

# Clocks in Kahn Process Networks \*

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## Abstract

The language Lustre was introduced to design and implement real-time control software, modeling it as a *continuous function* over streams of datas. A set of equations written in Lustre defines a restricted class of Kahn process networks which can be executed synchronously: all computations can be dated according to a global time scale so that when a value is produced, it is immediately consumed. This restriction is obtained by associating to every stream a *clock* that defines when a value is present or not according to a global time scale. A dedicated type system — the clock calculus — computes a clock for every expression and checks that its actual clock equals its expected clock and thus that intermediate buffers are not needed.

In these course notes,<sup>1</sup> we present a static and dynamic semantics of synchronous Kahn networks. We consider a first-order functional language of streams reminiscent of Lustre and Lucid Synchronic to which we give several denotational semantics. We show that without imposing restrictions, we get two kinds of bad behavior: some networks may deadlock and some cannot execute without unbounded FIFOs. We introduce a clocked semantics and show that the clocking rules correspond to a type system with dependent types. We then extend the language kernel with an explicit **buffer** operator to model communication through a FIFO. The clock calculus is extended with a subtyping rule that is applied where the buffer is used and whose size is inferred. To reduce the complexity of the resolution, we present an abstraction of clocks.

## 1 Introduction

Synchronous languages [3] were introduced about thirty years ago by the concurrent work on three academic languages: Signal [5], Esterel [7] and Lustre [20]. These *domain specific languages* targeted real-time control software, allowing to write modular and mathematically precise system specifications, to simulate, test and verify them, and to automatically translate them into embedded executable code. The environment SCADÉ,<sup>2</sup> based on a synchronous language [15], is now used routinely to develop various critical control software: in planes (fly-by-wire, engine control, emergency braking), trains (on-board control, interlocking), etc.

All these languages are founded on the synchronous model of time [6] where a system is modeled ideally, with communications and computations assumed to be instantaneous, with formal checks of important safety properties like determinism, deadlock freedom, execution in bounded time and space, and with a posteriori verification that a given implementation in software or hardware executes quickly enough.

Lustre is a data-flow language: it manipulates infinite streams of data that represent the evolution of an input, an output or a local variable, streams are defined by writing mutually

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\*The title refers to the article [10] by Paul Caspi.

<sup>1</sup>The present notes are based on one of the lectures on synchronous programming given by the author at the Marktoberdorf summer school in August 2018. It includes previous works by Louis Mandel, Florence Plateau and the author.

<sup>2</sup><http://www.esterel-technologies.com/products/scade-suite>

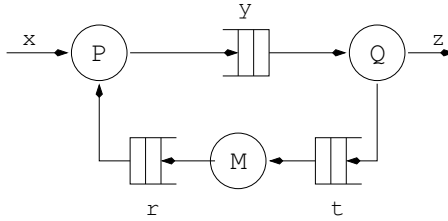


Figure 1: A Kahn Process Network with three processes

recursive equations over them, and a system is a function from streams to streams. Time is simply the index in a stream. After passing static checks, a stream function is compiled to sequential code (typically C). It can also serve as a functional model of a device or software for the purposes of formal verification ([19] summarises the different uses of Lustre).

A set of stream equations written in Lustre can be interpreted as a *Kahn Process Network* [21]: stream functions are the nodes, every stream defines a communication channel and a set of equations corresponds to a process network. Lustre is Kahnian because a stream function cannot dynamically test whether a signal is present or absent. The consequence is that all execution strategies for a network are guaranteed to compute the same set of streams. Nonetheless, as Lustre targets real-time applications, a function written in Lustre defines a particular subset for which the compiler ensures that it does not deadlock and can be compiled into statically scheduled code running in bounded time and space. This is achieved by imposing a set of static constraints to ensure that the network can be executed synchronously, that is, every computation in the network must be dated according to a global time scale so that when a value is produced, it can be immediately consumed. Hence, no intermediate buffers are needed. This synchronous interpretation is obtained by associating to every stream a *clock* that defines when a value is present or not according to a global time scale. Clocks may or may not be periodic and may depend on input values. A dedicated type system — the clock calculus — computes a clock for every expression and checks that it matches the expected clock.

In this text, we describe a static and dynamic semantics for Lustre from the perspective of Kahn process networks. We consider a simple first-order language of streams reminiscent of Lustre and Lucid Synchronic to which we give several denotational semantics. We show that the naive encoding of streams as lazy data structures gives rise to strange non-causal behaviors, highlighting the need for the prefix order introduced by Kahn. We then give a Kahn semantics to the language kernel. To account for the synchronous restriction, we introduce a clocked semantics and show that the clocking rules that a program must fulfill correspond to typing constraints in a type system with dependent types. We derive a simpler type system which reduces the equality of clocks to name equality. We then extend the language kernel with an explicit buffer operator to model communications via FIFOs. The clock calculus is extended with a subtyping rule that is applied where the buffer is used and whose size is inferred. To reduce the complexity of the resolution, we present an abstraction of clocks.

## 1.1 Kahn Process Networks

In the 1970s, Kahn studied the semantics of networks of deterministic parallel processes communicating asynchronously through FIFO channels. One may think, e.g., of a set of Unix processes communicating via pipes or of threads running asynchronously and synchronising through bounded FIFO queues. Kahn showed that in the case where elementary processes are deterministic, with blocking read on an empty channel and non-blocking writes, the overall network is deterministic — the result does not depend on the relative order in which nodes are activated — and delay insensitive — computation and communication times do not change the network semantics [21, 22]. In short, the model is one of the very few that conciliates parallelism and determinism. In a Kahn

network, a basic process can be programmed in a sequential language with two primitives: **push** to write a value to a channel and **pop** to read a value. Figure 1 depicts a network with three processes.<sup>3</sup> There is a single reader and writer per channel. A process may only read a single channel at a time and, once committed to reading, it must wait until a value is available. It may not test the channel for emptiness or impose a timeout; that is, it cannot test whether a value is present or absent. One cannot write, for instance:<sup>4</sup>

```
if is_empty a then ... or if not (is_empty a) or not (is_empty b) then ...
```

But, it is possible to conditionally read or write according to a value that has been read from a channel, e.g.:

```
let v = pop c in let w = if cond a then pop a else pop b in ...
```

or

```
let v = pop a in if cond v then push a (f v)
```

Figure 2 gives a few examples of elementary primitives: **lift2 f x y z** applies a function **f** pointwise to its two input channels **x** and **y**, and produces an output on channel **z**; the unit delay **fby x y z** concatenates the first element of its input channel **x** to the elements of its second input channel **y** and writes on channel **z**; **merge c x y z** conditionally reads an input channel **x** or **y** according to the value on channel **c** and writes on channel **z**; **split c y z** conditionally writes on channel **y** or **z** according to the value on channel **c**.

Kahn networks with bounded buffers can be implemented by adding a *back pressure* mechanism in order to avoid writes into a full buffer. Nonetheless, this may introduce artificial blocking if the size of buffers are underestimated. The size of buffers can be increased dynamically [28] but this solution cannot be used for real-time applications where overall memory use must be guaranteed at compile time.

Whether or not a Kahn network is deadlock free or can be executed in bounded memory is undecidable in general [9]. *Synchronous Data Flow* (or SDF) [24] and its variants (*Cyclo Static Data Flow* [27] among others) are restricted classes of networks where every node consumes and produces a fixed number of tokens at every step. The size of buffers can be computed at compile time and a periodic static schedule can be generated. This make SDF suitable for modeling and programming video intensive applications with periodic behavior [32].

To prove that determinism is preserved by composition, Kahn took an approach based on denotational semantics using the following interpretation of channels and processes. A communication channel that carries values of type  $T$  is interpreted as a (possibly infinite) sequence of values of type  $T$  that describe the *history* of values on the channel. Because a node has its own internal memory, it is interpreted as a function from the histories of its inputs to the histories of its outputs, that is, a stream function. We now recall a few basic properties of sequences, cpos and continuous functions.

### 1.1.1 Sequences and Continuous Functions over Sequences

Consider a set  $T$  of values.  $T^n$  denotes the set of sequences of length  $n$  made by concatenating elements from  $T$ . The sequence  $v.s$  comprises head  $v$  and tail  $s$ . The empty sequence is written  $\epsilon$ . The set of finite sequences is written  $T^* = \cup_{n=0}^{\infty} T^n$ . The set of finite and infinite sequences of elements of  $T$  is written  $T^\infty = T^* \cup T^\omega$ . We write  $\leq$  for the prefix order over sequences;  $s \leq s'$  means that  $s$  is a prefix of  $s'$ . For any  $s, s', \epsilon \leq s$  and if  $s \leq s'$ , then  $v.s \leq v.s'$ . A chain in  $T^\infty$  is any non-empty subset that is totally ordered by  $\leq$ .  $(T^\infty, \leq, \epsilon)$  is a complete partial order (CPO):  $\epsilon$  is its minimum element for the partial order  $\leq$  and every chain has a least upper bound. In the case of boolean sequences, where 0 stands for *false* and 1 for *true*,  $\epsilon \leq 0 \leq 0.1 \leq 0.1.0 \leq 0.1.0.0$  but not  $1.1 \leq 1.0$ .

<sup>3</sup>Figure 15 in Appendix A gives an implementation with threads.

<sup>4</sup>The concrete syntax is that of OCaml.

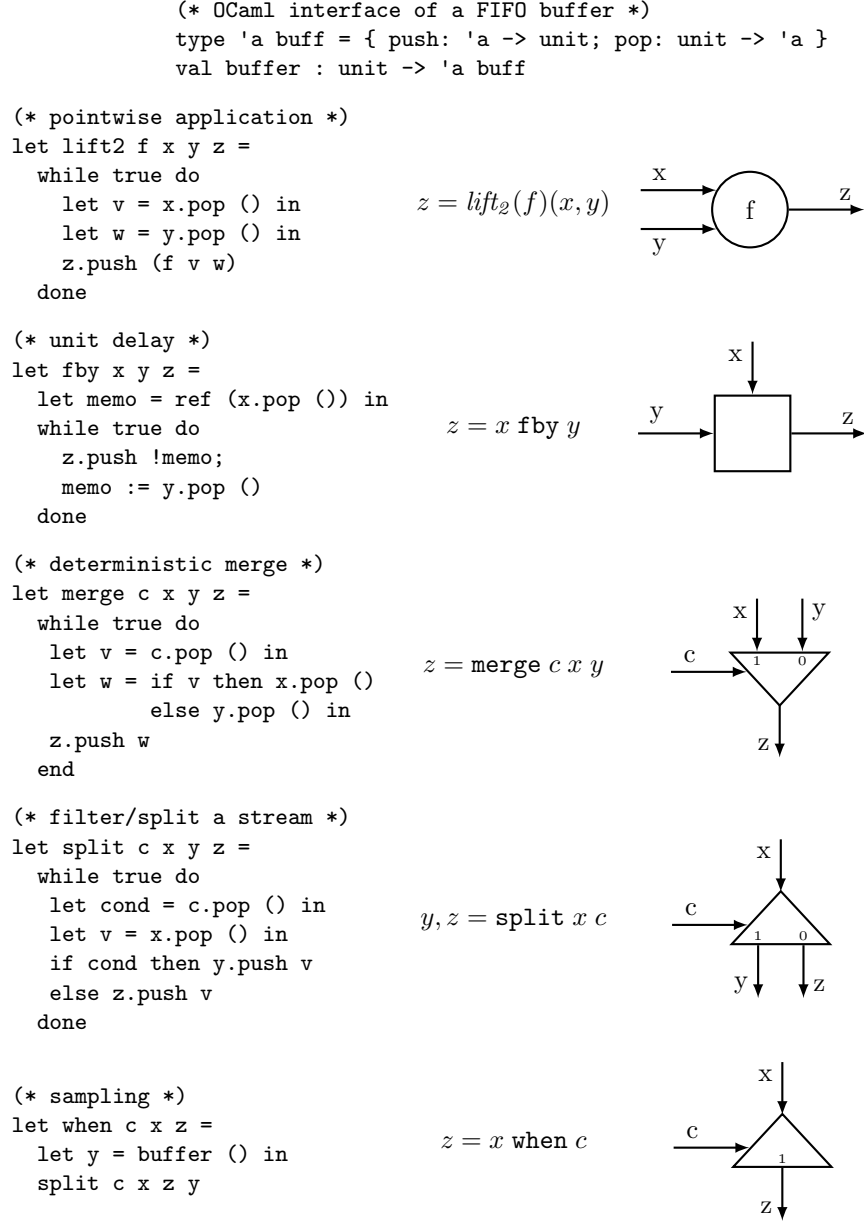


Figure 2: A set of data-flow primitives

$x$	$x_0$	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$
$y$	$y_0$	$y_1$	$y_2$	$y_3$	$y_4$	$y_5$
$x + y$	$x_0 + y_0$	$x_1 + y_1$	$x_2 + y_2$	$x_3 + y_3$	$x_4 + y_4$	$x_5 + y_5$
$x \text{ fby } y$	$x_0$	$y_0$	$y_1$	$y_2$	$y_3$	$y_4$
$h$	1	0	1	0	1	0
$x' = x \text{ when } h$	$x_0$		$x_2$		$x_4$	
$z$		$z_0$		$z_1$		$z_2$
$\text{merge } h \ x' \ z$	$x_0$	$z_0$	$x_2$	$z_1$	$x_4$	$z_2$

Figure 3: A set of primitives interpreted as stream functions

In the sequel, we shall sometimes write a sequence in a more traditional way. A sequence  $u = (u_i)_{i \in I}$ , finite or not, is a set indexed by an initial segment  $I$  of  $\mathbb{N}$ .  $I \subseteq \mathbb{N}$  is an initial segment when  $\forall n, m \in \mathbb{N}. (n \in I) \wedge (m \leq n) \Rightarrow (m \in I)$ .

For any subset  $A$  of  $\mathbb{N}$ , there exists a strictly increasing, one-to-one function  $\phi_A$  between an initial segment  $I_A$  of  $\mathbb{N}$  and  $A$ . An operation that builds a sub-sequence from a sequence by picking a subset of indices or merges two sequences to build another one corresponds to defining particular  $\phi$  functions. This picking does not have to be periodic, as in  $(u_{2i})_{i \in \mathbb{N}}$  that is made by taking one element of  $u$  every two. It can depend on the value of streams. We shall see concrete examples in the next section.

**General properties of a CPO** If  $D_1 = (A_1, \leq_1, \perp_1)$  and  $D_2 = (A_2, \leq_2, \perp_2)$  are two cpos, with respective minimum elements  $\perp_1$  and  $\perp_2$ , a function  $f : D_1 \rightarrow D_2$  is monotonic if and only if for any  $x, x' \in D_1$ ,  $x \leq_1 x' \Rightarrow f(x) \leq_2 f(x')$ . It is continuous if and only if for any chain  $C$  in  $D_1$ ,  $f(\sup(C)) = \sup(\{f(d), d \in C\})$ . Any continuous function  $f : D \rightarrow D$  on a CPO  $D = (A, \leq, \perp)$  has a least fix point  $\text{fix}(f) = \lim_{n \rightarrow \infty} (f^n(\perp))$ , with  $f^0(x) = x$  and  $f^{n+1}(x) = f(f^n(x))$  (Kleene theorem).

If  $A_1$  and  $A_2$  are CPOs, then  $(D_1 \times D_2, \leq', \perp')$  is also a CPO, with  $D_1 \times D_2$  being the set of pairs  $(x_1, x_2)$  comprising an element  $x_1$  from  $D_1$  and an element  $x_2$  from  $D_2$ , taking  $\perp' = (\perp_1, \perp_2)$  as the minimum element and  $\leq'$  such that  $(x_1, x_2) \leq' (y_1, y_2) \Leftrightarrow (x_1 \leq_1 y_1) \wedge (x_2 \leq_2 y_2)$ . The set  $D = (D_1 \mapsto D_2, \leq', \perp')$  where  $D_1 \mapsto D_2$  is the set of total continuous functions from  $D_1$  to  $D_2$ , with  $f \leq' g \Leftrightarrow \forall s \in D_1. f(s) \leq_2 g(s)$  and  $\perp' = (\lambda s. \perp_2)$  is the minimum element, is also a CPO.

### 1.1.2 Application to Kahn Process Networks

Following the formulation in [21], a network is represented by a set of equations built according to the two following rules:

- If  $x_1, \dots, x_k$  are the input channels of the network fed with the sequences  $i_1, \dots, i_k$ , add the equations

$$\{x_1 = i_1, \dots, x_n = i_n\}$$

- Interpret every node  $f$  with  $n$  input channels  $x_1, \dots, x_n$  and  $p$  output channels  $x'_1, \dots, x'_p$  as  $p$  continuous functions over sequences and add the equations

$$\{y'_1 = f_1(x_1, \dots, x_n), \dots, y'_p = f_p(x_1, \dots, x_n)\}$$

The example in Figure 1 is represented by the following set of equations, if  $p$  is the stream function associated to process  $P$ ;  $\langle q_1, q_2 \rangle$  is associated to  $Q$ ;  $m$  to  $M$ :

$$\{y = p(x, r), z = q_1(y), t = q_2(y), r = m(t)\}$$

Elementary nodes in the network are interpreted as *continuous functions* over sequences. Monotonicity corresponds to the intuition that as a process reads more inputs, it can only produce more outputs: it cannot contradict what has already been produced.

Since every node in a Kahn process network is a continuous function, a set of equations:

$$\{x_1 = f_1(x_1, \dots, x_n), \dots, x_n = f_n(x_1, \dots, x_n)\}$$

has a minimal solution which is  $x_1^\infty, \dots, x_n^\infty = \lim_{j \rightarrow \infty} (x_1^j, \dots, x_n^j)$  where for all  $1 \leq i \leq n$ ,  $x_i^0 = \epsilon$  and  $x_i^{j+1} = f_i(x_1^j, \dots, x_n^j)$ .

The primitives given in Figure 2<sup>5</sup> can be interpreted as stream functions as illustrated in Figure 3. An important consequence of the interpretation of elementary nodes as continuous functions is that any composition, where some variable may be made local, still defines a continuous function. For the network in Figure 1, if the channels  $y$ ,  $r$  and  $t$  are considered to be local, the network can be interpreted as a continuous function  $f$  of the input  $x$ , such that the output  $z$  satisfies  $z = f(x)$ .

<sup>5</sup>The operator **when** can also be programmed directly by removing the **else** branch of a **split**. This operator is itself a composition of two **when**.

$d$	$::=$	$\mid \text{let node } f \text{ } pat = e$ $\mid \text{let clock } c = ce$ $\mid d \text{ } d$	node definition clock definition sequence of definitions
$pat$	$::=$	$x \mid (pat, \dots, pat)$	pattern
$e$	$::=$	$\mid i$ $\mid x$ $\mid (e, \dots, e)$ $\mid \text{get}_i(e)$ $\mid e \text{ } op \text{ } e$ $\mid \text{if } e \text{ then } e \text{ else } e$ $\mid f \text{ } e$ $\mid e \text{ where rec } eqs$ $\mid e \text{ fby } e$ $\mid e \text{ when } ce \mid e \text{ whenot } ce$ $\mid \text{merge } ce \text{ } e \text{ } e$ $\mid \text{buffer } e$	constant flow flow variable tuple <i>i</i> -th component of a tuple imported operator mux operator node application local definitions initialized delay sampling merging buffering
$eqs$	$::=$	$pat = e \mid eqs \text{ and } eqs$	mutually recursive equations
$ce$	$::=$	$e$	clock expressions

Figure 4: Language kernel.

## 2 A Language of Streams and Stream Functions

We consider a first-order synchronous dataflow language reminiscent of Lustre and Lucid Synchrone but extended with an explicit buffering operator. The syntax is given in Figure 4. A program ( $d$ ) is a sequence of definitions of stream functions called *nodes* and definitions of clock names ( $c$ ). The inputs of a node are described by a pattern ( $pat$ ) and its body by an expression ( $e$ ). The operators are the basic ones of Lucid Synchrone and their intuitive semantics is detailed later. The expression  $e_1 \text{ } op \text{ } e_2$  denotes the pointwise application of a binary operator; **if**  $e_1$  **then**  $e_2$  **else**  $e_3$  is the pointwise application of a conditional;  $f \text{ } e$  is the application of a node  $f$  to an expression  $e$ ;  $e_1 \text{ fby } e_2$  conses the head of  $e_1$  to  $e_2$  and thus corresponds to an initialized delay;  $e \text{ when } ce$  samples a stream  $e$  according to a boolean expression  $ce$  (**whenot** samples when the expression is 0). We call this boolean expression a clock. The operator **merge**  $ce \text{ } e_1 \text{ } e_2$  merges two streams according to a clock. Finally **buffer**  $e$  buffers  $e$ . We write  $e \text{ where rec } eqs$  for an expression defined by a collection of mutually recursive equations ( $eqs$ ). The basic data-flow primitives of this language kernel are those of Figure 2. For the language of clocks  $ce$ , we take any boolean expression. We shall later consider particular cases of this language.

### 2.1 Denotational Semantics

We first give a denotational semantics based on possibly infinite sequences, following the interpretation given by Kahn. In this setting, the operator **buffer** is simply the identity function. We then define a synchronous semantics which characterises the evolution of the streams and the contents of the buffers.

**Notation for the semantics.** We write  $\rho$  for an environment and  $\rho(x)$  for the value associated to the variable  $x$  in the environment  $\rho$ . The environment  $\rho + [x \leftarrow v]$  is the environment  $\rho$  to which has been added the binding of  $x$  to  $v$ . The environment  $\rho + \rho'$  is the environment that contains the associations of the environment  $\rho$  and the associations of the environment  $\rho'$  provided that no single variable appears in both  $\rho$  and  $\rho'$ .

$$\begin{array}{ll}
op^\sharp(v_1.s_1, v_2.s_2) & = v.op^\sharp(s_1, s_2) \text{ where } v = op(v_1, v_2) \\
fby^\sharp(v_1.s_1, s_2) & = v_1.s_2 \\
when^\sharp(v_1.s_1, 1.w) & = v_1.when^\sharp(s_1, w) \\
when^\sharp(v_1.s_1, 0.w) & = when^\sharp(s_1, w) \\
whenot^\sharp(v_1.s_1, 0.w) & = v_1.whenot^\sharp(s_1, w) \\
whenot^\sharp(v_1.s_1, 1.w) & = whenot^\sharp(s_1, w) \\
merge^\sharp(1.w, v_1.s_1, s_2) & = v_1.merge^\sharp(w, s_1, s_2) \\
merge^\sharp(0.w, s_1, v_2.s_2) & = v_2.merge^\sharp(w, s_1, s_2)
\end{array}$$

Figure 5: Then Kahn semantics for the primitives

The interpretation of an expression  $e$  in an environment  $\rho$  is written  $\llbracket e \rrbracket_\rho$ . This notation will also be used for the denotation of equations and declarations.

Finally, when presenting the interpretation of the primitives as stream functions, we shall use the notation  $\sharp$  as an exponent to distinguish syntactic constructs from their interpretations.

### 2.1.1 A Kahn semantics

Every node declaration **let node**  $f \text{ pat} = e$  is interpreted as a continuous function over sequences. The Kahn semantics for the primitives of the language is given in Figure 5.

- **op** applies an imported operator pointwise on scalar values;
- **fby** is the unit delay; it appends the head of its first argument onto the value of its second argument;<sup>6</sup>
- **when** is the sampling operator: it passes its input to its output only if the condition is true (value 1) and otherwise does not produce any output;
- **whenot** is the complementary sampling operator which passes its input to its output only if the second input is false (value 0) and otherwise does not produce any output;
- **merge** merges two input streams according to a boolean condition. It passes its first input to its output when the boolean condition is true and the second input otherwise.

These definitions must be completed to deal with the empty sequence  $\epsilon$ . All operators return  $\epsilon$  if one of their arguments is the empty sequence ( $\epsilon$  is absorbing), except for the operator **fby** which is such that  $fby^\sharp(v.s, \epsilon) = v.\epsilon$ . We also have to deal with possible type errors. Several solutions can be taken: (1) complete all the definitions by returning  $\epsilon$  in case of a type error; (2) add a special **TypeError** value to the set of streams and transmit this value; (3) define the semantics for well-typed expressions only. For the sake of simplicity, we apply the first solution.

All the primitives are monotonic and continuous [10].

The semantics of expressions of the language is defined in Figure 6. The definition uses the interpretations of the primitives given previously. We write  $\llbracket e \rrbracket_\rho$  to denote the value of  $e$  in the environment  $\rho$ . We ensure that the language is first order by using two distinct namespaces: one that maps local variables to (stream) values or tuples of values, and a second that maps global variables to functions. The environment  $\rho$  is thus a pair  $(\rho_s, \rho_n)$  where  $\rho_s$  associates a value to every free variable of  $e$  and  $\rho_n$  associates a value to every function. Letting  $Var_s$  denote the set of variable names and  $Var_n$  the set of node names, we have

$$\begin{array}{lll}
Stream(T) & = & T^\infty & \text{sequences} \\
V & = & Stream(T_1) + \dots + Stream(T_n) + V \times \dots \times V & \text{values for local variables} \\
\rho_s & : & Var_s \rightarrow V & \text{local environment} \\
\rho_n & : & Var_n \rightarrow (V \rightarrow V) & \text{global environment}
\end{array}$$

<sup>6</sup>It corresponds to the **A** operator of [21]. The **fby** operator was introduced in Lucid [2] and used in [10].

$\llbracket i \rrbracket_\rho$	$= i.\llbracket i \rrbracket_\rho$
$\llbracket x \rrbracket_\rho$	$= \rho_s(x)$
$\llbracket (e_1, \dots, e_n) \rrbracket_\rho$	$= (\llbracket e_1 \rrbracket_\rho, \dots, \llbracket e_n \rrbracket_\rho)$
$\llbracket \text{get}_i(e) \rrbracket_\rho$	$= s_i \text{ if } \llbracket e \rrbracket_\rho = (s_1, \dots, s_n)$
$\llbracket e_1 \text{ op } e_2 \rrbracket_\rho$	$= \text{op}^\sharp(\llbracket e_1 \rrbracket_\rho, \llbracket e_2 \rrbracket_\rho)$
$\llbracket f e \rrbracket_\rho$	$= \rho_n(f) \llbracket e \rrbracket_\rho$
$\llbracket e \text{ where rec eqs} \rrbracket_\rho$	$= \llbracket e \rrbracket_{\rho+\rho'} \text{ where } \rho' = (\llbracket eqs \rrbracket_\rho, \emptyset)$
$\llbracket e_1 \text{ fby } e_2 \rrbracket_\rho$	$= \text{fby}^\sharp(\llbracket e_1 \rrbracket_\rho, \llbracket e_2 \rrbracket_\rho)$
$\llbracket e \text{ when } ce \rrbracket_\rho$	$= \text{when}^\sharp(\llbracket e \rrbracket_\rho, \llbracket ce \rrbracket_\rho^{\text{ce}})$
$\llbracket e \text{ whenot } ce \rrbracket_\rho$	$= \text{whenot}^\sharp(\llbracket e \rrbracket_\rho, \llbracket ce \rrbracket_\rho^{\text{ce}})$
$\llbracket \text{merge } ce \ e_1 \ e_2 \rrbracket_\rho$	$= \text{merge}^\sharp(\llbracket ce \rrbracket_\rho^{\text{ce}}, \llbracket e_1 \rrbracket_\rho, \llbracket e_2 \rrbracket_\rho)$
$\llbracket e \rrbracket_\rho^{\text{ce}}$	$= \llbracket e \rrbracket_\rho$
$\llbracket \text{buffer } e \rrbracket_\rho$	$= \llbracket e \rrbracket_\rho$

Figure 6: The Kahn semantics for the language expressions

If  $\rho = (\rho_s, \rho_n)$  and  $\rho' = (\rho'_s, \rho'_n)$  then  $\rho + \rho' = (\rho_s + \rho'_s, \rho_n + \rho'_n)$ . We write  $\rho + [z \leftarrow v]$  to add the association  $z \leftarrow v$  in the appropriate part of the pair  $\rho$ .

The interpretation of  $e \text{ where rec eqs}$  uses the interpretation of the set of equations  $eqs$  as the supplementary environment. If  $eqs$  is  $x_1 = e_1 \text{ and } \dots \text{ and } x_k = e_k$ , its interpretation is an environment that associates every variable  $x_i$  with the interpretation of  $e_i$ :

$$\begin{aligned} \llbracket x_1 = e_1 \text{ and } \dots \text{ and } x_k = e_k \rrbracket_\rho &= [x_1 \leftarrow x_1^\sharp, \dots, x_k \leftarrow x_k^\sharp] \\ \text{where } x_1^\sharp, \dots, x_k^\sharp &= \text{fix } (\lambda s_1, \dots, s_k. \llbracket e_1 \rrbracket_{\rho+[x_1 \leftarrow s_1, \dots, x_k \leftarrow s_k]}, \dots, \llbracket e_k \rrbracket_{\rho+[x_1 \leftarrow s_1, \dots, x_k \leftarrow s_k]}) \end{aligned}$$

The interpretation of the operators **when**, **whenot** and **merge** uses the interpretation of their clock argument  $ce$ . In this basic language, we consider that a clock expression can be any boolean expression, hence  $\llbracket ce \rrbracket_\rho = \llbracket e \rrbracket_\rho$ . In the second part of these notes, we introduce a dedicated sublanguage of boolean expressions.

The operation **buffer** copies its input into its output, possibly delaying it. Since the Kahn semantics is unable to express timing, the interpretation here is simply the identity function.

The semantics of a program is defined as follows:

$$\begin{aligned} \llbracket \text{let node } f \ x = e \rrbracket_\rho &= \rho + [f \leftarrow (\lambda s. \llbracket e \rrbracket_{\rho+[x \leftarrow s]})] \\ \llbracket \text{let clock } c = ce \rrbracket_\rho &= \rho + [c \leftarrow \llbracket ce \rrbracket_\rho^{\text{ce}}] \\ \llbracket d_1 \ d_2 \rrbracket_\rho &= \llbracket d_2 \rrbracket_{\rho+\rho_1} \text{ where } \rho_1 = \llbracket d_1 \rrbracket_\rho \end{aligned}$$

The evaluation of a program  $d$  having  $f$  as the main node in an environment where the input stream is  $I$  is defined by:

$$\rho_n(f) \ I \text{ where } (\rho_s, \rho_n) = \llbracket d \rrbracket_{(\emptyset, \emptyset)}$$

### 2.1.2 Reformulating the Kahn semantics using indexed streams

The semantics can also be formulated using the notation  $(u_i)_{i \in N}$  with  $N \subseteq \mathbb{N}$  being an initial section of  $\mathbb{N}$ . If  $A \subseteq \mathbb{N}$ , there exists a one-to-one function  $\phi_A$  between an initial section  $I_A$  and  $A$ .



```

(* sequences as functions *)
let lift0 v n = v
let constant = lift0

let lift1 op x n = op (x(n))
let lift2 op x y n = op (x(n)) (y(n))
let lift3 op x y z n = op (x(n)) (y(n)) (z(n))

let not1 x = lift1 not

let fby x y = if n = 0 then x 0 else y(n-1)

let when x h n = x(index(h)(n+1))
let merge h x y n = if h(n) then x(cumul(h)(n)) else y(cumul(not1 h)(n))

(* cumulative and index functions *)
let rec cumul(h)(n) = h(n) + (if n = 0 then 0 else cumul(h)(n-1))

let rec index(h)(n) = index_aux(h)(0)(n)
and index_aux(h)(i)(n) =
  if h(i) then if n = 1 then i else index_aux(h)(i+1)(n-1)
  else index_aux(h)(i+1)(n)

```

Figure 7: An implementation of sequences as functions in OCaml

$$\begin{aligned}
\text{lift}_0^\#(v) &= (u)_{n \in \mathbb{N}} \quad \text{with} \quad \forall n \in \mathbb{N}. u_n = v \\
\text{lift}_1^\#(\text{op})((u_n)_{n \in N}) &= (v_n)_{n \in N} \quad \text{with} \quad \forall n \in N. v_n = \text{op}(u_n) \\
\text{lift}_2^\#(\text{op})((u_n)_{n \in N}, (v_n)_{n \in N}) &= (w_n)_{n \in N} \quad \text{with} \quad \forall n \in N. w_n = \text{op}(u_n, v_n) \\
\text{fby}^\#((u_n)_{n \in N}, (v_n)_{n \in N}) &= (w_n)_{n \in N} \quad \text{with} \quad w_0 = u_0 \\
&\quad \text{and} \quad \forall n \in N \setminus \{0\}. w_n = v_{n-1}
\end{aligned}$$

If  $(h_n)_{n \in N}$  is a boolean sequence, define  $N_h$  and  $N_{\bar{h}}$  as a partition of  $N$ :

$$N_h = \{k \in N \mid h_k = 1\} \quad \text{and} \quad N_{\bar{h}} = \{k \in N \mid h_k = 0\}$$

The filter operator **when** and complement **merge** are defined in the following way:

$$\begin{aligned}
\text{when}^\#((u_n)_{n \in N}, (h_n)_{n \in N}) &= (v_n)_{n \in I_{N_h}} \quad \text{with} \quad v_n = u_{\phi_{N_h}(n)} \\
\text{merge}^\#((h_n)_{n \in N}, (u_n)_{n \in I_{N_h}}, (v_n)_{n \in I_{N_{\bar{h}}}}) &= (w_n)_{n \in N} \quad \text{with} \quad w_n = u_n \text{ if } n \in N_h \\
&\quad \text{and} \quad w_n = v_n \text{ if } n \in N_{\bar{h}}
\end{aligned}$$

A set of equations over sequences becomes a set of mutually recursive functions, from natural numbers to values. Figure 7 gives a possible implementation in OCaml.

We use the functions **index** and **cumul** which are, respectively, the index and cumulative functions, written  $I$  and  $O$  in [25]. If  $h$  is a boolean stream,  $O(h)(n)$  is the sum of 1s up to index  $n$ ;  $I(h)(n)$  is the index of the  $n$ th 1 in  $h$ .

$$O(h)(n) = \sum_{i=0}^n h(i) \quad I(h)(n) = \min(\{k \in \mathbb{N} \mid O_h(k) = n\})$$

This implementation, however, only addresses sequences that are total over  $\mathbb{N}$ . Trying to compute **index(x when (constant false))** for any  $n$  results in a stack overflow. This is no surprise since the domain of **when** can be finite. We shall see later how this problem can be addressed.

```

module Streams where

-- lifting constants
constant x = x : (constant x)

-- pointwise application
extend (f:fs) (x:xs) = (f x):(extend fs xs)

lift1 op xs = extend (constant op) xs
lift2 op xs ys = extend (extend (constant op) xs) ys
lift3 op xs ys zs = extend (extend (extend (constant op) xs) ys) zs

-- delays
(x:xs) `fby` y = x:y
pre x y = x : y

-- sampling
(x : xs) `when` (True : cs) = (x : (xs `when` cs))
(x : xs) `when` (False : cs) = xs `when` cs

merge (True : c) (x : xs) y = x : (merge c xs y)
merge (False : c) x (y : ys) = y : (merge c x ys)

```

Figure 8: A Haskell implementation with (lazy) lists

Moreover, even in the case where all sequences are infinite, the implementation is extremely inefficient. While it is useful for reasoning about programs, it is not a practical implementation. Indeed, the value of a sequence  $x$  at instant  $n$  is computed recursively, possibly back to index 0, with no reuse of previously computed values. It is possible, though, to program the initial Kahn semantics almost directly using infinite data structures and lazy evaluation.

### 2.1.3 An implementation in Haskell with lazy lists

The definitions in Figure 6 can be implemented with potentially infinite data structures and lazy evaluation. Figure 8 gives an implementation in Haskell using lists which can then be used to write many stream functions. For example,

```

plus1 x y = lift2 (+) x y
minus1 x y = lift2 (-) x y

-- integers greater than n
from n =
  let nat = n `fby` (plus1 nat (constant 1)) in nat

-- a resettable counter
reset_counter res input =
  let output = ifthenelse res (constant 0) v
      v = ifthenelse input
          (pre 0 (plus1 output (constant 1)))
          (pre 0 output)
  in output

-- a periodic clock
every n =
  let o = reset_counter (pre 0 o = plus1 n (constant 1)) (constant True)
  in o

filter n top = top `when` (every n)

```

```

hour_minute_second top =
  let second = filter (constant 10) top in
  let minute = filter (constant 60) second in
  let hour = filter (constant 60) minute in
  hour, minute, second

```

Yet, we have essentially just written Lustre functions that pass the compilation checks. The two following functions cannot be written in Lustre. The first one computes the sequence  $(o_n)_{n \in \mathbb{N}}$  from an input  $(x_n)_{n \in \mathbb{N}}$  such that  $o_{2n} = x_n$  and  $o_{2n+1} = x_n$ .

```

-- the half clock
half = (constant True) `fby` not1 half

-- double its input
stutter x =
  o = merge half x ((pre 0 o) when not1 half) in o

```

This is an example of an oversampling function: its internal rate is faster than the rate of its input. This program can be implemented in bounded memory and time. But the Lustre compiler rejects oversampling functions. Another example of an oversampling function is one that computes the root of an input  $x$  using the Newton method.<sup>7</sup> It mimics an internal while loop.

$$u_n = u_{n-1}/2 + x/2u_{n-1} \quad u_1 = x$$

```

eps = constant 0.001
`div` = lift2 (\x y -> x div y)
`minus` = lift2 (\x y -> x - y)
`lessthan` = (lift2 (\x y -> x <= y))

root input =
  let ic = merge ok input ((pre 0.0 ic) `when` (not1 ok))
      uc = ((pre 0.0 uc) `div` (constant 2.0)) `plus1`
            (ic `div` ((constant 2.0) `times` (pre 0.0 uc)))
      ok = (constant true) `->`
            ((uc `minus` (pre 0.0 uc)) `lessthan` eps)
      output = uc `when` ok
  in output

```

Of course, there are many other valid programs that cannot be written in Lustre, in particular those that exploit the expressiveness of Haskell and its type system, like the possibility to write higher order functions.

**Remark 2.1** (Finite values encoded as infinite ones). A classic way to represent  $\epsilon$  with a coinductive type that represents only infinite values, is to infinitely repeat a distinguished value. This approach is used, for example, in [8] and [29]. Instead of interpreting streams as lists, we can instead define streams as:

```

data Value a = Value a | Eps
data Stream a = Cons a (Stream a)
data StreamEps a = Stream (Value a)

eps = Cons Eps eps
one = Cons 1 eps

```

We shall see, however, that the encodings as lazy data-structures we have considered model neither synchrony nor causality.

---

<sup>7</sup>This example is due to Paul Le Guernic and was originally written in Signal.

### 2.1.4 Where are the monsters?

**Causality monsters** In the above encoding, a stream is represented as a lazy data structure. Laziness, however, allows streams to be built in a strange manner. The following definitions are perfectly valid and produce infinite streams for `one`, `x` and `output`.

```

hd (x:y) = x
tl (x:y) = y
incr (x:y) = (x+1) : incr y

one = 1 : one
x = (if hd(tl(tl(tl(x)))) = 5 then 3 else 4) : 1 : 2 : 3 : one
output = (hd(tl(tl(tl(x)))) : (hd(tl(tl(x)))) : (hd(x)) : x

```

The values are  $x = 4 : 1 : 2 : 3 : 1 : \dots$  and  $\text{output} = 3 : 2 : 4 : 3 : 2 : 4 : \dots$ . Streams are implemented as an inductive data structure, `x` and `output` are computed sequentially:

- $x^0 = \perp$ ,  $x^1 = \perp : 1 : 2 : 3 : \text{one}$ ,  $x^2 = 4 : 1 : 2 : 3 : \text{one}$ .
- $\text{output}^0 = \perp$ ,  $\text{output}^1 = 3 : 2 : 4 : \dots$

Another example:<sup>8</sup>

```

next x = tl x
nat = zero `fby` (incr nat)
ifn n x y = if n <= 9 then hd(x) : ifn (n+1) (tl(x)) (tl(y)) else y
if9 x y = ifn 0 x y

x = if9 (incr (next x)) nat

```

The output stream is  $x = 20 : 19 : 18 : 17 : 16 : 15 : 14 : 13 : 12 : 11 : 10 : 11 : 12 : 13 : 14 : 15 : \dots$ .

Are these reasonable programs? Streams are constructed in a reverse manner from the future to the past. They do not obey the intuition that we have of causality, that streams must be constructed from left to right. This is because the structural order between streams allows to fill in the holes in any order, e.g.:

$$(\perp : \perp) \leq (\perp : \perp : \perp : \perp) \leq (\perp : \perp : 2 : \perp) \leq (\perp : 1 : 2 : \perp) \leq (0 : 1 : 2 : \perp)$$

Note that it is possible to build streams with intermediate holes, that is, with undefined values in the middle, from which one can build other streams without holes:

$$\text{half} = 0 : \perp : 0 : \perp : \dots$$

```

fail = fail
half = 0:fail:half
fill x = (hd(x)) : fill (tl(tl x))
ok = fill half

```

The definition of streams in Figure 8 follows the structural order between data structures, which is also the order between functions:  $\perp \leq_{\text{struct}} v$  and the structure  $(v : w)$  is less defined than  $(v' : w')$  when  $v$  is less defined than  $v'$  and  $w$  is less defined than  $w'$ :  $(v : w) \leq_{\text{struct}} (v' : w') \Leftrightarrow v \leq_{\text{struct}} v' \wedge w \leq_{\text{struct}} w'$ . It does not model the intuition of causality that values in the stream must be computed from left to right. The prefix order is thus preferable, that is,  $\perp \leq x$  and  $v : x \leq v : y \Leftrightarrow x \leq y$ .

---

<sup>8</sup>This example is due to Paul Caspi [4]

```

module SStreams where
-- only consider streams where the head is always a value (not bot)
data ST a = Cons !a (ST a) deriving Show
constant x = Cons x (constant x)

extend (Cons f fs) (Cons x xs) = Cons (f x) (extend fs xs)

(Cons x xs) `fby` y = Cons x y

(Cons x xs) `when` (Cons True cs) = (Cons x (xs `when` cs))
(Cons x xs) `when` (Cons False cs) = xs `when` cs

merge (Cons True c) (Cons x xs) y = Cons x (merge c xs y)
merge (Cons False c) x (Cons y ys) = Cons y (merge c x ys)

```

Figure 9: A Haskell implementation with (lazy) lists that enforces causality

**Remark:** This order can be adapted to functions from natural numbers to values, allowing to have intermediate holes in results [4].

$$(x \leq' y) \Leftrightarrow (\forall n \in \mathbb{N}. x(n) \leq y(n) \Rightarrow \forall m \geq n. x(m) = \perp)$$

For example, the following sequence is ordered:

$$\perp \leq (1.\perp) \leq (1.\perp.2.\perp) \leq (1.\perp.2.\perp.3.\perp)$$

Under the prefix ordering, all the previous strange programs denote  $\perp$ .

**Causal function:** A function from streams to streams, is said to be *causal* when it is monotonic for the prefix order. This definition may seem too permissive as the function `next` (or `t1`), given below and presented like the following is considered to be causal.

$$\forall n \in \mathbb{N}. \text{next}(x)(n) = x(n+1)$$

Indeed, the operator `next` can be programmed and is perfectly valid (up to syntactic details) in Lucid Synchrone (and also Lucy-n), for example.

```
let node next x = x when (false fby true)
```

We shall see in the next section how the use of such functions must nonetheless be constrained.

Possibly non-causal streams can be proscribed by forbidding values of the form  $\perp.x$ . Figure 9 gives a simple modification of the previous definitions in Haskell. The annotation `!a` forces the first argument of the stream constructor `Cons` to be strict, that is, to evaluate to a value. Now all the previous strange, non-causal programs have value  $\perp$ .

**Some “synchrony” monsters** Another kind of strange behavior can occur. Consider the input sequence  $x = (x_i)_{i \in \mathbb{N}}$  and the function `even` such that  $\text{even}(x) = (x_{2i})_{i \in \mathbb{N}}$ . Define the equation  $y = x \& \text{even}(x)$ . It should define the sequence  $(x_i \& x_{2i})_{i \in \mathbb{N}}$ . In Haskell, given the definitions of Figure 9, we have:

```

even (Cons x (Cons y xs)) = Cons x (even xs)
and_gate (Cons x xs) (Cons y ys) = Cons (x && y) (and xs ys)

```

Figure 10 depicts the corresponding Kahn network. The fork on the left implicitly represents a simple duplication operator. Even though the `even` and `&` blocks are finite-memory processes, the

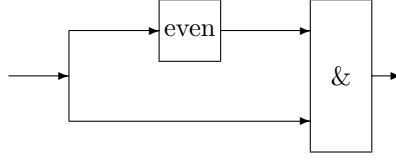


Figure 10: A non synchronous example

composition cannot be executed in bounded memory. As time goes by, the size of the FIFO of the bottom line increases and must eventually overflow.

In real-time applications, such compositions must be statically rejected. Moreover, all the synchronization is hidden in communication channels. Finally, even in the case where the overall memory can be statically bounded, our Haskell encoding needs a complicated runtime system, with allocation and deallocation of intermediate stream values at every step and a garbage collector. There are no real surprises here. The Kahn semantics models neither time nor the resources necessary to synchronise values. If bounded FIFOs are explicitly managed, their size has to be determined, and this can lead to possible deadlocks.

### 2.1.5 Clocked Streams

To account for precise synchronisations between nodes, we introduce a new semantics in which the use of data-flow primitives is restricted. We shall consider that all streams progress synchronously, each producing at global steps either a standard value or the special explicit value *abs* denoting that a value is *absent*, that is, not yet present. The size and content of buffers is also made explicit.

*AbsStreamT* defines the set of clocked sequences made of values from the set  $T_{abs} = T + \{abs\}$ .

$$\begin{aligned} T_{abs} &= T + \{abs\} \\ AbsStream(T) &= (T_{abs})^\infty \end{aligned}$$

It is a sequence of present and absent values that can be represented in Haskell as follows.

```
data maybe a = Present a | Absent
data AbsStream a = ST (maybe a)
```

The clock of a sequence  $s$  is a boolean sequence that indicates when a value is present. For that, we define the function *clock* between clocked sequences and boolean sequences:

$$\begin{aligned} \text{bool} &= \{0, 1\} \\ \text{Clock} &= \text{bool}^\infty \\ \text{clock}(\epsilon) &= \epsilon \\ \text{clock}(abs.x) &= 0.\text{clock}(x) \\ \text{clock}(v.x) &= 1.\text{clock}(x) \end{aligned}$$

We now make the link between the clock and the set of present/absent values more precise by defining:

$$ClockedStream(T)(c) = \{s \mid s \in (T^{abs})^\infty \wedge \text{clock}(s) \leq c\}$$

For a boolean sequence  $c$ ,  $ClockedStream(T)(c)$  is the set of sequences with clock  $c$ . It is prefix closed: if  $s$  is a prefix of  $s'$  with clock  $c$ , that is,  $s' \in ClockedStream(T)(c)$ ,  $s \in ClockedStream(T)(c)$ .

The *synchronous semantics* is defined by reinterpreting the basic primitives over clocked sequences. We can replay the Kahn semantics in Section 2.1.1. It is defined by  $\llbracket e \rrbracket_\rho^{abs}$  which computes the value of  $e$  in an environment  $\rho = (\rho_s, \rho_n)$ . The set of values is replaced by:

$$V = AbsStream(T_1) + \dots + AbsStream(T_n) + V \times \dots \times V \quad \text{values for local variables}$$

$\text{const}^\#(i, 1.w)$	$= i.\text{const}^\#(i, w)$
$\text{const}^\#(i, 0.w)$	$= \text{abs}.\text{const}^\#(i, w)$
$op^\#(\text{abs}.s_1, \text{abs}.s_2)$	$= \text{abs}.op^\#(s_1, s_2)$
$op^\#(v_1.s_1, v_2.s_2)$	$= v.op^\#(s_1, s_2) \text{ where } v = op(v_1, v_2)$
$\text{fby}^\#(\text{abs}.s_1, \text{abs}.s_2)$	$= \text{abs}.\text{fby}^\#(s_1, s_2)$
$\text{fby}^\#(v_1.s_1, v_2.s_2)$	$= v_1.\text{fby}^\#(v_2, s_1, s_2)$
$\text{fby}^\#(v, \text{abs}.s_1, \text{abs}.s_2)$	$= \text{abs}.\text{fby}^\#(v, s_1, s_2)$
$\text{fby}^\#(v, v_1.s_1, v_2.s_2)$	$= v.\text{fby}^\#(v_2, s_1, s_2)$
$\text{when}^\#(\text{abs}.s_1, \text{abs}.w)$	$= \text{abs}.\text{when}^\#(s_1, w)$
$\text{when}^\#(v_1.s_1, 1.w)$	$= v_1.\text{when}^\#(s_1, w)$
$\text{when}^\#(v_1.s_1, 0.w)$	$= \text{abs}.\text{when}^\#(s_1, w)$
$\text{whenot}^\#(\text{abs}.s_1, \text{abs}.w)$	$= \text{abs}.\text{whenot}^\#(s_1, w)$
$\text{whenot}^\#(v_1.s_1, 0.w)$	$= v_1.\text{whenot}^\#(s_1, w)$
$\text{whenot}^\#(v_1.s_1, 1.w)$	$= \text{abs}.\text{whenot}^\#(s_1, w)$
$\text{merge}^\#(\text{abs}.w, \text{abs}.s_1, \text{abs}.s_2)$	$= \text{abs}.\text{merge}^\#(w, s_1, s_2)$
$\text{merge}^\#(1.w, v_1.s_1, \text{abs}.s_2)$	$= v_1.\text{merge}^\#(w, s_1, s_2)$
$\text{merge}^\#(0.w, \text{abs}.s_1, v_2.s_2)$	$= v_2.\text{merge}^\#(w, s_1, s_2)$
$\text{buffer}^\#(v.s, n, \text{abs}.s_1, 1.w)$	$= v.\text{buffer}^\#(s, n+1, s_1, w)$
$\text{buffer}^\#(v.s, n, v_1.s_1, 1.w)$	$= v.\text{buffer}^\#(s.v_1, n, s_1, w)$
$\text{buffer}^\#(\epsilon, n, v_1.s_1, 1.w)$	$= v_1.\text{buffer}^\#(\epsilon, n, s_1, w)$
$\text{buffer}^\#(s, n, \text{abs}.s_1, 0.w)$	$= \text{abs}.\text{buffer}^\#(s, n, s_1, w)$
$\text{buffer}^\#(s, n, v_1.s_1, 0.w)$	$= \text{abs}.\text{buffer}^\#(s.v_1, n-1, s_1, w) \text{ if } n > 0$

Figure 11: The clocked semantics for the primitives

The semantics of expressions, equations and global definitions is essentially unchanged. What changes is the interpretation of primitives on which we concentrate now.

In the following, we write  $\epsilon$  for the empty sequence;  $v$  for a present value and  $\text{abs}$  for an absent value. Hence,  $v.s$  denotes a clocked sequence whose head is present and  $\text{abs}.s$  denotes a sequence whose head is absent.

The interpretation over clocked sequences for the primitives of the language is summarised in Figure 11. We start with the simplest operators, the generator of a constant sequence from a scalar value and the operator to lift a scalar function pointwise over input sequences.

To give a clocked semantics for the constant generator, we need an extra argument to determine whether the current value is present or not, that is:

$$\begin{aligned} \text{const}^\#(i, 1.w) &= i.\text{const}^\#(i, w) \\ \text{const}^\#(i, 0.w) &= \text{abs}.\text{const}^\#(i, w) \end{aligned}$$

Thus,  $\text{const}^\#(i, w)$  defines a constant stream with clock  $w$ , that is,  $\text{clock}(\text{const}^\#(i, w)) = w$ .

Consider now the semantics of  $s + s'$ , for example. At least two situations can occur. If the two inputs are absent, we propagate the absent on the output. If the two inputs are present, we output the sum of the two.

$$\begin{aligned} op^\#(\text{abs}.s_1, \text{abs}.s_2) &= \text{abs}.op^\#(s_1, s_2) \\ op^\#(v_1.s_1, v_2.s_2) &= v.op^\#(s_1, s_2) \text{ where } v = op(v_1, v_2) \end{aligned}$$

It is tempting to add:

$$\begin{aligned} op^\#(\text{abs}.s_1, v_2.s_2) &= \text{abs}.op^\#(s_1, v_2.s_2) \\ op^\#(v_1.s_1, \text{abs}.s_2) &= \text{abs}.op^\#(v_1.s_1, s_2) \end{aligned}$$

to complete with the default rule for absent values as in the initial Kahn semantics. A benefit of having added an absent value to the set of instantaneous values is that we no longer need to deal with both finite and infinite sequences. The empty sequence is simply represented as the infinite sequence  $abs^\omega$  and finite sequences are simply completed to infinite ones by suffixing them with  $abs^\omega$ . The synchronous monsters, however, have not been eradicated!

The synchronous aspect comes from the absence of certain definitions. For example, there is no definition to evaluate  $op^\sharp(v_1.s_1, abs.s_2)$  nor  $op^\sharp(abs.s_1, v_2.s_2)$ , that is, both inputs must be simultaneously present or absent. Otherwise, one of them should be buffered.

### 2.1.6 Dealing with partial definitions: the clock calculus

What happens when one element is present and the other is absent? One idea is to statically reject these cases by requiring  $+$  to have the following type:

$$(+) : \forall cl : Clock. ClockedStream(int)(cl) \times ClockedStream(int)(cl) \rightarrow ClockedStream(int)(cl)$$

In words,  $(+)$  expects its input streams to be on the same clock  $cl$  and guarantees to produce its output on that clock. These conditions are expressed in the form of a type that must be verified statically. This idea is exploited in [8] by defining clocked sequences in Coq as a coinductive dependent type: the type constraint for  $(+)$  and other operators, and the clock constraints for expressions, equations and functions are performed directly by the Coq type checker.

**Remark 2.2** (Two type systems versus a single one). There has been long debate about whether the so-called clock calculus for Lustre, Scade 6, Lucid Synchrone and Signal should merge both classical type information about data and presence/absence information. For Lustre and Signal, the clock calculus was not expressed as a type system and was applied after (classical) static typing. Separating the two, we have two signatures for  $(+)$ , computed by two different type systems:

$$\begin{aligned} (+) & : \text{int} \times \text{int} \rightarrow \text{int} && \text{type signature} \\ (+) & : \forall cl. cl \times cl \rightarrow cl && \text{clock signature} \end{aligned}$$

For Lucid Synchrone,<sup>9</sup> we also decided to separate the type system for data from that for clocks; the compiler thus calculates the two types given above. One of the reasons is that the compiler implements two other type systems, one that ensures the absence of instantaneous loops and another that analyses uses of the uninitialised delay **pre**. After much trial and error, we found it simpler to implement the various systems separately. Moreover, typing occurs sequentially (datatypes, clocks, causality, initialization) so that the information produced by earlier passes is reused by later ones. In particular, the skeleton for types is used to simplify the inference of clock types, causality types and initialization types.

Nonetheless, having several type systems adds useless redundancy in the implementation. It also complicates the formulation of correctness properties. Each of the systems precludes a particular kind of error. It also adds redundancy in interfaces, for example, if one wants to declare a data structure or function that requires a specific clock type. The debate is unfinished. In his thesis [17] and paper [18], Guatto follows an alternative approach that mixes regular type information and clock information, where the clocks express a form of modality in the spirit of guarded types.

In the following, we only consider clock types. Let us consider the case of the unit delay **fb**y.

$$\begin{aligned} \text{fb}y^\sharp(abs.s_1, abs.s_2) &= abs.\text{fb}y^\sharp(s_1, s_2) \\ \text{fb}y^\sharp(v_1.s_1, v_2.s_2) &= v_1.\text{fb}y1^\sharp(v_2, s_1, s_2) \\ \text{fb}y1^\sharp(v, abs.s_1, abs.s_2) &= abs.\text{fb}y1^\sharp(v, s_1, s_2) \\ \text{fb}y1^\sharp(v, v_1.s_1, v_2.s_2) &= v.\text{fb}y1^\sharp(v_2, s_1, s_2) \end{aligned}$$

Here again, the arguments and the result of the **fb**y operator must have the same clock. A **fb**y is a two-state machine: while its two arguments are initially absent, it returns an absent

<sup>9</sup><https://www.di.ens.fr/~pouzet/lucid-synchrone/>



value and remains in the initial state ( $\mathbf{fby}^\#$ ). When both are present, it returns the value of its first argument and enters the steady state ( $\mathbf{fby}1^\#$ ) which stores the previous value of its second argument, emitting it whenever both arguments are present.

$$(\mathbf{fby}) : \forall cl : \text{Clock}. cl \times cl \rightarrow cl \quad \text{clock signature}$$

**Remark 2.3** (Is  $\mathbf{fby}$  length preserving?). It may be surprising to consider that  $\mathbf{fby}$  is a length preserving function. In particular, if its second argument is empty but not the first one, it is able to return a value. But if its first argument is the empty sequence, its output is also empty.

The clock type signature does not express that the output at instant  $n$  does not depend on the second input at instant  $n$ . Hence, both the following two equations are well clocked:

$$x = x + 1 \quad \text{or} \quad x = 0 \mathbf{fby} (x + 1)$$

The causality information could be embedded in the clock type system as in [26], in the case of a simple Lustre-like language (or systems with guarded types) but this calls either for adding subtyping constraints or explicit conversions. This make the clock calculus more complicated or leads to programs, in the case of a Lustre-like language, that are inelegant and not very modular.

In Lustre, Scade 6 and Lucid Synchronic, the detection of instantaneous dependences is ensured by the causality analysis, which is performed after the clock calculus. The consequence is that some valid programs cannot be written. The Signal language mixes clock inference and causality analysis [1].

We now consider the filtering (sampling) operator  $\mathbf{when}$  and the combination operator  $\mathbf{merge}$ .

$$\begin{aligned} \mathbf{when}^\#(abs.s_1, abs.w) &= abs.\mathbf{when}^\#(s_1, w) \\ \mathbf{when}^\#(v_1.s_1, 1.w) &= v_1.\mathbf{when}^\#(s_1, w) \\ \mathbf{when}^\#(v_1.s_1, 0.w) &= abs.\mathbf{when}^\#(s_1, w) \\ \mathbf{whenot}^\#(abs.s_1, abs.w) &= abs.\mathbf{whenot}^\#(s_1, w) \\ \mathbf{whenot}^\#(v_1.s_1, 0.w) &= v_1.\mathbf{whenot}^\#(s_1, w) \\ \mathbf{whenot}^\#(v_1.s_1, 1.w) &= abs.\mathbf{whenot}^\#(s_1, w) \\ \\ \mathbf{merge}^\#(abs.w, abs.s_1, abs.s_2) &= abs.\mathbf{merge}^\#(w, s_1, s_2) \\ \mathbf{merge}^\#(1.w, v_1.s_1, abs.s_2) &= v_1.\mathbf{merge}^\#(w, s_1, s_2) \\ \mathbf{merge}^\#(0.w, abs.s_1, v_2.s_2) &= v_2.\mathbf{merge}^\#(w, s_1, s_2) \end{aligned}$$

The result of the sampling operator  $\mathbf{when}$  is present only when its first input is present and the sampling condition is present and true. The definition of  $\mathbf{merge}$  says that the first branch must be present and the second must be absent when the condition is true; the first branch must be absent and the second present when the condition is false. Again, some rules are lacking. What is the clock of the result?

We need to define an operator on clocks.

$$\begin{aligned} cl \text{ on } c &= \epsilon \text{ if } cl = \epsilon \text{ or } c = \epsilon \\ (1.cl) \text{ on } (1.c) &= 1.(cl \text{ on } c) \\ (1.cl) \text{ on } (0.c) &= 0.(cl \text{ on } c) \\ (0.cl) \text{ on } (abs.c) &= 0.(cl \text{ on } c) \end{aligned}$$

Using it, the clock type of  $\mathbf{when}$  and  $\mathbf{merge}$  can be expressed as:

$$\begin{aligned} \mathbf{when} &: \forall cl. \forall x : cl. \forall c : cl. cl \text{ on } c \\ \mathbf{merge} &: \forall cl. \forall c : cl. \forall x : cl \text{ on } c. \forall y : cl \text{ on } (\mathbf{not } c). cl \end{aligned}$$

The first signature says that, for any clock  $cl$ , if the first input of  $\mathbf{when}$  is  $x$  and it has clock  $cl$ , the second input  $c$  has clock  $cl$ , then the result of  $x \mathbf{when } c$  has clock  $cl \text{ on } c$ . The rule for  $\mathbf{whenot}$  is similar. The signature for  $\mathbf{merge}$  says that if the first input  $c$  has clock  $cl$ , the second input  $x$  has clock  $cl \text{ on } c$  and third input  $y$  has clock  $cl \text{ on } (\mathbf{not } c)$ , then the result of  $\mathbf{merge } c \ x \ y$  has clock  $cl$ .

The last operator we consider is the buffer. As for the definition of **const**, the production or not of a value by the operator **buffer** depends on the environment. The definition is given in Figure 11. The first parameter ( $s$ ) of the operator is the contents of the buffer, the second ( $n$ ) is the number of places remaining in the buffer, the third is the input stream, and the fourth is the clock ( $w$ ) of the output. The semantics only gives a meaning to programs that use bounded buffers. The operator returns a value when the output clock is 1, provided that there is at least one stored value or an input value, and it stores input values as they arrive, provided that the number of remaining places is greater than zero. Moreover, it is not possible to store a value when the buffer is full, nor to pop a value when the buffer is empty.

The rule must be completed to deal with the empty sequence  $\epsilon$ . As for the Kahn semantics, the operators  $op^\sharp$ , **when** $^\sharp$ , **whenot** $^\sharp$  and **merge** $^\sharp$  return  $\epsilon$  if one of their argument is  $\epsilon$ :  $\epsilon$  is absorbing. The definitions for the operators **fby** and **buffer** applied to at least one  $\epsilon$  argument are:

$$\begin{array}{llll} \mathbf{fby}^\sharp(\epsilon, s_2) & = & \epsilon & \mathbf{fby1}^\sharp(\epsilon, s_1, s_2) & = & \epsilon \\ \mathbf{fby}^\sharp(v_1.s_1, \epsilon) & = & v_1.\epsilon & \mathbf{fby1}^\sharp(v, \epsilon, s_2) & = & v.\epsilon \\ \mathbf{fby}^\sharp(abs.s_1, \epsilon) & = & abs.\epsilon & \mathbf{fby1}^\sharp(v, s_1, \epsilon) & = & v.\epsilon \\ \mathbf{buffer}^\sharp(s, n, \epsilon, w) & = & \epsilon & & & \end{array}$$

All these functions on clocked streams are continuous. In particular, the function **buffer** $^\sharp$  is monotonic: given a memory  $s$  and a number of remaining cells  $n$  (two parameters which are not inputs of the program), for any pair of inputs  $(s_1, w)$  and  $(s'_1, w')$  such that  $s_1 \leq s'_1$  and  $w \leq w'$ , we have  $\mathbf{buffer}^\sharp(s, n, s_1, w) \leq \mathbf{buffer}^\sharp(s, n, s'_1, w')$ . Continuity follows because **buffer** $^\sharp$  is length preserving.

The semantics is not directly defined on the language kernel but on a slight variation where each constant takes an extra argument specifying the clock of its result. The buffer operator also takes extra arguments: one giving the clock of its input — when a value must be pushed, — another giving the clock of its output — when a value must be popped, — and another for the size of the buffer. The following translation defines the passage from the source language:

$$\begin{array}{ll} i & \longrightarrow \mathbf{const}(i, w) \\ \mathbf{buffer}(e) & \longrightarrow \mathbf{buffer}(n, e', w) \text{ where } e \longrightarrow e' \end{array}$$

The semantics for expressions, equations and programs are defined in the same way as for the Kahn semantics, except for constants and the buffer for which we take:

$$\begin{array}{ll} \llbracket \mathbf{const}(i, w) \rrbracket_\rho^{\text{abs}} & = \mathbf{const}^\sharp(i, w) \\ \llbracket \mathbf{buffer}(n, e, w) \rrbracket_\rho^{\text{abs}} & = \mathbf{buffer}^\sharp(\epsilon, n, \llbracket e \rrbracket_\rho^{\text{abs}}, w) \end{array}$$

These operators produce or not according to the operators that consume their output. This is why we add an extra argument giving the expected clock of the result. Moreover the **buffer** operator is initialized with an empty memory (written  $\epsilon$ ). The maximum size  $n$  of this memory is synthesized by the clock calculus and passed as an extra argument.

**Checking Synchrony** The example given in Figure 10 is now wrongly typed according to the composition of operator typing rules. Let *half* be the infinite periodic sequence  $1.0.1.0 \dots = (1.0)$ . To fulfil the typing rule for pointwise function applications, the expression  $x \& (x \text{ when } half)$  is only correct if the clock of  $x$ , say  $cl$ , equals the clock of  $x \text{ when } half$ , that is  $cl \text{ on } half$ . This is impossible and this program must thus be rejected. The compiler for Lucid Synchrone [31] emits the error message:

```
let node even x = x when half
let node non_synchronous x = x & (even x)
~~~~~
This expression has clock 'a on half,
but is used with clock 'a
```

In the kernel language we consider, every stream  $s$  is associated to a boolean sequence or *clock* with value 1 at the instants where  $s$  is present and 0 otherwise. Two streams can be composed (e.g., added together) without any buffer when their clocks are equal. This is essentially a typing problem [11]. As we mentioned, it was later formulated as a *shallow embedding* in Coq, showing that clock type verification could be implemented by Coq type verification.

The successive versions of Lucid Synchrone experimented with different extensions of the initial type system. We realised that having a powerful equivalence between expressions when comparing clock types  $c \text{ on } e_1$  and  $c' \text{ on } e_2$  was not very useful. In version 2, we experimented with a very simple clock calculus reminiscent of the simple ML type system with polymorphism but extended with the rule of Laufer and Odersky [23] for existential quantification [14]. This was the basis of the clock calculus used in the Scade 6 language.

The clock calculus is not only used to reject programs. Once the clock calculus is performed, every expression is annotated with a clock type. Those clocks are then used to generate efficient imperative code, in particular to factorise control structures by grouping computations that are activated on the same clock.

**Remark 2.4** (Embedding the clock calculus in ML). It seems possible to do a shallow embedding of a language of streams similar to the one considered in these course notes, with an encoding of both the dynamic semantics and the static clock constraints by using the Generalized Abstract Data Types (GADTs) of OCaml. We do not know if such an experiment has been completed.

In essence, the rule for typing an expression  $e_1 + e_2$  is:

$$\frac{H \vdash e_1 : ck \quad H \vdash e_2 : ck}{H \vdash e_1 + e_2 : ck}$$

This rule states that under the typing environment  $H$ , if  $e_1$  has type  $ck$  and if  $e_2$  has type  $ck$ , then  $e_1 + e_2$  has type  $ck$ . Recall that a clock type for a stream is of the form:

$$ck \quad :: \quad \alpha \mid ck \text{ on } e$$

where  $\alpha$  is a clock variable and  $e$  is a boolean expression. In the synchronous case,  $ck_1 \text{ on } e_1 = ck_2 \text{ on } e_2$  if  $ck_1 = ck_2$  and  $e_1 = e_2$ . Equality of types ensures equality of clocks. Hence, the composition of two flows of the same type can be defined without buffering.

## 2.2 From synchrony to n-synchrony

In Lustre and its relatives, two input streams can be composed with a point-wise operator only when they have the same clock. This ensures that no buffer is need for the composition. This is quite constraining for video applications that are easy to describe as a Kahn process networks. If a buffer is needed, a synchronous compiler is of any help: the place where to put the buffer, its size, its input and output clock of the buffer must be determined by the programmer.

Consider for example a Picture-in-Picture as depicted in figure 12 which incrusts an image into another one. This kind of system is well modeled as a Kahn process network but the manual computation of buffer sizes is mostly manual and difficult to determine.

The PiP takes a high definition image (1920×1080 pixels), downscales it into a small definition image (720×480 pixels); it takes an other high definition image and merges it with the small definition one. The downscaler introduces a delay, hence a buffer is needed for the second image. We would like that this size be computed automatically as well as the delay (latency) for the first pixer of the output image to come up.

Can we compose non strictly synchronous streams provided their clocks are closed from each other? Can we allow for the communication between systems which are “almost” synchronous, e.g., for modeling bounded jittering or bounded delays? Can we relax the clocking rule to give more freedom to the compiler so that it can generate more efficient code, translate into regular synchronous code if necessary?

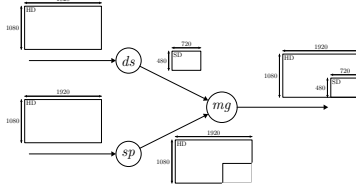


Figure 12: Picture in Picture

The  $n$ -synchronous model [12] relaxes the classical constraints of a synchronous language like to allow for the composition of streams whose clocks are not equal but can be synchronized through the introduction of a bounded buffer. It is obtained by relaxing the clock calculus with a subtyping rule. If a stream  $x$  with type  $ck$  can be consumed later with type  $ck'$  using a bounded buffer, we shall say that  $ck$  is a subtype of  $ck'$  and we write  $ck <: ck'$ . This allow to type a synchronous language extended with a **buffer** construct which indicates the points where the subtyping rule should be applied.

$$\frac{H \vdash e : ck \quad ck <: ck'}{H \vdash \mathbf{buffer} \, e : ck'}$$

In terms of sequences of values, **buffer**  $e$  is equivalent to  $e$  but it may delay its input using a bounded buffer. The **buffer** construct gives more freedom to the designer while preserving an execution in bounded memory.

Here, we consider a simple definition for  $<:$  allowing to compare two types if they are of the form  $\alpha \text{ on } w_1$  and  $\alpha \text{ on } w_2$  only, with  $w_1 <: w_2$ .  $w_1$  and  $w_2$  are two boolean expressions.

**Definition 2.1** (Ultimately periodic clocks). We consider a particular clock language  $ce$  that define ultimately periodic boolean sequences only:

$$\begin{aligned} ce &::= c \mid u(v) \\ u &::= \varepsilon \mid 0.u \mid 1.u \\ v &::= 0 \mid 1 \mid 0.v \mid 1.v \end{aligned}$$

It can be a variable name ( $c$ ) or a periodic word ( $u(v)$ ) made of a finite prefix ( $u$ ) followed by the infinite repetition of a binary word ( $v$ ). For example, (10) defines the half sequence 101010...

### 2.2.1 Clock Adaptability

Here is the intuition of adaptability: *a clock  $w_1$  is adaptable to clock  $w_2$  if any stream with clock  $w_1$  can be consumed with clock  $w_2$  up to the insertion of a bounded buffer.*

To properly define this relation, we introduce the *cumulative function* of a binary word: for any binary word  $w$ ,  $\mathcal{O}_w(i)$  counts the number of 1s up to the index  $i$ . Figure 13 shows the cumulative functions of  $w_1 = (11010)$  and  $w_2 = 0(00111)$ .

**Definition 2.2** (Elements and Cumulative Function of  $w$ ).

Let  $w = b.w'$  with  $b \in \{0, 1\}$ . We write  $w[i]$  for the  $i$ -th element of  $w$ :

$$\begin{aligned} w[1] &\stackrel{def}{=} b \\ \forall i > 1. w[i] &\stackrel{def}{=} w'[i-1] \end{aligned}$$

We write  $\mathcal{O}_w$  for the cumulative function of  $w$ :

$$\begin{aligned} \mathcal{O}_w(0) &\stackrel{def}{=} 0 \\ \forall i \geq 1. \mathcal{O}_w(i) &\stackrel{def}{=} \begin{cases} \mathcal{O}_w(i-1) & \text{if } w[i] = 0 \\ \mathcal{O}_w(i-1) + 1 & \text{if } w[i] = 1 \end{cases} \end{aligned}$$

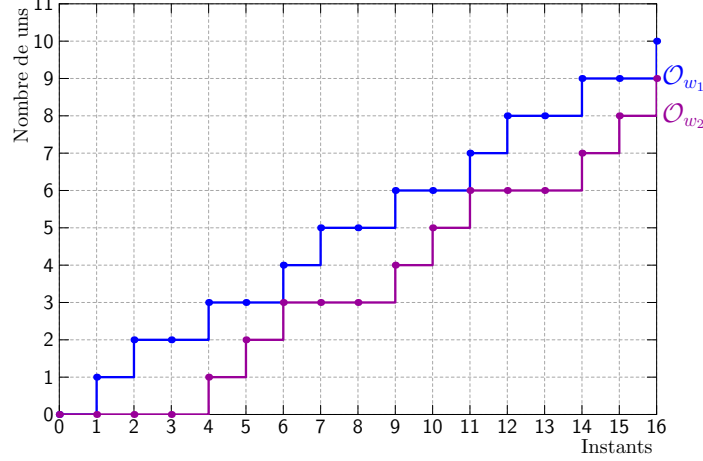


Figure 13: Cumulative functions for  $w_1 = (11010)$  and  $w_2 = 0(00111)$ .

Adaptability is the conjunction of two relations: *precedence* and *synchronizability*. Precedence ensures that there is no read in an empty buffer, that is at each instant, more values have been written than read in the buffer. Synchronizability ensures that the number of values present in the buffer during the execution is bounded.

**Definition 2.3** (Synchronizability  $\bowtie$ , Precedence  $\preceq$ , Adaptability  $<:$ ).

$$\begin{aligned}
w_1 \bowtie w_2 &\stackrel{\text{def}}{\iff} \exists b_1, b_2 \in \mathbb{Z}, \forall i \geq 0. b_1 \leq \mathcal{O}_{w_1}(i) - \mathcal{O}_{w_2}(i) \leq b_2 \\
w_1 \preceq w_2 &\stackrel{\text{def}}{\iff} \forall i > 0. \mathcal{O}_{w_1}(i) \geq \mathcal{O}_{w_2}(i) \\
w_1 <: w_2 &\stackrel{\text{def}}{\iff} w_1 \preceq w_2 \wedge w_1 \bowtie w_2
\end{aligned}$$

In Figure 13,  $w_1 \bowtie w_2$  since the vertical distance between the two curves is bounded and  $w_1 \preceq w_2$  since the curve  $\mathcal{O}_{w_1}$  is always above the one of  $\mathcal{O}_{w_2}$ .

**Buffer Size.** Consider a buffer with an input clock  $w_1$  and output clock  $w_2$ . For every instant  $i$ , the number of elements present in the buffer is:

$$size_i(w_1, w_2) = \mathcal{O}_{w_1}(i) - \mathcal{O}_{w_2}(i)$$

A negative value means that there were more reads than writes and this case should not appear. A sufficient size for the buffer is the maximal number of values present in the buffer during the execution:

$$size(w_1, w_2) = \max_{i \geq 1} (\mathcal{O}_{w_1}(i) - \mathcal{O}_{w_2}(i))$$

Thus, if  $w_1$  is adaptable to  $w_2$ , a stream with clock  $w_1$  can be safely consumed on the clock  $w_2$  by insertion of a bounded buffer. Otherwise, the size of the buffer may be infinite.

The purpose of the extended clock calculus is to check that bounds exist for buffer sizes and to compute them. To this aim, subtyping constraints have to be solved and it can be done for clock that are ultimately periodic (see 2.1) [12].

To reduce the algorithmic complexity of constraint resolutions and deal with non periodic clocks, it is possible to reason with *clock envelopes*. These clock envelopes are sets of concrete clocks which are not necessarily periodic. They can model various features that exist in embedded systems such as bounded jittering, logical execution time (lower and upper bounds on the numbers of atomic steps done by a process), latencies (between when an input data is read and a output is produced), scheduling resources (a process is activated a certain number of time during a period)

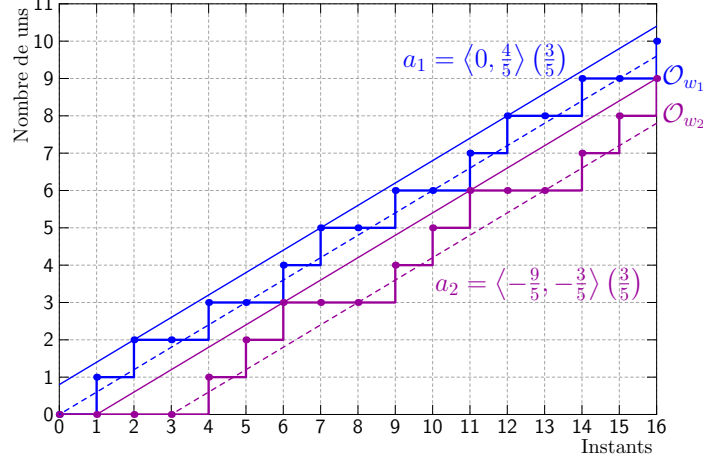


Figure 14: Envelopes of  $w_1$  and  $w_2$ .

and the communication through buffers. Said differently, an envelope is an over abstraction of the exact clocks of the system. Hence, instead of comparing two exact clocks, we compare envelopes.

The abstraction introduced in [13] consists in reasoning on sets of clocks (or *envelopes*) defined by an asymptotic rate and two shifts bounding the potential delay with respect to this rate. It was made more precise (in the sense that it over approximates less) in [25]. Then, subtyping constraints can be replaced by linear constraints on those rates and shifts, and solved with a tool such as Glpk. We only give here an intuition of this abstraction. It was implemented for a language called Lucy-n that includes an explicit **buffer** construction and whose syntax and semantics is exactly that of the language introduced in 2. On several examples such as the *Picture in Picture*, the over-estimation due to the abstraction is small with respect to the exact solution.

### 2.2.2 Abstraction of Binary Words

The idea behind abstraction is to reason on sets of binary words. An abstraction bounds the cumulative function of a set of words by two linear curves with the same slope. Thus, the abstraction of an infinite binary word  $w$  keeps only the asymptotic proportion  $r$  of 1s in  $w$  and two values  $b^0$  and  $b^1$  which give the minimum and maximum shift of 1s in  $w$  compared to  $r$ . This abstract information is called an *envelope* and noted  $\langle b^0, b^1 \rangle (r)$ .

**Definition 2.4** (Concretization).

$$\text{concr} (\langle b^0, b^1 \rangle (r)) \stackrel{\text{def}}{=} \left\{ w \mid \forall i \geq 1, \quad \wedge \begin{array}{ll} w[i] = 1 & \Rightarrow \mathcal{O}_w(i) \leq r \times i + b^1 \\ w[i] = 0 & \Rightarrow \mathcal{O}_w(i) \geq r \times i + b^0 \end{array} \right\}$$

with  $b^0, b^1, r \in \mathbb{Q}$  and  $0 \leq r \leq 1$ .

The words  $w_1 = (11010)$  and  $w_2 = 0(00111)$  seen previously are respectively in envelopes  $a_1 = \langle 0, \frac{4}{5} \rangle (\frac{3}{5})$  and  $a_2 = \langle -\frac{9}{5}, -\frac{3}{5} \rangle (\frac{3}{5})$  shown in Figure 14. In chronograms, an abstract value  $\langle b^0, b^1 \rangle (r)$  is represented by two lines  $\Delta^1: r \times i + b^1$  and  $\Delta^0: r \times i + b^0$  that bound the cumulative functions of a set of binary words. The definition states that any rising edge must be below the line  $\Delta^1$  (solid line) and any absence of a rising edge must be above the line  $\Delta^0$  (dashed line).

For the set of words defined by an envelope to be non-empty, the line  $\Delta^1$  must be above the line  $\Delta^0$ . At each instant, there must be a discrete value between the two lines. It is the case if the distance between them respects the following constraint.

**Proposition 2.1** (Non-empty envelope).

$$\forall a = \left\langle \frac{k^0}{\ell}, \frac{k^1}{\ell} \right\rangle \left( \frac{n}{\ell} \right) \cdot \frac{k^1}{\ell} - \frac{k^0}{\ell} \geq 1 - \frac{1}{\ell} \Rightarrow \text{concr}(a) \neq \emptyset$$

The abstraction of a periodic binary word can be computed automatically.

**Definition 2.5** (Abstraction of a Periodic Word).

Let  $p = u(v)$  a periodic binary word. We define  $abs(p) \stackrel{def}{=} \langle b^0, b^1 \rangle(r)$  with:

$$\begin{aligned} r &= rate(p) = \frac{|v|_1}{|v|} \\ b^0 &= \min_{i=1..|u|+|v| \text{ with } p[i]=0} (\mathcal{O}_p(i) - r \times i) \\ b^1 &= \max_{i=1..|u|+|v| \text{ with } p[i]=1} (\mathcal{O}_p(i) - r \times i) \end{aligned}$$

where  $|u|$  is the length of  $u$  and  $|u|_1$  its number of 1s.

The asymptotic rate  $r$  corresponds to the ratio between the number of 1s in the periodic pattern and its length. To compute  $b^0$  and  $b^1$ , the word must be traversed. The shift  $b^0$  is the minimum difference when a 0 occurs between the number of 1s seen at instant  $i$  and the ideal value  $r \times i$ . The shift  $b^1$  is the maximal difference between these values when a 1 occurs.

The interest of the abstraction is to reduce the complexity of exact computations and decisions on binary words by transforming them into arithmetic manipulations on rational numbers. For example, the computation of  $\mathbf{on}$  on envelopes only needs three multiplications and two additions:

**Definition 2.6** ( $\mathbf{on}^\sim$  Operator). Let  $b^0_1 \leq 0$  and  $b^0_2 \leq 0$ .<sup>10</sup> We define:

$$\begin{aligned} \mathbf{on}^\sim & \langle \begin{matrix} b^0_1 & , & b^1_1 \\ b^0_2 & , & b^1_2 \end{matrix} \rangle \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} \\ & \stackrel{def}{=} \langle b^0_1 \times r_2 + b^0_2, b^1_1 \times r_2 + b^1_2 \rangle (r_1 \times r_2) \end{aligned}$$

The elements of  $w_1 \mathbf{on} w_2$  are the elements of  $w_1$  filtered by the elements of  $w_2$ . The rate of 1 in  $w_1 \mathbf{on} w_2$  is thus the product of the rate of  $w_1$  and the one of  $w_2$ . When  $w_1$  is sampled by  $w_2$ , its shifts are multiplied by  $r_2$ . The shifts of  $w_2$  are added to those of  $w_1$ .

All the proofs on algebraic properties of binary sequences and abstractions have been done in Coq [25] and are available publicly. Full proofs on paper are available in the PhD. thesis of Florence Plateau [30].

### 3 Conclusion

In these course notes, we considered a simple first-order functional language that manipulates streams and functions that transform streams into streams. This language is reminiscent of the language Lustre invented by Caspi and Halbwachs, which was the basis for the development of the industrial language and environment SCADE, now regularly used in the development of critical control software, as well as the academic language Lucid Synchronic.

We showed that this language corresponds to a particular kind of Kahn process network that can be executed synchronously. This is expressed by associating a clock to every stream to indicate when the current value is present or not. Stream functions must then fulfill certain static rules to ensure that when a value is expected to be present (or absent), it is indeed present (or absent). Clocks can be understood as types and the associated static constraints as typing constraints in a type system with dependent types. Finally, we relax the synchronous constraint to allow communications through bounded buffers by adding sub-typing rules for when buffers are used. A relaxed clock calculus can infer the size of these communication buffers.

These course notes are far from exhaustive. In particular, they do not detail the actual clock calculus for the language, and notably the restrictions made in both Lucid Synchronic and SCADE about clock equality. In these two languages, the clock language  $ce$  is limited essentially to names so that clock equality reduces to name equality. These notes also sweep under the carpet the important question of causality. E.g., equations like  $x = x$  or  $x = x + 1$  are perfectly valid

<sup>10</sup>We can always lose precision on the envelopes to satisfy this condition. More details are given in [30].

from a clock calculus point-of-view but must be rejected because  $x$  depends instantaneously on itself and no sequential code can be generated: we say that  $x$  is not causal. The detection of instantaneous loops or dependencies can also be handled by static typing. Finally, these notes did not address the important question of generating sequential code. Clocks are also fundamental to code generation [16]. The clock constraints can be interpreted as dedicated techniques to ensure the perfect fusion of all the intermediate streams.

## Acknowledgment

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## A Some examples

**Question A.1** (Reasonning about processes). As motivated by Kahn in [21], the denotational interpretation of processes as stream function can be used to reason and prove properties about stateful systems. Consider the example in Figure 15.

- Propose an interpretation for processes `p`, `q`, `m` and `main` as continuous functions. Propose an alternative implementation of the network that use, for example, the primitives given in Figure 2. How would you prove it equivalent to the initial one?
- What does it change to remove line `(* init *)`?
- Prove that the program `main` is non blocking, i.e, if input `x` is an infinite stream, `z` is an infinite stream. It can be done with a length argument, taking  $|\epsilon| = 0$  and  $|v.s| = 1 + |s|$ .
- Propose a sequential, equivalent implementation of `main`, made of a single elementary process with an input channel `x` and output channel `z`.

```

type 'a buff = { push: 'a -> unit; pop: unit -> 'a }

let buffer () =
  let b = Queue.create () in
  let t = Mutex.create () in
  let push v = Mutex.lock t; Queue.push v b; Mutex.unlock t in
  let pop () = Mutex.lock t; Queue.pop b; Mutex.unlock t in
  { push = push; pop = pop }

(* Process P *)
let process_p x r y () =
  y.push 0; (* init *)
  let memo = ref 0 in
  while true do
    let v = x.pop () in
    let w = r.pop () in
    memo := if v then 0 else !memo + w;
    y.push !memo
  done

(* Process Q *)
let process_q y t z () =
  while true do
    let v = y.pop () in
    t.push v; z.push v
  done

(* Process R *)
let process_m t r () =
  while true do
    let v = t.pop () in
    r.push (v + 1)
  done

(* Put them in parallel. *)
let main x z () =
  let r = buffer () in
  let y = buffer () in
  let t = buffer () in
  ignore (Thread.create (process_p x r y) ());
  ignore (Thread.create (process_q y t z) ());
  ignore (Thread.create (process_m t r) ())

```

Figure 15: A simple implementation of KPN with threads in OCaml