# Hybrid System Modeling and Programming

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Current Practice
- Hybrid Systems Modelers
- Some issues

Interlude: interacting with a numerical solver

Key elements of our approach
- Non Standard Synchronous Semantics
- Typing
- Compilation

# Trends for building safe and complex embedded systems

Write executable mathematical specifications in a high-level language so that a model is:

A reference semantics independent of any implementation.

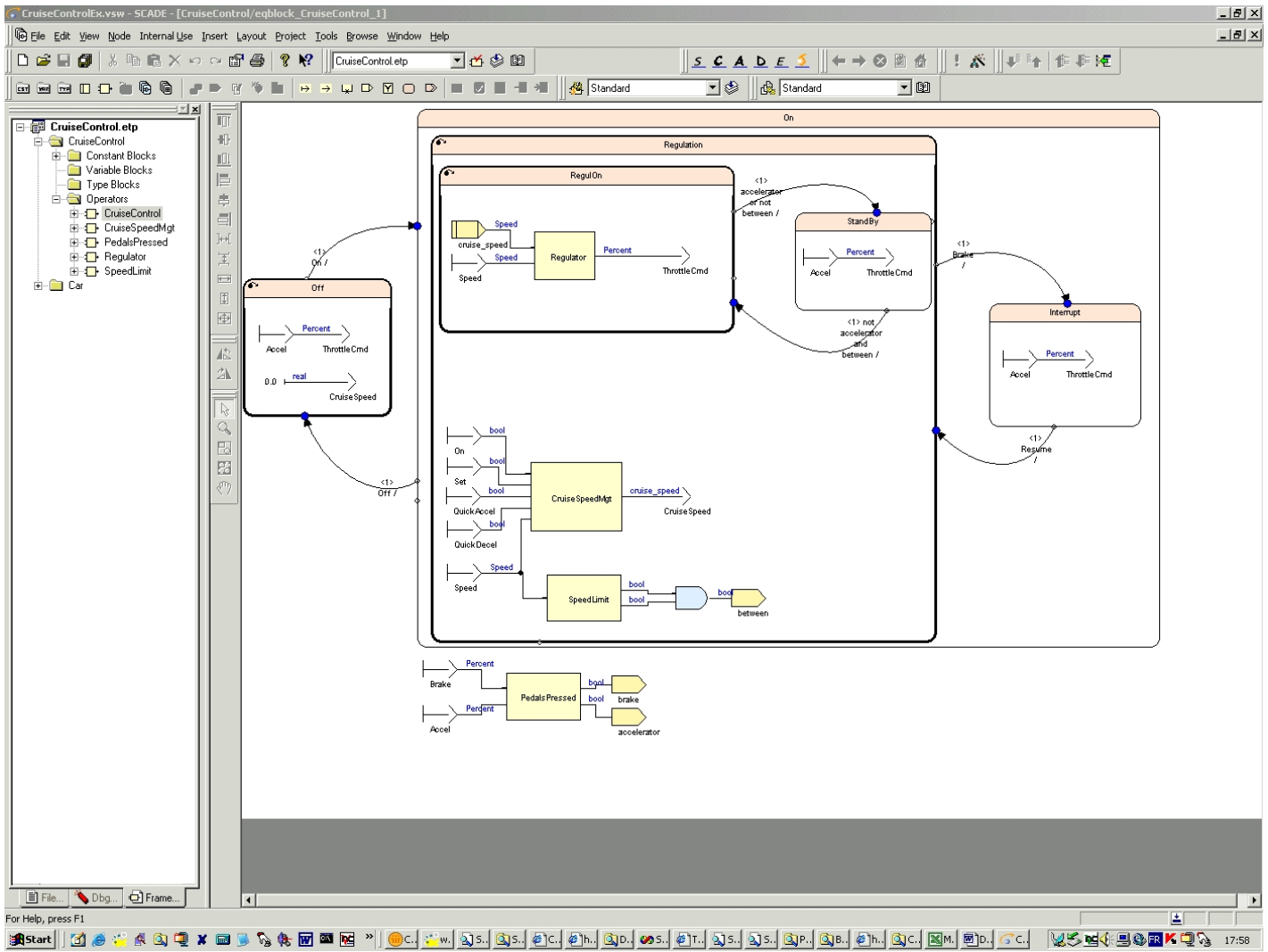
A base for simulation, testing, formal verification.

Then compiled into executable code, sequential or parallel.

A way to achieve correct-by-construction software.

# Synchronous Block Diagram Languages[1]

E.g., The Cruise control in SCADE 6 (Esterel-Technologies/ANSYS).




[1] Cf. previous courses by Gérard Berry.

# A good match for programming discrete-time controllers

Their semantics is simple and mathematically precise:

- Difference equations; hierarchical automata; parallel composition.
- Simulate/test/verify throughout the development process.

Then compiler ensures strong safety properties.

- The program is deterministic.
- The generated code runs in bounded time and memory.

Efficient and fully traceable code generation.

- The code is correct w.r.t the source model.
- Meets the highest quality level of civil avionics (DO178C, level A).[2]

SCADE 6 is used for programming various critical control software.[3]

[2] Cf. Seminar by Bruno Pagano, in Spring 2013.

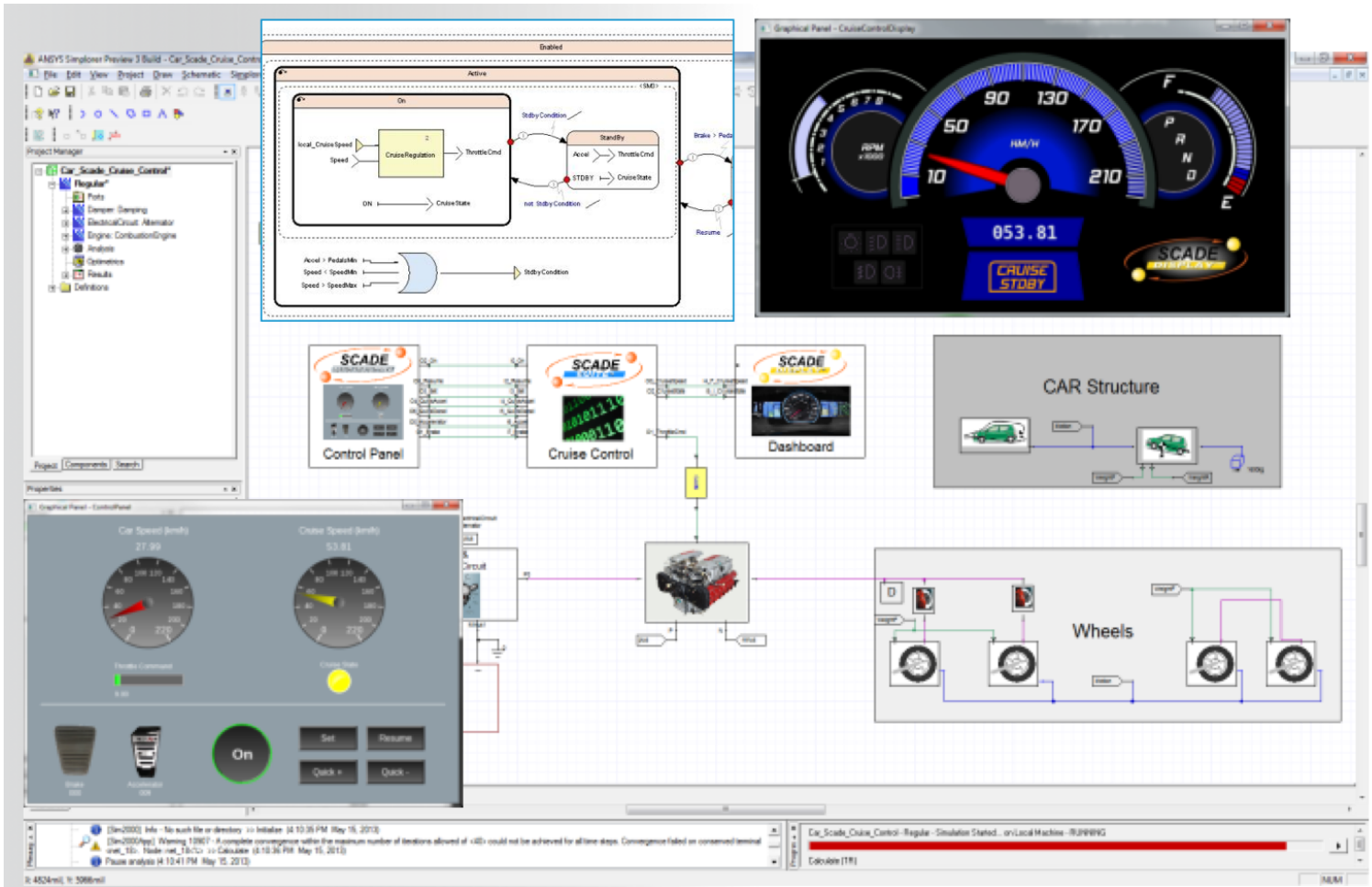
[3] Cd. Seminar by Emmanuel Ledinot, in Spring 2013.

# But modern systems need more. . .

# The Current Practice of Hybrid Systems Modeling

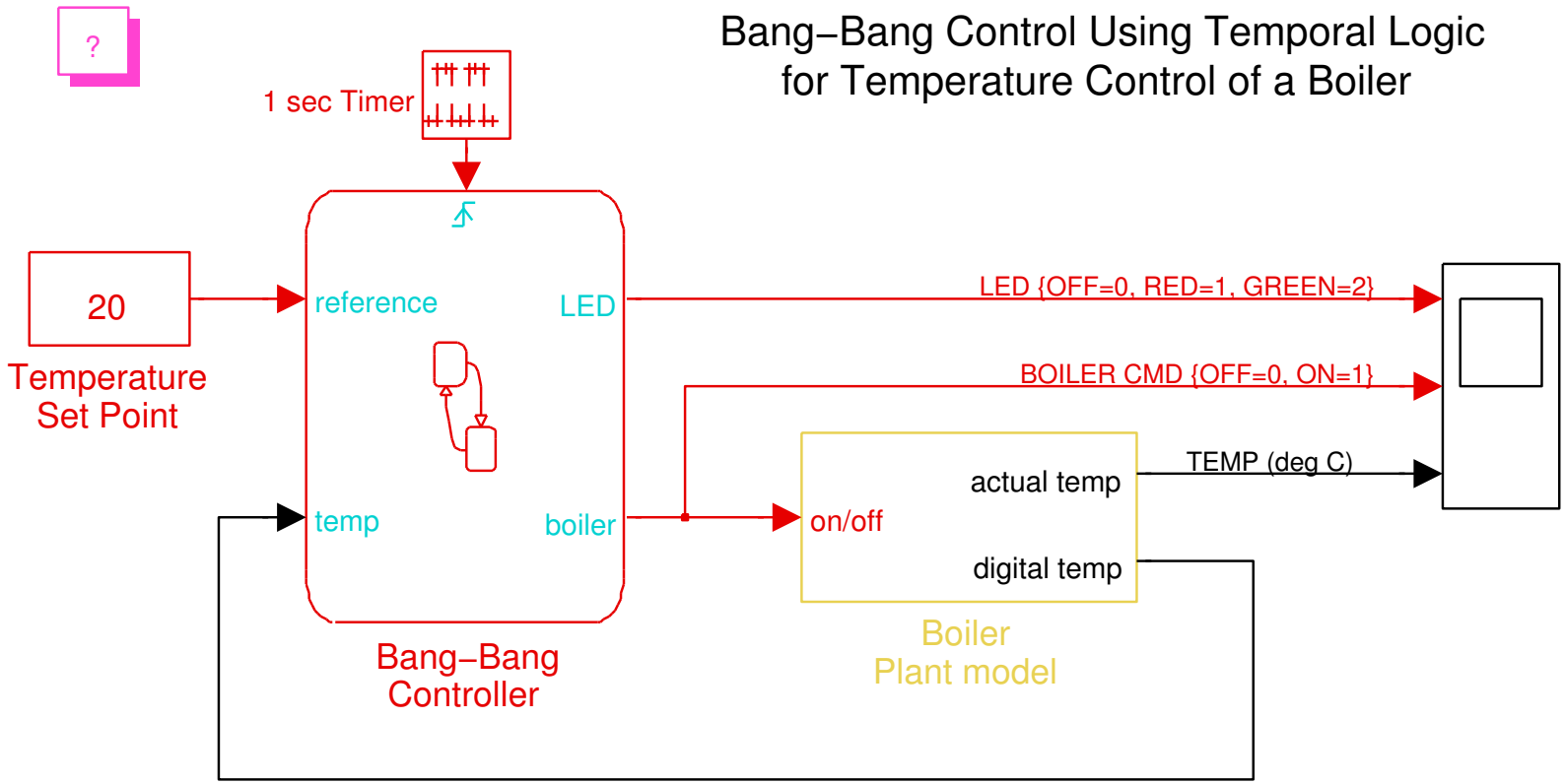
Embedded software interacts with physical devices.

The whole system has to be modeled: the controller and the plant.[4]




[4] Image by Esterel-Technologies/ANSYS.

# Example: a Bang-bang controller [demo].



# A Wide Range of Hybrid Systems Modelers Exist

## Ordinary Differential Equations + discrete time:

Simulink/Stateflow ($\geq 10^6$ licences), LabView, Ptolemy II, etc.

## Differential Algebraic Equations + discrete time:

Modelica, VHDL-AMS, VERILOG-AMS, etc.

## Dedicated tools for multi-physics:

Mechanics, electro-magnetics, fluid, etc.

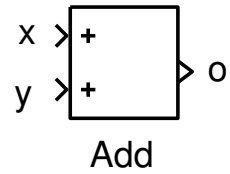
## Co-simulation/combination of tools:

Agree on a common format/protocol: FMI/FMU, S-functions, etc.

Convert the model of one tool into another.

# Underlying Mathematical Models

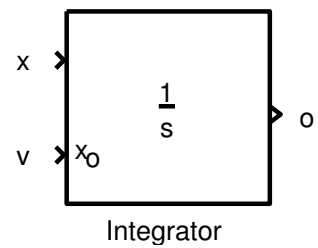
## Synchronous parallelism, sequence equations:[5]




Time is discrete and logical (indices in $\mathbb{N}$)

Equation $o = x + y$ means $\forall n \in \mathbb{N}.o(n) = x(n) + y(n)$

## Ordinary Differential Equations (ODEs):[6]




Time is continuous (indices in $\mathbb{R}$)

Equation $o = \frac{1}{s}(x)\ \mathit{init}\ v$ means $\forall t \in \mathbb{R}.o(t) = v(0) + \int_0^t x(\tau)\, d\tau$

[5] Cf. Course by Gerard Berry, Spring 2013.

[6] Cf. Seminar by Juliette Leblond, Feb. 2014.

# Is there anything left to do?

We know how to build tools for discrete-time models.

We know how to build tools for continuous-time models.

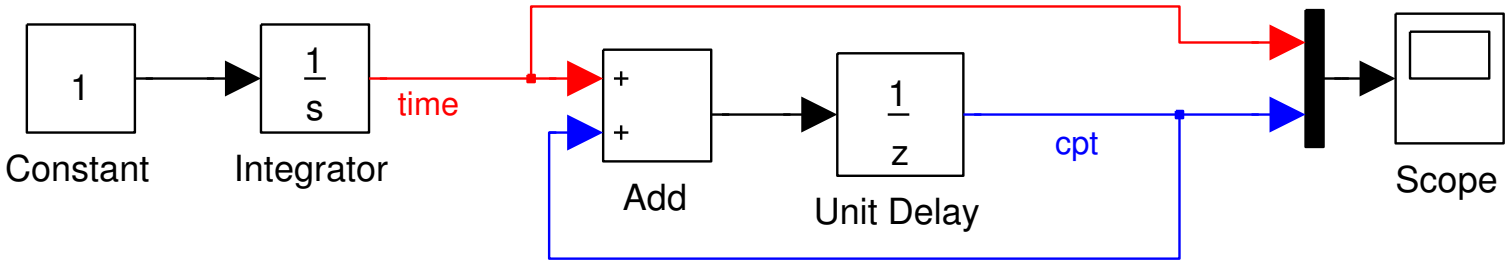
But what if the two are mixed together?

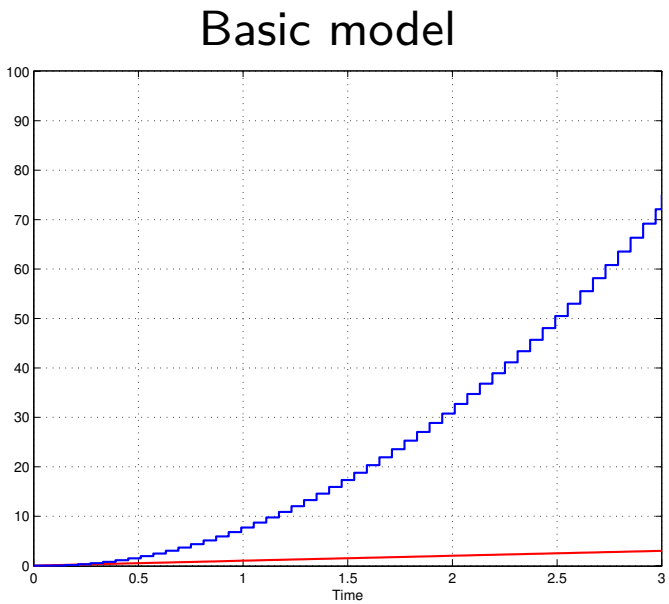
Is it enough to connect one to the other?

Can you trust code automatically generated from such tools?

# Strange beasts. . .

# Typing issue 1: Mixing continuous & discrete components



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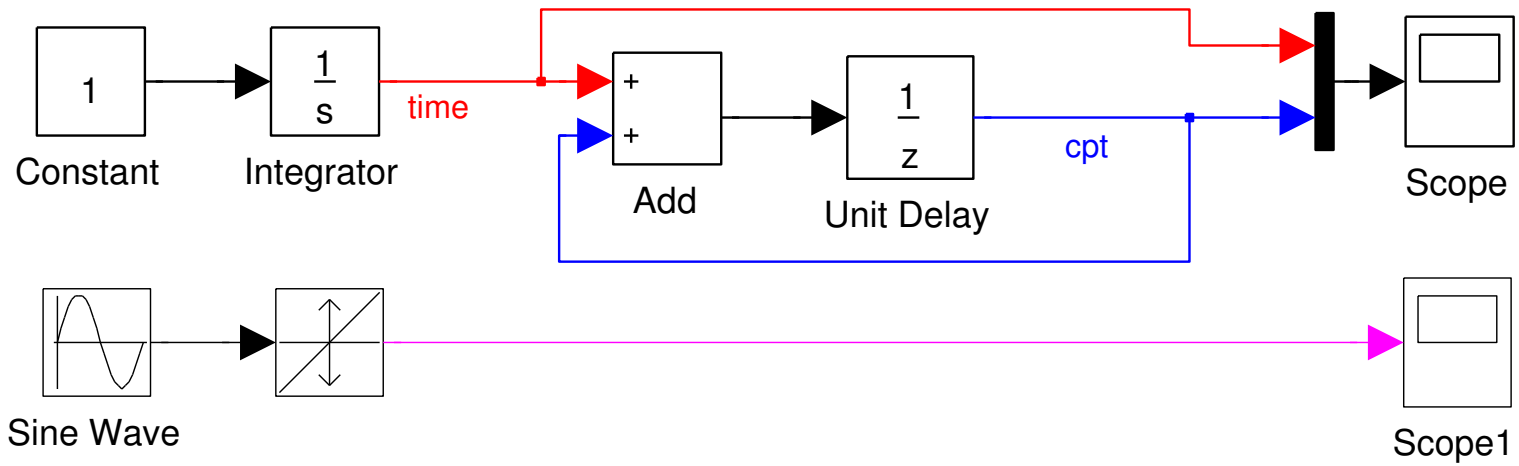
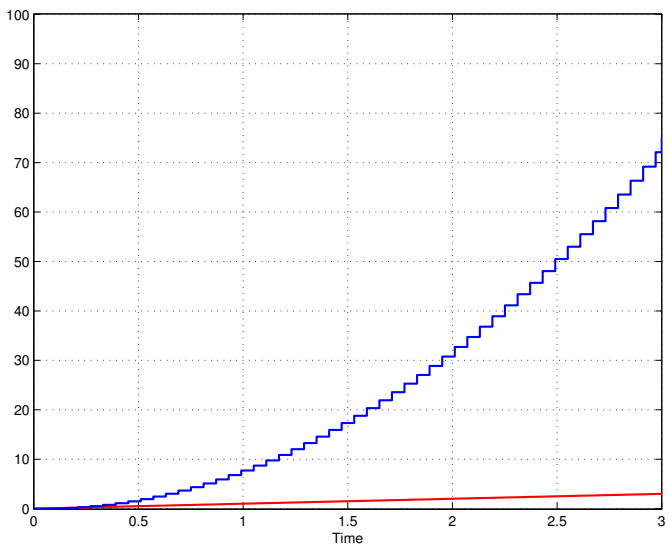
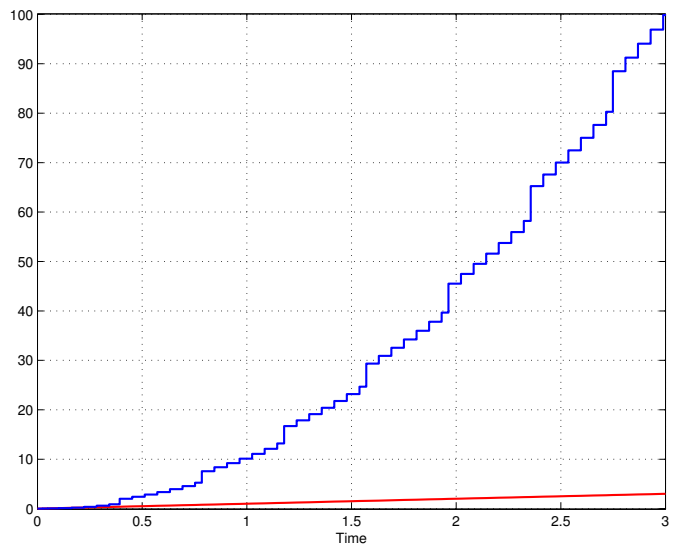



Basic model

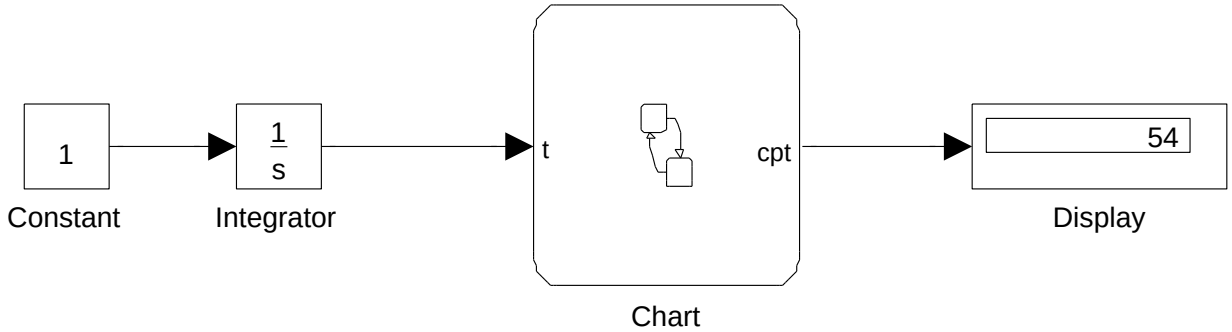


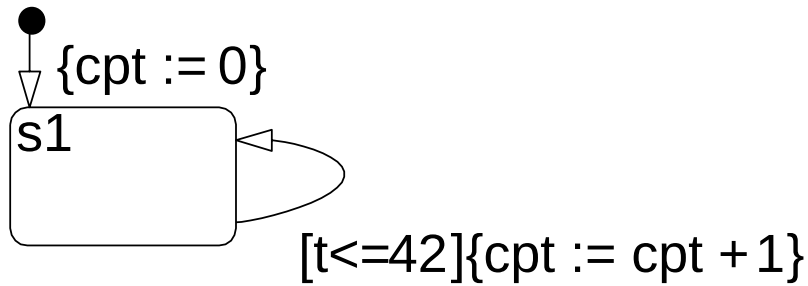

with Sine Wave




- The shape of cpt depends on the steps chosen by the solver.
- Putting another component in parallel can change the result.

# Typing issue 2: Boolean guards in continuous automata



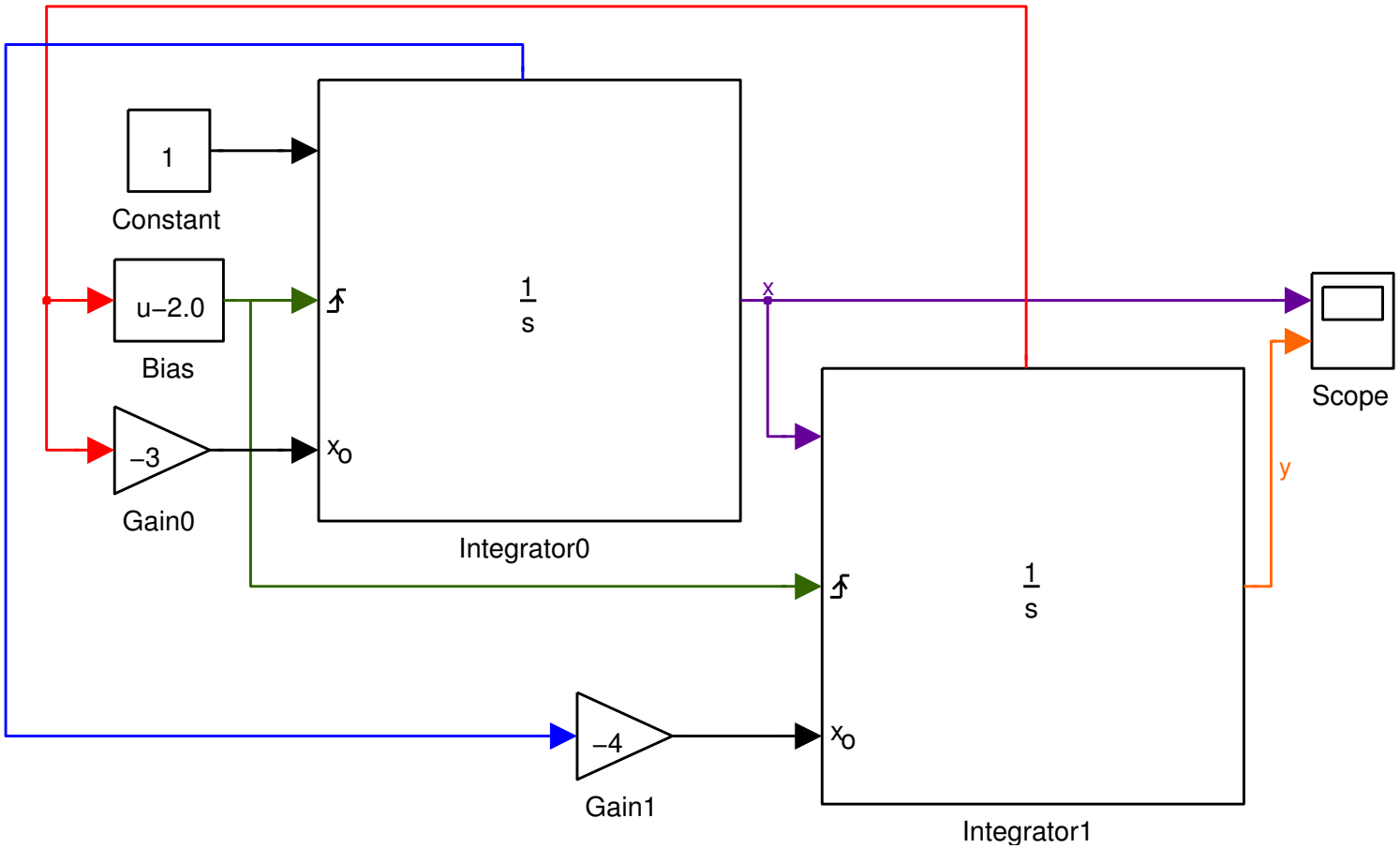




## How long is a discrete step?

- Adding a parallel component changes the result.
- No warning by the compiler.
- The manual says: “A single transition is taken per major step”.

Discrete time is not logical: it is that of the simulation engine.

# Causality issue: the Simulink state port



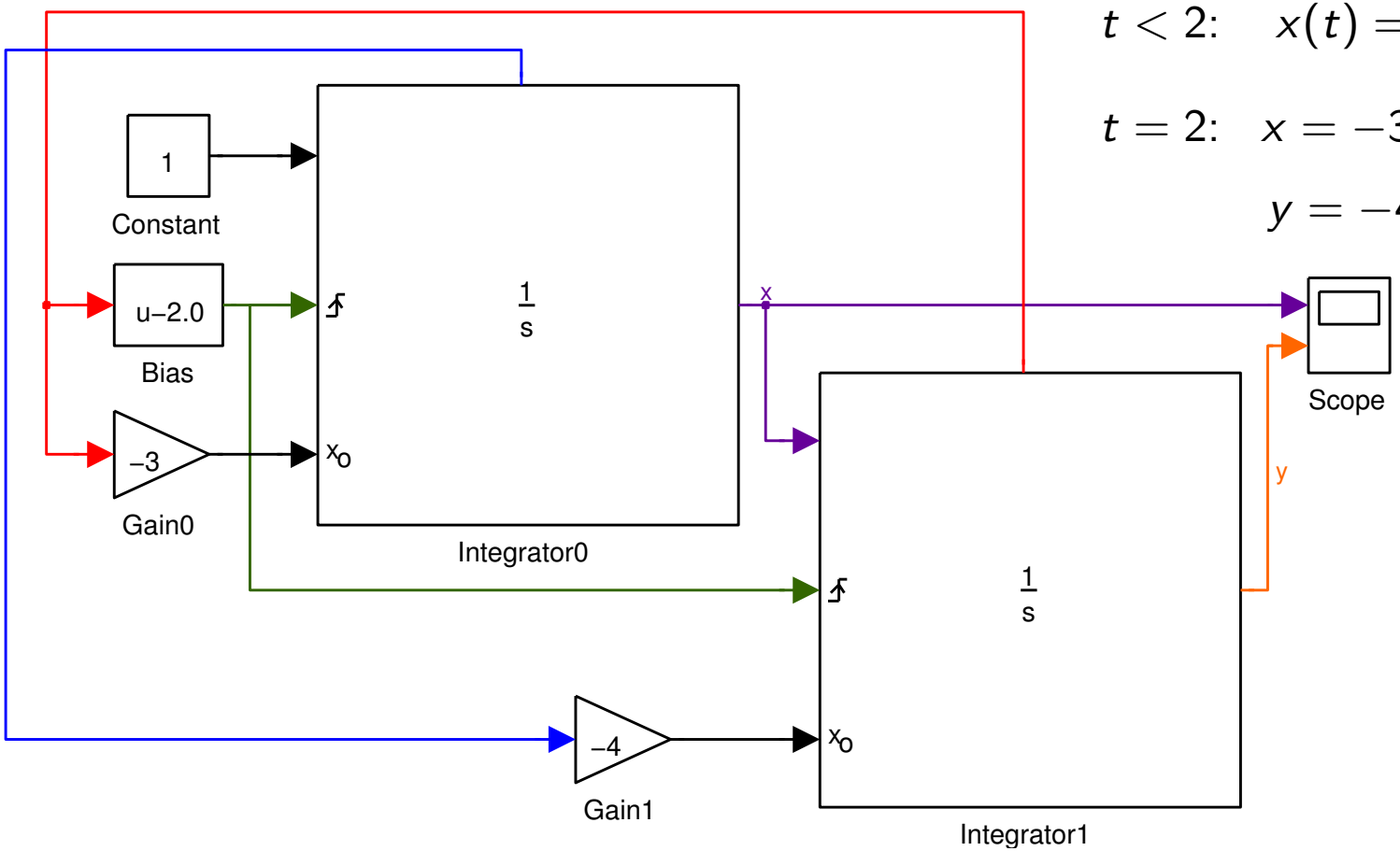

The output of the state port is the same as the output of the block's standard output port except for the following case. If the block is reset in the current time step, the output of the state port is the value that would have appeared at the block's standard output if the block had not been reset.
–Simulink Reference (2-685)

# Causality issue: the Simulink state port

$$t < 2: \quad x(t) = t,\ y(t) = \frac{t^2}{2}$$

$$t = 2: \quad x = -3 \cdot \text{last}\, y = -6,$$

$$y = -4 \cdot \text{last}\, x = -8$$

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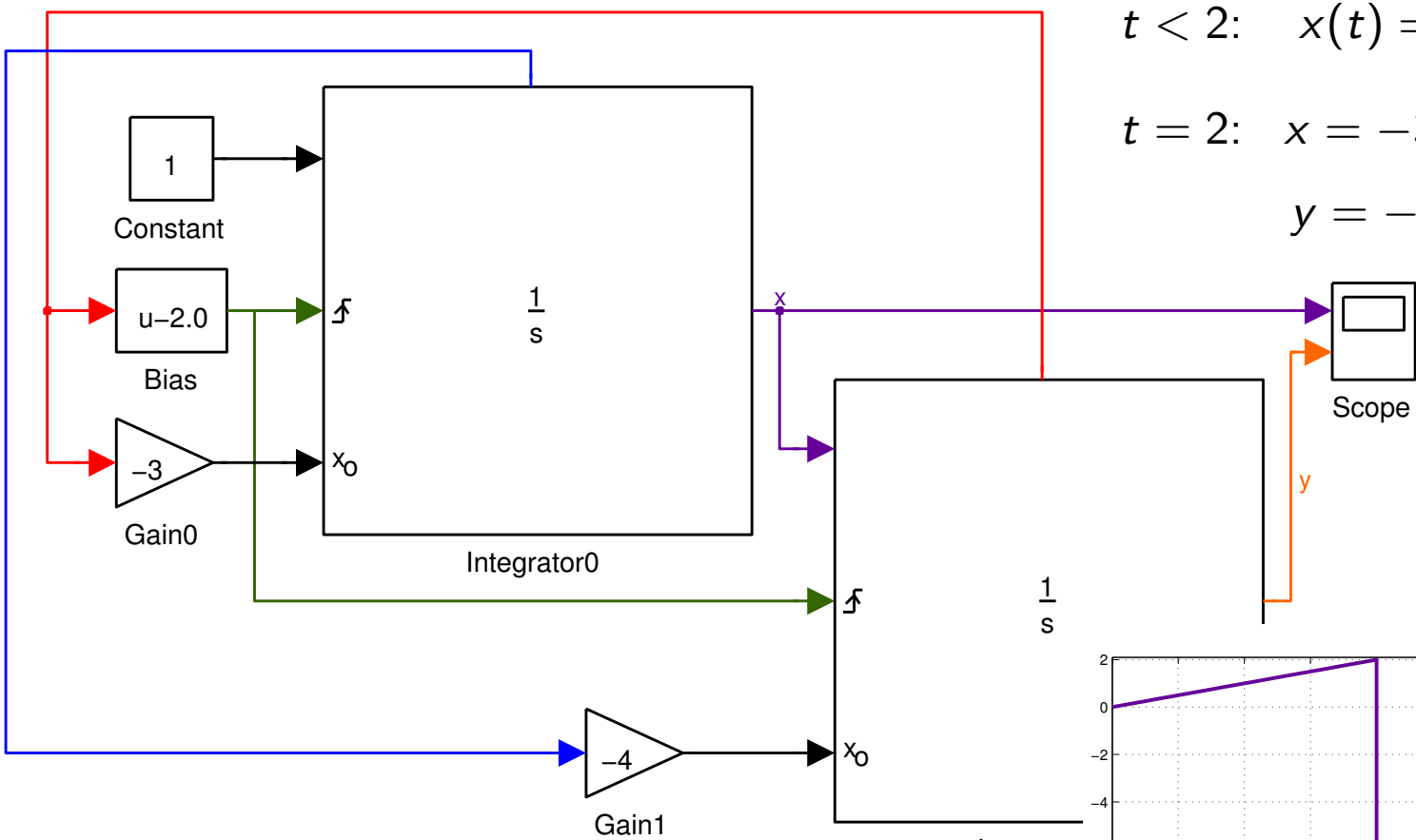


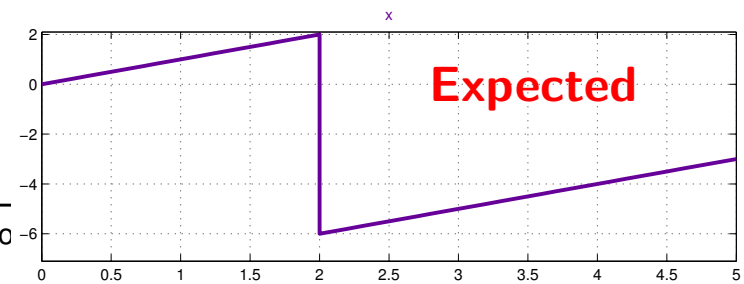


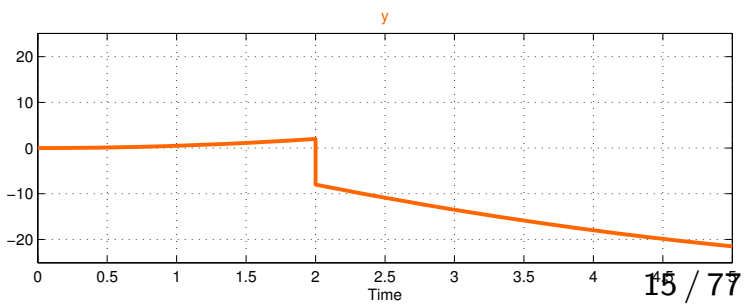


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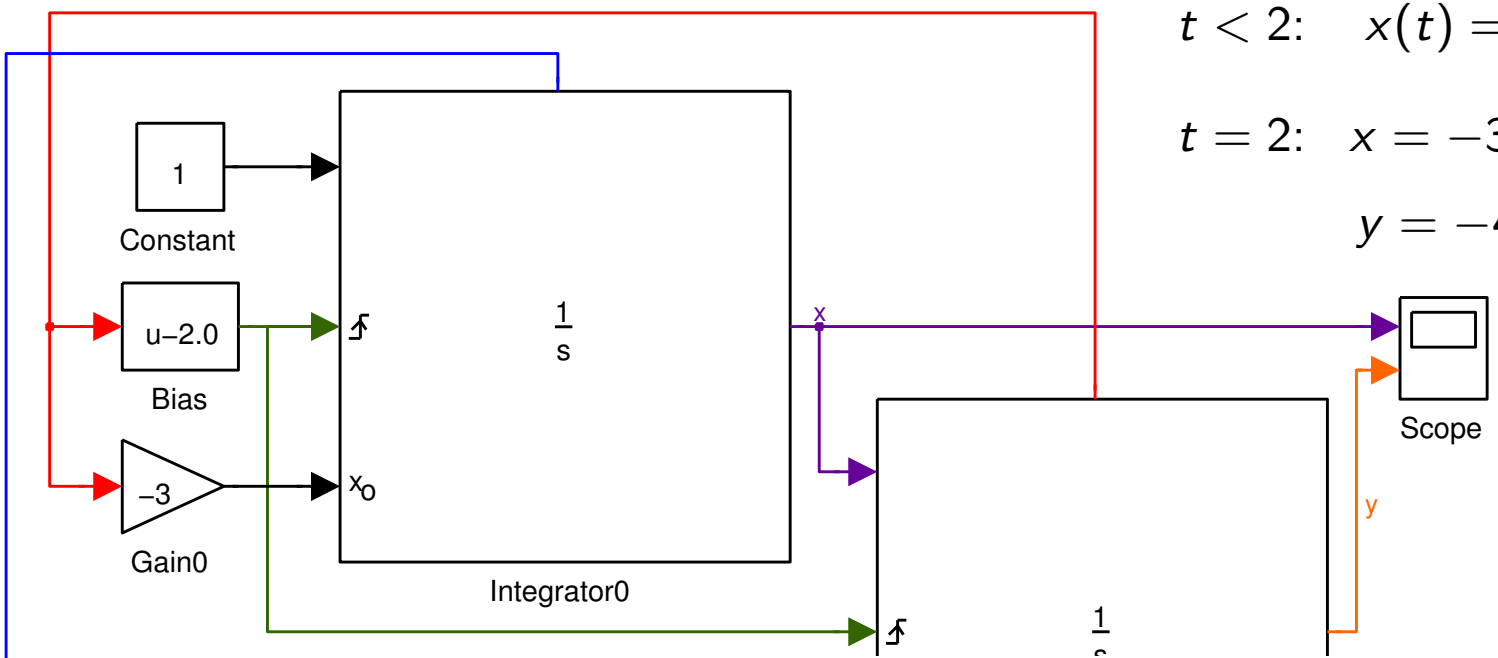


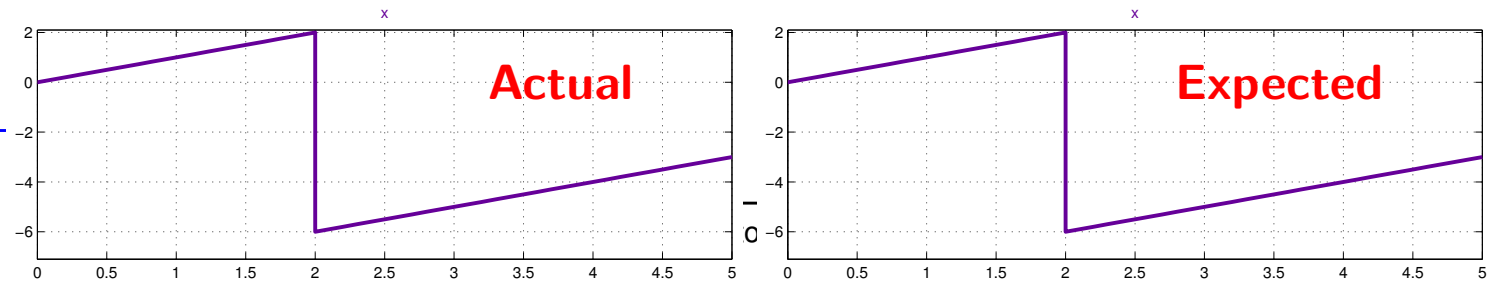


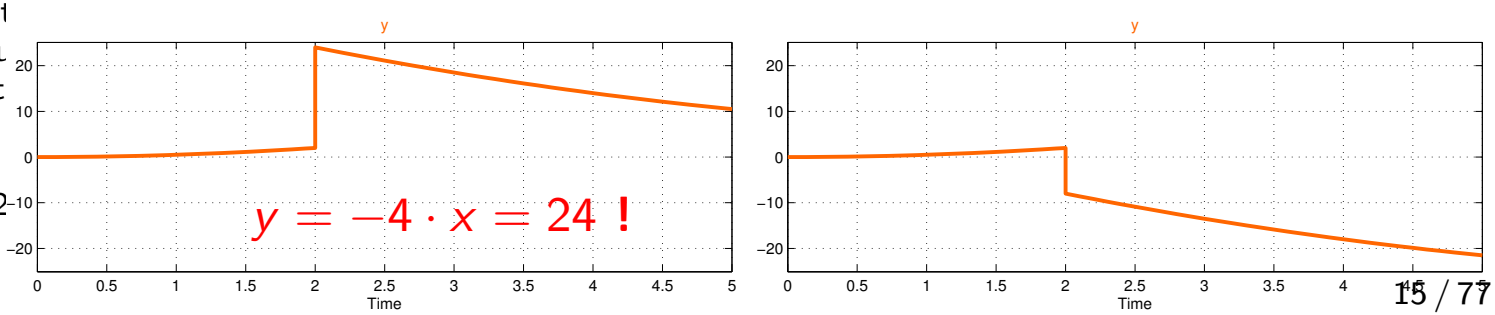


The output of the stat
block's standard outpu
the block is reset in t
state port is the value
standard output if the
–Simulink Reference (2

# Excerpt of C code produced by RTW (release R2009)

```
static void mdlOutputs(SimStruct * S, int_T tid)
{ _rtX = (ssGetContStates(S));
  ...
  _rtB = (_ssGetBlockIO(S));
  _rtB->B_0_0_0 = _rtX->Integrator1_CSTATE + _rtP->P_0;
  _rtB->B_0_1_0 = _rtP->P_1 * _rtX->Integrator1_CSTATE;
  if (ssIsMajorTimeStep (S))
    { ...
      if (zcEvent || ...)
        { (ssGetContStates (S))->Integrator0_CSTATE =
            _ssGetBlockIO (S))->B_0_1_0;
        }
      ...
  (_ssGetBlockIO (S))->B_0_2_0 =
    (ssGetContStates (S))->Integrator0_CSTATE;
    _rtB->B_0_3_0 = _rtP->P_2 * _rtX->Integrator0_CSTATE;
    if (ssIsMajorTimeStep (S))
    { ...
      if (zcEvent || ...)
       { (ssGetContStates (S))-> Integrator1_CSTATE =
            (ssGetBlockIO (S))->B_0_3_0;
       }
      ... } ... }
```

Before assignment: integrator state contains 'last' value

$x = -3 \cdot \text{last } y$

After assignment: integrator state contains the new value

$y = -4 \cdot x$

So, $y$ is updated with the new value of $x$

There is a problem in the treatment of causality.

# Causality: Modelica example

```
model scheduling
  Real x(start = 0);
  Real y(start = 0);
equation

  der(x) = 1;
  der(y) = x;

  when x >= 2 then
    reinit(x, -3 * y)
  end when;
  when x >= 2 then
    reinit(y, -4 * x);
  end when;

end scheduling;
```

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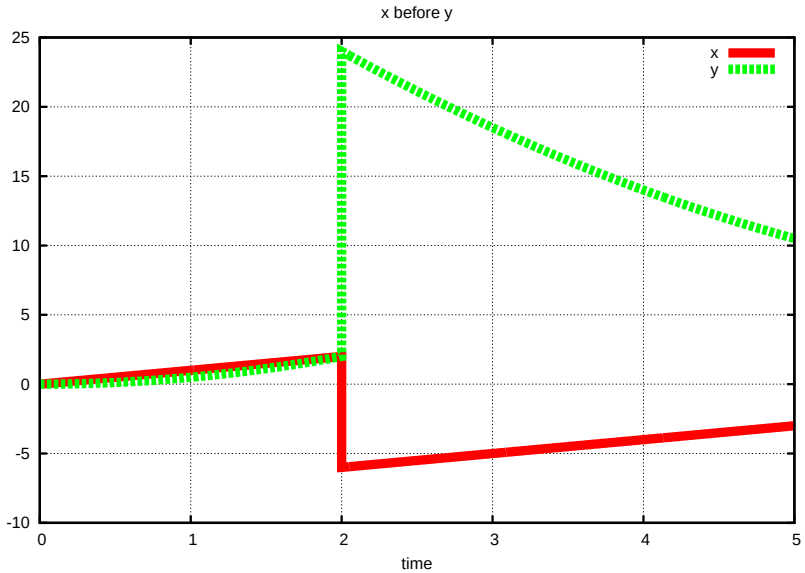

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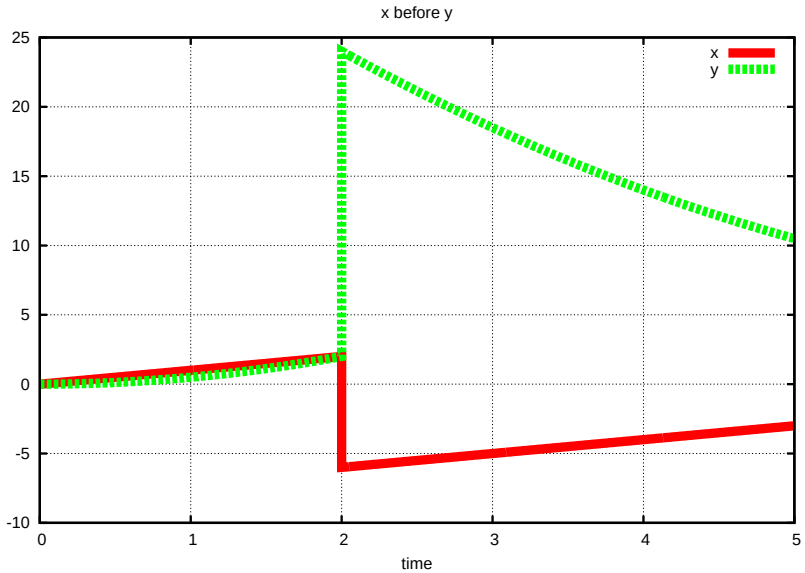
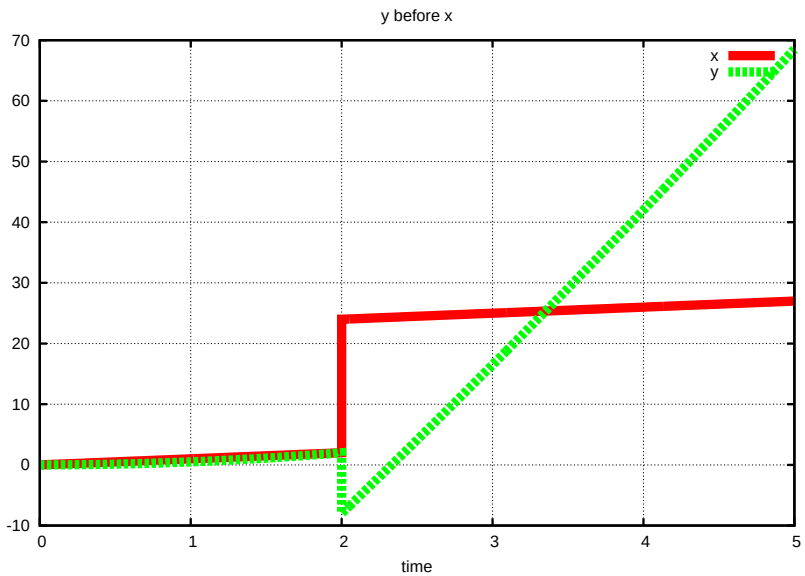



OpenModelica 1.9.2beta1 (r24372)
Also in Dymola

# Integrating a discontinuous signal

## Solver Reset

Resetting the solver at every zero-crossing event degrades performance and precision.

Yet, integrating a discontinuous signal gives non faithfull results, e.g.:

```
der x = if floor(t mod 2.0) = 0.0
        then 1.0 else 0.0 init 0.0
```

E.g., SUNDIALS CVODE, the ones provided by Simulink.

Can we impose a strong type discipline to either reject this program or indicate that the result of `x` is fragile?

# Current Practice: conclusion

What is the semantics of these tools?

When the manual and implementions diverge, which is right?

There are side effects, global variables, backtracking.

Hard to judge whether the generated code is correct.

What more could we want?

An cleaner integration of discrete and continous time.

Static rejection of bizarre programs.

Current Practice
- Hybrid Systems Modelers
- Some issues

Interlude: interacting with a numerical solver

Key elements of our approach
- Non Standard Synchronous Semantics
- Typing
- Compilation

# Interacting with a numerical solver

It is not always feasible, nor even possible, to calculate the behaviour of a hybrid model analytically.

All major tools thus calculate approximate solutions numerically.

## Numerical solvers (e.g., LLNL Sundials CVODE)

Designed by experts!

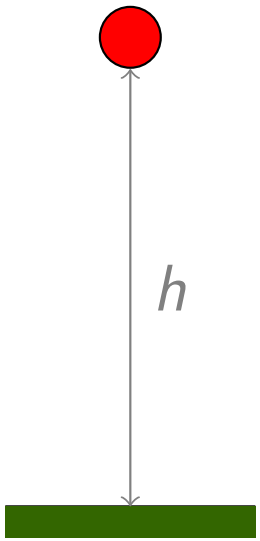
Compute a discrete-time approximations of continuous-time signals.

Subtle: variable step, change order dynamically, explicit/implicit.

Define compilation schemes with solver's internals kept abstract.

# Bouncing ball

## model




$$F = m \cdot a$$

$$m \cdot -g = m \cdot \frac{d^2 h(t)}{dt^2}$$

$$\frac{d^2 h(t)}{dt^2} = -g$$

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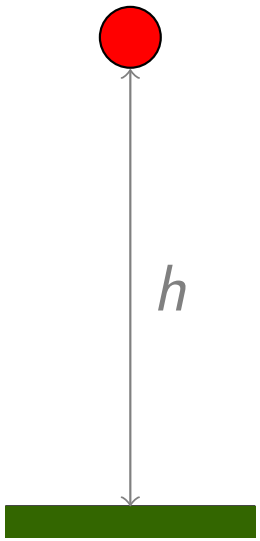


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$$\dot{h} = v \qquad h(0) = h_0$$

First-order ODE

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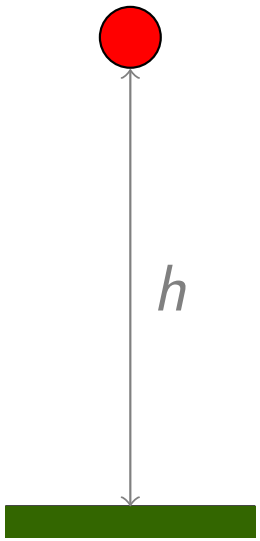


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$$\dot{h} = v \qquad h(0) = h_0$$

$$v(t) = v_0 + \int_0^t -g \, .d\tau$$

$$h(t) = h_0 + \int_0^t v(\tau) \, .d\tau$$

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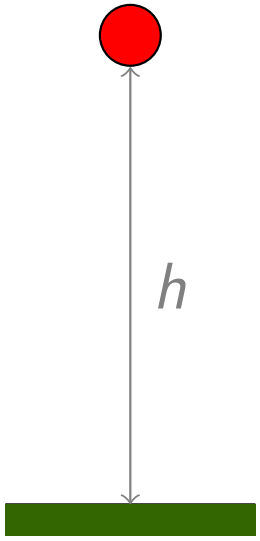


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$[\dot{v}; \dot{h}] = \mathbf{f}(t, [v; h])$

**Solver**

**approximation**

$\mathbf{y_i} = [v_0; h_0]$

$$\begin{matrix} \dot{v} = -g \\ \dot{h} = v \end{matrix} \qquad \begin{matrix} v(0) = v_0 \\ h(0) = h_0 \end{matrix} \qquad \begin{matrix} v(t) = v_0 + \int_0^t -g \, .d\tau \\ h(t) = h_0 + \int_0^t v(\tau) \, .d\tau \end{matrix}$$

First-order ODE

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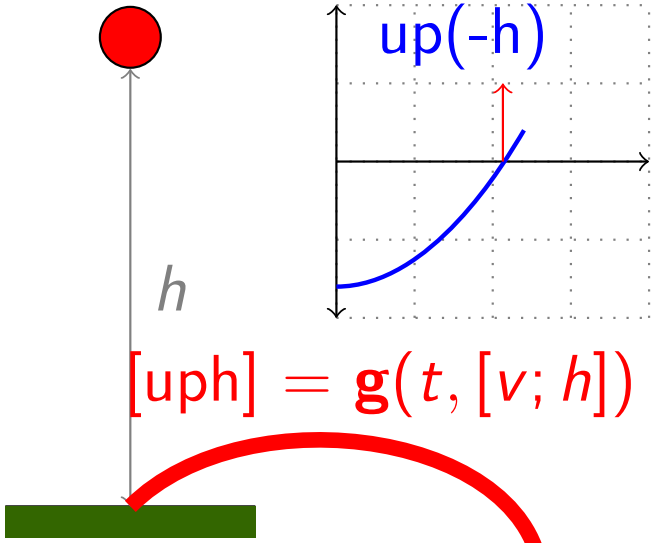


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$[\text{uph}] = \mathbf{g}(t, [v; h])$

$[\dot{v}; \dot{h}] = \mathbf{f}(t, [v; h])$

**Solver**

**event!**

**approximation**

$\mathbf{y_i} = [v_0; h_0]$

$$\begin{aligned} \dot{v} &= -g \\ \dot{h} &= v \end{aligned}$$

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First-order ODE

# Solver execution (e.g., LLNL Sundials CVODE)

Give solver two functions: $\dot{y} = f_\sigma(t, y)$, $upz = g_\sigma(t, y)$


- Bigger and bigger steps (bound by $h_{min}$ and $h_{max}$)
- $t$ does not necessarily advance monotonically
  - No side-effects within $f$ or $g$

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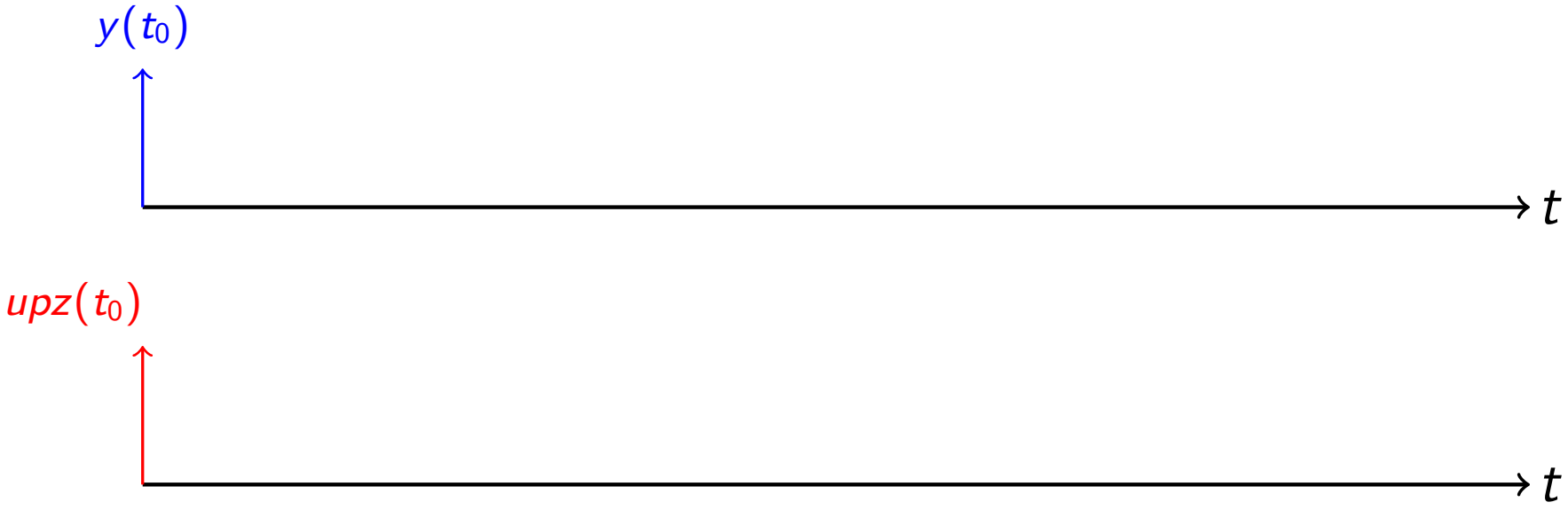


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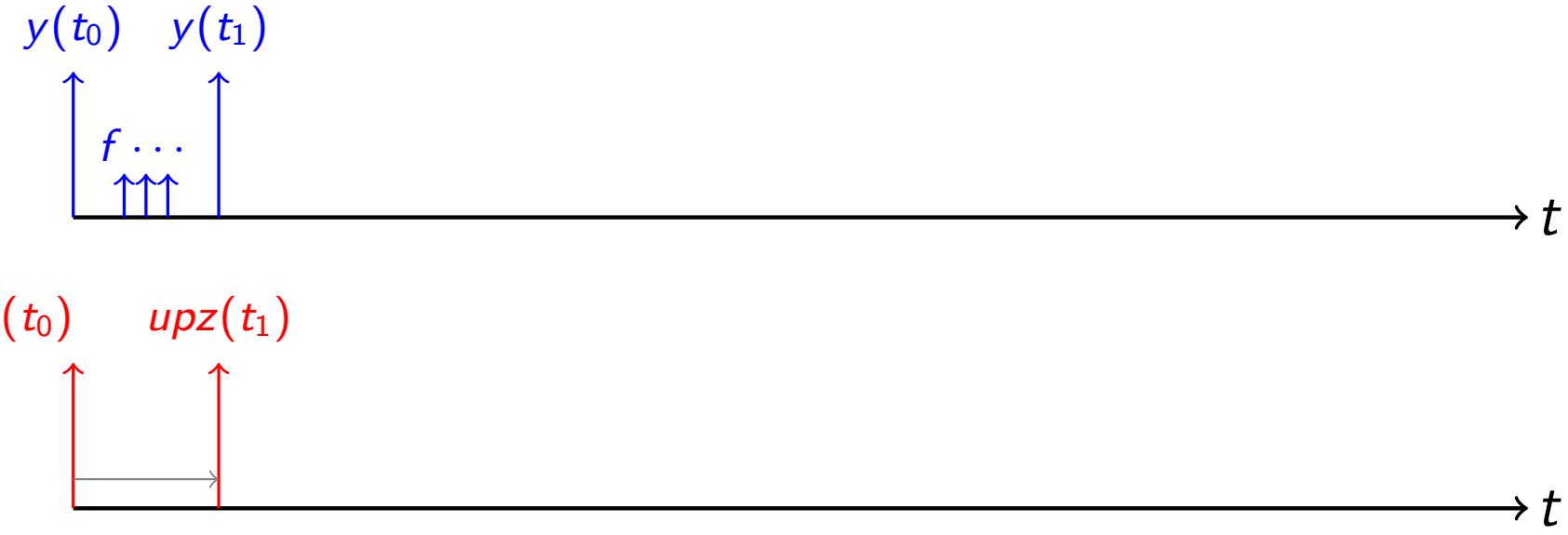


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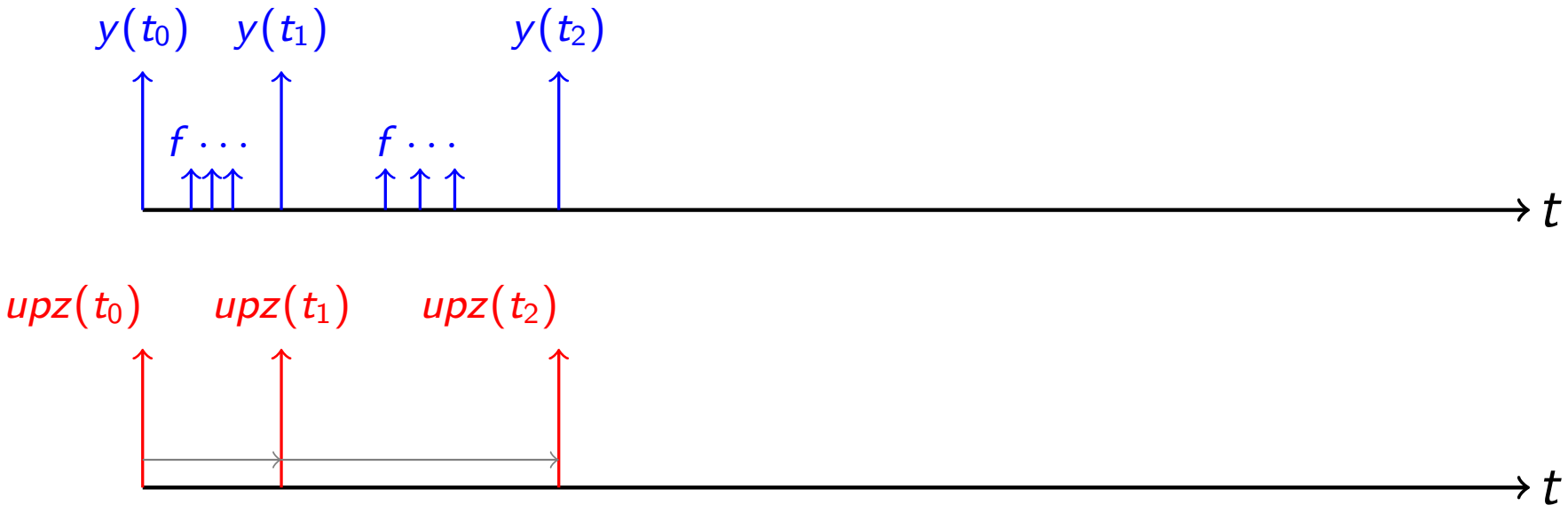


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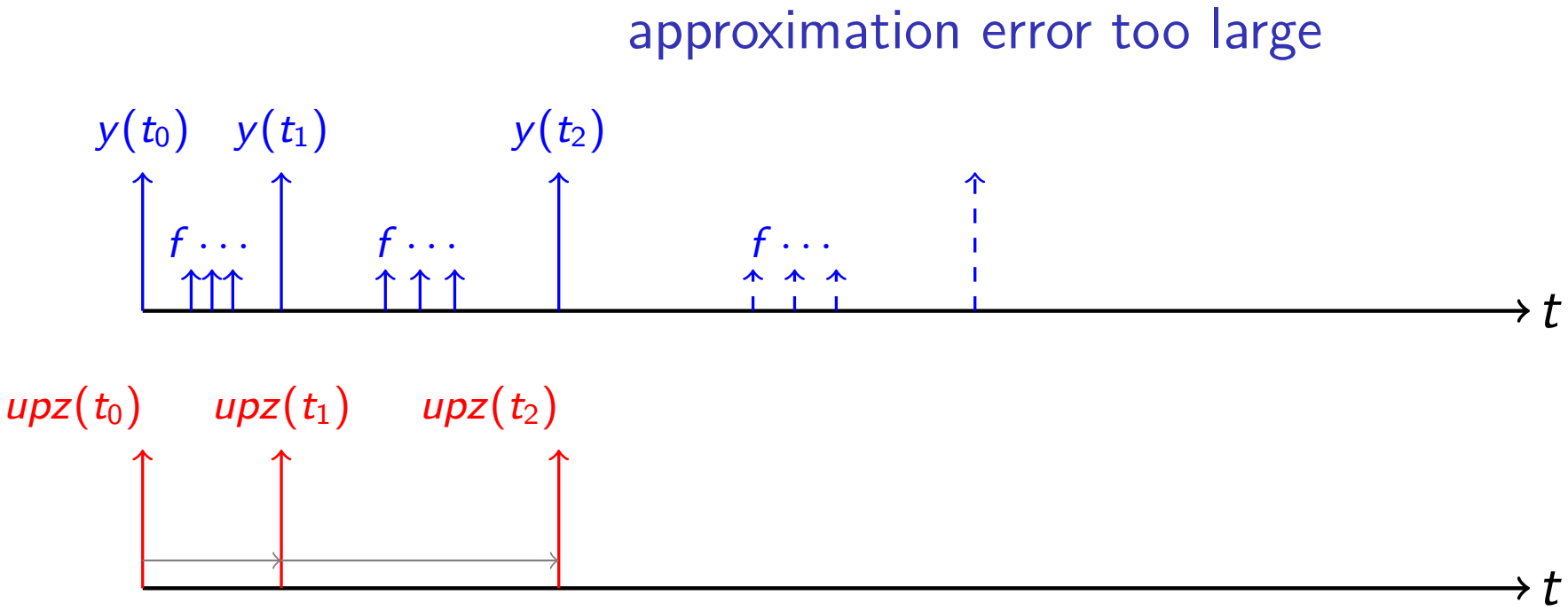
Give solver two functions: $\dot{y} = f_\sigma(t, y)$, $upz = g_\sigma(t, y)$

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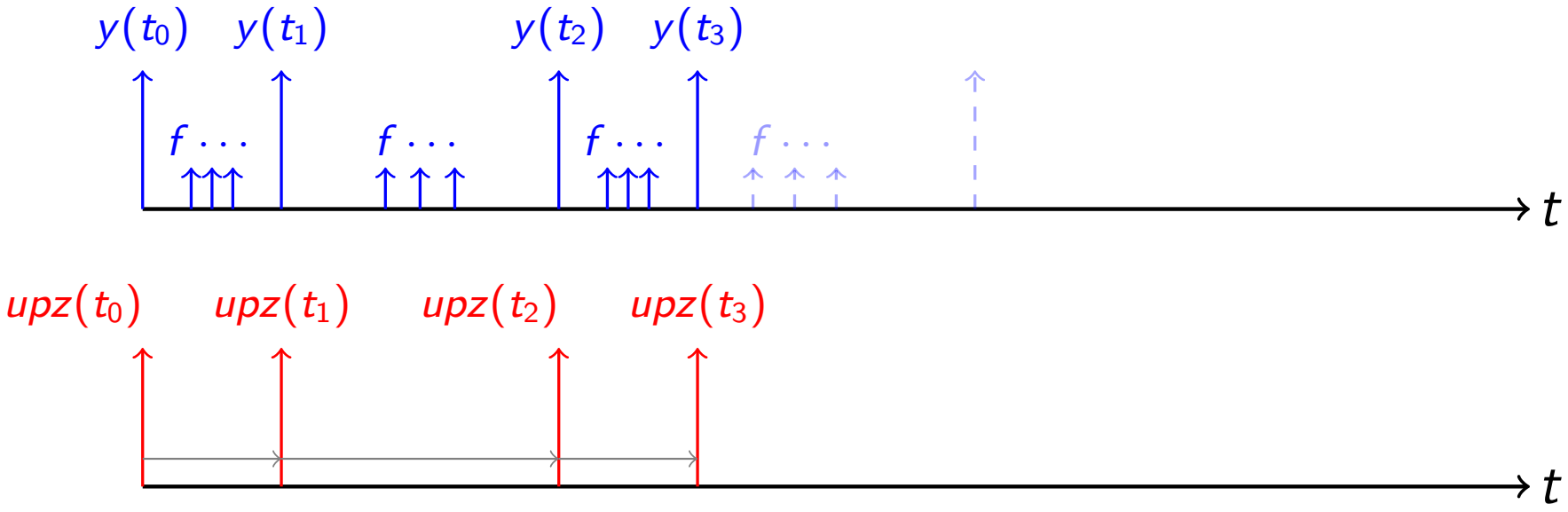


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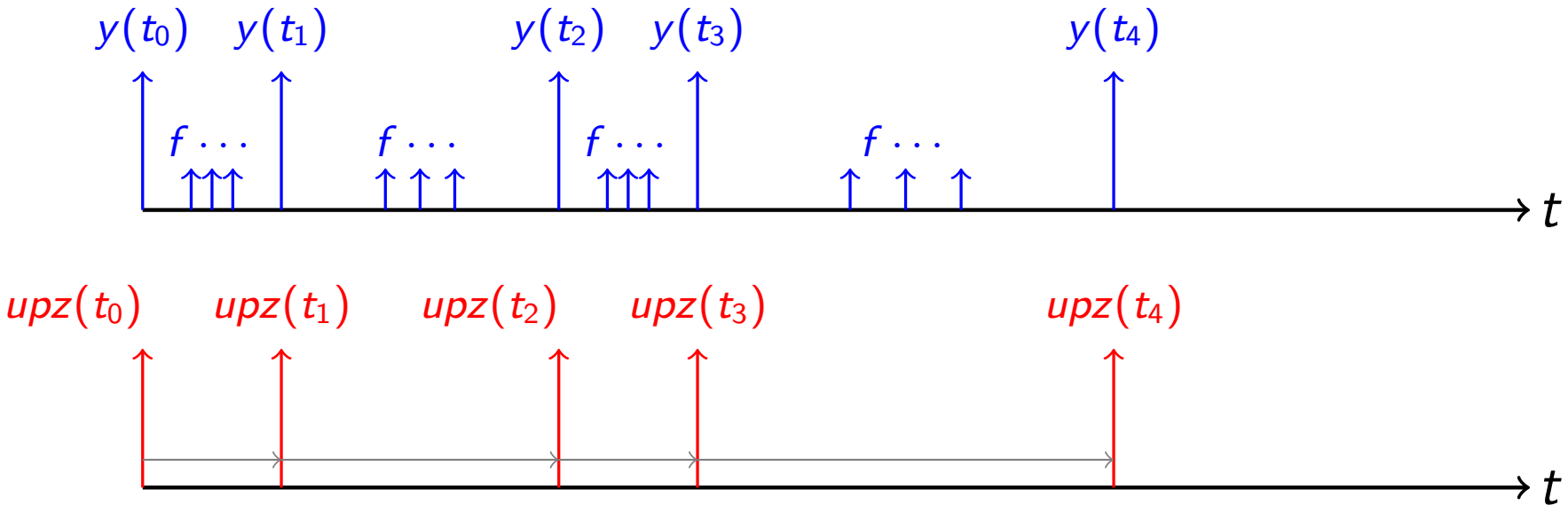


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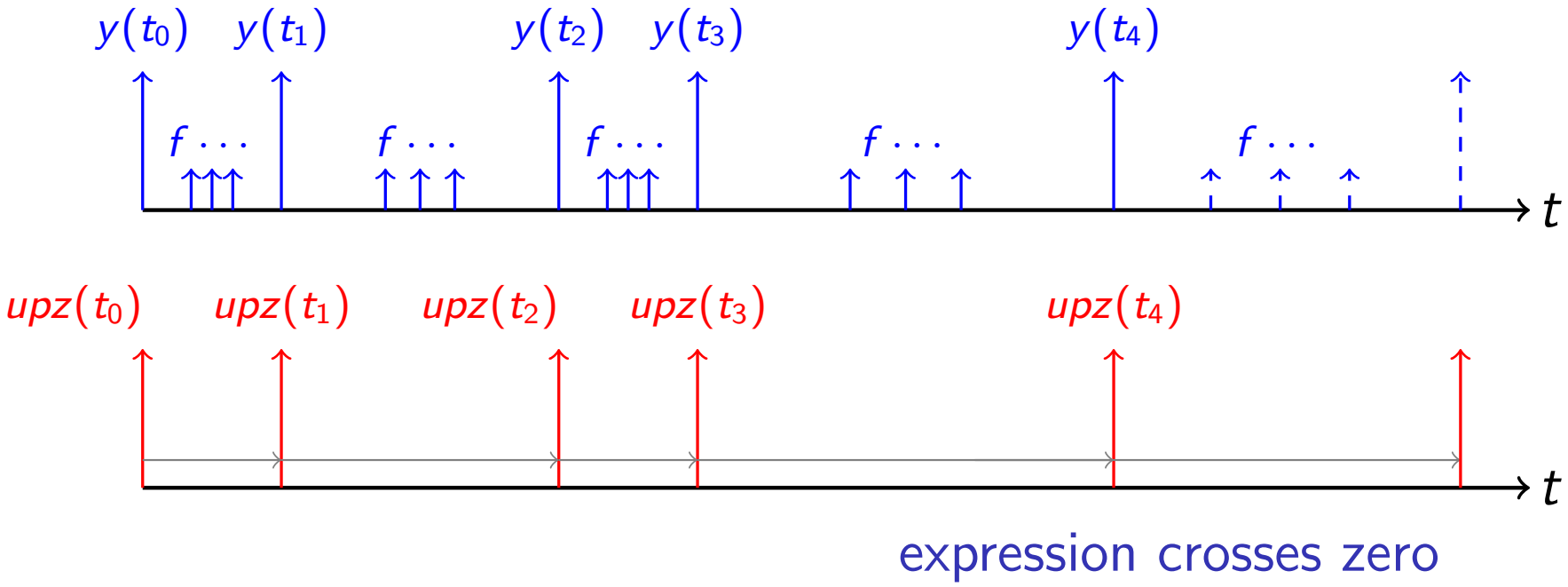


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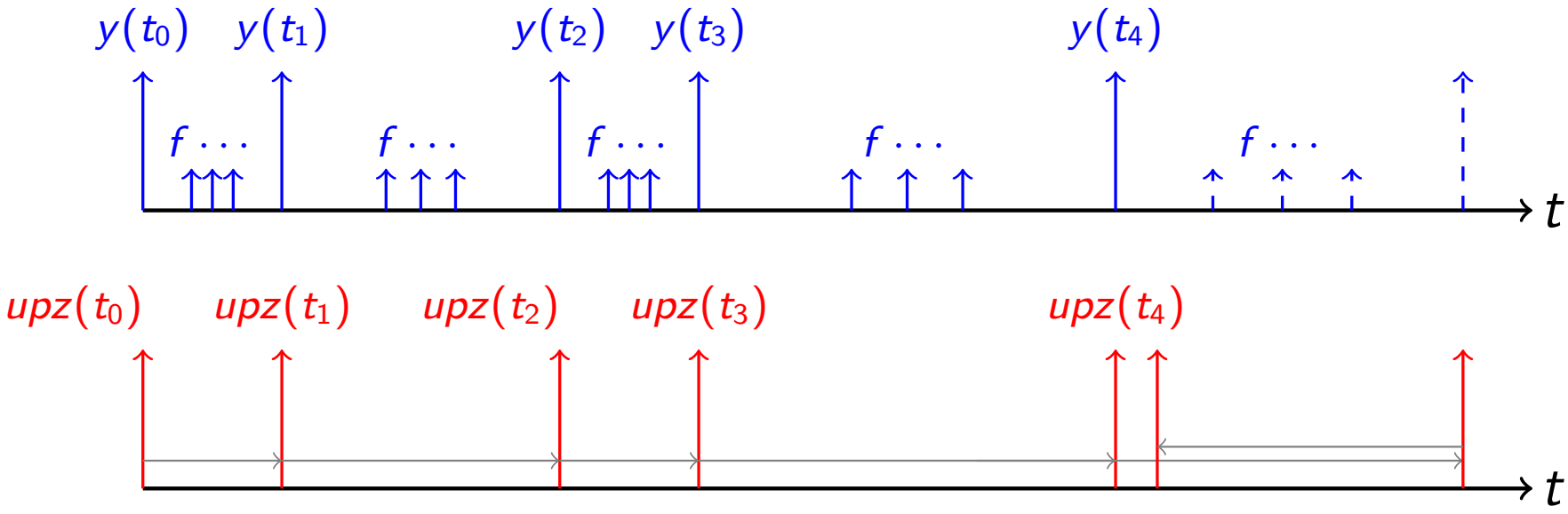


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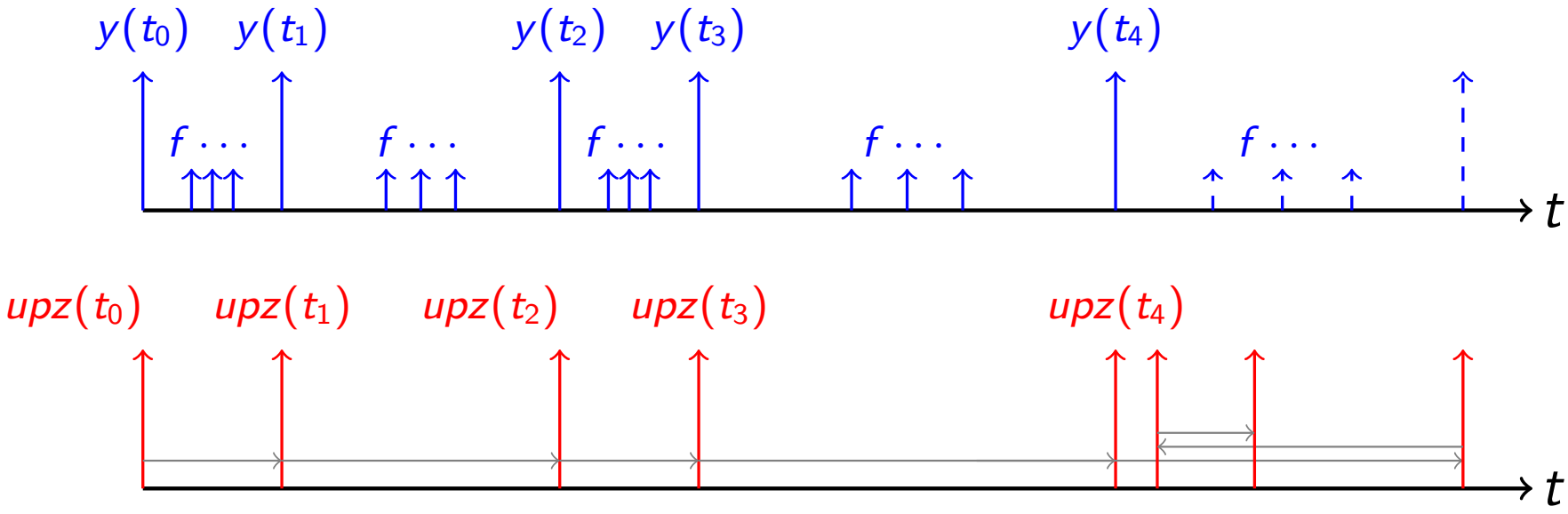


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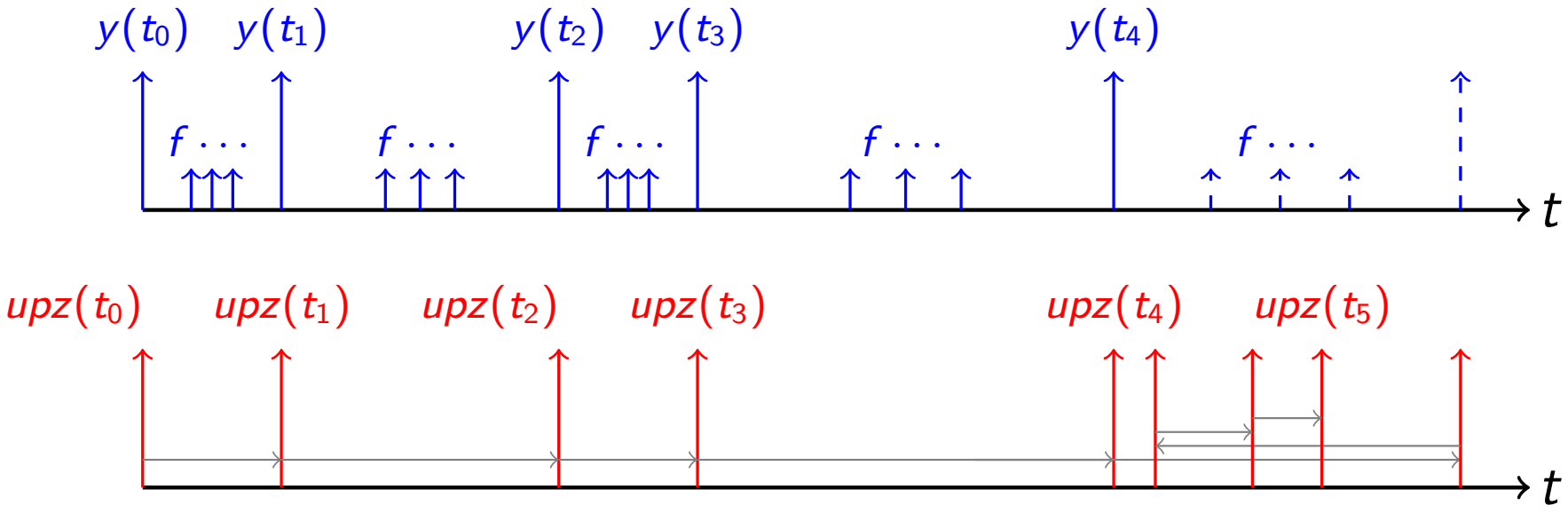


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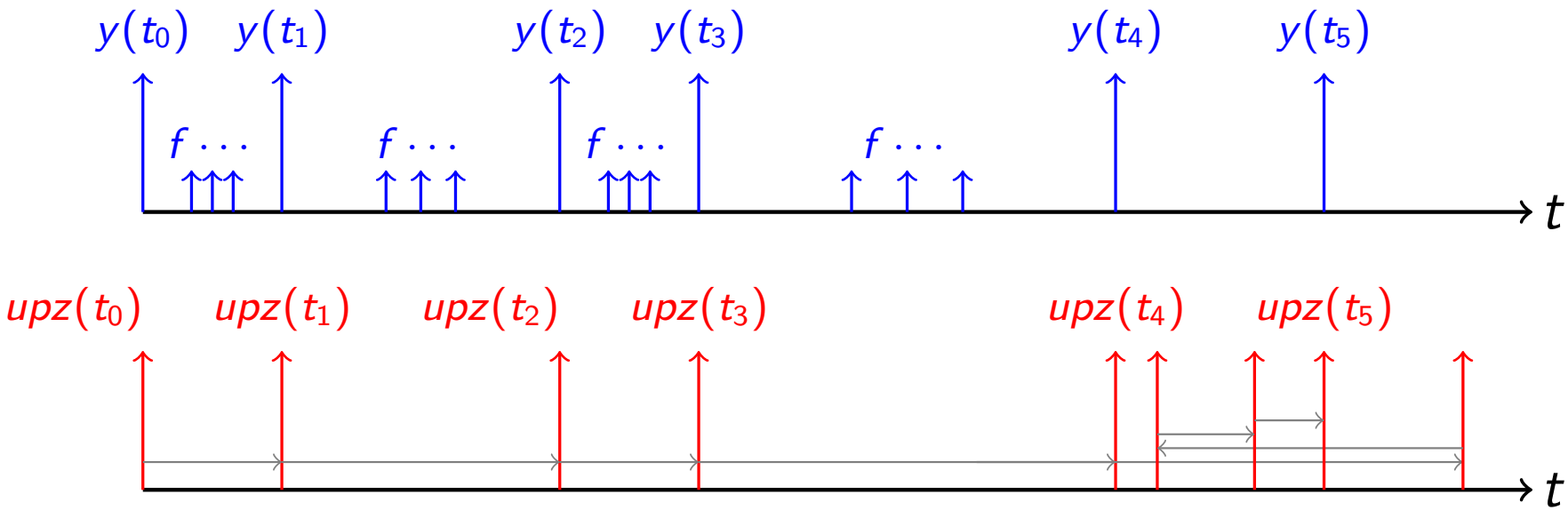


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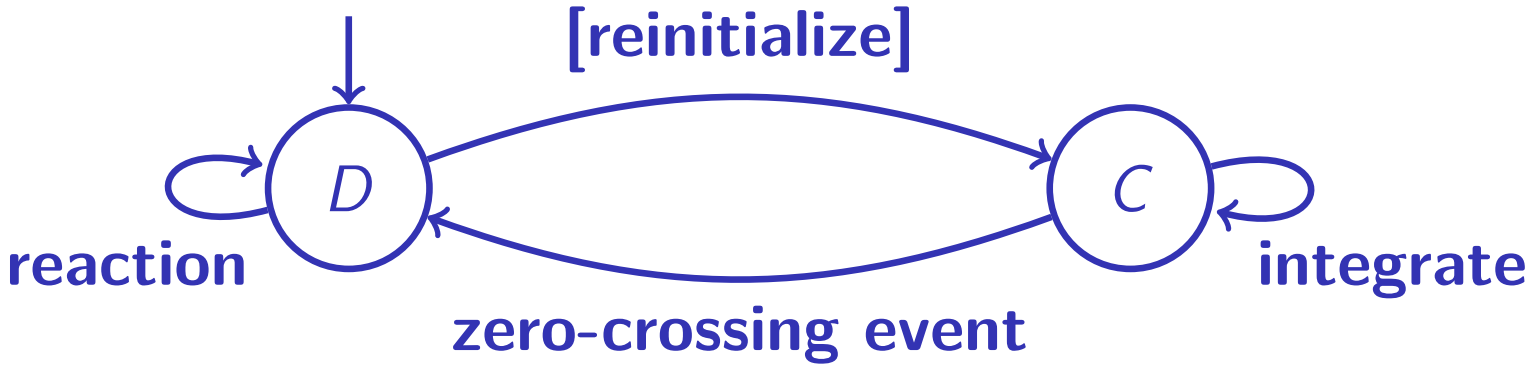
# The Simulation Engine of Hybrid Systems

## Alternate discrete steps and integration steps

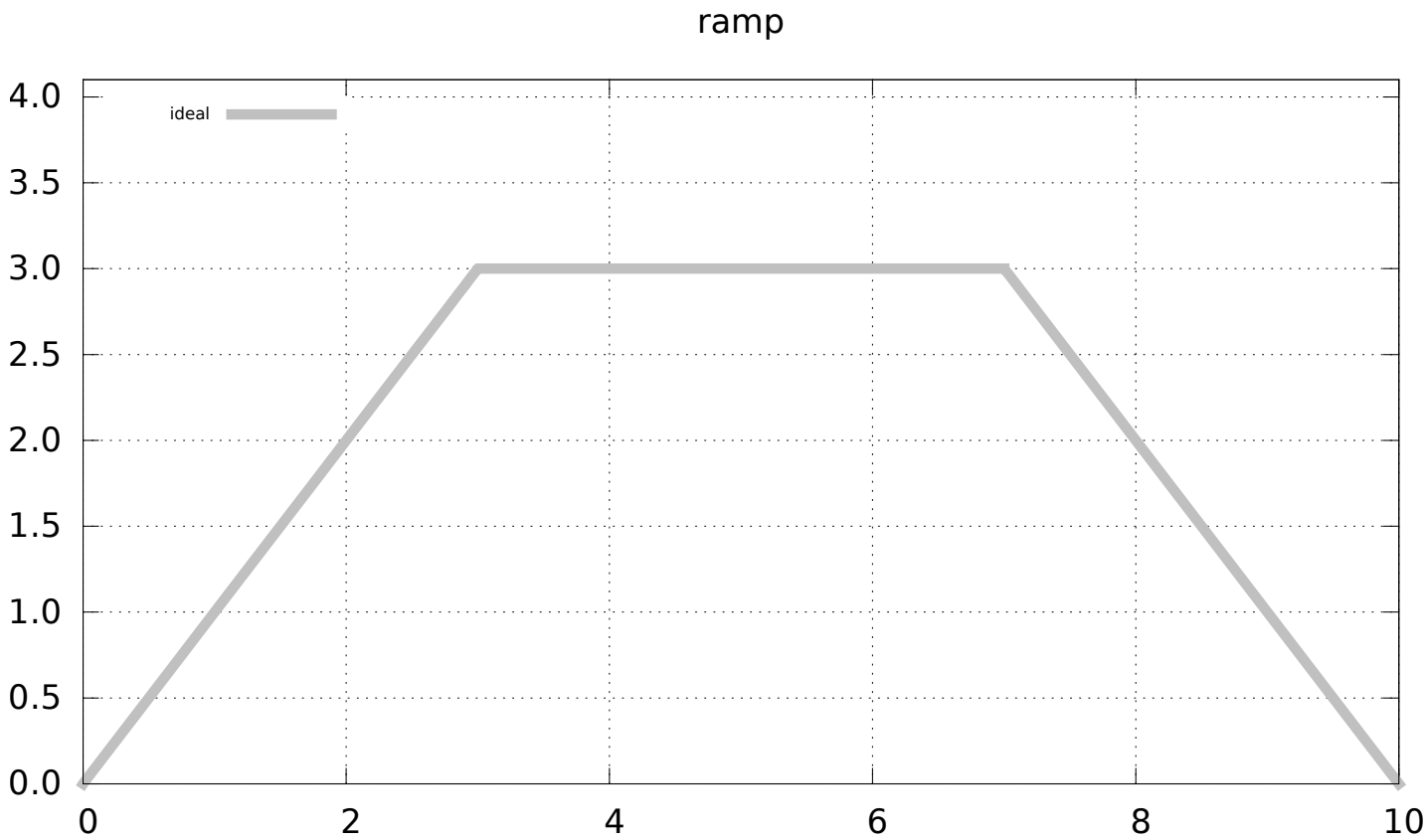



$$\sigma', y' = d_\sigma(t, y) \qquad upz = g_\sigma(t, y) \qquad \dot{y} = f_\sigma(t, y)$$

## Properties of the three functions

- $d_\sigma$ gathers all discrete changes.
- $g_\sigma$ defines signals for zero-crossing detection.
- $f_\sigma$ and $g_\sigma$ should be free of side effects and, better, continuous.

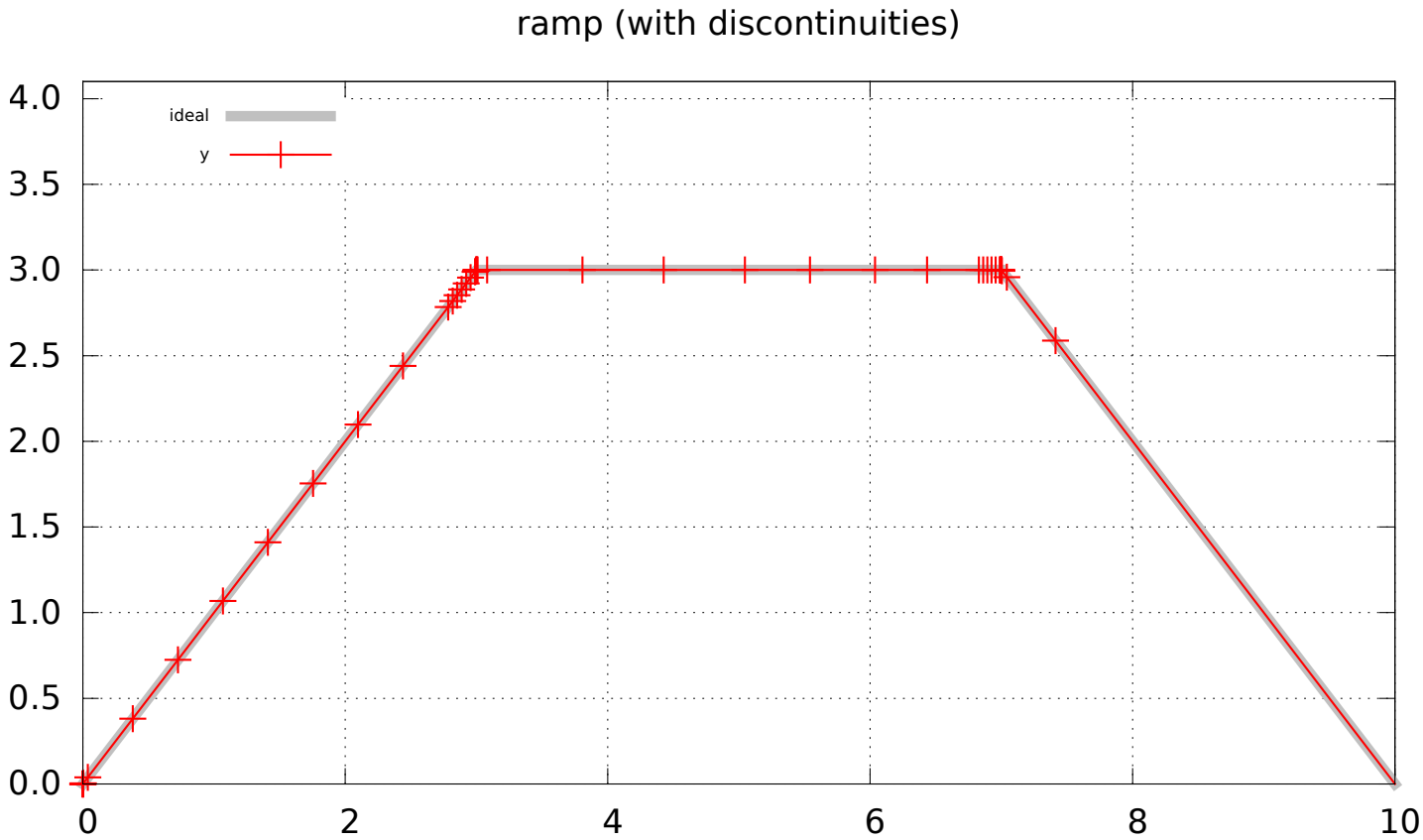
# Numerical Integration (Sundials CVODE)




$$\dot{y}(t) = \begin{cases} 1 & \text{if } t < 3 \\ 0 & \text{if } 3 \leq t \leq 7 \\ -1 & \text{if } 7 < t \end{cases}$$

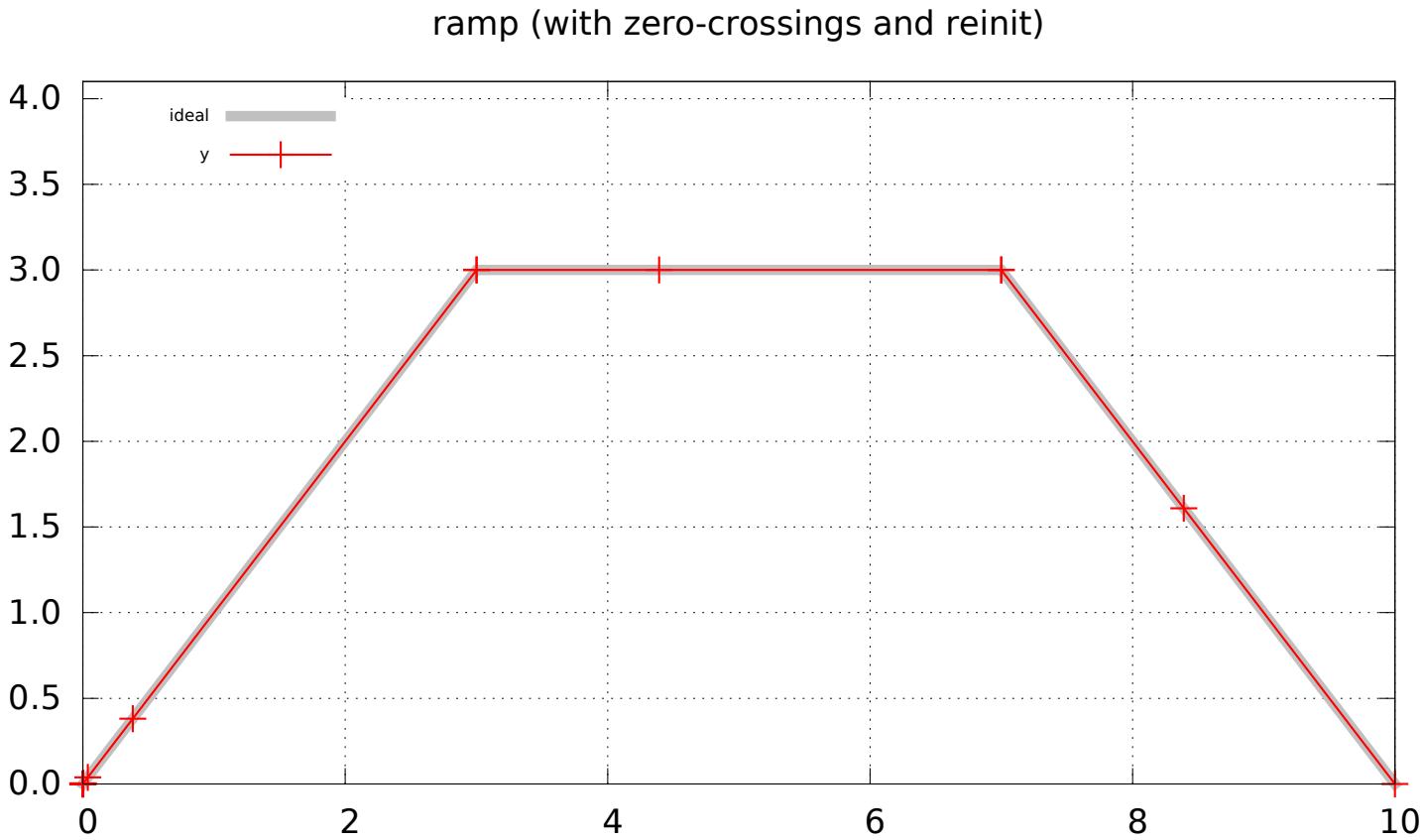
$$y(0) = 0$$

# Numerical Integration: a derivative with a discontinuity




```
let f (t, y) =
  if t < 3.0 then 1.0
  else if t <= 7.0 then 0.0
  else -1.0
```

# Numerical Integration: discrete state with three modes



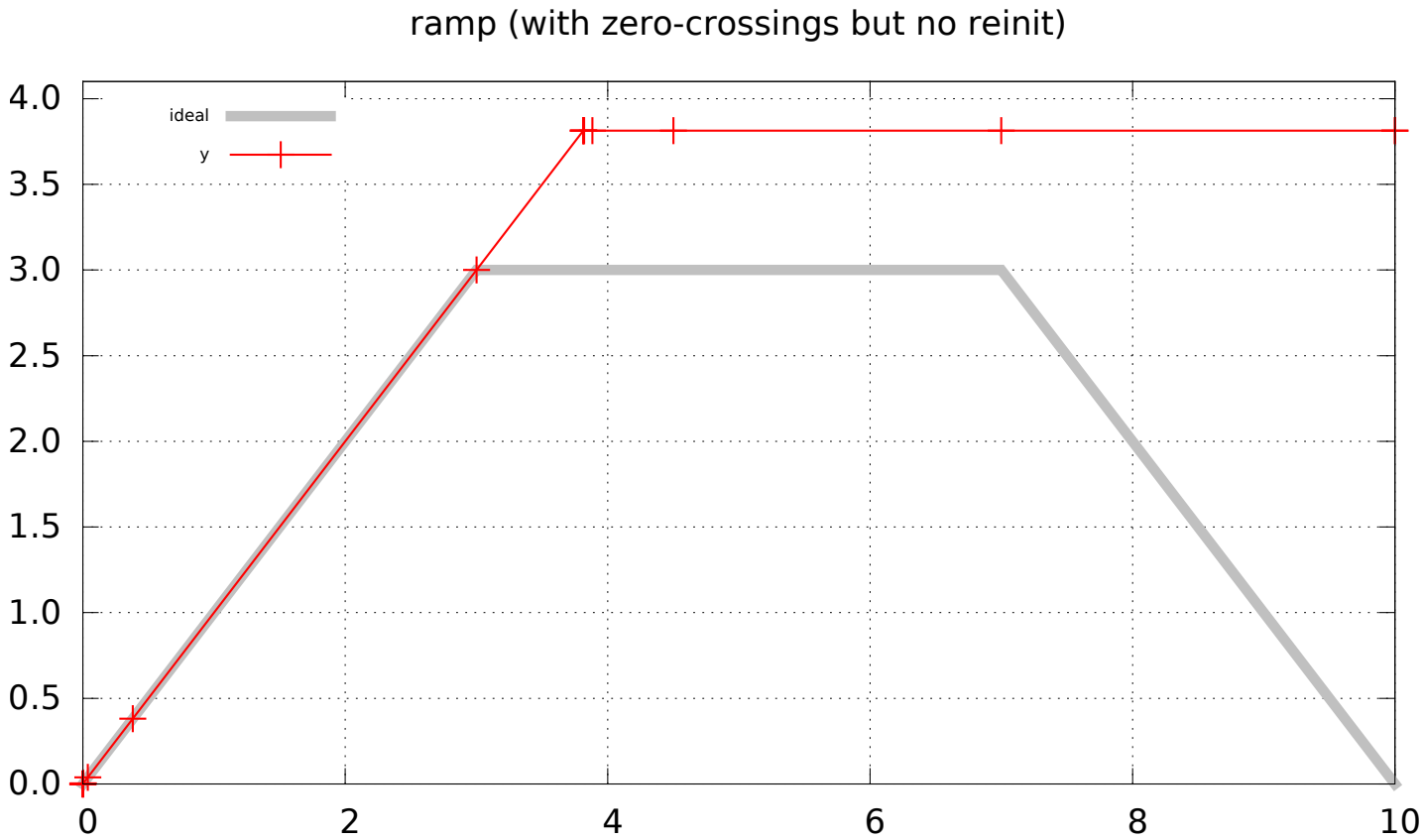

```
let f (discrete_state) (t, y) =
  match discrete_state with
  | RampingUp → 1.0
  | Flat → 0.0
  | RampingDown → −1.0

let g (discrete_state) (t, y) =
  match discrete_state with
  | RampingUp → t −. 3.0
  | Flat → t −. 7.0
  | _ → t
```

```
let discrete_state_i = RampingUp

let d (discrete_state) z (t, y) =
  match discrete_state with
  | RampingUp when z → Flat
  | Flat when z → RampingDown
  | s → s
```

# Numerical Integration: with no reinit of the solver



Current Practice
- Hybrid Systems Modelers
- Some issues

Interlude: interacting with a numerical solver

Key elements of our approach
- Non Standard Synchronous Semantics
- Typing
- Compilation

# Key elements of our approach

Build a hybrid modeler on top of a synchronous language.

Use synchronous constructs for arbitray mix of discrete and continuous.

Divide and Recycle

Recycle existing synchronous languages techniques.

Semantics, static checking, code-generation.

Divide from the code what is for the solver.

Simulate with off-the-shelf numerical solvers.

Be conservative: any synchronous program must be compiled, optimized, and executed as per usual.

These elements are experimented within the language `Zélus`.

# Zélus

`zelus.di.ens.fr`

# Combinatorial and sequential functions

A signal is a sequence of values. Nothing is said about the actual time to go from one instant to another.

```
let add (x,y) = x + y

let node min_max (x, y) = if x < y then (x, y) else (y, x)

let node after (n, t) = (c = n) where
  rec c = 0 → pre(min(tick, n))
  and tick = if t then c + 1 else c
```

When fed into the compiler, we get:

```
val add : int × int  --A--> int
val min_max : α× α --D--> α× α
val after : int × bool  --D--> bool
```

x, y, etc. are infinite sequences of values.

The counter can be instantiated twice in a two state automaton,

```
let node blink (n, m, t) = x where
 automaton
 | On → do x = true   until (after(n, t)) then Off
 | Off → do x = false  until (after(m, t)) then On
```

which returns a value for x that alternates between true for n occurrences of t and false for m occurrences of t.

```
let node blink_reset (r, n, m, t) = x where
 reset
   automaton
   | On  → do x = true  until (after(n, t)) then Off
   | Off → do x = false until (after(m, t)) then On
 every r
```

The type signatures inferred by the compiler are:

```
val blink : int × int × bool -D-> bool
val blink_reset : bool × int × int × bool -D-> bool
```

# Examples

Up to syntactic details, these are `Scade 6` or `Lucid Synchrone` programs. E.g., a simple heat controller.[7]

```
(* an hysteresis controller for a heater *)
let hybrid heater(active) = temp where
  rec der temp = if active then c -. k *. temp else -. k *. temp init temp0

let hybrid hysteresis_controller(temp) = active where
  rec automaton
      | Idle →   do active = false until (up(t_min -. temp)) then Active
      | Active → do active = true until (up(temp -. t_max)) then Idle

let hybrid main() = temp where
  rec active = hysteresis_controller(temp)
  and temp = heater(active)
```

[7] This is the hybrid version of one of Nicolas Halbwachs' examples with which he presented Lustre at the Collège de France, in January 2010.

# The Bouncing ball [demo]

```
let hybrid bouncing(x0,y0,x'0,y'0) = (x,y) where
 rec
    der x = x' init x0
 and
    der x' = 0.0 init x'0
 and
    der y = y' init y0
 and
    der y' = -. g init y'0 reset up(-. y) -> -0.9 *. last y'
```

Its type signature is:

$$\texttt{val bouncing} : \texttt{float} \times \texttt{float} \times \texttt{float} \times \texttt{float} \xrightarrow{c} \texttt{float} \times \texttt{float}$$

- When $-\texttt{y}$ crosses zero, re-initialize the speed $\texttt{y'}$ with $-0.9 * \texttt{last y'}$.
- When $\texttt{y'}$ is continuous, $\texttt{last y'}$ is the left limit of $\texttt{y'}$.
- As $\texttt{y'}$ is immediately reset, writing $\texttt{last y'}$ is mandatory —otherwise, $\texttt{y'}$ would instantaneously depend on itself.

# Summary of Programming Constructs

- Synchronous constructs: data-flow equations/automata.
- An ODE with initial condition: der x = e init e0
- last x is the left limit of x.
- Detect a zero-crossing (from negative to positive): up(x).
- This defines a discrete instant, that is, an event.
- All discrete changes must occur on an event. E.g.,:

```
let hybrid f(x, y) = (v, z1, z2) where
  rec v = present z1 → 1 | z2 → 2 init 0
  and z1 = up(x)
  and z2 = up(y)
```

$$\texttt{val f : float} \times \texttt{float} \xrightarrow{\texttt{c}} \texttt{int} \times \texttt{zero} \times \texttt{zero}$$

- If x = up(e), all handlers using x are governed by the same event.

# Three difficulties

## Semantics

- An ideal semantics to say which program make sense;
- useful to prove that compilation is correct.

## Ensure that continuous and discrete time signals interfere correctly.

- Discrete time should stay logical and independent on when the solver decides to stop.
- Otherwise, we get the bizarre behaviors seen previously.

## Ensure that fix-points exist and code can be scheduled.

- Algebraic loops must be statically detected.
- Restrict the use of last x so that signals are proved to be continuous during integration.

# A Core Hybrid Systems Language

## Syntax

$$d ::= \texttt{const}\, x = e \mid k\, f(p) = e\, \texttt{where}\, E \mid d; d$$

$$e ::= x \mid v \mid op(e) \mid e\, \texttt{fby}\, e \mid \texttt{last}\, x \mid f(e) \mid (e, e) \mid \texttt{up}(e)$$

$$p ::= (p, p) \mid x$$

$$\begin{aligned} E ::= () &\mid x = e \mid \texttt{init}\, x = e \mid \texttt{next}\, x = e \mid \texttt{der}\, x = e \\ &\mid E\ \texttt{and}\ E \mid \texttt{local}\, x\, \texttt{in}\, E \mid \texttt{if}\, e\, \texttt{then}\, E\, \texttt{else}\, E \\ &\mid \texttt{present}\, e\, \texttt{then}\, E\, \texttt{else}\, E \end{aligned}$$

$$k ::= \texttt{node} \mid \texttt{hybrid} \mid \texttt{A}$$

- Lustre + ODEs
- Core language of Zélus [HSCC'13]
- The full language is extended with hierarchical automata.

# A Non-standard Semantics for Hybrid Modelers [JCSS'12]

We proposed to build the semantics on non-standard analysis.

der y = z init 4.0 and z = 10.0 −. 0.1 ∗. y and k = y +. 1.0

defines signals $y$, $z$ and $k$, where for all $t \in \mathbb{R}^+$:

$$\frac{dy}{dt}(t) = z(t) \quad y(0) = 4.0 \quad z(t) = 10.0 - 0.1 \cdot y(t) \quad k(t) = y(t) + 1$$

What would be the value of $y$ if it were computed by an ideal solver taking an infinitesimal step of duration $\partial$?

${}^\star y(n)$ stands for the values of $y$ at instant $n\partial$, with $n \in {}^\star\mathbb{N}$ a non-standard integer.

$$\begin{aligned} {}^\star y(0) &= 4 & {}^\star z(n) &= 10 - 0.1 \cdot {}^\star y(n) \\ {}^\star y(n+1) &= {}^\star y(n) + {}^\star z(n) \cdot \partial & {}^\star k(n) &= {}^\star y(n) + 1 \end{aligned}$$

# Non standard semantics [JCSS'12]

Let ${}^\star\mathbb{R}$ and ${}^\star\mathbb{N}$ be the non-standard extensions of $\mathbb{R}$ and $\mathbb{N}$.

Let $\partial \in {}^\star\mathbb{R}$ be an infinitesimal, i.e., $\partial > 0, \partial \approx 0$.

Let the global time base or base clock be the infinite set of instants:

$$\mathbb{T}_\partial = \{t_n = n\partial \mid n \in {}^\star\mathbb{N}\}$$

$\mathbb{T}_\partial$ inherits its total order from ${}^\star\mathbb{N}$. A sub-clock $T \subset \mathbb{T}_\partial$.

What is a discrete clock?

> *A clock $T$ is termed discrete if it is the result of a zero-crossing or a sub-sampling of a discrete clock. Otherwise, it is termed continuous.*

If $T \subseteq \mathbb{T}$, we write ${}^\bullet T(t)$ for the immediate predecessor of $t$ in $T$ and $T^\bullet(t)$ for the immediate successor of $t$ in $T$.

# Signals and clocks

## Signals

Let $V$ a set. $V_\perp = V + \{\perp\}$ with $\forall v \in V, \perp \leq v$. $S(V) = \mathbb{T} \mapsto V_\perp$ is the set of signals.

A signal $x : T \mapsto V_\perp$ is a total function from $T \subseteq \mathbb{T}$ to $V_\perp$. Moreover, for all $t \notin T, x(t) = \perp$.

The clock of $x$ is $\mathit{clock}\ x = \{t \in \mathbb{T} \mid x(t) \neq \perp\}$

## Sampling

Let $\texttt{bool} = \{\texttt{false}, \texttt{true}\}$ and $x : T \mapsto \texttt{bool}_\perp$. The sampling of $T$ according to $x$, written $T\ \texttt{on}\ x$ is the subset of instants:

$$T\ \texttt{on}\ x = \{t \mid (t \in T) \wedge (x(t) = \texttt{true})\}$$

Note that $T\ \texttt{on}\ x \subseteq T$, it is also totally ordered.

# Semantics of basic operations

Replay the classical semantics of a synchronous language.

An ODE with reset on clock $T$:

$\texttt{der}\, x = e\, \texttt{init}\, \texttt{reset}\, e_0\, \texttt{every}\, z \longrightarrow e_1$

$$^\star x(t_0) = {}^\star e_0(0) \text{ if } t_0 = \min T$$

$$^\star x(t) = \texttt{if}\, {}^\star z(t)\, \texttt{then}\, {}^\star e_1(t)\, \texttt{else}\, {}^\star x(^\bullet T(t)) + \partial \cdot {}^\star e(^\bullet T(t)) \text{ if } t \in T$$

$\texttt{last}\, x$ if $x$ is defined on clock $T$

$$^\star \texttt{last}\, x(t) = {}^\star x(^\bullet T(t))$$

Zero-crossing $\texttt{up}(x)$ on clock $T$

$$^\star \texttt{up}(x)(T)(t_0) = \texttt{false} \text{ if } t_0 = \min T$$

$$^\star \texttt{up}(x)(T)(t) = ({}^\star x(^\bullet T(t) < 0) \wedge ({}^\star x(t) > 0) \text{ if } t \in T$$

# Fixpoint Semantics: Principle

Define semantics as mutual least fixpoint of set of monotonous operators (one for each expression or definition). Semantics of expression $e$:

$$^{\star}[\![e]\!]^{\rho}_{G}(T)(t) \quad = \quad (v, z)$$

With:

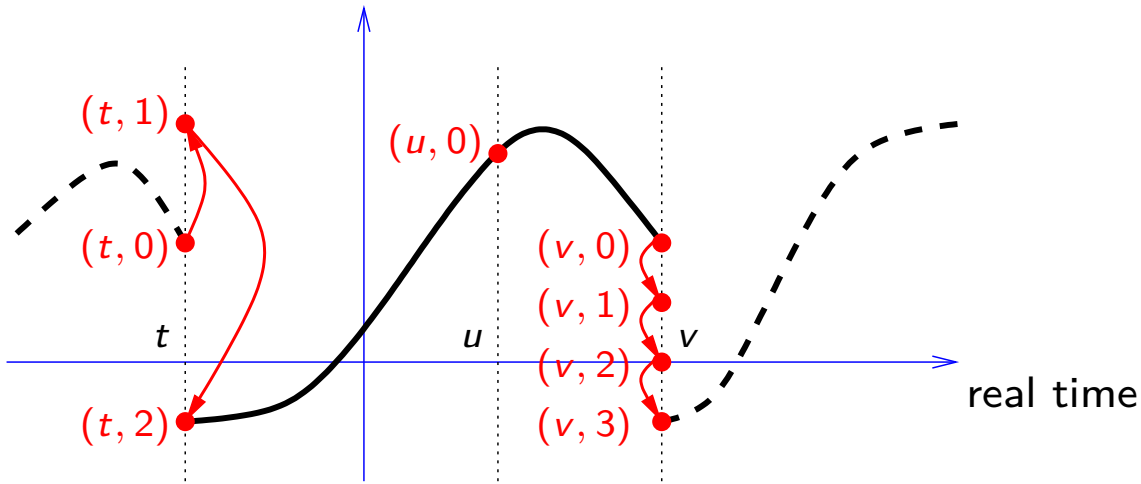
- $t \in \mathbb{T} = \{n\partial | n \in {}^{\star}\mathbb{N}\}$ non standard date
- $T \subseteq \mathbb{T}$ : set of dates of evaluation of expression $T$ is a discrete clock for a `Lustre` expression.
- $v \in {}^{\star}V \uplus \{\bot\}$ : $\bot$ if undefined, $\bot < v \in V$ (flat order)
- $z \in \mathbb{B}$ : true iff zero-crossings occurs in $e$ at instant $t$
- Signals: $S({}^{\star}V) = \mathbb{T} \to {}^{\star}V_{\bot}$
- $G : L_g \to S({}^{\star}V) \to S({}^{\star}V)$ maps global function names to semantics
- $\rho \quad : \quad L \to S({}^{\star}V)$ maps local variable names to semantics

# Semantics: Expressions

$$
\begin{array}{lcl}
{}^{\star}[\![e]\!]^{\rho}_{G}(T)(t) & = & \bot \text{ if } t \notin T \\
{}^{\star}[\![v]\!]^{\rho}_{G}(T)(t) & = & v, \texttt{false} \\
{}^{\star}[\![x]\!]^{\rho}_{G}(T)(t) & = & \rho(x)(t), \texttt{false} \\
{}^{\star}[\![op(e)]\!]^{\rho}_{G}(T)(t) & = & \mathit{let}\ v, z = {}^{\star}[\![e]\!]^{\rho}_{G}(T)(t)\ \mathit{in} \\
 & & op(v), z \\
{}^{\star}[\![\texttt{last}\ x]\!]^{\rho}_{G}(T)(t) & = & \rho(x)(^{\bullet}\mathit{clock}\ x(t)), \texttt{false} \\
{}^{\star}[\![f(e)]\!]^{\rho}_{G}(T)(t) & = & \mathit{let}\ s(t'), z(t') = {}^{\star}[\![e]\!]^{\rho}_{G}(T)(t')\ \mathit{in} \\
 & & \mathit{let}\ v', z' = G(f)(s)(t)\ \mathit{in} \\
 & & v', z(t) \vee z' \\
{}^{\star}[\![\texttt{up}(e)]\!]^{\rho}_{G}(T)(t) & = & \mathit{let}\ s(t'), z(t') = {}^{\star}[\![e]\!]^{\rho}_{G}(T)(t')\ \mathit{in} \\
 & & \mathit{let}\ v' = up(s)(T)(t)\ \mathit{in} \\
 & & v', z(t) \vee v'
\end{array}
$$

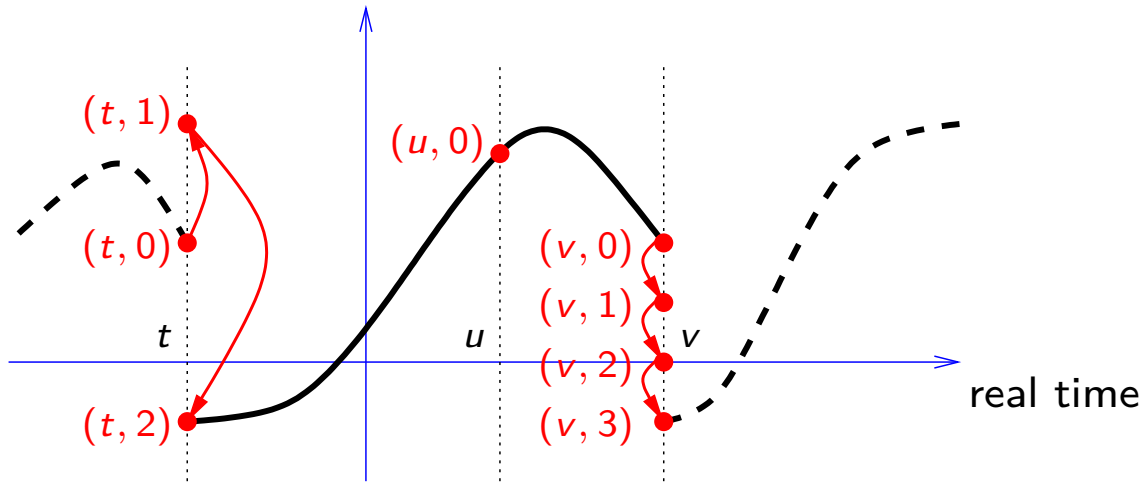
# Non-standard time vs. Super-dense time

- Maler et al., Lee et al. super-dense time modeling $\mathbb{R} \times \mathbb{N}$

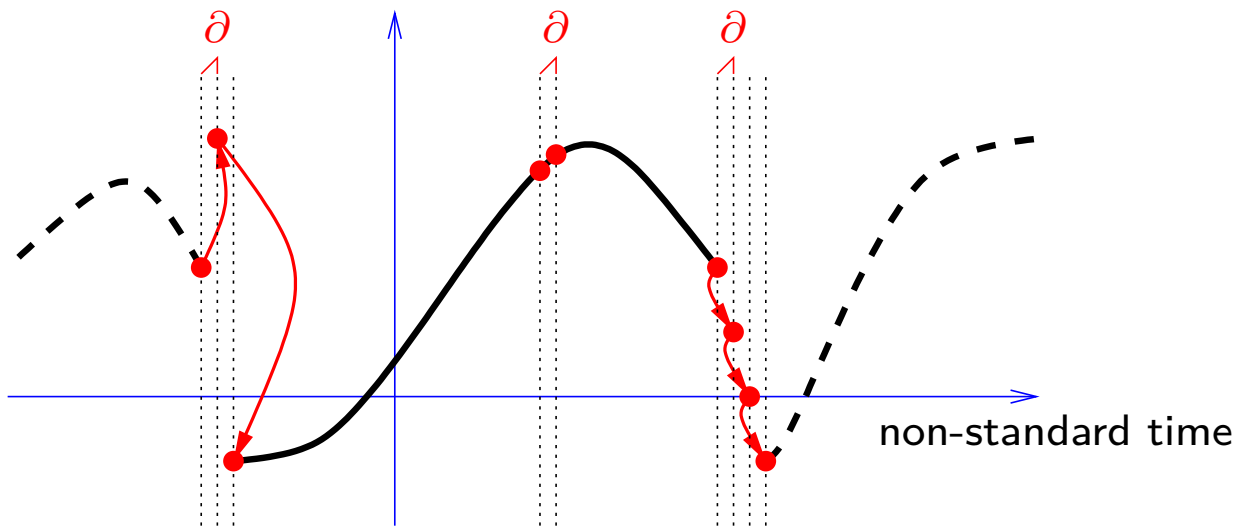


# Non-standard time vs. Super-dense time

- Edward Lee & al. super-dense time modeling $\mathbb{R} \times \mathbb{N}$




- non-standard time modeling $\mathbb{T}_\partial = \{n\partial \mid n \in {}^\star\mathbb{N}\}$



# Standardisation

## Super-dense time

Define the time index $\mathbb{S} = \mathbb{R} \times \mathbb{N}$. A signal as a total function $\mathbb{R} \times \mathbb{N} \mapsto V_\perp$.

Instants are lexically ordered: $(t, n) <_{\mathbb{S}} (t', n')$ iff $t <_{\mathbb{R}} t'$, or $t = t'$ and $n <_{\mathbb{N}} n'$.

For any $(t, n)$ and $(t, n')$ where $n \leq_{\mathbb{N}} n'$, if $x(t, n') \neq \perp$ then $x(t, n) \neq \perp$.

## Timeline

A timeline for a signal $x$ is a function $N_x : \mathbb{R}_+ \mapsto \mathbb{N}_\perp$. $N_x(t)$ is the number of instants of $x$ that occur at a real date $t$.

$$\mathbb{S}_{N_x} = \{(t, n) \in \mathbb{S} \mid n \leq_{\mathbb{N}} N_x(t)\}$$

# Standardisation

If $N_x$ is always 0, then $S_{\mathbb{N}_x}$ is isomorphic to $R_+$. For $t \in \mathbb{R}$ and $T \leq \mathbb{T}$, define:

$$set(T)(t) =_{\text{def}} \{t' \in T \mid t' \approx t \wedge t \in \mathbb{R}\} \subseteq \mathbb{T}$$

that is, the set of all instants infinitely close to $t$. $T$ is totally ordered, hence so is $set(T)(t)$.

The standardisation of $x$, written $st(x) : \mathbb{R} \times \mathbb{N} \mapsto V_\perp$, such that $st(x)(t, n) = \perp$ for $n > N_x(t)$.

# Standardisation

Let $T' =_{\text{def}} set(T)(t)$ and $st(x(T')) =_{\text{def}} \{st(x(t')) \mid t' \in T'\}$.

i. If $st(x(T')) = \{v\}$ then, at instant $t$, $x$'s timeline is $N_x(t) = 0$ and its standardization is $st(x)(t, 0) = v$.

ii. If $st(x(T'))$ is not a singleton set, then let

$$Z \quad =_{\text{def}} \quad \{t' \mid t' \in T' \wedge x(t') \not\approx x(T'^{\bullet}(t'))\}$$

$Z$ collects the instants at which $x$ experiences a non-infinitesimal change. $Z$ is either finite or infinite:

i. If $Z = \{t_{z_0}, \ldots, t_{z_m}\}$ is finite, timeline $N_x(t) = m$ and the standard value of signal $x$ at time $t$ is:

$$\forall n \in \{0, \ldots, m\}.st(x)(t, n) = st(x(t_{z_n}))$$

ii. If $Z$ is infinite (it may even lack a minimum element), let

$$N_x(t) = \perp \text{ and } \forall n.st(x)(t, n) = \perp$$

which corresponds to a Zeno behavior.

# Typing: mixing discrete (logical) time and continuous time

The following two parallel composition make sense.

## Discrete time: the clock should be discrete

```
let node sum(x) = cpt where
  rec cpt = 0 → pcpt
  and pcpt = pre(cpt) + x
```

## Continuous time: the clock should be continuous

```
let hybrid bouncing(y0, y'0) = o where
  rec der y = y' init y0
  and der y' = −.g init y'0
  and o = y +. 10.0
```

## The following do not make sense

At what clock should we compute cpt?

```
rec der t = 1.0 init 0.0
and cpt = 0.0 → pre(cpt) + t
```

# Intuition

Distinguish functions with three kinds A/D/C.

- Combinatorial function get kind A (for "any").
- Discrete-time (synchronous) functions get kind D (for "discrete").
- Continuous-time (hybrid) functions get kind C (for "continuous").

## Explicitly relate simulation and logical time

All discontinuities and side effects must be aligned with a zero-crossing instant.

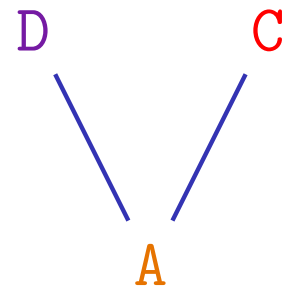
```
let hybrid correct (z) = (time, y) where
  rec der time = 1.0 init 0.0
  and y = present up(z) → sum(time) init 0.0
```

# Basic typing [LCTES'11]

A simple ML type system with effects.

## The type language

$$
\begin{array}{lcl}
bt & ::= & \texttt{float} \mid \texttt{int} \mid \texttt{bool} \mid \texttt{zero} \\
t & ::= & bt \mid t \times t \mid \beta \\
\sigma & ::= & \forall \beta_1, ..., \beta_n . t \xrightarrow{k} t \\
k & ::= & \texttt{D} \mid \texttt{C} \mid \texttt{A}
\end{array}
$$




## Initial conditions

$$
\begin{array}{lcl}
(+) & : & \texttt{int} \times \texttt{int} \xrightarrow{\texttt{A}} \texttt{int} \\
\texttt{if} & : & \forall \beta . \texttt{bool} \times \beta \times \beta \xrightarrow{\texttt{A}} \beta \\
(=) & : & \forall \beta . \beta \times \beta \xrightarrow{\texttt{D}} \texttt{bool} \\
\texttt{pre}(\cdot) & : & \forall \beta . \beta \xrightarrow{\texttt{D}} \beta \\
\cdot\,\texttt{fby}\,\cdot & : & \forall \beta . \beta \times \beta \xrightarrow{\texttt{D}} \beta \\
\texttt{up}(\cdot) & : & \texttt{float} \xrightarrow{\texttt{C}} \texttt{zero}
\end{array}
$$

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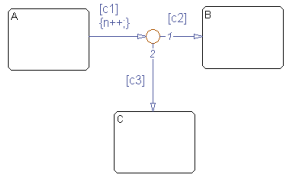


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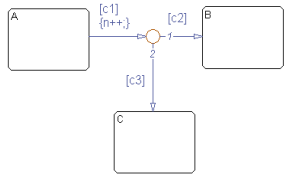


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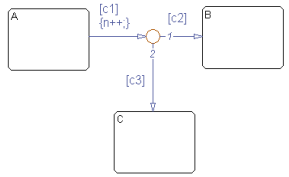


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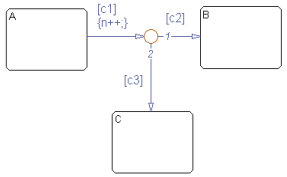


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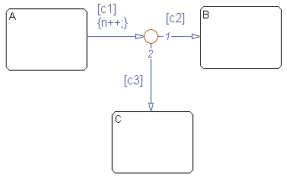


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- 'Restricted subset of Stateflow chart semantics'
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- Our `D`/`C`/`A`/`zero` system extends naturally for the same effect.

# Causality loops

Some programs have causality loops.

## Which programs should we accept?

- OK to reject (no solution).

  let hybrid f () = x where rec x = x + 1

- OK as an algebraic constraint (e.g., Simulink and Modelica).

  let hybrid f () = x where rec x = 1 − x

  But NOK for sequential code generation.

- last x does not necessarily break causality loops!

  let hybrid f () = x where rec x = last x + 1

**Can we find a simple and uniform justification?**

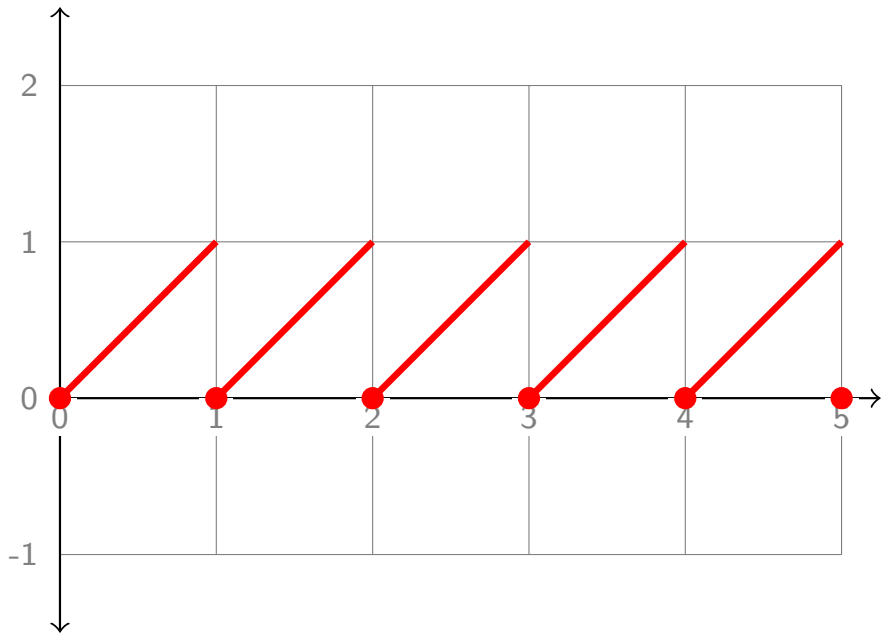
# ODEs with reset

Consider the sawtooth signal $y : \mathbb{R}^+ \mapsto \mathbb{R}^+$ such that:

$$\frac{dy}{dt}(t) = 1 \quad y(t) = 0 \text{ if } t \in \mathbb{N}$$

written:

```
let hybrid f () = y where rec
der y = 1.0 init 0.0 reset up(y -. 1.0) → 0.0
```



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The ideal non-standard semantics is:

$$\begin{aligned}
{}^\star y(0) &= 0 & {}^\star y(n) &= \texttt{if } {}^\star z(n) \texttt{ then } 0.0 \texttt{ else } {}^\star ly(n) \\
{}^\star ly(n) &= {}^\star y(n-1) + \partial & {}^\star c(n) &= ({}^\star y(n) - 1) \geq 0 \\
{}^\star z(0) &= \texttt{false} & {}^\star z(n) &= {}^\star c(n) \wedge \neg {}^\star c(n-1)
\end{aligned}$$

This set of equation is not causal: ${}^\star y(n)$ depends on itself.

# Accessing the "left limit" of a signal (last y)

## Two ways to break this cycle

- consider that the effect of the zero-crossing is delayed by one cycle, that is, the test is made on ${}^\star z(n-1)$ instead of on ${}^\star z(n)$, or,
- distinguish the current value of ${}^\star y(n)$ from the value it would have had were there no reset, namely ${}^\star ly(n)$.

## Testing a zero-crossing of *ly* (instead of *y*)

$$ {}^\star c(n) = ({}^\star ly(n) - 1) \geq 0, $$

gives a program that is causal since ${}^\star y(n)$ no longer depends instantaneously on itself.

## Write

der y = 1.0 init 0.0 reset up(last y − 1.0) → 0.0

# An explanation of the causality 'issue' of Simulink

## The source program

```
let hybrid f () = y where
rec der x = 1.0 init 0.0 reset z → −3.0 ∗. last y
and der y = x init 0.0 reset z → −4.0 ∗. last x
and z = up(last x −. 2.0)
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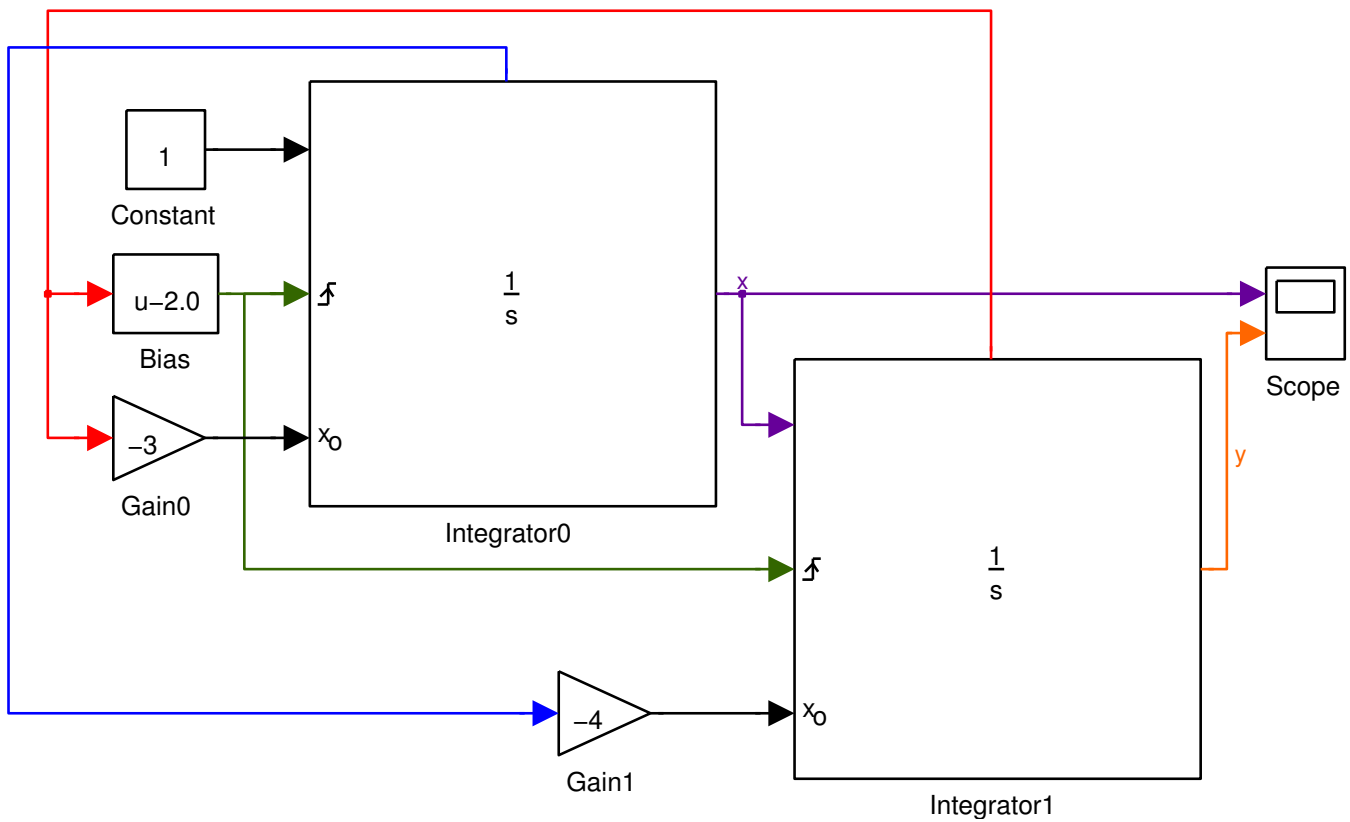

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```

## Its non-standard interpretation

$$
\begin{array}{rcl}
{}^{\star}x(n) & = & \texttt{if}\ {}^{\star}z(n)\ \texttt{then}\ -3 \cdot {}^{\star}y(n-1)\ \texttt{else}\ {}^{\star}x(n-1) + \partial \\
{}^{\star}y(n) & = & \texttt{if}\ {}^{\star}z(n)\ \texttt{then}\ -4 \cdot {}^{\star}x(n-1)\ \texttt{else}\ {}^{\star}y(n-1) + \partial \cdot {}^{\star}x(n-1) \\
 & \vdots &
\end{array}
$$

## Explanation

- The first two equations are scheduled this way so ${}^{\star}x(n-1)$ is lost.
- This is a scheduling bug: the sequential code lacks a copy variable.

# Causality Analysis [HSCC'14]

Every feedback loop must cross a delay.

**Intuition:** associate a 'time stamp' to every expression and require that the relation $<$ between time stamps is a strict partial order.

## The type language

$$\begin{array}{lcl} \sigma & ::= & \forall \alpha_1, ..., \alpha_n : C.\ ct \xrightarrow{k} ct \\ ct & ::= & ct \times ct \mid \alpha \\ k & ::= & \texttt{node} \mid \texttt{hybrid} \mid \texttt{A} \end{array}$$

## Precedence relation:

$$C ::= \{\alpha_1 < \alpha_1', ..., \alpha_n < \alpha_n'\}$$

$C \vdash ct_1 < ct_2$ means that $ct_1$ precedes $ct_2$ according to $C$.

# The Type System

## Type Judgments

$$\frac{}{C \mid G, H \vdash_k e : ct}\ (\text{TYP-EXP}) \qquad \frac{}{C \mid G, H \vdash_k E : H'}\ (\text{TYP-ENV})$$

$$G ::= [\sigma_1/f_1, ..., \sigma_k/f_k] \qquad H ::= [ct_1/x_1, ..., ct_n/x_n]$$

## Initial Conditions

$$\begin{array}{lcl}
(+), (-), (*), (/) & : & \forall\alpha.\ \alpha \times \alpha \xrightarrow{\texttt{A}} \alpha \\
\texttt{pre}(\cdot) & : & \forall\alpha_1, \alpha_2 : \{\alpha_2 < \alpha_1\}.\ \alpha_1 \xrightarrow{\texttt{node}} \alpha_2 \\
\cdot\,\texttt{fby}\,\cdot & : & \forall\alpha_1, \alpha_2 : \{\alpha_1 < \alpha_2\}.\ \alpha_1 \times \alpha_2 \xrightarrow{\texttt{node}} \alpha_1 \\
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\end{array}$$

# The Typing Rules

$$\frac{C, ct_1 \xrightarrow{k} ct_2 \in \mathit{Inst}(G(f)) \qquad C \mid G, H \vdash_k e : ct_1}{C \mid G, H \vdash_k f(e) : ct_2} \text{(APP)}$$

$$C \mid G, H + x : ct \vdash_k x : ct \text{(VAR)}$$

$$\frac{C \vdash ct_2 < ct_1}{C \mid G, H + x : ct_1 \vdash_{\texttt{node}} \texttt{last}\, x : ct_2} \text{(LAST)}$$

$$\frac{C \mid G, H \vdash_k p : ct \qquad C \mid G, H \vdash_k e : ct}{C \mid G, H \vdash_k p = e : [ct/p]} \text{(EQ)}$$

$$\frac{C \mid G, H \vdash_{\texttt{hybrid}} e : ct_1 \qquad C \vdash ct_2 < ct_1}{C \mid G, H \vdash_{\texttt{hybrid}} \texttt{der}\, x = e : [ct_2/x]} \text{(DER)}$$

$$\frac{C \mid G, H \vdash_k e : ct \qquad C \vdash ct < ct'}{C \mid G, H \vdash_k e : ct'} \text{(SUB)}$$

Associate a type that express input/output dependences. E.g.,

```
let node plus(x, y) = x + 0 → pre y
```

We get: $f : \forall \alpha_1, \alpha_2 . \alpha_1 \times \alpha_2 \xrightarrow{\texttt{node}} \alpha_1$

- der x breaks a loop: der temp = c −. temp init 20.0 is correct.
- last(x) breaks a loop in a discrete context.

The following is rejected; the next is accepted.

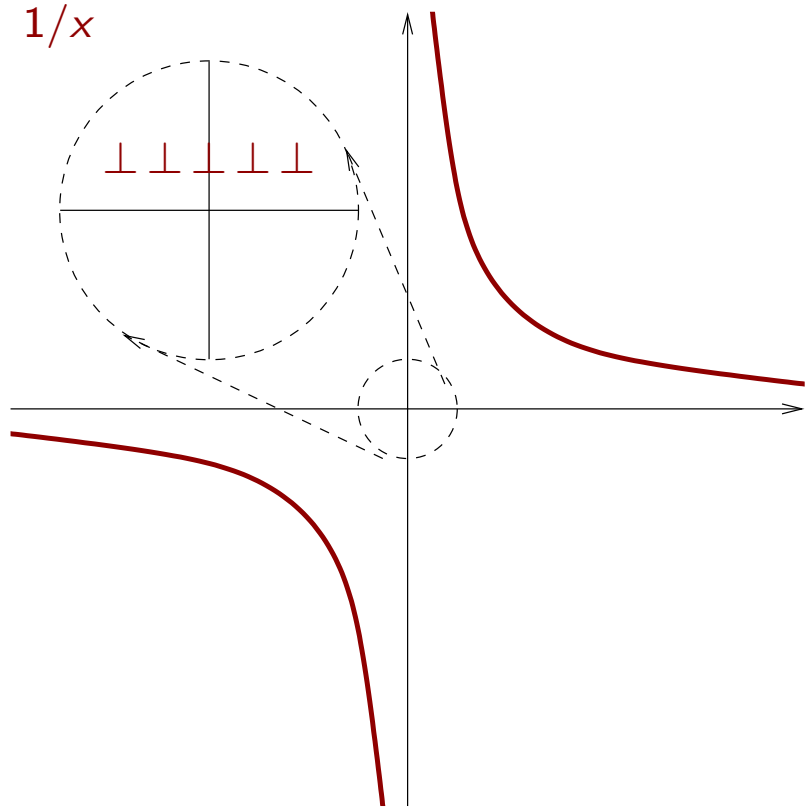
```
let g = −9.8 let y0 = 0.0
let hybrid f () = y where
rec der y' = −. g init 0.0 reset up(−.y) → −0.9 ∗. y'
and der y = y' init y0
```

```
let g = −9.8 let y0 = 0.0
let hybrid f () = y where
rec der y' = −. g init 0.0 reset up(−.y) → −0.9 ∗. last y'
and der y = y' init y0
```

# Assumption on operators and external functions

## Assumption 1

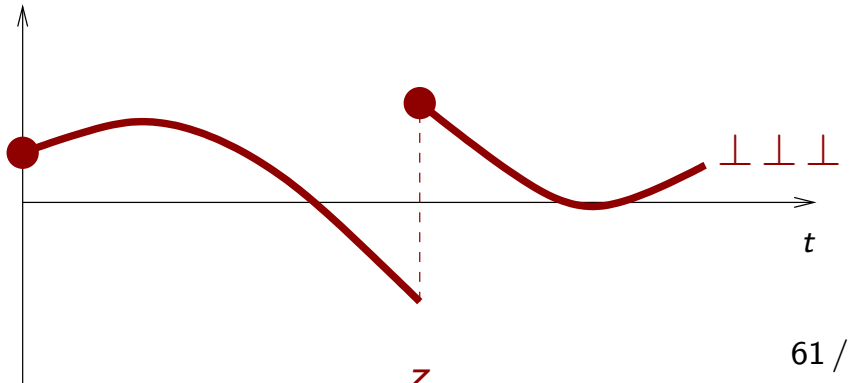
The semantics of operators of kind `hybrid` or `A` is undefined ($\perp$) on every infinitesimal neighborhood of a discontinuity or singularity




## Assumption 2

For every compact set of dates $K \subseteq \mathbb{T}$, and every input $u : S(^\star V)$, continuous and without zero-crossing on K, the semantics of every external function of kind `hybrid` or `A` is either:

1. continuous on $K$ or,
2. triggers a zero-crossing at some $t \in K$ or,
3. is undefined ($\perp$) at some



# Main Theorem

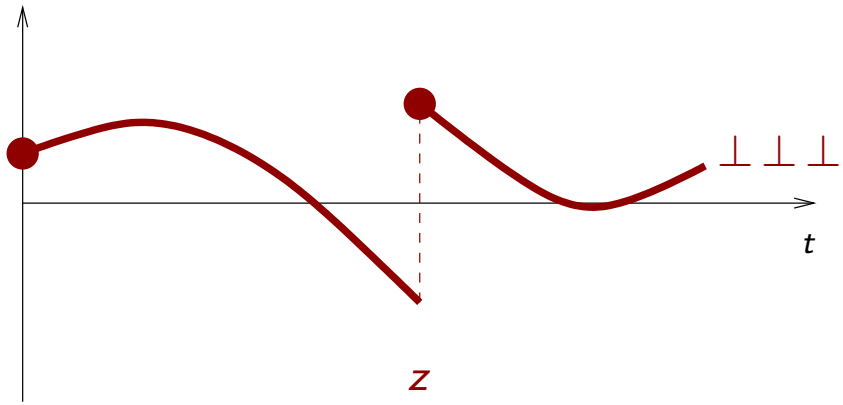
## Theorem

The semantics of every causally correct programm is:

1. standardizable,
2. independent of $\partial$,
3. continuous

on every compact set of dates not containing:

1. a zero crossing, or
2. an undefined value ($\perp$)




The proof deeply rely on the use of the non-standard synchronous semantics.

# Outline

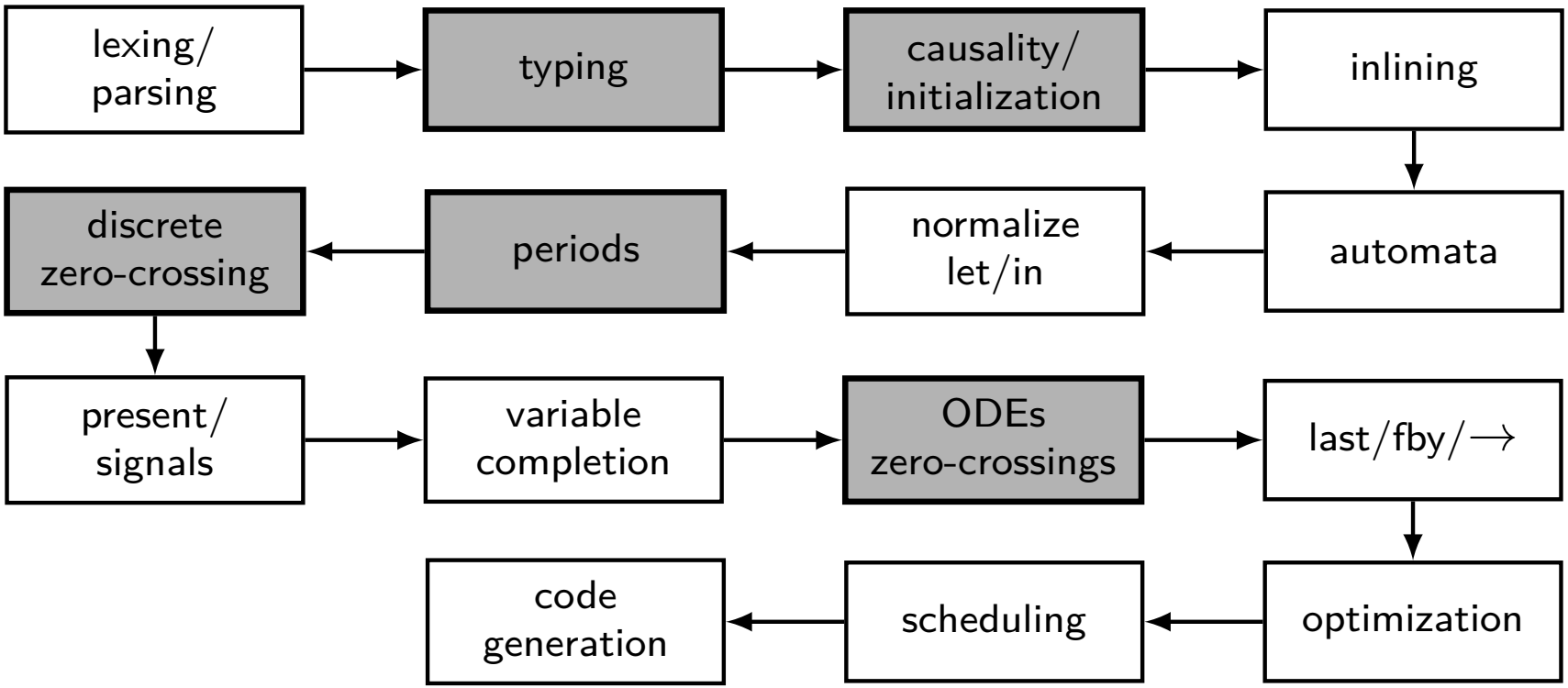
# Compiler Architecture

Two implementations: `Zélus` and KCG 6.4 (Release 2014) or SCADE.

## SCADE Hybrid

- An industrial prototype built on KCG 6.4 (release July 2014). Now KCG 6.6 (2017).
- Generates FMI 1.0 model-exchange FMUs for Simplorer.
- Only 5% of the compiler modified. Small changes in:
  - static analysis (typing, causality).
  - automata translation; code generation.
  - FMU generation (XML description, wrapper).
- FMU integration loop: about 1000 LoC.

# Compiler architecture (Zelus)




## Built on an existing synchronous compiler

- Source-to-source and traceable transformations
- Resulting program is synchronous and translated to sequential code

# A SCADE-like Input Language

Essentially SCADE with three syntax extensions (in red).

$$
\begin{array}{lcl}
d & ::= & \texttt{const}\, x = e \mid k\, f(pi) = pi\, \texttt{where}\, E \mid d; d \\
k & ::= & \texttt{fun} \mid \texttt{node} \mid \texttt{hybrid} \\
e & ::= & x \mid v \mid op(e, \ldots, e) \mid v\, \texttt{fby}\, e \mid \texttt{last}\, x \mid f(e, \ldots, e) \mid \texttt{up}(e) \\
p & ::= & x \mid (x, \ldots, x) \\
pi & ::= & xi \mid xi, \ldots, xi \\
xi & ::= & x \mid x\ \texttt{last}\ e \mid x\ \texttt{default}\ e \\
E & ::= & p = e \mid \texttt{der}\, x = e \\
 & & \mid \texttt{if}\, e\, \texttt{then}\, E\, \texttt{else}\, E \\
 & & \mid \texttt{reset}\ E\ \texttt{every}\ e \\
 & & \mid \texttt{local}\ pi\ \texttt{in}\ E \mid \texttt{do}\ E\ \texttt{and}\ \ldots E\ \texttt{done}
\end{array}
$$

# A Clocked Data-flow Internal Language

The internal language is extended with three extra operations.
Translation based on Colaco et al. [EMSOFT'05].

$$
\begin{array}{lcl}
d & ::= & \texttt{const}\, x = c \mid k\, f(p) = a\, \texttt{where}\, C \mid d; d \\
k & ::= & \texttt{fun} \mid \texttt{node} \mid \texttt{hybrid} \\
C & ::= & (x_i = a_i)_{x_i \in I} \text{ with } \forall i \neq j . x_i \neq x_j \\
a & ::= & e^{ck} \\
e & ::= & x \mid v \mid op(a, ..., a) \mid v\, \texttt{fby}\, a \mid \texttt{pre}(a) \\
 & & \mid f(a, ..., a) \\
 & & \mid \texttt{merge}(a, a, a) \mid a\, \texttt{when}\, a \\
 & & \mid \texttt{integr}(a, a) \mid \texttt{up}(a) \\
p & ::= & x \mid (x, ..., x) \\
ck & ::= & \texttt{base} \mid ck\ \texttt{on}\ a
\end{array}
$$

# Clocked Equations Put in Normal Form

Name the result of every stateful operation. Separate into syntactic categories.

- *se*: strict expressions
- *de*: delayed expressions
- *ce*: controlled expressions.

Equation $lx = \texttt{integr}(x', x)$ defines $lx$ to be the continuous state variable; possibly reset with $x$.

$$
\begin{array}{lcl}
eq & ::= & x = ce^{ck} \mid x = f(sa, \ldots, sa)^{ck} \mid x = de^{ck} \\
sa & ::= & se^{ck} \\
ca & ::= & ce^{ck} \\
se & ::= & x \mid v \mid op(sa, \ldots, sa) \mid sa \texttt{ when } sa \\
ce & ::= & se \mid \texttt{merge}(sa, ca, ca) \mid ca \texttt{ when } sa \\
de & ::= & \texttt{pre}(ca) \mid v \texttt{ fby } ca \mid \texttt{integr}(ca, ca) \mid \texttt{up}(ca)
\end{array}
$$

# Well Scheduled Form

Equations are statically scheduled.

$Read(a)$: set of variables read by $a$.

Given $C = (x_i = a_i)_{x_i \in I}$, a valid schedule is a one-to-one function

$$Schedule(.) : I \to \{1 \dots |I|\}$$

such that, for all $x_i \in I, x_j \in Read(a_i) \cap I$:

1. if $a_i$ is strict, $Schedule(x_j) < Schedule(x_i)$ and
2. if $a_i$ is delayed, $Schedule(x_i) \leq Schedule(x_j)$.

From the data-dependence point-of-view, $\texttt{integr}(ca_1, ca_2)$ and $\texttt{up}(ca)$ break instantaneous loops.

# A Sequential Object Language (SOL)

- Translation into an intermediate imperative language [Colaco et al., LCTES'08]
- Instead of producing two methods `step` and `reset`, produce more.
- Mark memory variables with a kind $m$

$$
\begin{array}{lll}
md & ::= & \mid \texttt{const}\, x = c \\
 & & \mid \texttt{const}\, f = \texttt{class}\langle M,\ I,\ (method_i(p_i) = e_i\ \texttt{where}\ S_i)_{i\in[1..n]} \\
M & ::= & [x : m[= v]; ...; x : m[= v]] \\
I & ::= & [o : f; ...; o : f] \\
m & ::= & \mathit{Discrete} \mid \mathit{Zero} \mid \mathit{Cont} \\
e & ::= & v \mid lv \mid op(e, ..., e) \mid o.method(e, ..., e) \\
S & ::= & () \mid lv \leftarrow e \mid S\ ;\ S \mid \texttt{var}\ x, ..., x\ \texttt{in}\ S \mid \texttt{if}\ c\ \texttt{then}\ S\ \texttt{else}\ S \\
R, L & ::= & S; ...; S \\
lv & ::= & x \mid lv.field \mid \texttt{state}(x)
\end{array}
$$

# State Variables

## Discrete State Variables (sort *Discrete*)

- Read with `state`$(x)$;
- modified with `state`$(x) \leftarrow c$

## Zero-crossing State Variables (sort *Zero*)

- A pair with two fields.
- The field `state`$(x).zin$ is a boolean, true when a zero-crossing on $x$ has been detected, false otherwise.
- The field `state`$(x).zout$ is the value for which a zero-crossing must be detected.

## Continuous State Variables (sort *Cont*)

- `state`$(x).der$ is its instantaneous derivative;
- `state`$(x).pos$ its value

# Example: translation of the bouncing ball

```
let bouncing = machine(continuous) {
  memories disc init_25 : bool = true;
           zero result_17 : bool = false;
           cont y_v_15 : float = 0.; cont y_14 : float = (

  method reset  =
    init_25 <- true; y_v_15.pos <- 0.

  method step time_23 y0_9 =
    (if init_25 then (y_14.pos <- y0_9; ()) else ());
    init_25 <- false;
    result_17.zout <- (~-.) y_14.pos;
    if result_17.zin
     then (y_v_15.pos <- ( *. ) 0.8 y_v_15.pos);
    y_14.der <- y_v_15.pos;
    y_v_15.der <- (~-.) g; y_14.pos }
```

# Finally

1. Translate as usual to produce a function `step`.
2. For hybrid nodes, **copy-and-paste** the step method.
3. Either into a `cont` method activated during the continuous mode, or two extra methods `derivatives` and `crossings`.
4. Apply the following:
   - During the continuous mode (method `cont`), all zero-crossings (variables of type *zero*, e.g., `state`$(x).zin$) are surely false. All zero-crossing outputs (`state`$(x).zout \leftarrow \ldots$) are useless.
   - During the discrete step (method `step`), all derivative changes (`state`$(x).der \leftarrow \ldots$) are useless.
   - Remove dead-code by calling an existing pass.
5. That's all!

Examples (both Zélus and SCADE) at: `zelus.di.ens.fr/cc2015`

# Example: translation of the bouncing ball

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let bouncing = machine(continuous) {
  memories disc init_25 : bool = true;
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  method reset  =
    init_25 <- true; y_v_15.pos <- 0.
  method step time_23 y0_9 =
    (if init_25 then (y_14.pos <- y0_9; ()) else ());
    init_25 <- false;
    if result_17.zin
     then (y_v_15.pos <- ( *. ) 0.8 y_v_15.pos);
    y_14.pos
  method cont time_23 y0_9 =
    result_17.zout <- (~-.) y_14.pos;
    y_14.der <- y_v_15.pos;
    y_v_15.der <- (~-.) g }
```

# Some conclusion

## Two experiments

- The `Zélus` academic langage and compiler.
- The industrial **KCG 6.7** (Release 2016) code generator of SCADE.
- For KCG, **less than 5%** of extra LOC, in all.
- The extension is **fully conservative** w.r.t existing SCADE.
- The very same code is used both for simulation and embedded code.

Yet, is this type discipline too constraining for writting real applications?

# Comparison with existing tools

## Simulink/Stateflow (Mathworks)

- Integrated treatment of automata *vs* two distinct languages
- More rigid separation of discrete and continuous behaviors

## Modelica

- Do not handle DAEs
- Our proposal for automata has been integrated into version 3.3

## Ptolemy (E.A. Lee et al., Berkeley)

- A unique computational model: synchronous
- Everything is compiled to sequential code (not interpreted)

Synchronous languages should and can properly treat hybrid systems

http://zelus.di.ens.fr

Zélus | Download | Examples | Publications | Contact


## Compiler

Zélus is a synchronous language extended with Ordinary Differential Equations (ODEs) to model systems with complex interaction between discrete-time and continuous-time dynamics. It shares the basic principles of Lustre with features from Lucid Synchrone (type inference, hierarchical automata, and signals). The compiler is written

## Research

Zélus is used to experiment with new techniques for building hybrid modelers like Simulink/Stateflow and Modelica on top of a synchronous language. The language exploits novel techniques for defining the semantics of hybrid modelers, it provides dedicated type systems to ensure the absence of discontinuities during integration and the