Clocks in Kahn Process Networks

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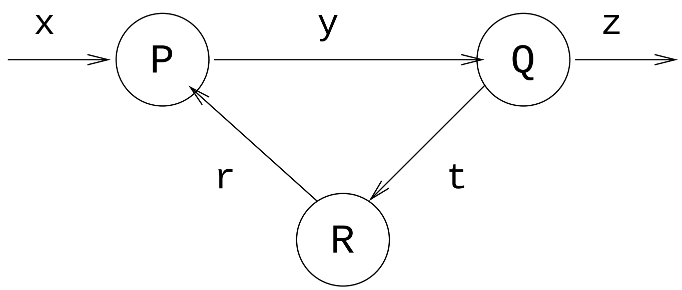
Dataflow Semantics

Kahn Principle : The semantics of process networks communicating through unbounded FIFOs (e.g., Unix pipe, sockets) ?



- message communication into FIFOs (`send`/`wait`)
- reliable channels, bounded communication delay
- blocking wait on a channel. The following program is **forbidden**

```
if (A is present) or (B is present) then ...
```

- a process = a continuous function $(V^\infty)^n \rightarrow (V'^\infty)^m$.

Lustre :

- Lustre has a **Kahn semantics** (no test of absence)
- A dedicated **type system** (clock calculus) to guaranty the existence of an execution with no buffer (no synchronization)

Pros and Cons of KPN

(+) : Simple semantics : a process defines a function (determinism) ; composition is function composition

(+) : Modularity : a network is a continuous function

(+) : Asynchronous distributed execution : easy ; no centralized scheduler

(+/-) : Time invariance : no explicit timing ; but impossible to state that two events happen at the same time.

x	$=$	x_0	x_1		x_2	x_3	x_4	x_5			$\dots$
$f(x)$	$=$	y_0	y_1		y_2	y_3	y_4	y_5			$\dots$
$f(x)$	$=$	y_0		y_1	y_2		y_3		y_4	y_5	$\dots$

This appeared to be a useful model for video apps (TV boxes) : Sally (Philips NatLabs), StreamIt (MIT), Xstream (ST-micro) with various "synchronous" restriction *à la SDF* (Edward Lee)

A small dataflow kernel

A small kernel with minimal primitives

$$
\begin{array}{rcl}
e & ::= & e \,\texttt{fby}\, e \mid op(e, ..., e) \mid x \mid i \\
  &   & \mid \texttt{merge}\; e\; e\; e \mid e \,\texttt{when}\, e \\
  &   & \mid \lambda x.e \mid e\; e \mid \texttt{rec}\; x.e \\
op & ::= & + \mid - \mid \texttt{not} \mid ...
\end{array}
$$

- function ($\lambda x.e$), application ($e\ e$), fix-point ($\texttt{rec}\ x.e$)
- constants i and variables (x)
- dataflow primitives : x `fby` y is the unitary delay ; $op(e_1, ..., e_n)$ the point-wise application ; sub-sampling/oversampling (`when`/`merge`).

Dataflow Primitives

x	x_0	x_1	x_2	x_3	x_4	x_5
y	y_0	y_1	y_2	y_3	y_4	y_5
$x + y$	$x_0 + y_0$	$x_1 + y_1$	$x_2 + y_2$	$x_3 + y_3$	$x_4 + y_4$	$x_5 + y_5$
$x \texttt{ fby } y$	x_0	y_0	y_1	y_2	y_3	y_4
h	1	0	1	0	1	0
$x' = x \texttt{ when } h$	x_0		x_2		x_4	
z		z_0		z_1		z_2
$\texttt{merge } h\ x'\ z$	x_0	z_0	x_2	z_1	x_4	z_2

Sampling :

- if h is a boolean sequence, $x \texttt{ when } h$ produces a sub-sequence of x
- $\texttt{merge } h\ x\ z$ combines two sub-sequences

Kahn Semantics

Every operator is interpreted as a stream function ($V^\infty = V^* + V^\omega$). E.g., if $x \mapsto s_1$ and $y \mapsto s_2$ then the value of $x + y$ is $+^\# (s_1, s_2)$

$$
\begin{array}{rcl}
i^\# & = & i.i^\# \\[1em]
+^\# (x.s_1, y.s_2) & = & (x + y).+^\# (s_1, s_2) \\[1em]
(x.s_1)\, \mathtt{fby}^\#\, s_2 & = & x.s_2 \\[1em]
x.s\, \mathtt{when}^\#\, 1.c & = & x.(s\, \mathtt{when}^\#\, c) \\
x.s\, \mathtt{when}^\#\, 0.c & = & s\, \mathtt{when}^\#\, c \\[1em]
\mathtt{merge}^\#\, 1.c\, x.s_1\, s_2 & = & x.\mathtt{merge}^\#\, c\, s_1\, s_2 \\
\mathtt{merge}^\#\, 0.c\, s_1\, y.s_2 & = & y.\mathtt{merge}^\#\, c\, s_1\, s_2
\end{array}
$$

All this can be simulated in a few lines of Haskell

```
module Streams where

-- lifting constants
constant x = x : (constant x)

-- pointwise application
extend (f:fs) (x:xs) = (f x):(extend fs xs)

-- delays
(x:xs) `fby` y = x:y
pre x y = x : y

-- sampling
(x : xs) `when` (True : cs)  = (x : (xs `when` cs))
(x : xs) `when` (False : cs) = xs `when` cs

merge (True : c)  (x : xs) y  = x : (merge c xs y)
merge (False : c) x (y : ys)  = y : (merge c x ys)
```

After all, why do not use Haskell (or existing FP) ?

We can write many usefull examples and benefit from powerfull type/module systems for free. Some of them are clearly real-time.

```
lift2 f x y = extend (extend (constant f) x) y
plusl x y = lift2 (+) x y

-- integers greaters than n
from n =
  let nat = n `fby` (plusl nat (const 1)) in
  nat

-- resetable counter
reset_counter res input =
  let output = ifthenelse res (const 0) v
      v = ifthenelse input
                     (pre 0 (plusl output (constant 1)))
                     (pre 0 output)
  in output
```

Multi-periodic systems

```
every n =
  let o = reset_counter (pre 0 o = n - 1)
                        (const True)
  in o

filter n top = top when (every n)

hour_minute_second top =
  let second = filter (const 10) top in
  let minute = filter (const 60) second in
  let hour = filter (const 60) minute in
  hour,minute,second
```

Over-sampling (with fixed step)

Compute the sequence $(o_n)_{n \in I\!N}$ such that $o_{2n} = x_n$ and $o_{2n+1} = x_n$.

```
-- the half clock
half = (const True) ‘fby‘ notl half


-- double its input
stutter x =
  o = merge half x ((pre 0 o) when notl half) in o
```

- over-sampling : the internal rate is faster than the rate of inputs
- this is still a real-time program
- why is it rejected in LUSTRE ?

Over-sampling with variable step

Compute the root of an input x (using Newton method)

$$u_n = u_{n-1}/2 + x/2u_{n-1}$$

$$u_1 = x$$

```
eps = const 0.001
root input =
  let ic = merge ok input (pre 0 ic) when notl ok)
      uc = (pre 0 uc) / 2 + (ic / 2 * pre 0 uc)
      ok = true -> uc - pre 0 uc <= eps
      output = uc when ok
  in output
```

This example mimics an internal while loop (example due to Paul Le Guernic)

Some Programs generate monsters !

A stream is represented as a lazy data-structure. Nonetheless, lazyness allows streams to be build in a strange manner.

Structural (Scott) order :

$\perp \leq_{struct} v$, $(v : w) \leq_{struct} (v' : w')$ iff $v \leq_{struct} v'$ and $w \leq_{struct} w'$.

The following programs are perfectly correct in Haskell (with a unique non-empty solution)

```
first (x:y) = x
next (x:y) = y
incr (x:y) = (x+1) : incr y


one = 1 : one
x = (if hd(tl(tl(tl(x)))) = 5 then 3 else 4) : 1 : 2 : 3 : one
output = (hd(tl(tl(tl(x))))) : (hd(tl(tl(x)))) : (hd(x)) : output
```

The values are :

— $x = 4 : 1 : 2 : 3 : 1 : ...$

— $output = 3 : 2 : 4 : 3 : 2 : 4 : ...$

These stream may be constructed lazilly :

- $x^0 = \perp, x^1 = \perp : 1 : 2 : 3 : un, x^2 = 4 : 1 : 2 : 3 : one.$
- $output^0 = \perp, output^1 = 3 : 2 : 4 : ...$

An other example (due to Paul Caspi) :

```
nat = zero ‘fby‘ (incr nat)
ifn n x y = if n <= 9 then hd(x) : if9(n+1) (tl(x)) (tl(y)) else y
if9 x y = ifn 9 x y


x = if9 (incr (next x)) nat
```

We have $x = 18, 17, 16, 15, 14, 13, 12, 11, 10, 9, 10, 11, ...$

Are they reasonnable programs ? Streams are constructed in a reverse manner from the future to the past and are not “causal”.

This is because the structural order between streams allows to fill the holes in any order, e.g. :

$$(\perp : \perp) \leq (\perp : \perp : \perp : \perp) \leq (\perp : \perp : 2 : \perp) \leq (\perp : 1 : 2 : \perp) \leq (0 : 1 : 2 : \perp)$$

It is also possible to build streams with intermediate holes (undefined values in the middle) through the final program is correct :

$$half = 0.\perp.0.\perp...$$

```
fail = fail
half = 0:fail:half
fill x = (hd(x)) : fill (tl(tl x))
ok = fill half
```

We need to model **causality**, that is, stream should be produced in a sequential order. We take the **prefix order** introduced by Kahn :

Prefix order :

$x \leq y$ if x is a prefix of y, that is : $\perp \leq x$ and $v.x \leq v.y$ if $x \leq y$

Causal function :

A function is causal when it is monotonous for the prefix order :

$$x \leq y \Rightarrow f(x) \leq f(y)$$

All the previous program will get the value $\perp$ in the Kahn semantics.

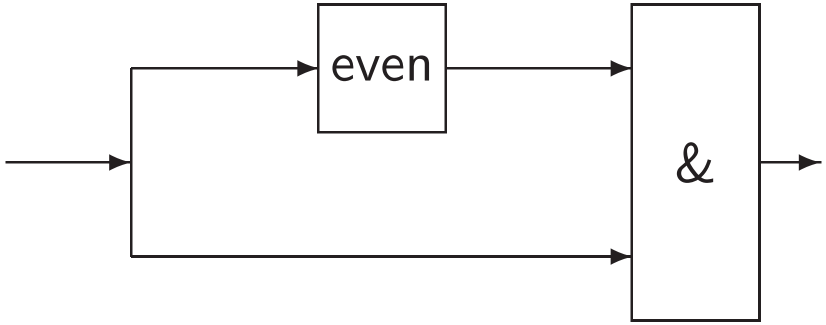
Kahn Semantics in Haskell

It is possible to remove possible non causal streams by forbidding values of the form $\perp.x$. In Haskell, the annotation !a states that the value with type a is strict ($\neq \perp$).

```
module SStreams where
-- only consider streams where the head is always a value (not bot)
data ST a = Cons !a (ST a) deriving Show
constant x = Cons x (constant x)

extend (Cons f fs) (Cons x xs) = Cons (f x) (extend fs xs)

(Cons x xs) `fby` y = Cons x y

(Cons x xs) `when` (Cons True cs)  = (Cons x (xs `when` cs))
(Cons x xs) `when` (Cons False cs) = xs `when` cs

merge (Cons True c)  (Cons x xs) y  = Cons x (merge c xs y)
merge (Cons False c) x (Cons y ys)  = Cons y (merge c x ys)
```

This time, all the previous non causal programs have value $\perp$ (stack overflow).

Some “synchrony” monsters



If $x = (x_i)_{i \in \mathbb{N}}$ then $\texttt{even}(x) = (x_{2i})_{i \in \mathbb{N}}$ and $x \& \texttt{even}(x) = (x_i \& x_{2i})_{i \in \mathbb{N}}$.

Unbounded FIFOs !

- must be rejected statically
- every operator is finite memory through the composition is not : all the complexity (synchronization) is hidden in communication channels
- the Kahn semantics does not model time, i.e., impossible to state that two event arrive **at the same time**

Synchronous (Clocked) streams

Complete streams with an explicit representation of absence (abs).

$$x : (V^{abs})^{\infty}$$

Clock : the clock of x is a boolean sequence

$$
\begin{array}{l}
\mathbb{B} = \{\mathtt{0}, \mathtt{1}\} \\
\mathcal{CLOCK} = \mathbb{B}^{\infty} \\
\begin{array}{lcl}
\mathtt{clock}\ \epsilon & = & \epsilon \\
\mathtt{clock}\ (abs.x) & = & \mathtt{0.clock}\ x \\
\mathtt{clock}\ (v.x) & = & \mathtt{1.clock}\ x
\end{array}
\end{array}
$$

Synchronous streams :

$$ClStream(V, cl) = \{s/s \in (V^{abs})^{\infty} \wedge \mathtt{clock}\ s \leq_{prefix} cl\}$$

An other possible encoding : $x : (V \times \mathbb{N})^{\infty}$

Dataflow Primitives

Constant :

$$\begin{array}{rcl} i^{\#}(\epsilon) & = & \epsilon \\ i^{\#}(\mathtt{1}.cl) & = & i.i^{\#}(cl) \\ i^{\#}(\mathtt{0}.cl) & = & abs.i^{\#}(cl) \end{array}$$

Point-wise application :

Synchronous arguments must be constant, i.e., having the same clock

$$\begin{array}{rcl} +^{\#}(s_1, s_2) & = & \epsilon \text{ if } s_i = \epsilon \\ +^{\#}(abs.s_1, abs.s_2) & = & abs.+^{\#}(s_1, s_2) \\ +^{\#}(v_1.s_1, v_2.s_2) & = & (v_1 + v_2).+^{\#}(s_1, s_2) \end{array}$$

Partial definitions

What happens when one element is present and the other is absent ?

Constraint their domain :

$$(+) : \forall cl : \mathcal{CLOCK}.ClStream(\texttt{int}, cl) \times ClStream(\texttt{int}, cl) \rightarrow ClStream(\texttt{int}, cl)$$

i.e., $(+)$ expect its two input stream to be on the same clock cl and produce an output on the same clock

These extra conditions are **types** which must be statically verified

Remark (notation) : Regular types and clock types can be written separately :

- $(+) : \texttt{int} \times \texttt{int} \rightarrow \texttt{int}$ $\leftarrow$ **its type signature**
- $(+) :: \forall cl.cl \times cl \rightarrow cl$ $\leftarrow$ **its clock signature**

In the following, we only consider the clock type.

Sampling

$$
\begin{array}{lcl}
s_1 \, \texttt{when}^{\#} \, s_2 & = & \epsilon \text{ if } s_1 = \epsilon \text{ or } s_2 = \epsilon \\
(abs.s) \, \texttt{when}^{\#} \, (abs.c) & = & abs.s \, \texttt{when}^{\#} \, c \\
(v.s) \, \texttt{when}^{\#} \, (1.c) & = & v.s \, \texttt{when}^{\#} \, c \\
(v.s) \, \texttt{when}^{\#} \, (0.c) & = & abs.x \, \texttt{when}^{\#} \, c
\end{array}
$$

$$
\begin{array}{lcl}
\texttt{merge} \, c \, s_1 \, s_2 & = & \epsilon \text{ if one of the } s_i = \epsilon \\
\texttt{merge} \, (abs.c) \, (abs.s_1) \, (abs.s_2) & = & abs.\texttt{merge} \, c \, s_1 \, s_2 \\
\texttt{merge} \, (1.c) \, (v.s_1) \, (abs.s_2) & = & v.\texttt{merge} \, c \, s_1 \, s_2 \\
\texttt{merge} \, (0.c) \, (abs.s_1) \, (v.s_2) & = & v.\texttt{merge} \, c \, s_1 \, s_2
\end{array}
$$

Examples

$base = (1)$	1	1	1	1	1	1	1	1	1	1	1	1	...
x	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	...
$h = (10)$	1	0	1	0	1	0	1	0	1	0	1	0	...
$y = x\,\texttt{when}\,h$	x_0		x_2		x_4		x_6		x_8		x_{10}	x_{11}	...
$h' = (100)$	1		0		0		1		0		0	1	...
$z = y\,\texttt{when}\,h'$	x_0						x_6					x_{11}	...
k			k_0		k_1				k_2		k_3		...
$\texttt{merge}\ h'\ z\ k$	x_0		k_0		k_1		x_6		k_2		k_3		...

```
let clock five =
  let rec f = true fby false fby false fby false fby f in f
let node stutter x = o where
  rec o = merge five x ((0 fby o) whenot five) in o
```

$\texttt{stutter}(nat) = 0.0.0.0.1.1.1.1.2.2.2.2.3.3...$

Sampling and clocks

- $x \, \texttt{when}^{\#} \, y$ is defined when x and y have the same clock cl
- the clock of $x \, \texttt{when}^{\#} \, c$ is written $cl \, \mathsf{on} \, c$: "c moves at the pace of cl"

$$
\begin{array}{lcl}
s \, \mathsf{on} \, c & = & \epsilon \text{ if } s = \epsilon \text{ or } c = \epsilon \\
(1.cl) \, \mathsf{on} \, (1.c) & = & 1.cl \, \mathsf{on} \, c \\
(1.cl) \, \mathsf{on} \, (0.c) & = & 0.cl \, \mathsf{on} \, c \\
(0.cl) \, \mathsf{on} \, (abs.c) & = & 0.cl \, \mathsf{on} \, c
\end{array}
$$

We get :

$$\texttt{when} : \forall cl. \forall x : cl. \forall c : cl. cl \, \mathsf{on} \, c$$

$$\texttt{merge} : \forall cl. \forall c : cl. \forall x : cl \, \mathsf{on} \, c. \forall y : cl \, \mathsf{on} \, \textit{\textbf{not}} \; c.cl$$

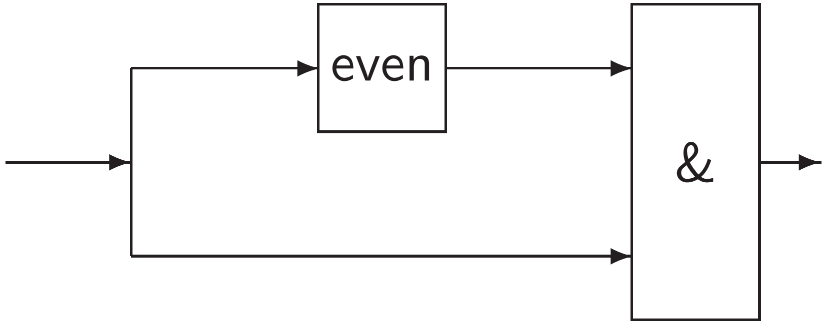
Written instead :

$$\texttt{when} : \forall cl.cl \rightarrow (c : cl) \rightarrow cl \, \mathsf{on} \, c$$

$$\texttt{merge} : \forall cl.(c : cl) \rightarrow cl \, \mathsf{on} \, c \rightarrow cl \, \mathsf{on} \, \textit{\textbf{not}} \; c \rightarrow cl$$

Checking Synchrony

The previous program is now rejected.



This is a now a **typing error**

```
let even x = x when half
let non_synchronous x = x & (even x)
                            ^^^^^^^^

This expression has clock 'a on half,
but is used with clock 'a
```

Final remarks :

- We only considered **clock equality**, i.e., "two streams are either synchronous or not"
- Clocks are used extensively to generate **efficient sequential code**

Lucid Synchrone (sept. 96 –)

How to extend Lustre in a conservative way (without breaking it) ?

Build a "laboratory" language

- a (quasi-dogmatic) attachment to the basic principles : stream Kahn semantics, clocks, functions
- study (implement) extensions of Lustre
- experiment things, manage all the compilation chain and write programs !
- Version 1 (1995), Version 2 (2001), V3 (2006)

Quite fruitful :

- start of a close colloboration with the SCADE team at Esterel-Technologies
- the new SCADE 6 language (Oct. 2008) incorporates several features from Lucid Synchrone
- the LCM language at Dassault-Systèmes (Delmia Automation) based on the same principles

From Synchrony to Relaxed Synchrony

Joint work with Albert Cohen, Marc Duranton, Louis Mandel and Florence Plateau (PhD. Thesis at `https://www.lri.fr/~mandel/lucy-n/~plateau/`)

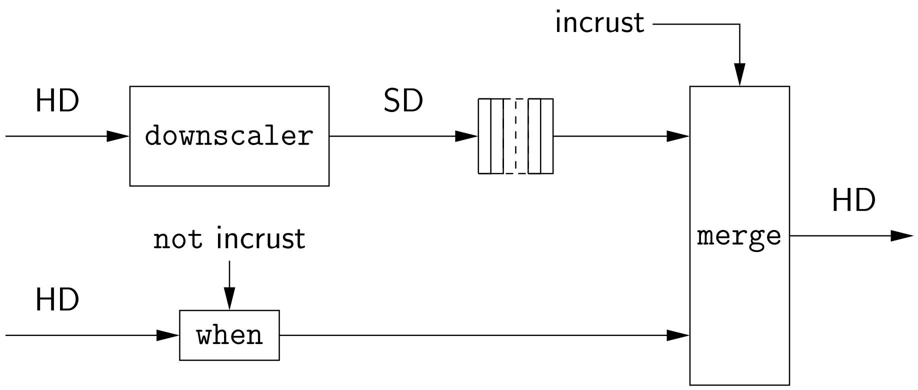
— can we compose non strictly synchronous streams provided their clocks are closed from each other ?

— communication between systems which are "almost" synchronous

— model jittering, bounded delays

— Give more freedom to the compiler, generate more efficient code, translate into regular synchronous code if necessary

A typical example : Picture in Picture



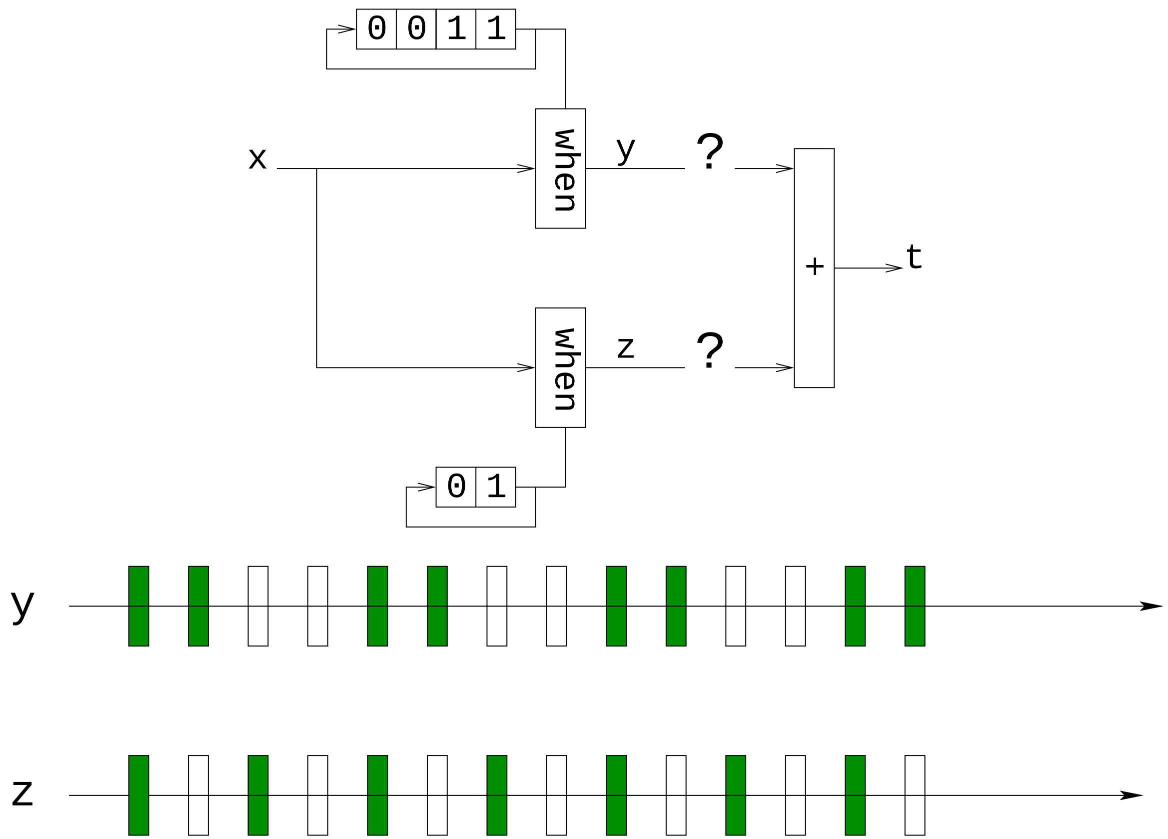
Incrustation of a Standard Definition (SD) image in a High Definition (HD) one

- `downscaler` : reduction of an HD image (1920×1080 pixels) to an SD image (720×480 pixels)
- `when` : removal of a part of an HD image
- `merge` : incrustation of an SD image in an HD image

Question :

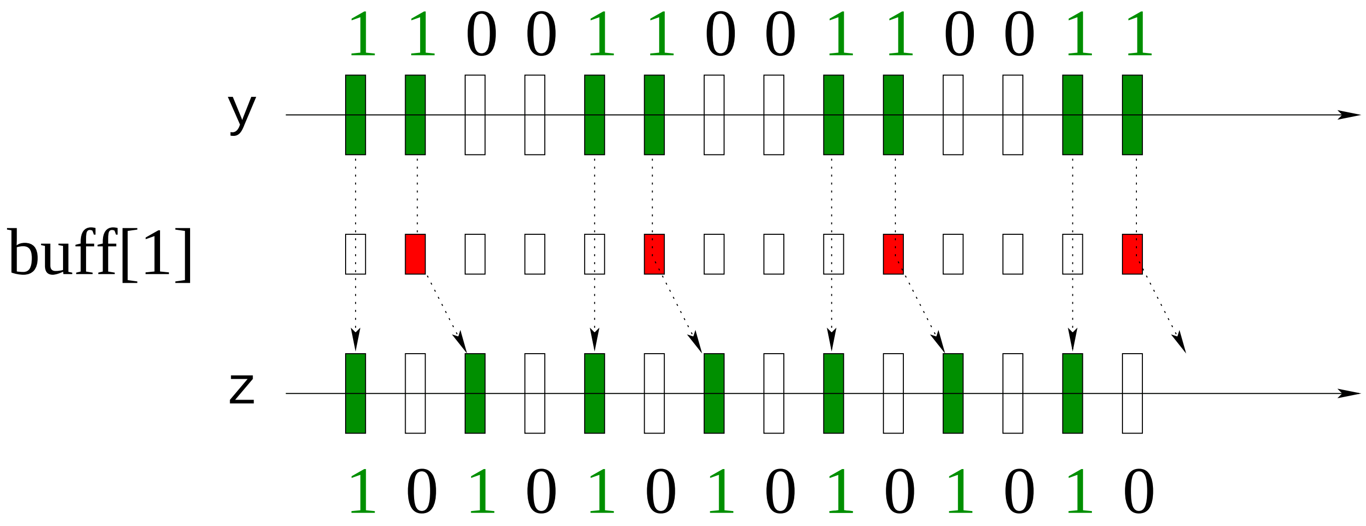
- buffer size needed between the `downscaler` and the `merge` nodes ?
- delay introduced by the picture in picture in the video processing chain ?

Too restrictive for video applications



- streams should be synchronous
- adding buffer (by hand) difficult and error-prone
- compute it automatically and generate synchronous code

relax the associated clocking rules



- based on the use of *infinite ultimately periodic sequences*
- a precedence relation $cl_1 <: cl_2$

Ultimately periodic sequences

$\mathbb{Q}_2$ for the set of infinite periodic binary words.

$$\begin{array}{rcl} (01) & = & 01\ 01\ 01\ 01\ 01\ 01\ 01\ 01\ 01 \ldots \\ 0(1101) & = & 0\ 1101\ 1101\ 1101\ 1101\ 1101\ 1101\ 1101 \ldots \end{array}$$

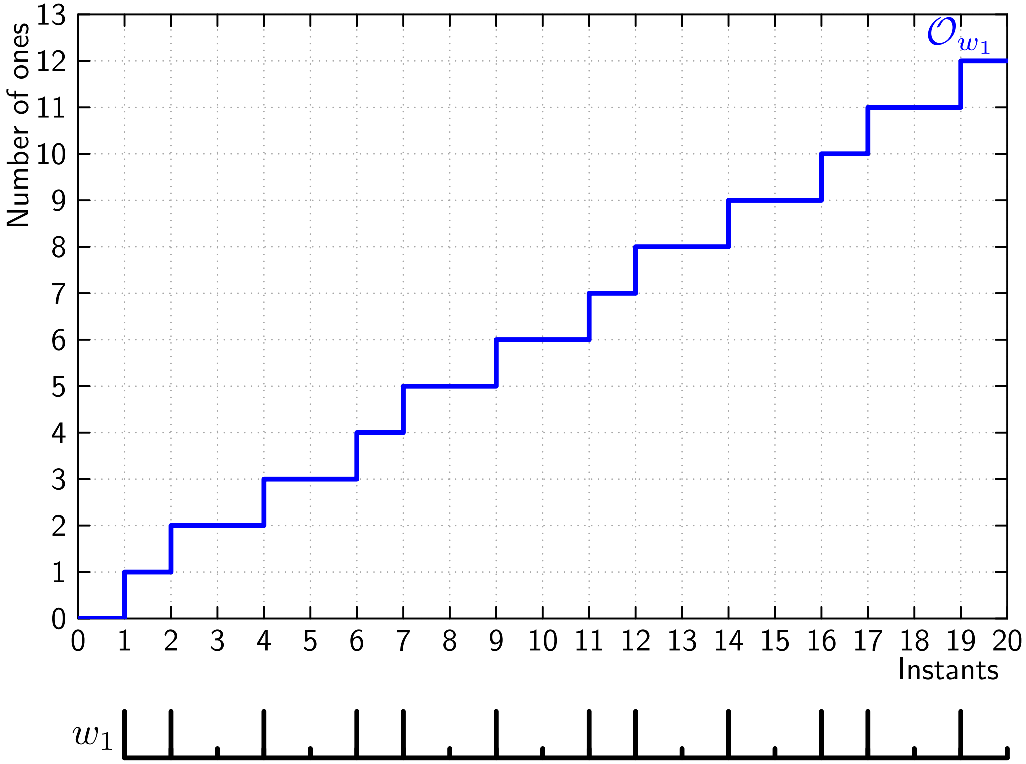
— 1 for presence

— 0 for absence

Definition :

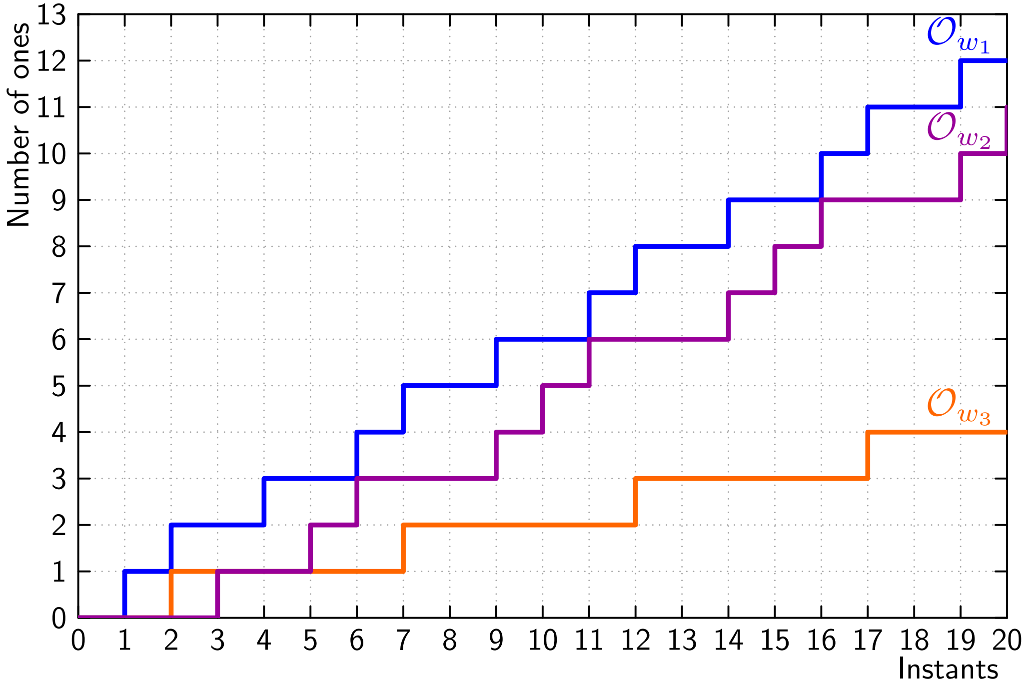
$$w ::= u(v) \quad \text{where } u \in (0+1)^* \text{ and } v \in (0+1)^+$$

Clocks and infinite binary words



$\mathcal{O}_w(i)$ = cumulative function of 1 from w

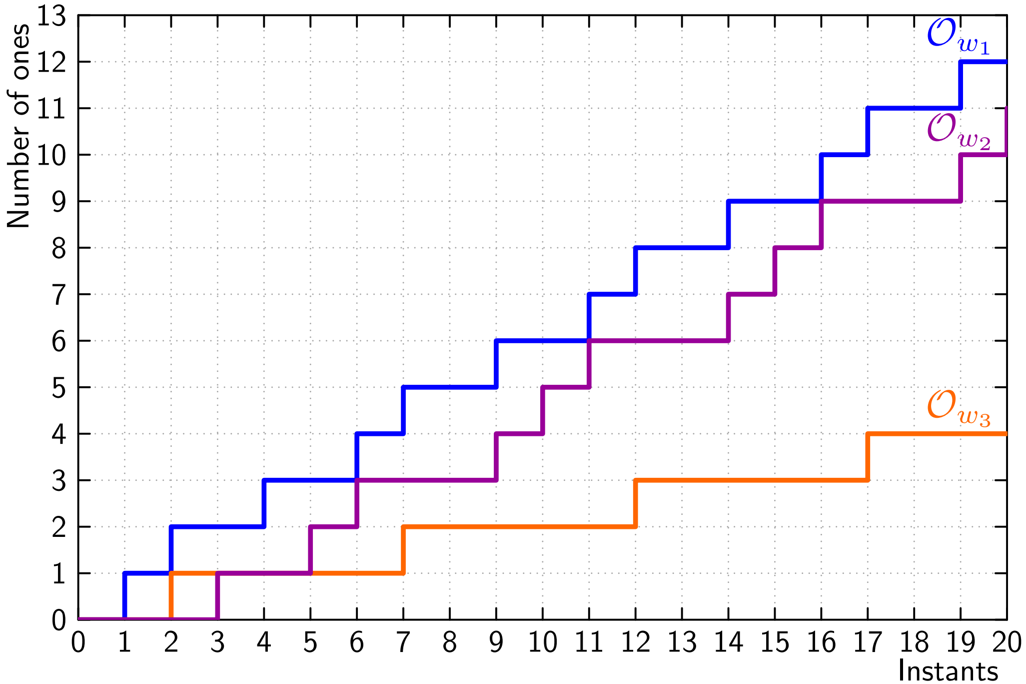
Clocks and infinite binary words



buffer $size(w_1, w_2) = \max_{i \in \mathbb{N}}(\mathcal{O}_{w_1}(i) - \mathcal{O}_{w_2}(i))$

sub-typing $w_1 <: w_2 \overset{def}{\Leftrightarrow} \exists n \in \mathbb{N}, \forall i,\ 0 \leq \mathcal{O}_{w_1}(i) - \mathcal{O}_{w_2}(i) \leq n$

Clocks and infinite binary words



buffer	$size(w_1, w_2) = \max_{i \in \mathbb{N}}(\mathcal{O}_{w_1}(i) - \mathcal{O}_{w_2}(i))$
sub-typing	$w_1 <: w_2 \overset{def}{\Leftrightarrow} \exists n \in \mathbb{N}, \forall i,\ 0 \leq \mathcal{O}_{w_1}(i) - \mathcal{O}_{w_2}(i) \leq n$
synchronizability	$w_1 \bowtie w_2 \overset{def}{\Leftrightarrow} \exists b_1, b_2 \in \mathbb{Z}, \forall i,\ b_1 \leq \mathcal{O}_{w_1}(i) - \mathcal{O}_{w_2}(i) \leq b_2$
precedence	$w_1 \preceq w_2 \overset{def}{\Leftrightarrow} \forall i,\ \mathcal{O}_{w_1}(i) \geq \mathcal{O}_{w_2}(i)$

Multi-clock

$$c \quad ::= \quad w \mid c \texttt{ on } w \qquad w \in (0+1)^\omega$$

c on w is a **sub-clock** of c, by moving in w at the pace of c. E.g., $1(10)$ on $(01) = (0100)$.

base	1	1	1	1	1	1	1	1	1	1	...	(1)
p_1	1	1	0	1	0	1	0	1	0	1	...	1(10)
base on p_1	1	1	0	1	0	1	0	1	0	1	...	1(10)
p_2	0	1		0		1		0		1	...	(01)
(base on p_1) on p_2	0	1	0	0	0	1	0	0	0	1	...	(0100)

For ultimately periodic clocks, precedence, synchronizability and equality are decidable (but expensive)

Come-back to the language

Pure synchrony :

- close to an ML type system (e.g., SCADE 6)
- structural equality of clocks

$$\frac{H \vdash e_1 : ck \qquad H \vdash e_2 : ck}{H \vdash op(e_1, e_2) : ck}$$

Relaxed Synchrony :

- we add a **sub-typing** rule :

$$(\text{SUB}) \quad \frac{H \vdash e : ck \ \mathtt{on}\ w \qquad w <: w'}{H \vdash \mathtt{buffer}(e) : ck \ \mathtt{on}\ w'}$$

- defines synchronization points when a buffer is inserted
- the basis of the language Lucy-N (Plateau and Mandel).

What about non periodic systems ?

- The same idea : synchrony + properties between clocks. Insuring the absence of deadlocks and bounded buffering.

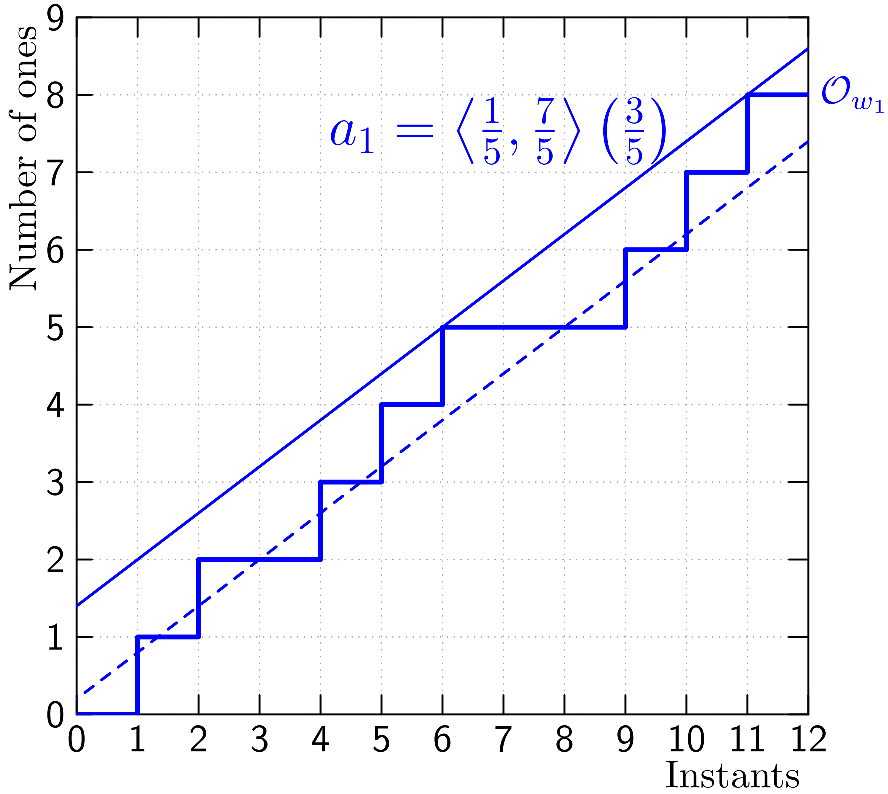
- The **exact** computation with periodic clocks is expensive. E.g., (10100100) $\texttt{on}$ $0^{3600}(1)$ $\texttt{on}$ $(101001001) =$ $0^{9600}(10^4 10^7 10^7 10^2)$

- Motivations :

  1. To treat long periodic patterns. To avoid an exact computation.
  2. To deal with almost periodic clocks. E.g., α $\texttt{on}$ w where $w = 00.(\ (10) + (01)\ )^*$ (e.g. $w = 00\ 01\ 10\ 01\ 01\ 10\ 01\ 10 \ldots$)

Idea : manipulate sets of clocks ; turn questions into arithmetic ones

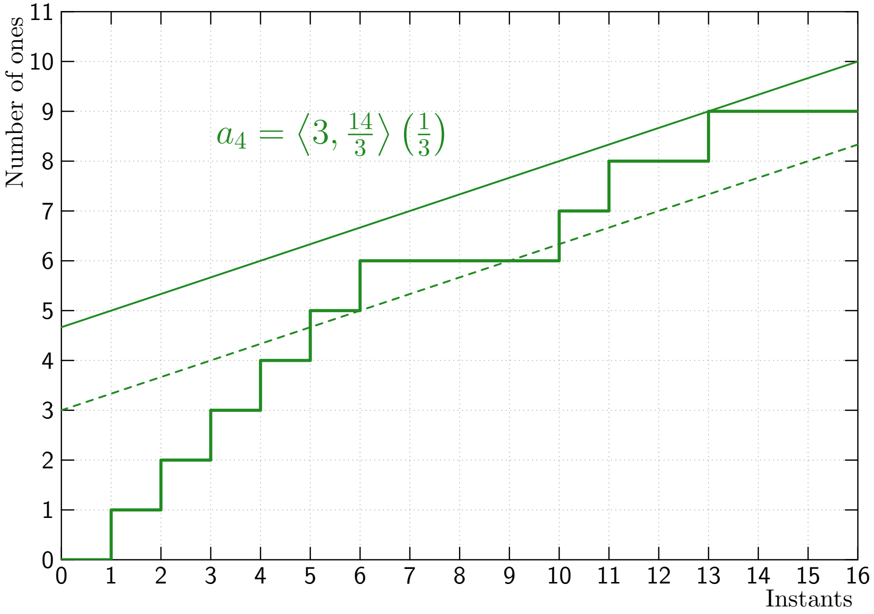
Abstraction of Infinite Binary Words

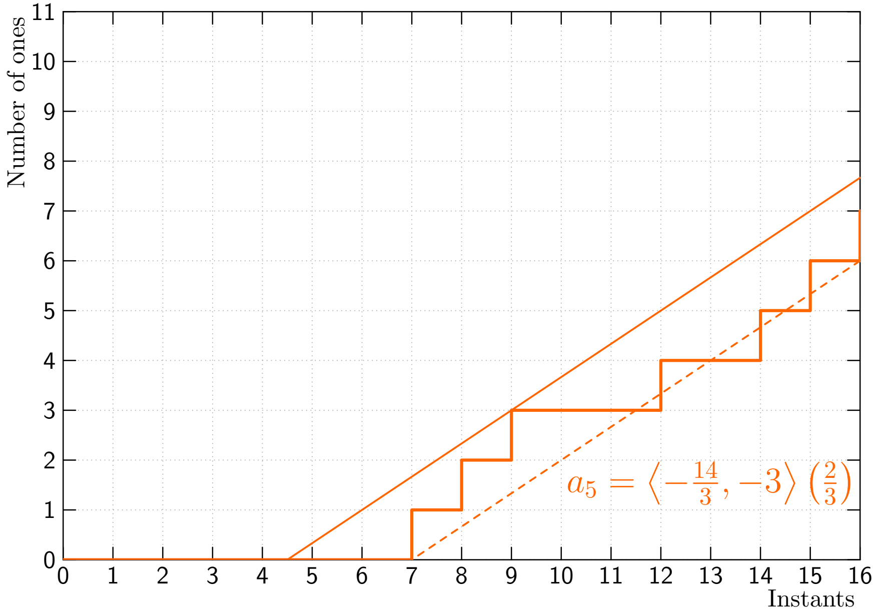


A word w can be abstracted by two lines : $abs(w) = \left\langle b^0, b^1 \right\rangle (r)$

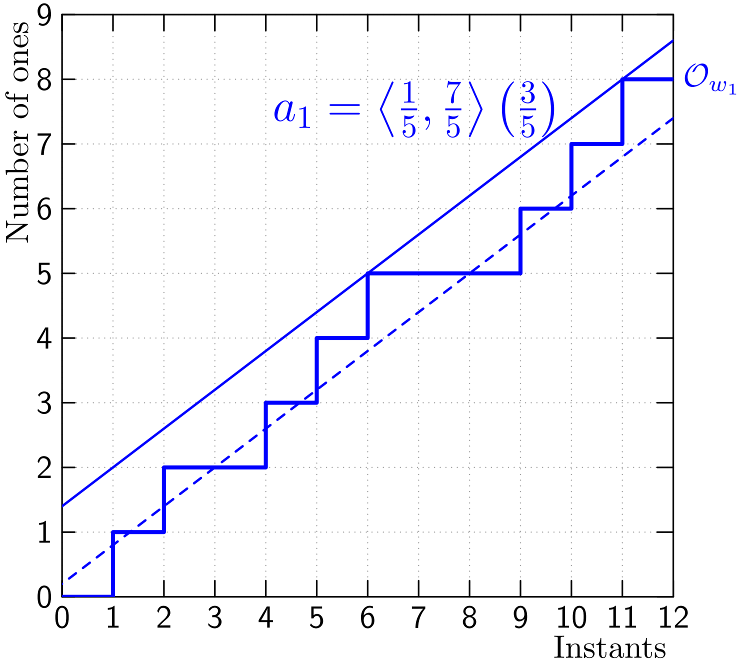
$$concr\left(\left\langle b^0, b^1 \right\rangle (r)\right) \overset{def}{\Leftrightarrow} \left\{ w,\ \forall i \geq 1,\ \wedge \begin{array}{lcl} w[i] = 1 & \Rightarrow & \mathcal{O}_w(i) \leq r \times i + b^1 \\ w[i] = 0 & \Rightarrow & \mathcal{O}_w(i) \geq r \times i + b^0 \end{array} \right\}$$

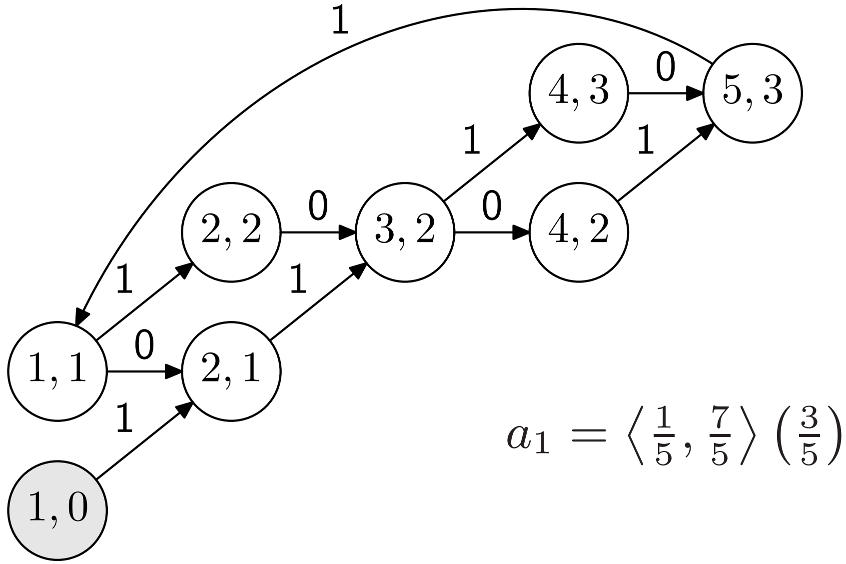
Abstraction of Infinite Binary Words





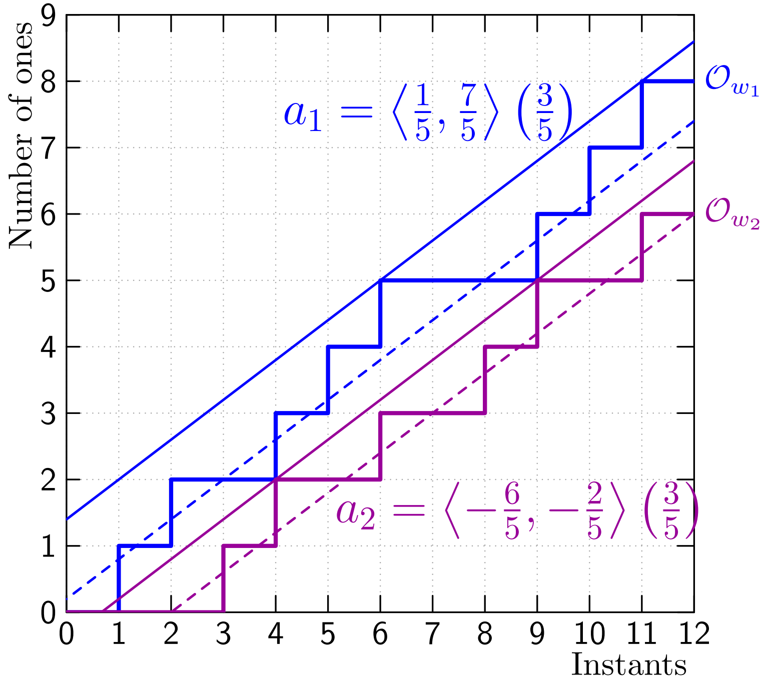
Abstract Clocks as Automata





- set of states $\{(i, j) \in \mathbb{N}^2\}$: coordinates in the 2D-chronogram
- finite number of state equivalence classes
- transition function δ : $\begin{cases} \delta(1,(i,j)) = nf(i+1, j+1) & \text{if } j+1 \leq r \times i + b^1 \\ \delta(0,(i,j)) = nf(i+1, j+0) & \text{if } j+0 \geq r \times i + b^0 \end{cases}$
- allows to check/generate clocks

Abstract Relations



Synchronizability : $r_1 = r_2 \Leftrightarrow \left\langle {b^0}_1, {b^1}_1 \right\rangle (r_1) \bowtie^{\sim} \left\langle {b^0}_2, {b^1}_2 \right\rangle (r_2)$

Precedence : ${b^1}_2 - {b^0}_1 < 1 \Rightarrow \left\langle {b^0}_1, {b^1}_1 \right\rangle (r) \preceq^{\sim} \left\langle {b^0}_2, {b^1}_2 \right\rangle (r)$

Subtyping : $a_1 <:^{\sim} a_2 \Leftrightarrow a_1 \bowtie^{\sim} a_2 \wedge a_1 \preceq^{\sim} a_2$

- proposition : $abs(w_1) <:^{\sim} abs(w_2) \Rightarrow w_1 <: w_2$
- buffer : $size(a_1, a_2) = \left\lfloor {b^1}_1 - {b^0}_2 \right\rfloor$

Abstract Operators

Composed clocks : $c ::= w \mid \mathtt{not}\, w \mid c\ \mathtt{on}\ c$

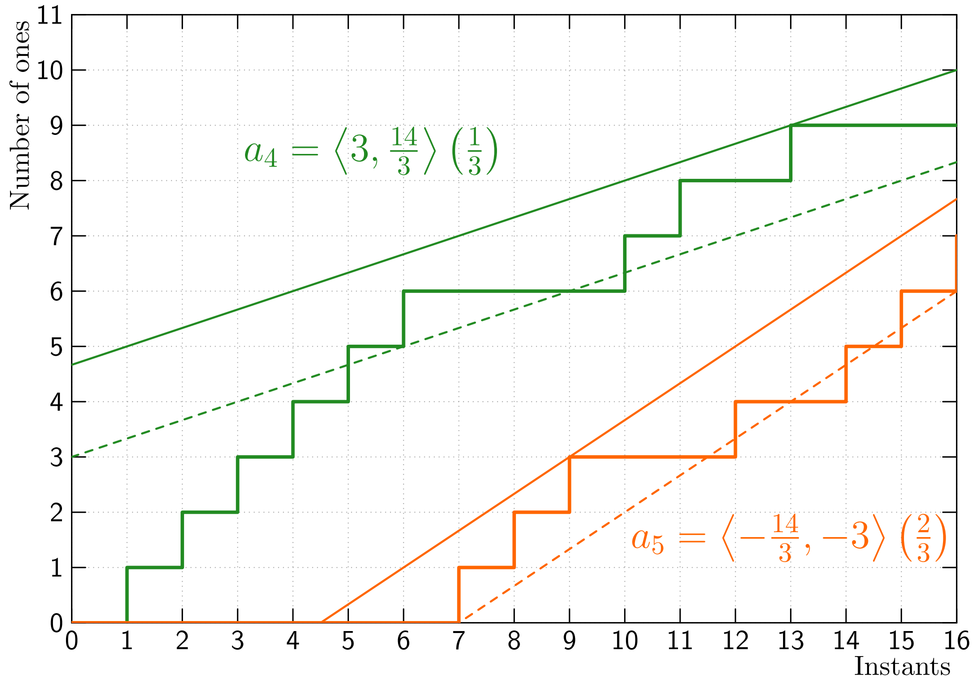
Abstraction of a composed clock :

$$\begin{array}{rcl} abs(\mathtt{not}\, w) & = & \mathtt{not}^{\sim}\ abs(w) \\ abs(c_1\ \mathtt{on}\ c_2) & = & abs(c_1)\ \mathtt{on}^{\sim}\ abs(c_2) \end{array}$$

Operators correctness property :

$$\begin{array}{rcl} \mathtt{not}\, w & \in & concr(\mathtt{not}^{\sim}\ abs(w)) \\ c_1\ \mathtt{on}\ c_2 & \in & concr(abs(c_1)\ \mathtt{on}^{\sim}\ abs(c_2)) \end{array}$$

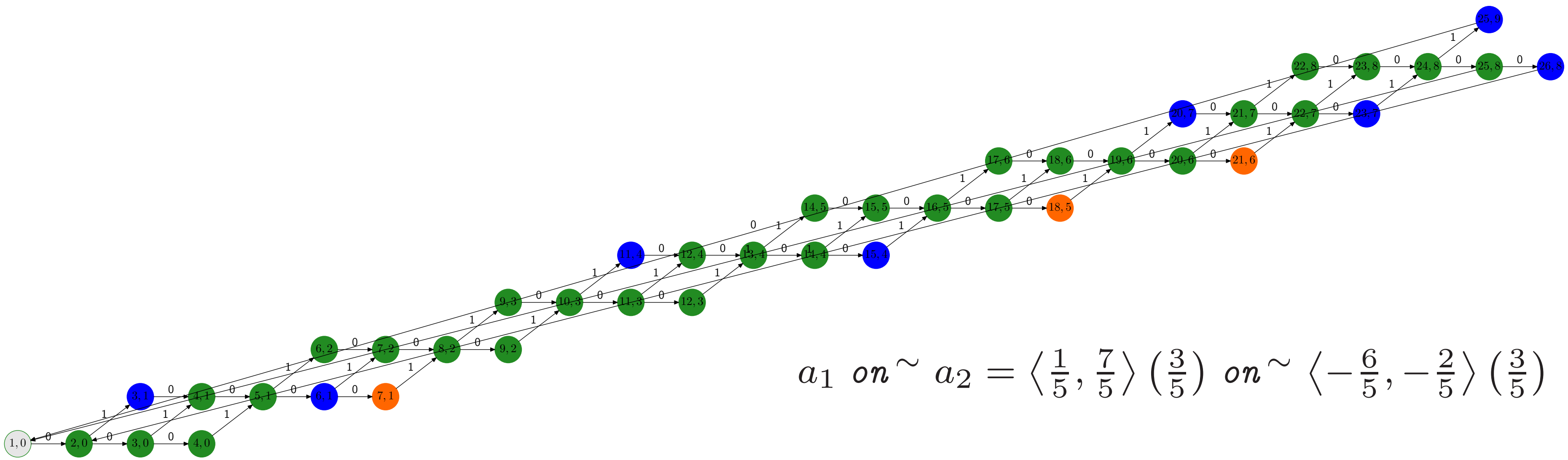
Abstract Operators



$\mathit{not}^{\sim}$ operator definition :

- $\mathit{not}^{\sim} \left\langle b^0, b^1 \right\rangle (r) = \left\langle -b^1, -b^0 \right\rangle (\ 1 - r)$

Abstract Operators



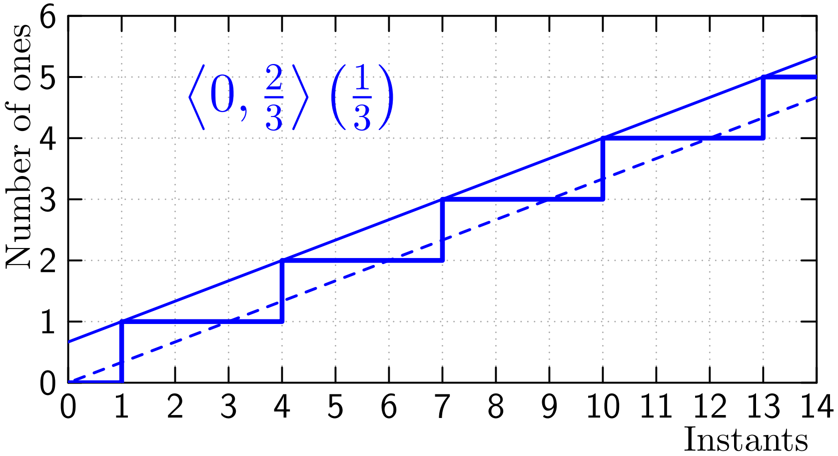
$$a_1 \ \mathit{on}^{\sim} \ a_2 = \langle \tfrac{1}{5}, \tfrac{7}{5} \rangle \left(\tfrac{3}{5}\right) \ \mathit{on}^{\sim} \ \langle -\tfrac{6}{5}, -\tfrac{2}{5} \rangle \left(\tfrac{3}{5}\right)$$

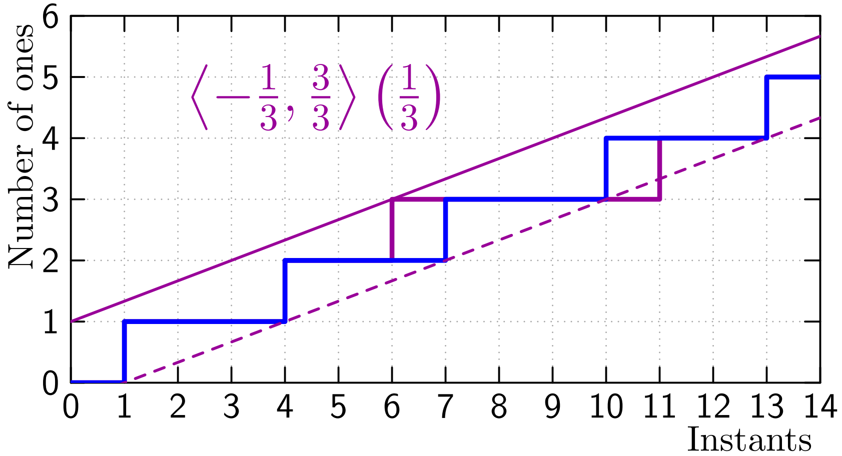
$\mathit{on}^{\sim}$ operator definition :

$$\begin{array}{rl} & \langle \quad {b^0}_1 \quad , \quad {b^1}_1 \quad \rangle (\quad r_1 \quad) \\ \mathit{on}^{\sim} & \langle \quad {b^0}_2 \quad , \quad {b^1}_2 \quad \rangle (\quad r_2 \quad) \\ = & \langle \, {b^0}_1 \times r_2 + {b^0}_2 \ , \ {b^1}_1 \times r_2 + {b^1}_2 \, \rangle \, (\, r_1 \times r_2 \,) \end{array}$$

with $\quad {b^0}_1 \leq 0, \quad {b^0}_2 \leq 0$

Modeling Jitter





- set of clock of rate $r = \frac{1}{3}$ and jitter 1 can be specified by $\langle -\frac{1}{3}, \frac{3}{3} \rangle \left(\frac{1}{3}\right)$
- $\langle -\frac{1}{3}, \frac{3}{3} \rangle \left(\frac{1}{3}\right) = \langle -1, 1 \rangle (1)\ \textit{\textbf{on}}^{\sim}\ \langle 0, \frac{2}{3} \rangle \left(\frac{1}{3}\right)$
- $f :: \forall \alpha . \alpha \to \alpha\ \texttt{on}^{\sim}\ \langle -\frac{1}{3}, \frac{3}{3} \rangle \left(\frac{1}{3}\right)$

Formalization in a Proof Assistant

By Louis Mandel and Florence Plateau

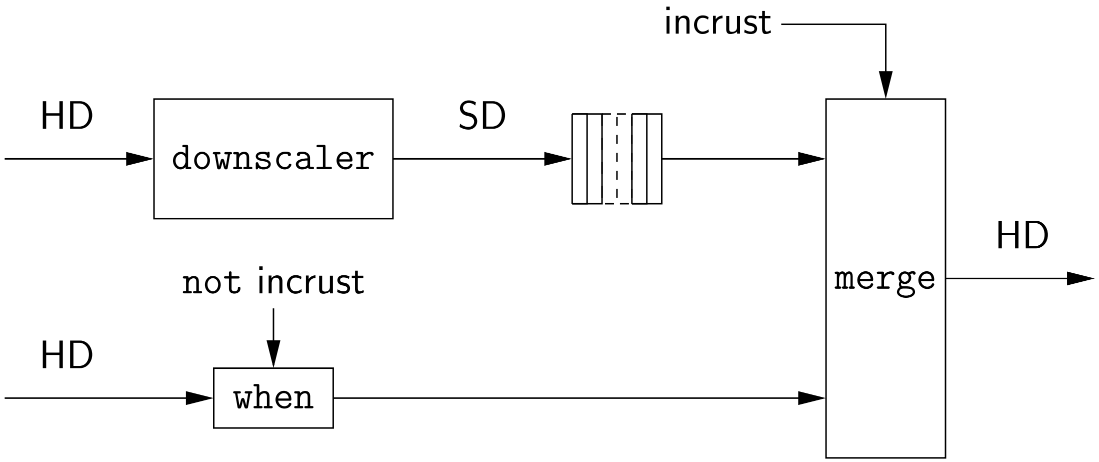
Most of the properties have been proved in Coq

- example of property

```
Property on_absh_correctness:
  forall (w1:ibw) (w2:ibw),
  forall (a1:abstractionh) (a2:abstractionh),
  forall H_wf_a1: well_formed_abstractionh a1,
  forall H_wf_a2: well_formed_abstractionh a2,
  forall H_a1_eq_absh_w1: in_abstractionh w1 a1,
  forall H_a2_eq_absh_w2: in_abstractionh w2 a2,
  in_abstractionh (on w1 w2) (on_absh a1 a2).
```

- number of Source Lines of Code
  - specifications : about 1600 SLOC
  - proofs : about 5000 SLOC

Back to the Picture in Picture Example



- abstraction of downscaler output :

  $abs((10100100)\ \mathit{on}\ 0^{3600}(1)\ \mathit{on}\ (1^{720}0^{720}1^{720}0^{720}0^{720}1^{720}0^{720}0^{720}1^{720}))$

  $= \langle 0, \frac{7}{8} \rangle \left(\frac{3}{8}\right)\ \mathit{on}^{\sim}\ \langle -3600, -3600 \rangle\ (1)\ \mathit{on}^{\sim}\ \langle -400, 480 \rangle \left(\frac{4}{9}\right) = \langle -2000, -\frac{20153}{18} \rangle \left(\frac{1}{6}\right)$

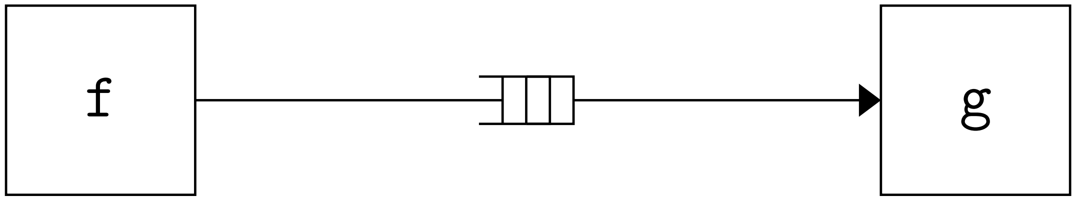
- minimal delay and buffer :

	delay	buffer size
exact result	$9\,598$ (≈ time to receive 5 HD lines)	$192\,240$ (≈ 267 SD lines)
abstract result	$11\,995$ (≈ time to receive 6 HD lines)	$193\,079$ (≈ 268 SD lines)

This is implemented in Lucy-N `http://lucy-n.org` by Louis Mandel.

Parallel implementation and integer clocks

Parallel processes communicating through a buffer




```
int f_out;
while (1) {
  f_step (f_mem, &f_out);
  fifo.push(f_out);
}
```

```
int g_in;
while (1) {
  fifo.pop(&g_in);
  v = g_step (g_mem, g_in);
}
```

Buffers allow to desynchronize the execution

FIFO with batching

To pop, the consumer has to check for the availability of data. This check is expensive. It is better to communicate by chunks.

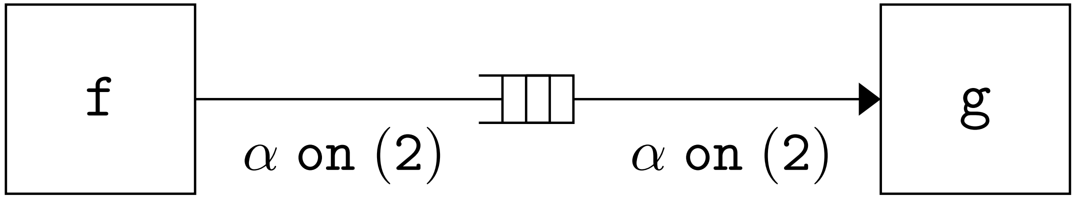
Batch :

- the consumer can read in the fifo only when *batch* values are available
- the producer can write in the fifo only when *batch* rooms are available

Batch size :	001	Cycles/push :	23.07	Bandwidth :	589.45 MB/s
Batch size :	002	Cycles/push :	15.79	Bandwidth :	861.40 MB/s
Batch size :	004	Cycles/push :	12.06	Bandwidth :	1127.83 MB/s
Batch size :	008	Cycles/push :	10.00	Bandwidth :	1359.69 MB/s
Batch size :	016	Cycles/push :	7.51	Bandwidth :	1810.58 MB/s
Batch size :	032	Cycles/push :	7.33	Bandwidth :	1855.32 MB/s
Batch size :	064	Cycles/push :	7.33	Bandwidth :	1855.20 MB/s

Batching : reduce the synchronization with the FIFO

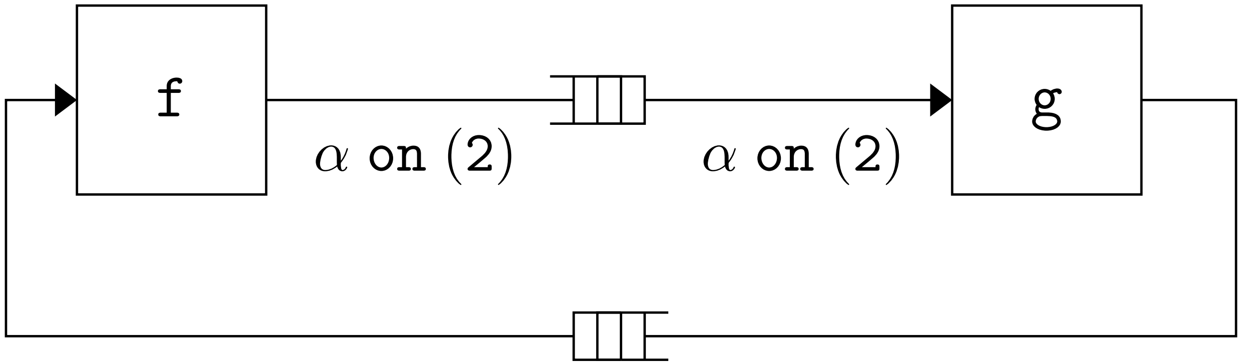
Integer clocks



Burst :

- allows to compute and communicate several values within one instant
- formulas can be easily lifted to integers

Integer clocks



Burst :

- allows to compute several values into one instant
- formulas can be easily lifted to integers
- impacts causality

This has been studied by Adrien Guatto in his PhD. thesis (2016).