p, p) | x \\
    E & ::= () | x = e | \text{init } x = e | \text{next } x = e | \text{der } x = e \\
    & \quad \mid E \ and \ \bar{E} | \text{local } x \ in \ E | \text{if } e \ then \ E \ else \ E \\
    k & ::= \mathbb{D} | \mathbb{C} | \mathbb{A}
\end{align*}
\]

A program is a sequence of definitions (\(d\)), that either bind the value of expression \(e\) to \(x\) (\texttt{let } \(x = e\)) or define a function (\texttt{let } \(k \ f(p) = e \ where \ E\)). In a function definition, \(k\) is the kind of the function \(f\), \(p\) denotes formal parameters, and the result is the value of an expression \(e\) which may contain variables defined in the auxiliary equations \(E\). There are three kinds of function: \(k = \mathbb{A}\) means that \(f\) is a \texttt{combinational} function (typically a function imported from the host language, e.g., addition); \(k = \mathbb{D}\) means that \(f\) is a \texttt{sequential} function that must be activated at discrete instants (typically a Lustre function with an internal discrete state); \(k = \mathbb{C}\) denotes a hybrid function that may contain ODEs and which must be activated continuously.

An expression \(e\) can be a variable \(x\), an immediate value \(v\), e.g., a boolean, integer or floating point value, the point-wise application of an imported function \(\text{op}(e)\) such as +, * or \texttt{not}(), an initialized unit delay \((e_1 \ \text{fb} \ e_2)\), the left-limit of a signal \((\text{last}(x))\), a function application \(f(e)\), a pair \((e, e)\) or a rising zero-crossing detection \((\text{up}(e))\), which, in this language kernel, is the only basic construct to produce an event from a continuous-time signal \(e\). A pattern \(p\) is a tree structure of identifiers \((x)\); an equality stating that a pattern equals the value of an expression at every instant \((x = e)\); the initialization of a state variable \(x\) with a value \(e\) \((\text{init } x = e)\); the value of a state variable \(x\) at the next instant \((\text{next } x = e)\); or, the current value of the derivative of \(x\) \((\text{der } x = e)\). An equation can also be the conjunction of two sets of equations \((E_1 \ and \ E_2)\); the declaration that a variable \(x\) is defined within, and local to, a set of equations \((\text{local } x \ in \ E)\); a conditional that activates a branch according to the value of a boolean expression \((if \ e \ then \ E_1 \ else \ E_2)\), and a variant that operates on an event expression \((present \ e \ then \ E_1 \ else \ E_2)\).

Notational abbreviations:

\[(a) \text{ if } e \ then \ E \quad def \ if \ e \ then \ E_1 \ else \ ().
\[(b) \text{ present } e \ then \ E \quad def \ present \ e \ then \ E \ else \ ().
\[(c) \text{ der } x = e \ \text{init } e_0 \quad def \ \text{init } x = e_0 \ and \ \text{der } x = e
\[(d) \text{ der } x = e \ \text{init } e_0 \ \text{reset } z \to e_1 \quad def \ \text{init } x = e_0 \ and \ \text{present } z \ then \ x = e_1 \ else \ \text{der } x = e
\]

Equations \((E)\) must be in Static Single Assignment (SSA) form: every variable has a unique definition at every instant.

3. NON-STANDARD SEMANTICS

3.1 Semantics

Let \(\mathbb{R}\) and \(\mathbb{N}\) be the non-standard extensions of \(\mathbb{R}\) and \(\mathbb{N}\). \(\mathbb{N}\) is totally ordered and every set bounded from above (resp. below) has a unique maximal (resp. minimal) element. Let \(\partial \in \mathbb{R}\) be an infinitesimal, i.e., \(\partial > 0, \partial = 0\). Let the global time base or base clock be the infinite set of instants:

\[T_0 = \{t_n = n \partial | n \in \mathbb{N}\}\]

\(T_0\) inherits a total order from \(\mathbb{N}\); in addition, for each element of \(\mathbb{R}\) there exists an infinitesimally close element of \(T_0\). Whenever possible we leave \(\partial\) implicit and write \(T\) instead of \(T_0\). Let \(T = \{t_n | n \in \mathbb{N}\} \subseteq T\). \(T(i)\) stands for \(t_i\), the \(i\)-th element of \(T\). In the sequel, we only consider subsets of the time base \(T\) obtained by sampling a time base on a boolean condition or a zero-crossing event. Any element of a time base will thus be of the form \(k \partial\) where \(k \in \mathbb{N}\). If \(T \subseteq T\), we write \(\ast T(t)\) for the immediate predecessor of \(t\) in \(T\) and \(T^*(t)\) for the immediate successor of \(t\) in \(T\). For an instant \(t\), we write its immediate predecessor and successor as, respectively, \(\ast t\) and \(\ast t^*\), rather than as \(\ast T(t)\) and \(T^*(t)\). For \(t \in T \subseteq T\), neither \(\ast t\) nor \(\ast t^*\) necessarily belong to \(T\). \(\min(T)\) is the minimal element of \(T\) and \(t \leq_T t'\) means that \(t\) is a predecessor of \(t'\) in \(T\).

**Definition 1 (Signals).** Let \(V_\perp = V + \{\perp\}\) where \(V\) is a set. \(S(V) = \perp \mapsto V_\perp\) is the set of signals. A signal \(x : T \mapsto V_\perp\) is a notation on from a time base \(T \subseteq R\) to \(V_\perp\). Moreover, for all \(t \in T\), \(x(t) = \perp\). If \(T\) is a time base, \(x(T(n))\) and \(x(t_n)\) are the value of \(x\) at instant \(t_n\) where
n ∈ ℤ is the n-th element of T. The clock of a signal x is clock(x) = \{ t ∈ T \mid x(t) \neq \perp \}. “

Sampling: Let bool = \{false, true\} and x : T → bool. The sampling of T according to x, written T on x, is the subset of instants defined by:

\[ T \mid x = \{ t \mid (t \in T) \land x(t) = \text{true} \}. \]

Note that as T on x ⊆ T, it is also totally ordered. The zero-crossing of x : T → ℝ is up(x) : T → bool. To emphasize that up(x) is defined only for t ∈ T, we write its value at time t as up(x)(T)(t). For t ∉ T, up(x)(T)(t) = ⊥. In the definition below < is the total order on ℝ.

\[ \text{up}(x)(T)(t_0) = \text{false} \text{ where } t_0 = \min(T) \]
\[ \text{up}(x)(T)(t) = \exists n \in \mathbb{N}, n \geq 1. \land (x(t-n\delta) < 0) \land \ldots \land (x(t-(n-1)\delta) = 0) \land (x(t-n\delta) > 0) \]
where t ∈ T

The above definition means that a zero-crossing on x occurs when x goes from a strictly negative to a strictly positive value, possibly with intermediate values equal to 0.

Let V be a set of values closed under product and sum. V is its non-standard extension such that “V₁ × V₂” = “V₁ × V₂” = “V” for any finite set V. “⊆” is V = {⊥} with ⊥ as the minimum element. Let L = \{x₁, x₂, ..., xₙ\} be a set of local variables and L₀ = \{f₁, ..., fₙ\} a set of global variables. An environment associates names to values. A local environment ρ and a global environment G map names to signals and function values:

\[ \rho : L \rightarrow S(V) \quad G : L₀ \rightarrow (S(V) \rightarrow S(V)) \]

Operations on environments: Consider ρ₁ and ρ₂.

- \((ρ₁ + ρ₂)(x)(t) = ρ₁(x)(t) \text{ if } ρ₂(x)(t) = ⊥, \quad ρ₂(x)(t) \text{ if } ρ₁(x)(t) = ⊥, \quad \text{and } \emptyset \text{ otherwise.} \]

- \(ρ = \text{merge}(T)(s)(ρ₁)(ρ₂) \) is the merge of two environments according to a signal s ∈ S(bool). The value of a signal x at instant t ∈ T is the one given by ρ₁ if s(t) is true and that of ρ₂ otherwise. Nonetheless, in case x is not defined in ρ₁ (respectively ρ₂), it implicitly keeps its previous value, that is \(ρ₁(\text{clock}(x)(t)). \) This corresponds to adding an equation \(x = \text{last}(x)\) when no equation is given in one branch of a conditional.

For all x and t ∈ T, \(ρ(x)(t) = ρ₁(x)(t) = ⊥, \) and \(ρ₂(x)(t) = ρ₂(x)(t) \text{ otherwise.} \)

Expressions: Expressions are interpreted as signals and node definitions as functions from signals to signals. For expressions, we define \(*[e]_{G}(T)(t) = \perp, \) if t ∉ T
\[ *[v]_{G}(T)(t) = v, \text{false} \]
\[ *[x]_{G}(T)(t) = ρ(x)(t), \text{false} \]
\[ *[\text{op}(u)]_{G}(T)(t) = \text{let } v, z = *[e]_{G}(T)(t) \text{ in } \text{op}(v), z \]
\[ *[e₁, e₂]_{G}(T)(t) = *[e₁]_{G}(T)(t) \text{ in } \text{let } v₁, z₁ = *[e₁]_{G}(T)(t) \text{ in } \text{let } v₂, z₂ = *[e₂]_{G}(T)(t) \text{ in } (v₁ ∨ v₂, (z₁ ∨ z₂) \]
\[ *[e₁ \text{ fby } e₂]_{G}(T)(t₀) = *[e₁]_{G}(T)(t₀) \text{ if } t₀ = \min(T) \]
\[ *[e₁ \text{ fby } e₂]_{G}(T)(t) = *[e₂]_{G}(T)(T(t)) \text{ otherwise } \]
\[ *[\text{last}(x)]_{G}(T)(t) = ρ(x)(\text{clock}(x)(t)), \text{false} \]
\[ *[f(e)]_{G}(T)(t) = \text{let } s(t'), z(t') = *[e]_{G}(T)(t') \text{ in } \text{let } v', z' = G(f(s)(t)) \text{ in } v', z(t) \lor z' \]
\[ *[\text{up}(e)]_{G}(T)(t) = \text{let } s(t'), z(t') = *[e]_{G}(T)(t') \text{ in } \text{let } v' = \text{up}(s)(T)(t) \text{ in } v', z(t) \lor v' \]

Figure 3: The non-standard semantics of expressions

op(e) is obtained by applying the operation op to e at every instant, an event is raised only if e raises one. An expression \((e₁, e₂)\) returns a pair at every instant and raises an event if either of e₁ or e₂ does. The initial value of a delay e₁ fby e₂ is that of e₁. Afterward, it is the previous value of e₂ according to clock T. E.g., the value of t fby x on clock T is the value x had at the previous instant that T was active. This is not necessarily the previous value of x. On the contrary, last(x) is the previous value of x the last time x was defined. The semantics of f(x) is the application of the function f to the signal value of e, which raises an event when either e or the body of f does. Finally, the semantics of up(e) is given by operator up(\_), which raises a zero-crossing event when either e does or \(\text{up}(s)(T)(t)\) is true.

Equations: If E is an equation, G is a global environment, \(ρ\) is a local environment and T is a time base, \(*[E]_{G}(T) = ρ, \) means that the evaluation of E on the time base T returns a local environment ρ’ and a zero-crossing signal z. As for expressions, the value of E is undefined outside of T, that is, for all t ∉ T, \(ρ’(x)(t) = ⊥ \) and \(z(t) = ⊥ \). For all t ∈ T, \(z(t) = \text{true} \) signals that a zero-crossing occurs at instant t and \(z(t) = \text{false} \) means that no zero-crossing occurred at that instant. The semantics of equations is given in Figure 4, where the following notation is used: if z₁ : T → bool₁ and z₂ : T → bool₂ then \(z_1 ∨ z_2 : T → \text{bool}_1 \land \text{val} \in T, (z₁ ∨ z₂)(t) = z₁(t) ∨ z₂(t) \) if \(z₁(t) \neq ⊥ \) and \(z₂(t) \neq ⊥ \), and otherwise, \(z₁ ∨ z₂)(t) = ⊥ \).

Function definitions: Function definition is our final concern: we must show the existence of fixed points in the sense of Kahn process network semantics based on Scott domains.

The prefix order on signals S(V) indexed by \(\perp\), is defined as: signal x is a prefix of signal y, written \(x \lessdot \perp\) y, if
\[ x(t) \neq y(t) \text{ implies } x'(t) = \bot \text{ for all } t' \text{ such that } t \leq t'. \] The minimum element is the undefined signal \( \bot \) for which \( \forall t \in T, \bot \leq \bot \). When possible, we write \( \bot \) for \( \bot \) and \( x \leq y \) for \( x \leq y \). The symbol \( \lor \) denotes a supremum in the prefix order. A function \( f : S(V) \rightarrow S(V) \) is continuous if \( \forall x \in S(V) \), \( f(x) = f(\bot) \). For every increasing chain of signals, where increasing refers to the prefix order. If \( f \) is continuous, then equation \( x = f(x) \) has a least solution denoted by \( \mathcal{L}(f) \), and equal to \( \mathcal{L}(f) \lor \bot \). We name such continuity on the prefix order Kahn continuity [14].

The prefix order is lifted to environments so that \( \rho \leq \rho' \) iff \( \rho(x) \leq \rho'(x) \). It is lifted to pairs such that \( (x, y) \leq (x', y') \) iff \( x \leq x' \) and \( y \leq y' \).

**Property 1 (Kahn continuity).** Let \( \rho/p \) be an environment, \( G \) a global environment of Kahn-continuous functions and \( T \) a clock. The function:

\[
F : (L \rightarrow S(V)) \times S(\text{bool}) \rightarrow (L \rightarrow S(V)) \times S(\text{bool})
\]

such that:

\[
F(\rho, z) = \rho', z' = [E[G]^{\rho/p}(T) \in \rho', z \text{ or } z']
\]

is Kahn continuous, that is, for any sequence \( (\rho_i, z_i) \) such that:

\[
F(\bigvee_{i \in I} (\rho_i, z_i)) = \bigvee_{i \in I} (F(\rho_i, z_i))
\]

**Proof:** We only provide a sketch. We first need to prove the result for expressions listed in Figure 3. We only review the expressions involving the non-standard semantics in a nontrivial way, as the other cases are routine. Consider \( e_1 \mathbf{fby} e_2 \) and \( \mathbf{last}(x) \). None of these expressions contributes to the second (zero-crossing) field of the semantics, so only the first field matters. In fact, the Kahn continuity of \( e_1 \mathbf{fby} e_2 \) is proved exactly as for \textsc{Lustre} [7], since only the total ordering of the underlying time index matters and the argument lifts without change from \( N \) to \( T \). The same holds for \( \mathbf{last}(x) \), which corresponds to \( \mathbf{pre}(x) \) in \textsc{Lustre} in the lifting from \( N \) to \( T \). The expression \( u/p \) contributes to the second field of the semantics. Formula (1) defining \( u/p \) is causal and thereby Kahn continuous. We then consider the equations of Figure 4. We discuss only next \( x = e \) and \( \mathbf{der} x = e \) since the other cases as handled as in \textsc{Lustre} (including the composition of equations \( E_1 \) and \( E_2 \)). Consider the first field of the semantics. If \( e \) returns the value \( v \) at the considered instant \( t \), then the first field of \( x \) returns \( v \) at the next instant \( t^* \), i.e., \( x(t^*) = v(t) \). Kahn continuity follows directly. The same reasoning holds for \( \mathbf{der} x = e \).

As a consequence, an equation \( (\rho, z) = F(\rho, z) \) admits a least fixed point \( \mathcal{L}(F) = \mathcal{L}(F(\bot, \bot)) \).

The declaration of \( \mathbf{let} k f(p) = e \text{ where } E[G]^{T}(s)(t) \) defines a Kahn-continuous function \( f \) such that:

\[
\mathbf{let} k f(p) = e \text{ where } E[G]^{T}(s)(t) = f(T)(s)(t)
\]

where

\[
E[G]^{T}(s)(t) = \mathbf{let} s'(t'), z'(t') = [E[G]^{\rho/p}(T) \in s'(t'), z(t)] \text{ in } s'(t'), z(t) \lor z'(t)
\]

and with

\[
(p', z') = \mathcal{L}(\rho/p, z) \Rightarrow [E[G]^{\rho/p}(T)]
\]

Yet, Kahn-continuity of \( f \) does not mean that the function computes anything interesting. In particular, the semantics gives a meaning to functions that become 'stuck', like:

\[
\mathbf{let} \mathbf{hybrid} f(x) = y \text{ where } \mathbf{rec} y = y + x
\]

The semantics of \( f \) is \( f(x) = \bot \) since the minimal solution of equation \( y = y + x \) is \( \bot \). The purpose of the causality analysis is to statically reject this kind of program.

---

1. The keyword hybrid stands for \( k = \mathbb{C} \) and node for \( k = \mathbb{D} \).
3.2 Standardization

We now relate the non-standard semantics to the usual super-dense semantics of hybrid systems. Following [18], the execution of a hybrid system alternates between integration steps and discrete steps. Signals are now interpreted as total functions from the time index \( \mathbb{S} = \mathbb{R} \times \mathbb{N} \) to \( V_\perp \). This time index is called super-dense time [18, 15] and is ordered lexically, \( (t, n) \ll (t', n') \) if \( t < t' \) or \( t = t' \) and \( n < n' \). Moreover, for any \( (t, n) \) and \( (t, n') \) where \( n \leq n' \), if \( x(t, n') \neq \perp \) then \( x(t, n) \neq \perp \). A timeline for a signal \( x \) is a function \( N_x : \mathbb{R}_+ \rightarrow \mathbb{N}_\perp \). \( N_x(t) \) is the number of instants of \( x \) that occur at a real date \( t \) and such timelines thus specify a subset of super-dense time \( \mathbb{S}_N_x = \{(t, n) \in \mathbb{S} \mid n \leq N_x(t)\} \). In particular, if \( N_x \) is always 0, then \( \mathbb{S}_N_x \) is isomorphic to \( \mathbb{R}_+ \). For \( t \in \mathbb{R} \) and \( T \subseteq \mathbb{T} \), define:

\[
\text{set}(T)(t) \overset{\text{def}}{=} \{ t' \in T \mid t' \approx t \land t \in \mathbb{T} \} \subseteq \mathbb{T}
\]

that is, the set of all instants infinitely close to \( t \). \( T \) is totally ordered and hence so is \( \text{set}(T)(t) \). Let \( x : T \rightarrow V_\perp \), we now proceed to the definition of the timeline \( N_x \) of \( x \) and the standardization of \( x \), written:

\[
st(x) : \mathbb{R} \times \mathbb{N} \rightarrow V_\perp,
\]

such that \( st(x)(t, n) = \perp \) for \( n > N_x(t) \).

Let \( T' \overset{\text{def}}{=} \text{set}(T)(t) \) and consider:

\[
st(x(T')) \overset{\text{def}}{=} \{ st(x(t')) \mid t' \in T' \}.
\]

(a) If \( st(x(T')) = \{v\} \) then, at instant \( t \), \( x \)'s timeline is \( N_x(t) = 0 \) and its standardization is \( st(x(t, 0)) = v \).

(b) If \( st(x(T')) \) is not a singleton set, then let

\[
Z \overset{\text{def}}{=} \{ t' \mid t' \in T' \land x(t') \not\approx x(T'^\ast(t')) \}
\]

i.e., \( Z \) collects the instants at which \( x \) experiences a non-infinitesimal change. \( Z \) is either finite or infinite:

(i) If \( Z = \{t_{m}, \ldots, t_{s}\} \) is finite, timeline \( N_x(t) = m \) and the standard value of signal \( x \) at time \( t \) is:

\[
\forall n \in \{0, \ldots, m\}, st(x)(t, n) = st(x(t_{m})).
\]

(ii) If \( Z \) is infinite (it may even lack a minimum element), let

\[
N_x(t) = \perp \land \forall n, st(x)(t, n) = \perp
\]

which corresponds to a Zeno behavior.

Our approach differs slightly from [15], where the value of a signal is frozen for \( n > N(t) \). We decide instead to set it to the value \( \perp \). Each approach has its merits. For ours, parts of signals that do not experience jumps are simply indexed by \( (t, 0) \) which we identify with \( t \). In turn, we squander the undefined value \( \perp \) which is usually devoted to Scott-Kahn semantics and causality issues.

3.3 Key properties

We now define two main properties that reasonable programs should satisfy. The first one states that discontinuities do not occur outside of zero-crossing events, that is, signals are continuous during integration. The second one states that the semantics should not depend on the choice of the infinitesimal. These two invariants are sufficient conditions to ensure that a standardization exists.

Invariant 1 (Zero-crossings). An expression \( e \) evaluated under \( G \), \( \rho \) and a base time \( T \) has no discontinuity outside of zero-crossing events. Formally, define \( st(t), z(t) = \{st(t^{\ast})(t), z(t^{\ast})(t)\} \), then \( \forall t', t \in T \) such that \( t \leq t' \):

\[
t \approx t' \Rightarrow (\exists \forall t'' \in T, t \leq t'' \leq t' \land z(t'')) \lor st(t) \approx st(t')
\]

This invariant states that all discontinuities are aligned on zero-crossings, that is, signals must evolve continuously during integration. Discrete changes must be announced to the solver using the construct \( \text{up}(\_\_\_\_) \). Not all programs satisfy the invariant, e.g.,

let hybrid \( f(x\ast y\) where rec \( y = \text{last } y + 1 \) and init \( y = 0 \)

\( f \) takes a single argument \( () \) of type unit and returns a value \( y \). Writing \( y(n) \) for the value of \( y \) at instant \( n \), with \( n \in \mathbb{N} \), we get \( y(0) = 0 \) and \( y(n) = y(n-1) + 1 \). Yet, \( y(n) \neq y(n-1) \) while no zero-crossing is registered for any instant \( n \in \mathbb{N} \). This program will be statically rejected by using the type system developed in the next section.

Invariant 2 (Independence from \( \partial \)). The semantics of \( e \) evaluated under \( G \), \( \rho \) and a base time \( T \) is independent of the infinitesimal time step. Formally, define \( st(t) = fst(\{st(G^{\ast}(T_0)(t))\}) \) and \( st(s) = fst(\{st(G^{\ast}(T_0)(t))\}) \), then:

\[
\forall t \in \mathbb{R}, n \in \mathbb{N}, st(s)(t, n) = st(s')(t, n)
\]

When satisfied, this invariant ensures that properties and values on non-standard time carry over to standard time and values.

4. A LUSTRE-LIKE CAUSALITY

Programs are statically typed. We adopt, for our language, the type system presented in [3]. Well-typed programs may still exhibit causality issues, that is, the definition of a signal at instant \( t \) may instantaneously depend on itself. A classical causality analysis is to reject loops which do not cross a delay. This ensures that outputs can be computed sequentially from current inputs and an internal state. This simple solution is used in the academic Lustre compiler [12], LUCID SYNCHRONIC [22] and SCADE 6.13 We propose generalizing it to a language mixing stream equations, ODEs and their synchronous composition. The causality analysis essentially amounts to checking that every loop is broken either by a unit delay or an integrator.

The analysis gives sufficient conditions for invariants 1 and 2. We adopt the convention quoted below [3, 4]. A signal is termed discrete if it only changes on a discrete clock:

A clock is termed discrete if it has been declared so or if it is the result of a zero-crossing or a sub-sampling of a discrete clock. Otherwise, it is termed continuous.

A discrete change on \( x \) at instant \( t \in \mathbb{T} \) means that \( x(t') \not\approx x(t) \) or \( x(t) \not\approx x(t^{\ast}) \). Said differently, all discontinuities have to be announced using the programming construct \( \text{up}(\_\_\_\_) \).

Two classes of approaches exist to formalize causality analyses. In the first, causality is defined as an abstract preorder relation on signal names. The causality preorder evolves dynamically at each reaction. A program is causally correct if its associated causality preorder is provably a partial order 13 http://www.esterel-technologies.com/scafe
at every reaction. In the second class, causality is defined as the tagging of each event by a ‘stamp’ taken from some preordered set. The considered program is causally correct if its set of stamps can be partially ordered—similarly to Lamport vector clocks. Previous works \[1, 4\] belong to the first class whereas this paper belongs to the second.

Our analysis associates a type to every expression and function via two predicates: \((\text{typ-exp})\) states that, under constraints \(C\), global environment \(G\), local environment \(H\), and kind \(k\in \{A, D, C\}\), an expression \(e\) has type \(ct\); \((\text{typ-env})\) states that under constraints \(C\), global environment \(G\), local environment \(H\), and kind \(k\), the equation \(E\) produces the type environment \(H'\).

\[
\begin{align*}
(\text{typ-exp}) & : C \mid G, H \vdash_k e : ct \\
(\text{typ-env}) & : C \mid G, H \vdash_E E : H'
\end{align*}
\]

The type language is

\[
\sigma ::= \forall \alpha_1, \ldots, \alpha_n : C. ct \overset{b}{\to} ct
\]

\[
ct ::= ct \times ct \mid \alpha
\]

\[
k ::= D \mid C \mid A
\]

where \(\sigma\) defines type schemes, \(\alpha_1, \ldots, \alpha_n\) are type variables and \(C\) is a set of constraints. A type is either a pair \((ct \times ct)\) or a type variable \((\alpha)\). The typing rules for causality are defined with respect to an environment of causality types. \(G\) is a global environment mapping each function name to a type scheme \((\sigma)\). \(H\) is a local environment mapping each variable \(x\) to its type \(ct\):

\[
G ::= [\sigma_1/f_1, \ldots, \sigma_k/f_k] \quad H ::= [ct_1/x_1, \ldots, ct_n/x_n]
\]

If \(H_1\) and \(H_2\) are environments, \(H_1 + H_2\) is their disjoint union. \(H_1 \cdot H_2\) is their concatenation; and \(H_1 \ast H_2\) is a new environment such that \((H_1 + [x : ct]) \ast (H_2 + [x : ct]) = (H_1 \ast H_2) + [x : ct]\) where \(+\) and \(\ast\) are associative and commutative.

**Precedence relation:** \(C\) is a precedence relation between variables with the following intuition. If \(C \mid G, H \vdash_k e : \alpha_1\) holds and \(\alpha_1 < \alpha_2\), the current value of \(e\) is ready at \(\alpha_1\) and also later, within the execution of the same reaction, at \(\alpha_2\). \(<\) must be a strict partial order: it must not be possible to deduce both \(\alpha_1 < \alpha_2\) and \(\alpha_2 < \alpha_1\) from the transitive closure of \(C\).

\[
C :: \{ \alpha_1 < \alpha_1', \ldots, \alpha_n < \alpha_n' \}
\]

The predicate \(C \vdash ct_1 < ct_2\), defined in Figure 5, means that \(ct_1\) precedes \(ct_2\) according to \(C\). All rules are simple distribution rules.

The initial environment \(G_0\) gives type signatures to imported operators, synchronous primitives and the zero-crossing function.

\[
(\text{TAUT}) \quad C + \alpha_1 < \alpha_2 \vdash \alpha_1 < \alpha_2
\]

\[
(\text{TRANS}) \quad C \vdash ct_1 < ct_2 \quad C \vdash ct' < ct_2 \quad C \vdash ct_1 < ct_2
\]

\[
(\text{PAIR}) \quad C \vdash ct_1 < ct'_1 \quad C \vdash ct_2 < ct'_2
\]

\[
C \vdash ct_1 \times ct_2 < ct'_1 \times ct'_2
\]

\[
(\text{ENV}) \quad \forall i \in \{1, \ldots, n\}, C \vdash ct_i < ct_i'
\]

\[
C \vdash [x_1 : ct_1; \ldots; x_n : ct_n] < [x_1 : ct'_1; \ldots; x_n : ct'_n]
\]

Figure 5: Constraints between types

With the first one, the effect of a zero-crossing is delayed by one cycle. Hence, \(up(x)\) does not depend instantaneously on \(x\). With the second, the effect is instantaneous.

**Instantiation/Generalization** The types of global definitions are generalized to type schemes \((\sigma)\) by quantifying over free variables.

\[
Gen(C)(ct_1 \overset{b}{\to} ct_2) = \forall \alpha_1, \ldots, \alpha_n : C. ct_1 \overset{b}{\to} ct_2
\]

where \(\{\alpha_1, \ldots, \alpha_n\} = \text{Vars}(C) \cup \text{Vars}(ct_1) \cup \text{Vars}(ct_2)\). The variables in a type scheme \(\sigma\) can be instantiated, \(ct \in \text{Inst}(\sigma)\) means that \(ct\) is an instance of \(\sigma\). For \(\alpha^r\) and \(k \leq k'\):

\[
C[\alpha^r / \alpha^r], ct_1[\alpha^r / \alpha^r] \overset{ct_2[\alpha^r / \alpha^r]}{\to} ct_2[\alpha^r / \alpha^r] \in \text{Inst}(\forall \alpha^r : C. ty_1 \overset{b}{\to} ty_2)
\]

The typing relation is defined in Figure 6:

**Rule (var).** A variable \(x\) inherits the declared type \(ct\).

**Rule (const).** A constant \(v\) has any causality type.

**Rule (app).** An application \(f(e)\) has type \(ct_2\) if \(f\) has function type \(ct_1 \overset{ct_2}{\to}\) from the instantiation of a type scheme giving a new set of constraints \(C\), and \(e\) has type \(ct_1\).

**Rule (last).** \(\text{last}(x)\) is the previous value of \(x\). In this system, we only allow \(\text{last}(x)\) to appear during a discrete step (of kind \(D\)).

**Rule (eq).** An equation \(x = e\) defines an environment \([ct/x]\) if \(x\) and \(e\) are of type \(ct\).

**Rule (sub).** If \(e\) is of type \(ct\) and \(ct < ct'\) then \(e\) can also be given the type \(ct'\).

**Rule (eqx).** An integrator has a similar role as a unit delay: it breaks dependencies during integration. If \(e : ct_1\) then any use of \(x\) does not depend instantaneously on the computation of \(e\) and can thus be given a type \(ct_2\).

**Rules (present) and (iv).** The present statement returns an environment \(H_1 + H_2\). The first handler is activated during discrete steps and the second one has kind \(C\). The rule for conditionals is the same except that the handlers and condition must all be of kind \(k\).

**Rule (local).** The declaration of a local variable \(x\) is valid if \(E\) gives an equation for \(x\) which itself is causal.
The following program is a classic synchronous

\[
\begin{align*}
\text{let hybrid } f(z) &= y \\
\text{let der } y &= 1.0 \text{ init } 0.0
\end{align*}
\]

is rejected: If \( \text{up}(.) \) is considered to instantaneously depend on its input, \( \text{loop} \) is rejected:

\[
\text{let hybrid } g(x) = o \text{ where} \\
\text{rec } y = 1.0 \text{ init } 0.0 \\
\text{and } x = \text{last } x + y \text{ and init } x = 0.0
\]

Indeed, \( f \) has signature \( \forall \alpha, \alpha. \rightarrow \alpha \). For \( \text{loop} \) to be well-typed, we would need to be able to state an equation \( \alpha < \alpha < \alpha \) where \( y : \alpha \).

\textbf{Type simplification:} The ZÉLUS compiler implements a simplification algorithm to eliminate useless constraints. It follows the algorithm [23] to partition type variables according to \textit{Input-Output (IO)} relations. Moreover, as causality analysis is performed after typing, some relations can be removed. E.g., the actual signature of the unit delay is simply:

\[
\begin{align*}
\text{pre}(.) & : \forall \alpha_1, \alpha_2 : \alpha_1 \rightarrow \alpha_2
\end{align*}
\]

\textbf{The State Port:} The present causality analysis restricts the use of \( \text{last}(x) \) to appear only under a discrete context. An extension is to allow \( \text{last}(x) \) to appear in a continuous context provided \( x \) is a continuous state variable, i.e., it is defined by an equation \( \text{der } x = e \). Indeed, during integration \( \text{last}(x) \) and \( x \) are infinitely close to each other \( (x(n - 1) \approx x(n)) \). The ZÉLUS compiler implements this minor extension.
5. THE MAIN THEOREM

We can now state the main result of this paper: The semantics of well-typed programs satisfies Invariants 1 and 2. This theorem requires assumptions on primitive operators and imported functions, as the following example shows.

A Nonsmooth Model \(4\): It comprises several modules (written in ZÉLUS syntax). The first two are an integrator and a time base with a parameterized initial value \(t_0\):

\[
\text{let hybrid integrator}(y_0, x) = y \text{ where}
\text{rec init } y = y_0 \text{ and } \text{der } y = x
\]
\[
\text{let hybrid time}(t_0) = \text{integrator}(t_0, 1.0)
\]

Then a function producing a quasi-Dirac (Dirac with a width strictly greater than 0) yields a function \(\text{dirac}(d, t)\) such that \(\int_{-\infty}^{\infty} \text{dirac}(d, t) dt = 1\) for every constant \(d > 0\). It yields a function\(\text{dirac}(d, t)\) that is defined, finite and microcontinuous on \(K\), such that \(\text{dirac}(d, t)\) standardizes as a Dirac: \(\int_{d}^{t} \approx 1\) for every constant \(d > 0\). This can be achieved by integrating a pulse of magnitude 1, but of infinitesimal width. Such a pulse can be produced using a variable that is reset twice by the successive occurrences, separated by a \(\partial\), of two zero-crossings:

\[
\text{let hybrid doublecrossing}(t) = (x + 1.0) / 2.0 \text{ where}
\text{rec init } x = -1.0
\text{and present up}(t) \text{ then do next } x = 1.0 \text{ done else}
\text{present up}(x) \text{ then do next } x = -1.0 \text{ done else}
\text{do der } x = 0.0 \text{ done}
\]
\[
\text{let hybrid infinitesimal}(t, t) = \text{integrator}(0.0, \text{doublecrossing}(t))
\]

The first zero-crossing in \(\text{doublecrossing}\) occurs when \(t\) crosses zero and causes an immediate reset of \(x\) from \(-1\) to \(+1\), this in turn triggers an immediate zero-crossing on \(x\) and a reset of \(x\) back to \(-1\). The input of the integrator is thus one for one \(\partial\)-step; the output of the integrator, initially \(0\), becomes \(0\) at time \(t_1 + \partial\).

The main program is the following, where \(t_0 < t_1 < t_2\):

\[
\text{let hybrid nonsmooth}(t_0, t_1, t_2) = x \text{ where}
\text{rec t = time}(t_0) \text{ and } d = \text{infinitesimal}(t - t_1)
\text{and } x = \text{integrator}(0.0, \text{dirac}(d, (t - t_2)))
\]

What is the point of this example? It is causally correct and yet its standardization has a discontinuity at \(t_2\) though no zero-crossing occurs. This is because \(\text{dirac}\) standardizes to a Dirac mass.

Discussion: In the previous example, the problem arises with the function \(\text{dirac}\), that is not defined when \(t = 0\) and \(d = 0\). However, it is defined everywhere when \(d \neq 0\). In particular, it is defined for \(d = 0\). The solution seems clear: if a standard function \(f(x)\) of a real variable \(x\) is such that \(f(x_0) = \perp\), then the semantics must enforce \(f(x) = \perp\) for any \(x \approx x_0\). Applying this to the function \(d \rightarrow \frac{d}{\sqrt{\epsilon}}\) where \(t = 0\) is fixed gives \(\frac{d}{\sqrt{\epsilon}} = \perp\). This is formalized through the assumptions on operators and functions given below.

Given \(x, y \in *\mathbb{R}\), relation \(x \approx y\) holds iff \(st(x - y) = 0\). Recall that function \(f : *\mathbb{R} \rightarrow *\mathbb{R}\) is microcontinuous iff for all \(x, y \in *\mathbb{R}\),

\[
x \approx y \implies f(x) \approx f(y).
\]

Recall that the microcontinuity of \(f\) implies the uniform continuity of \(st(f) : *\mathbb{R} \rightarrow *\mathbb{R}\) [17]. Denote \([t_0, t_1]_T = \{t \in T \mid t_0 \leq t \leq t_1\}\), with \(t_0, t_1 \in T\) finite.

Assumption 1. Operators \(\text{op}(\cdot)\) of kind \(\mathcal{C}\) are standard and satisfy the following definedness, finiteness and continuity properties:

\[
\begin{cases}
\text{op}(\perp) = \perp \\
\forall v, \text{op}(v) \neq \perp \implies \text{fin}(\text{op}(v)) \\
\forall u, v, u \approx v \text{ and } \text{op}(u) \neq \text{op}(v) \implies \text{fin}(\text{op}(u)) = \perp
\end{cases}
\]

Assumption 2. Environment \(G\) is assumed to satisfy the following assumption, for all external functions \(f\) of kind \(\mathcal{C}\): for any bounded interval \(K = [t_1, t_2]_T\) for any input \(u\) that is defined, finite and microcontinuous on \(K\), if function \(G(f)(u)\) is defined and produces no zero-crossing in \(K\), then it is assumed to be finite and microcontinuous on \(K\):

\[
\forall t \in K, \begin{cases}
\text{fin}(G(f)(u)(t)) \neq \perp \\
\text{fin}(G(f)(u)(t)) \text{ finite, and } \\
\forall t', t' \in K, t \approx t' \implies \\
\text{fin}(G(f)(u)(t)) \approx \text{fin}(G(f)(u)(t'))
\end{cases}
\]

Assumption 1 has several implications on the definitions of the usual operators.

- For the square root function: \(\sqrt{\epsilon} = \sqrt{-\epsilon} = \sqrt{0}\), for any \(\epsilon \approx 0\), which yields two meaningful solutions: \(\sqrt{\epsilon} = \perp\) or \(\sqrt{\epsilon} = 0\)
- For the inverse: \(1/\epsilon = \perp\) for any infinitesimal \(\epsilon\) is the only solution.
- Consequently, the function \(\text{sgn}(x) = x/\sqrt{\epsilon}\) returning the sign of \(x\) must satisfy \(\text{sgn}(\epsilon) = \text{sgn}(-\epsilon) = \text{sgn}(0) = \perp\), for any infinitesimal \(\epsilon\).

THEOREM 1. Under Assumptions 1 and 2, the semantics of every causally correct equation \(E\) (wrt. typing rules of Section 4) satisfies Invariants 1 and 2 and is standardizable.

This is a direct consequence of the following lemmas.

**Lemma 1.** Assume that Assumptions 1 and 2 hold. For any activation clock \(T \subseteq \mathbb{R}\), for any bounded interval \(K = [t_1, t_2]_T\), for any environment \(\rho\) that is defined, finite and microcontinuous on \(K\), if expression \(e\), of kind \(\mathcal{A}\) or \(\mathcal{C}\), is defined and produces no zero-crossing on \(K\), then it is finite and microcontinuous on \(K\):

\[
\forall t \in K, \begin{cases}
\text{fin}(\text{st}(e)_G(T)(t)) \neq \perp \\
\text{fin}(\text{st}(e)_G(T)(t)) \text{ finite, and } \\
\forall t', t' \in K, t \approx t' \implies \\
\text{fin}(\text{st}(e)_G(T)(t)) \approx \text{fin}(\text{st}(e)_G(T)(t'))
\end{cases}
\]

**Proof:** Since \(\text{st}(e)_G(T)(t) = \perp\perp\) for all \(t \not\in T\), we can assume that \(K \subseteq T\). The Lemma is proved by induction on the structure of expression \(e\). We prove that it holds for all atomic expressions.
The semantics of constant \( v \), is a constant function \( \llbracket v \rrbracket_C(T)(t) = v \), false. Thus it is finite and microcontinuous.

The semantics of expression \( x \) is a function of time defined in environment \( \rho \): \( \llbracket x \rrbracket_C(T)(t) = \rho(x)(t) \), false, which is by assumption defined, finite and microcontinuous on \( K \).

\[ \llbracket \text{last}(x) \rrbracket_C(T)(t): \text{last}(x) \text{ expressions are of kind } \mathbb{D} \] which is ruled out by assumption.

Then, we assume that the Lemma holds for all causally correct expressions \( e \), \( e_1 \) and \( e_2 \) of kind \( A \) or \( C \), and prove that it holds for expressions built from \( e \), \( e_1 \) and \( e_2 \), using one of the following constructors:

\[ \llbracket \text{if}(e_1,e_2) \rrbracket_C(T)(t) \text{ is finite and microcontinuous if and only if } \llbracket e_1 \rrbracket_C(T)(t), i = 1, 2 \text{ are defined and microcontinuous.} \]

Consider the application of operator \( \text{op} \) on expression \( e \). Two cases must be distinguished: First case, \( \text{op} \) is of kind \( B \), in which case expression \( \text{op}(e) \) has the same kind, which is ruled out by assumption. Second case, \( \text{op} \) is of kind \( A \). By Assumption 1, if defined, the semantics of \( \text{op} \) is a finite and microcontinuous function. Using the induction hypothesis, \( \llbracket \text{op}(e) \rrbracket_C(T)(t) = \text{op}(v), z \) where \( v, z = \llbracket e \rrbracket_C(T)(t) \) is also finite and microcontinuous.

\[ e_1 \text{ fby } e_2 \text{ expressions have kind } \mathbb{D}. \] Thus they can only appear in expression of the same kind.

In \( A \) or \( C \) expressions of the form \( f(e), f \) and \( e \) can not be of the kind \( \mathbb{D} \). Therefore Assumption 2 applies to function \( f \) and the induction hypothesis applies to expression \( e \). Assume \( \llbracket f(e) \rrbracket_C(T)(t) = v', z(t) \forall z \), where \( v', z' = G(f)(t)(t) \) and \( \llbracket f \rrbracket_C(T)'(t) = \llbracket e \rrbracket_C(T)(t)' \) is defined and produces no zero-crossing in \( K \). It is then the composition of two finite and microcontinuous functions. It is, therefore, microcontinuous over \( K \).

\[ \llbracket \text{up}(e) \rrbracket_C(T)(t) \text{ defined and produces no zero crossing for all } t \in K \] implies that it is constant and therefore microcontinuous over \( K \).

Which proves that the induction hypothesis holds for all causally correct expression of kind \( A \) or \( C \). □

Given a bounded interval \( T = [t_0, t_1] \), define the following nonstandard dynamical system on \( T \):

\[ \begin{align*}
  x(t_0) &= x_0 \text{ finite} \\
  x(t + \partial) &= x(t) + \partial \times f(t, x(t))
\end{align*} \]

**LEMMA 2.** If the solution \( x : T \rightarrow \mathbb{R} \) of the dynamical system defined above is infinite or discontinuous at \( t \), then there exists \( t' < t \) such that \( f(t', x(t')) \) is infinite.

**Proof.** We will be using the following property, for any \( t_1 < t_2 \):

\[ |x(t_2) - x(t_1)| \geq \frac{|x(t_2) - x(t_1)|}{t_2 - t_1} \]

First case: Assume \( x(t) \) is infinite, for some \( t \in T \). Recall \( x(t_0) \) is finite. Applying (3) with \( t_1 = t_0, t_2 = t \) yields the existence of \( t', t_0 \leq t' \leq t \) such that \( |f(t', x(t'))| \geq \frac{|x(t') - x(t_0)|}{t_1 - t_0} \) is infinite.

Second case: Assume \( x(t) \) is not continuous for some \( t \in T \). There exists a \( t' \in T, t' \approx t \), such that \( x(t') \not= x(t) \). Assume wlog that \( t' < t \). Observe that \( \frac{|x(t') - x(t)|}{t - t_0} \) is infinite since \( x(t) \not= x(t') \) and \( t \approx t' \). Applying (3) with \( t_1 = t', t_2 = t \) yields the existence of \( t', t_0 \leq t' \leq t \) such that \( f(t', x(t')) \) is also infinite. □

The corollary of this lemma, is that under Assumptions 1 and 2, the semantics of \( \text{de} \) \( x = e \) is smooth provided that expression \( e \) is defined and triggers no zero-crossing:

**COROLLARY 1.** Assume that Assumptions 1 and 2 hold, and that \( e \) is a causally correct expression of kind \( k \) or \( C \).

For any activation clock \( T \subseteq \mathbb{T} \), for any bounded interval \( K = [t_1, t_2]^\mathbb{N} \), for any environment \( \rho \) that is defined, finite and microcontinuous on \( K \), if the least fixed point of the operator \( \rho' \), \( \rho' \not= \rho \) false on \( K \), then \( \rho' \) is microcontinuous on \( K \).

**Proof.** Assume \( \rho' \) defined and \( \rho' \) false on \( K \). Assume \( \rho' \) is infinite or discontinuous at \( t \in K \). Using Lemma 2, there exists \( t' \in K \), \( t' \approx t \), where the semantics of expression \( e \), \( \llbracket e \rrbracket_C(T')(t) \) is infinite, which contradicts Lemma 1. □

**LEMMA 3.** Assume that Assumptions 1 and 2 hold. For any activation clock \( T \subseteq \mathbb{T} \), for any bounded interval \( K = [t_1, t_2]^\mathbb{N} \), for any environment \( \rho \) that is defined, finite and microcontinuous on \( K \), if the semantics of \( E \), a causally correct equation of kind \( C \), is defined and produces no zero-crossing on \( K \), then it is finite and microcontinuous on \( K \):

\[ \forall x, \forall t \in K, \{ \text{fst}(\llbracket E \rrbracket_C(T))(x)(t) \not= \perp \text{ and } \text{snd}(\llbracket E \rrbracket_C(T))(t) \not= \text{false} \} \]

\[ \forall x, \forall t \in K, \{ \text{fst}(\llbracket E \rrbracket_C(T))(x)(t) \text{ finite, and } \forall t', t \in K, t \approx t' \text{ implies } \text{fst}(\llbracket E \rrbracket_C(T))(x)(t') \} \]

**Proof.** By induction on the structure of expression \( E \).

Consider a causally correct equation of kind \( C \) and of the form \( x = e \). The finiteness and microcontinuity of \( \text{fst}(\llbracket x = e \rrbracket_C(T)) = \text{fst}(\llbracket e \rrbracket_C(T)) \) is a direct consequence of Lemma 1.

**Equations** \( \text{init} \) \( x = c \) and \( \text{next} \) \( x = c \) can not be typed with the sort \( C \). Hence these cases are ruled out.

Consider a causally correct equation of kind \( C \) and of the form \( \text{de} \) \( x = e \). That \( \text{fst}(\llbracket \text{de} \rrbracket_C(T)) = \text{finite and microcontinuous} \) is a direct consequence of Corollary 1.

We now proceed with compositions of equations. Assuming the lemma is valid for equations \( E_1 \) and \( E_2 \), we show that it holds for equations \( E_1 \) and \( E_2 \), present \( e \) then \( E_1 \) else \( E_2 \) and \( e \). If \( e \) then \( E_1 \) else \( E_2 \):

Consider a causally correct equation \( E = E_1 \) and \( E_2 \) of kind \( C \). The finiteness and microcontinuity of the equation \( \text{fst}(\llbracket E \rrbracket_C(T)) = \text{fst}(\llbracket E_1 \rrbracket_C(T)) + \text{fst}(\llbracket E_2 \rrbracket_C(T)) \) are consequences of the induction hypothesis.
• Assume the equation $E = \textbf{present } e \textbf{ then } E_1 \textbf{ else } E_2$ to be causally correct. Since $\text{snd}(\langle E \rangle_{CT}(T))$ is equal to $\text{false}$ at every $t \in K$, $\text{fst}(\langle E \rangle_{CT}(T)) = \text{fst}(\langle E \rangle_{CT}(T))$ is finite and microcontinuous by induction hypothesis.

• Assume that $E = \textbf{if } e \textbf{ then } E_1 \textbf{ else } E_2$ is a causally correct equation of kind $\mathbb{C}$. Type correctness implies that expression $e$ is a causally correct expression with the same kind. By Lemma 1, it is microcontinuous on $K$, and since its values are boolean, it is constant. Wlog, assume the expression evaluates to true. Hence, $\text{fst}(\langle E \rangle_{CT}(T)) = \text{fst}(\langle E \rangle_{CT}(T))$ is finite and microcontinuous by induction hypothesis. $\Box$

6. DISCUSSION AND RELATED WORK

The present work continues that of Benveniste et al. [4], by exploiting non-standard semantics to define causality in a hybrid program. The proposed analysis gives a sufficient condition for the program to be statically scheduled.

The present work is related to Ptolemy [10] and the use of synchronous language concepts to define the semantics of hybrid modelers [16]. We follow the same path, replacing super-dense semantics by non-standard semantics that we found more helpful to explain causality constraints and generalize solutions adopted in synchronous compilers. The presented material is implemented in ZéLUS, a synchronous language extended with ODES [6]. It is more single-minded than Ptolemy but ZéLUS programs are turned into sequential code whereas Ptolemy only provides an interpreter.

Causality has been extensively studied for the synchronous languages SIGNAL [1] and ESTEREL [5]. Instead of imposing that every feedback loop crosses a delay, constructive causality checks that the corresponding circuit is constructive. A circuit is constructive if its outputs stabilize in bounded time when inputs are fed with a constant input. In the present work, we adapted the simpler causality of LUSTRE and LUCID SYNCHRONE based on a precedence relation in order to focus on specific issues raised when mixing discrete and continuous-time signals. Schneider et al. [2] have considered the causality problem for a hybrid extension of Quartz, a variant of ESTEREL, with ODES. But, they did not address issues due to the interaction of discrete and continuous behaviors.

Regarding tools like SIMULINK, we think that the synchronous interpretation of signals where time advances by infinitesimal steps can be helpful to define causality constraints and safe interactions between mixed signals.

Finally, type signatures can express the way a component may be used. To specify that an output instantaneously depends on an input — the direct feedthrough port of a SIMULINK function, — it suffices to give them the same type variable. E.g., the signature $\alpha_1 \times \alpha_1 \rightarrow \alpha_1 \times \alpha_2$ states that the first output depends on the two inputs and the second output does not depend on any input.

7. CONCLUSION

Causality in system modelers is a sufficient condition for ensuring that a hybrid system can be implemented: general fix-point equations may have solutions or not, but the subset of causally correct systems can definitely be computed sequentially using off-the-shelf solvers. The notion of causality we propose is that of a synchronous language where instantaneous feedback loops are statically rejected. An integrator plays the role of a unit delay for continuous signals as the previous value is infinitesimally close to the current value.

We introduced the construction last(x) which stands for the previous value of a signal and coincides with the left limit when the signal is left continuous. Then, we introduced a causality analysis to check for the absence of instantaneous algebraic loops. Finally, we established the main result: causally correct programs have no discontinuous changes during integration.

The proposed material has been implemented in ZéLUS, a conservative extension of a synchronous language with ODEs.

8. REFERENCES


