Provable Security in the Computational Model

III – Signatures

David Pointcheval

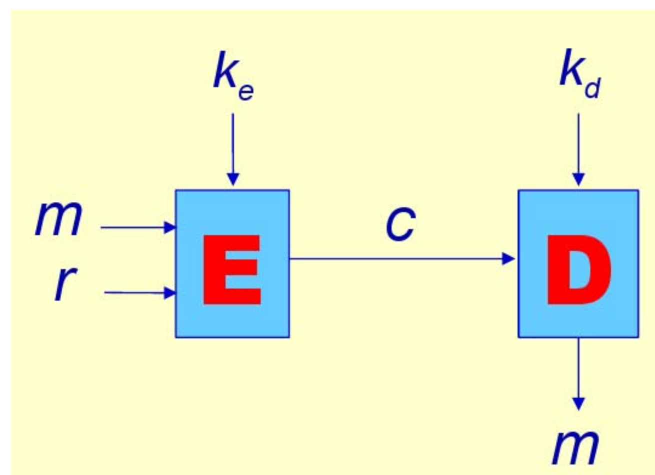
Ecole normale supérieure, CNRS & INRIA

MPRI – Paris

Outline

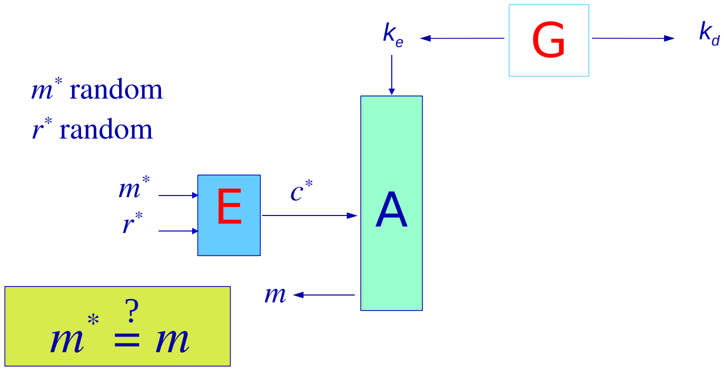
Outline

Public-Key Encryption



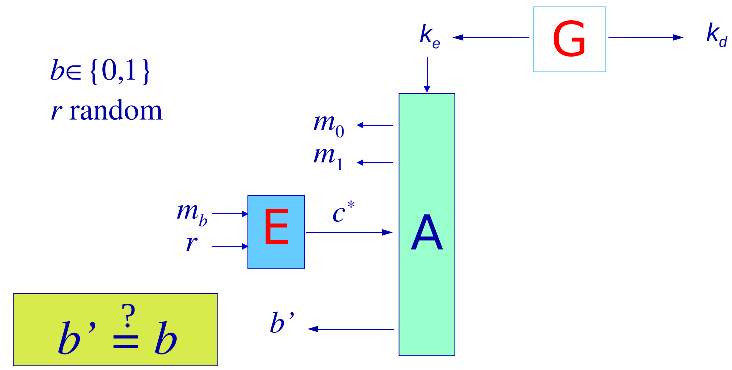
Goal: Privacy/Secrecy of the plaintext

OW − CPA Security Game



$$\mathbf{Succ}^{\mathsf{ow}}_{\mathcal{S}}(\mathcal{A}) = \Pr[(sk, pk) \leftarrow \mathcal{K}(); m \stackrel{R}{\leftarrow} \mathcal{M}; c = \mathcal{E}_{pk}(m) : \mathcal{A}(pk, c) \rightarrow m]$$

IND − CPA Security Game

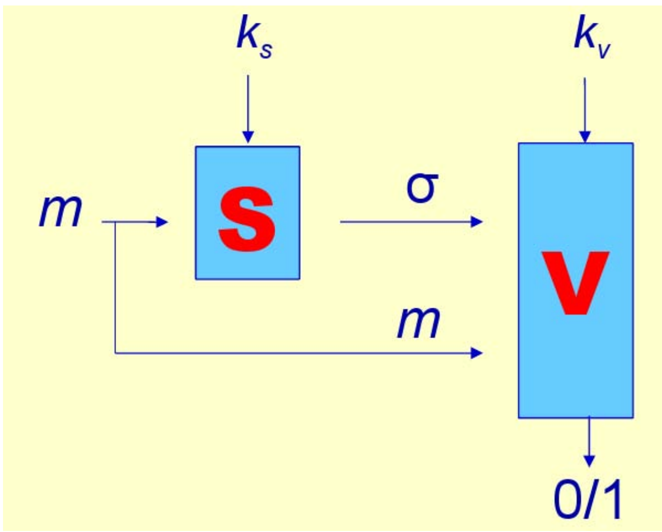


$$(sk, pk) \leftarrow \mathcal{K}(); (m_0, m_1, \text{state}) \leftarrow \mathcal{A}(pk);$$
$$b \stackrel{R}{\leftarrow} \{0,1\}; c = \mathcal{E}_{pk}(m_b); b' \leftarrow \mathcal{A}(\text{state}, c)$$

$$\mathbf{Adv}^{\mathsf{ind-cpa}}_{\mathcal{S}}(\mathcal{A}) = \Pr[b' = 1 | b = 1] - \Pr[b' = 1 | b = 0] = 2 \times \Pr[b' = b] - 1$$

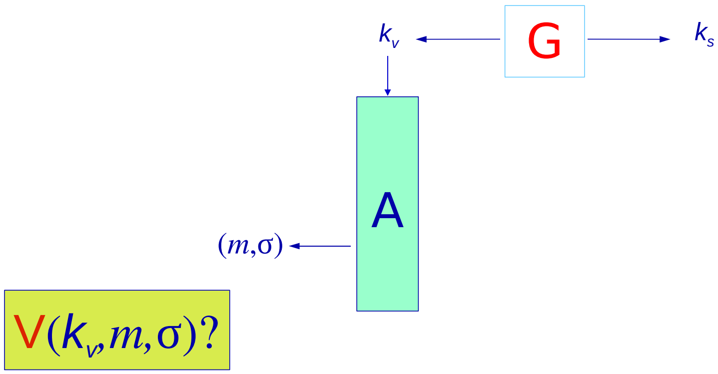
Outline

Signature



Goal: Authentication of the sender

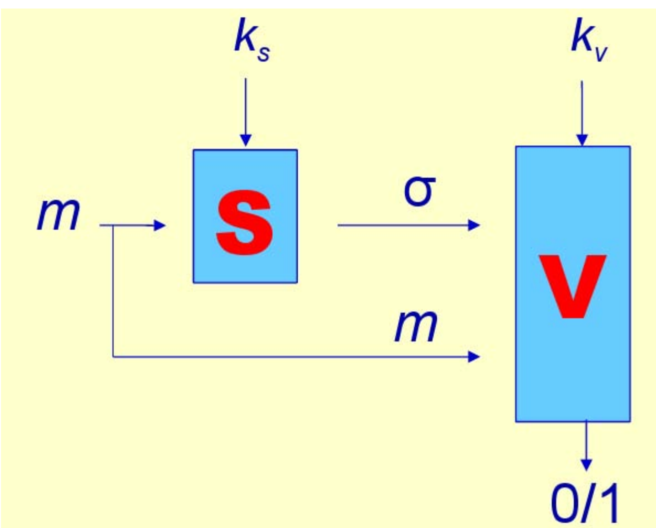
EUF − NMA



$$\mathbf{Succ}^{\text{euf}}_{\mathcal{SG}}(\mathcal{A}) = \Pr[(sk,pk)\leftarrow\mathcal{K}(); (m,\sigma)\leftarrow\mathcal{A}(pk) : \mathcal{V}_{pk}(m,\sigma)=1]$$

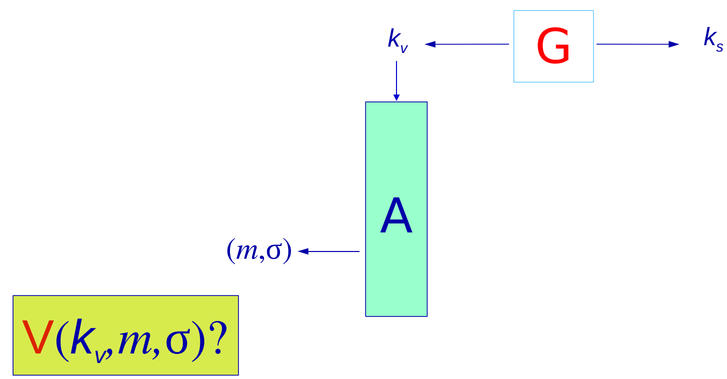
Outline

Signature



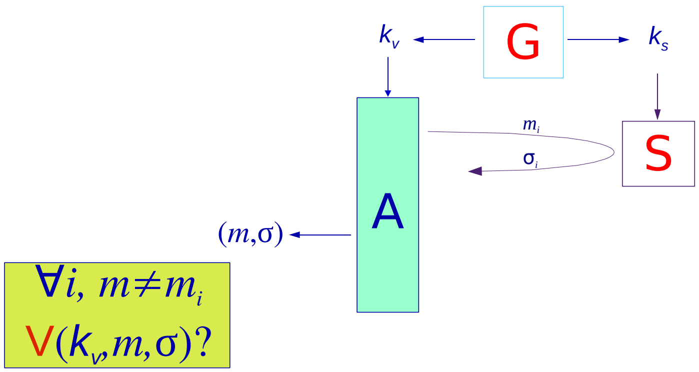
Goal: Authentication of the sender

EUF − NMA



The adversary knows the public key only, whereas signatures are not private!

EUF − CMA

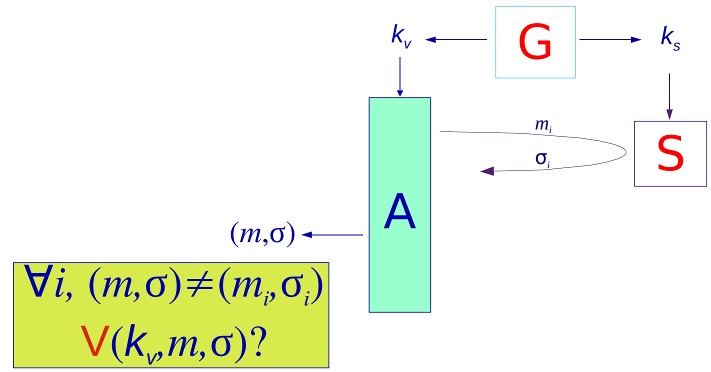


The adversary has access to any signature of its choice:
Chosen-Message Attacks (oracle access):

$$\mathbf{Succ}_{\mathcal{SG}}^{\mathsf{euf-cma}}(\mathcal{A}) = \Pr\left[\begin{array}{l}(sk, pk) \leftarrow \mathcal{K}(); (m,\sigma) \leftarrow \mathcal{A}^{\mathcal{S}}(pk) : \\ \forall i, m \neq m_i \wedge \mathcal{V}_{pk}(m,\sigma) = 1\end{array}\right]$$

SUF − CMA

[Stern-Pointcheval-Malone-Lee-Smart – Crypto '01]



The notion is even stronger (in case of probabilistic signature):
also known as non-malleability:

$$\mathbf{Succ}_{\mathcal{SG}}^{\mathsf{suf-cma}}(\mathcal{A}) = \Pr\left[\begin{array}{l}(sk, pk) \leftarrow \mathcal{K}(); (m,\sigma) \leftarrow \mathcal{A}^{\mathcal{S}}(pk) : \\ \forall i, (m,\sigma) \neq (m_i,\sigma_i) \wedge \mathcal{V}_{pk}(m,\sigma) = 1\end{array}\right]$$

Outline

Full-Domain Hash Signature

[Bellare-Rogaway – Eurocrypt '94]

Signature Scheme

- Key generation: the public key $f \stackrel{R}{\leftarrow} \mathcal{P}$ is a trapdoor one-way bijection from X onto Y; the private key is the inverse $g : Y \rightarrow X$;
- Signature of $M \in Y$: $\sigma = g(M)$;
- Verification of (M, σ): check $f(\sigma) = M$

Full-Domain Hash (Hash-and-Invert)

$$\mathcal{H} : \{0,1\}^\star \rightarrow Y$$

- in order to sign m, one computes $M = \mathcal{H}(m) \in Y$, and $\sigma = g(M)$
- and the verification consists in checking whether $f(\sigma) = H(m)$

Random Oracle Model

Random Oracle

- $\mathcal{H}$ is modelled as a truly random function, from $\{0,1\}^*$ into Y.
- Formally, $\mathcal{H}$ is chosen at random at the beginning of the game.
- More concretely, for any new query, a random element in Y is uniformly and independently drawn

Any security game becomes:

$$\mathbf{Succ}^{\mathsf{euf-cma}}_{\mathcal{SG}}(\mathcal{A}) = \Pr\left[\begin{array}{l}\mathcal{H} \stackrel{R}{\leftarrow} Y^{\infty}; (sk, pk) \leftarrow \mathcal{K}(); (m, \sigma) \leftarrow \mathcal{A}^{\mathcal{S},\mathcal{H}}(pk) : \\ \forall i, m \neq m_i \wedge \mathcal{V}_{pk}(m, \sigma) = 1\end{array}\right]$$

Security of the FDH Signature

Theorem

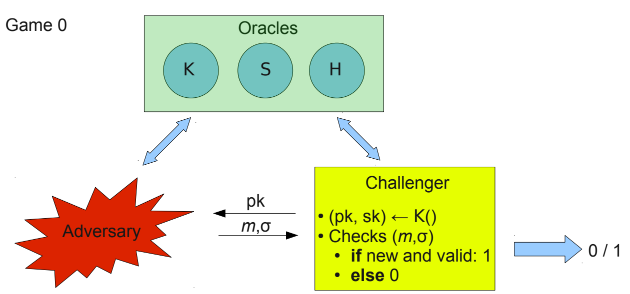
The FDH signature achieves $\mathbf{EUF-CMA}$ *security, under the One-Wayness of* $\mathcal{P}$*, in the Random Oracle Model:*

$$\mathbf{Succ}^{\mathsf{euf-cma}}_{\mathcal{FDH}}(t) \leq q_H \times \mathbf{Succ}^{\mathsf{ow}}_{\mathcal{P}}(t + q_H \tau_f)$$

Assumptions:

- any signing query has been first asked to $\mathcal{H}$
- the forgery has been asked to $\mathcal{H}$
- τ_f is the maximal time to evaluate $f \in \mathcal{P}$

Real Attack Game



Random Oracle

$\mathcal{H}(m)$: $M \stackrel{R}{\leftarrow} Y$, output M

Key Generation Oracle

$\mathcal{K}()$: $(f, g) \stackrel{R}{\leftarrow} \mathcal{P}$, $sk \leftarrow g$, $pk \leftarrow f$

Signing Oracle

$\mathcal{S}(m)$: $M = \mathcal{H}(m)$, output $\sigma = g(M)$

Simulations

- **Game**$_0$: use of the oracles $\mathcal{K}$, $\mathcal{S}$ and $\mathcal{H}$
- **Game**$_1$: use of the *simulation of the Random Oracle*

Simulation of $\mathcal{H}$

$\mathcal{H}(m)$: $\mu \stackrel{R}{\leftarrow} X$, output $M = f(\mu)$

$\Longrightarrow$ **Hop-D-Perfect**: $\Pr_{\mathbf{Game}_1}[1] = \Pr_{\mathbf{Game}_0}[1]$

- **Game**$_2$: use of the *simulation of the Signing Oracle*

Simulation of $\mathcal{S}$

$\mathcal{S}(m)$: find μ such that $M = \mathcal{H}(m) = f(\mu)$, output $\sigma = \mu$

$\Longrightarrow$ **Hop-S-Perfect**: $\Pr_{\mathbf{Game}_2}[1] = \Pr_{\mathbf{Game}_1}[1]$

$\mathcal{H}$-Query Selection

- **Game**$_3$: random index $t \stackrel{R}{\leftarrow} \{1, \ldots, q_H\}$

Event Ev

If the t-th query to $\mathcal{H}$ is not the output forgery

We terminate the game and output 0 if **Ev** happens
$\Longrightarrow$ **Hop-S-Non-Negl**
Then, clearly

$$\Pr_{\textbf{Game}_3}[1] = \Pr_{\textbf{Game}_2}[1] \times \Pr[\neg \textbf{Ev}] \qquad \Pr[\textbf{Ev}] = 1 - 1/q_H$$

$$\Pr_{\textbf{Game}_3}[1] = \Pr_{\textbf{Game}_2}[1] \times \frac{1}{q_H}$$

OW Instance

- **Game**$_4$: $\mathcal{P} - \mathbf{OW}$ instance (f, y) (where $f \stackrel{R}{\leftarrow} \mathcal{P}, x \stackrel{R}{\leftarrow} X, y = f(x)$)
  Use of the *simulation of the Key Generation Oracle*

Simulation of $\mathcal{K}$

$\mathcal{K}()$: set $pk \leftarrow f$

Modification of the *simulation of the Random Oracle*

Simulation of $\mathcal{H}$

If this is the t-th query, $\mathcal{H}(m)$: $M \leftarrow y$, output M

The unique difference is for the t-th simulation of the random oracle, for which we cannot compute a signature.
But since it corresponds to the forgery output, it cannot be queried to the signing oracle:
$\Longrightarrow$ **Hop-S-Perfect**: $\Pr_{\textbf{Game}_4}[1] = \Pr_{\textbf{Game}_3}[1]$

Summary

In **Game**$_4$, when the output is 1, $\sigma = g(y) = g(f(x)) = x$
and the simulator computes one exponentiation per hashing:

$$\begin{aligned}
\Pr_{\textbf{Game}_4}[1] &\leq \mathbf{Succ}^{\mathsf{ow}}_{\mathcal{P}}(t + q_H\tau_f) \\
\Pr_{\textbf{Game}_4}[1] &= \Pr_{\textbf{Game}_3}[1] \\
\Pr_{\textbf{Game}_3}[1] &= \Pr_{\textbf{Game}_2}[1] \times \frac{1}{q_H} \\
\Pr_{\textbf{Game}_2}[1] &= \Pr_{\textbf{Game}_1}[1] \\
\Pr_{\textbf{Game}_1}[1] &= \Pr_{\textbf{Game}_0}[1] \\
\Pr_{\textbf{Game}_0}[1] &= \mathbf{Succ}^{\mathsf{euf-cma}}_{\mathcal{FDH}}(\mathcal{A})
\end{aligned}$$

$$\mathbf{Succ}^{\mathsf{euf-cma}}_{\mathcal{FDH}}(\mathcal{A}) \leq q_H \times \mathbf{Succ}^{\mathsf{ow}}_{\mathcal{P}}(t + q_H\tau_f)$$

Key Size

$$\mathbf{Succ}^{\mathsf{euf-cma}}_{\mathcal{FDH}}(\mathcal{A}) \leq q_H \times \mathbf{Succ}^{\mathsf{ow}}_{\mathcal{P}}(t + q_H\tau_f)$$

- If one wants $\mathbf{Succ}^{\mathsf{euf-cma}}_{\mathcal{FDH}}(t) \leq \varepsilon$ with $t/\varepsilon \approx 2^{80}$
- If one allows q_H up to 2^{60}

Then one needs $\mathbf{Succ}^{\mathsf{ow}}_{\mathcal{P}}(t) \leq \varepsilon$ with $t/\varepsilon \geq 2^{140}$.

If one uses FDH-RSA: at least 3072 bit keys are needed.

Improvement

[Coron – Crypto '00]

In the case that f is homomorphic (as RSA): $f(ab) = f(a)f(b)$

- **Game**$_0$: use of the oracles $\mathcal{K}$, $\mathcal{S}$ and $\mathcal{H}$
- **Game**$_1$: use of the *simulation of the Random Oracle*

Simulation of $\mathcal{H}$

$\mathcal{H}(m)$: $\mu \stackrel{R}{\leftarrow} X$, output $M = f(\mu)$

$\Longrightarrow$ **Hop-D-Perfect**: $\Pr_{\mathbf{Game}_1}[1] = \Pr_{\mathbf{Game}_0}[1]$

- **Game**$_2$: use of the *homomorphic property*

$\mathcal{P} - \mathbf{OW}$ instance (f, y) (where $f \stackrel{R}{\leftarrow} \mathcal{P}, x \stackrel{R}{\leftarrow} X, y = f(x)$)

Simulation of $\mathcal{H}$

$\mathcal{H}(m)$: flip a biased coin b (with $\Pr[b = 0] = p$), $\mu \stackrel{R}{\leftarrow} X$.
If $b = 0$, output $M = f(\mu)$, otherwise output $M = y \times f(\mu)$

$\Longrightarrow$ **Hop-D-Perfect**: $\Pr_{\mathbf{Game}_2}[1] = \Pr_{\mathbf{Game}_1}[1]$

Signature Oracle

- **Game**$_3$: use of the *simulation of the Signing Oracle*

Simulation of $\mathcal{S}$

$\mathcal{S}(m)$: find μ such that $M = \mathcal{H}(m) = f(\mu)$, output $\sigma = \mu$

Fails (with output 0) if $\mathcal{H}(m) = M = y \times f(\mu)$:
but with probability p^{q_S}

$\Longrightarrow$ **Hop-S-Non-Negl**: $\Pr_{\mathbf{Game}_3}[1] = \Pr_{\mathbf{Game}_2}[1] \times p^{q_S}$

Summary

In **Game**$_3$, when the output is 1, with probability $1 - p$:

$$\sigma = g(M) = g(y \times f(\mu)) = g(y) \times g(f(\mu)) = g(f(x)) \times \mu = x \times \mu$$

$$\begin{aligned}
\Pr_{\mathbf{Game}_3}[1] &\le \mathbf{Succ}^{\mathsf{ow}}_{\mathcal{P}}(t + q_H\tau_f)/(1-p) \\
\Pr_{\mathbf{Game}_3}[1] &= \Pr_{\mathbf{Game}_2}[1] \times p^{q_S} \\
\Pr_{\mathbf{Game}_2}[1] &= \Pr_{\mathbf{Game}_1}[1] \\
\Pr_{\mathbf{Game}_1}[1] &= \Pr_{\mathbf{Game}_0}[1] \\
\Pr_{\mathbf{Game}_0}[1] &= \mathbf{Succ}^{\mathsf{euf-cma}}_{\mathcal{FDH}}(\mathcal{A})
\end{aligned}$$

$$\mathbf{Succ}^{\mathsf{euf-cma}}_{\mathcal{FDH}}(\mathcal{A}) \le \frac{1}{(1-p)p^{q_S}} \times \mathbf{Succ}^{\mathsf{ow}}_{\mathcal{P}}(t + q_H\tau_f)$$

Key Size

$$\mathbf{Succ}^{\mathsf{euf-cma}}_{\mathcal{FDH}}(\mathcal{A}) \le \frac{1}{(1-p)p^{q_S}} \times \mathbf{Succ}^{\mathsf{ow}}_{\mathcal{P}}(t + q_H\tau_f)$$

The maximal for $p \mapsto (1-p)p^{q_S}$ is reached for

$$p = 1 - \frac{1}{q_S + 1} \quad \rightarrow \quad \frac{1}{q_S+1} \times \left(1 - \frac{1}{q_S+1}\right)^{q_S} \approx \frac{e^{-1}}{q_S}$$

- If one wants $\mathbf{Succ}^{\mathsf{euf-cma}}_{\mathcal{FDH}}(t) \le \varepsilon$ with $t/\varepsilon \approx 2^{80}$
- If one allows q_S up to 2^{30}

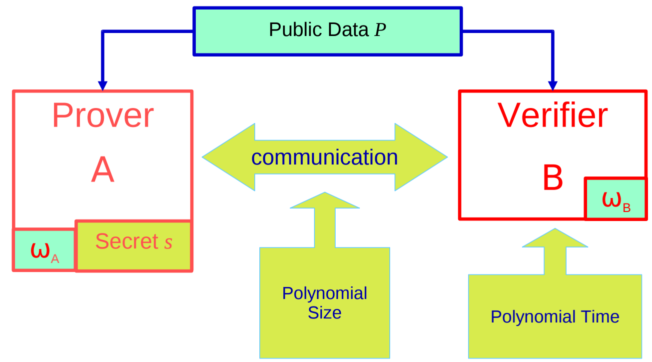
Then one needs $\mathbf{Succ}^{\mathsf{ow}}_{\mathcal{P}}(t) \le \varepsilon$ with $t/\varepsilon \ge 2^{110}$.

If one uses FDH-RSA: 2048 bit keys are enough.

Outline

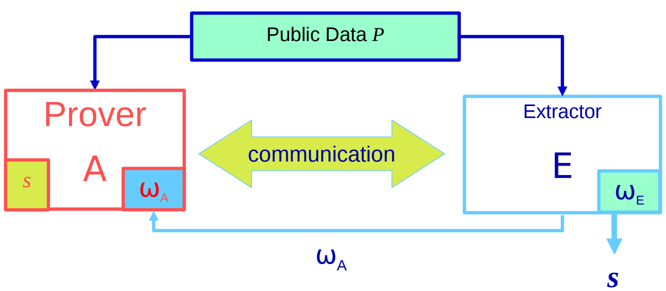
Proof of Knowledge

How do I prove that I know a solution *s* to a problem *P*?



Proof of Knowledge: Soundness

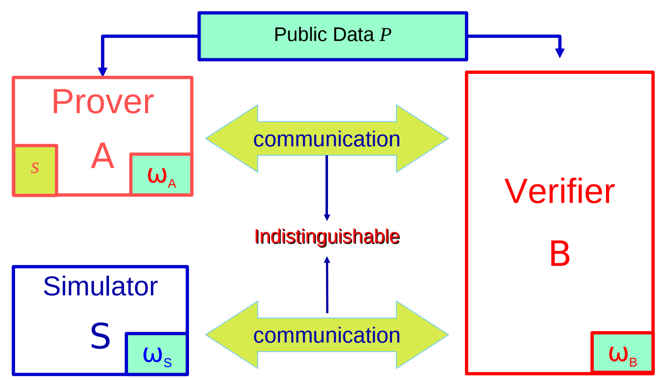
If I can be accepted, I really know a solution: extractor



Proof of Knowledge: Zero-Knowledge

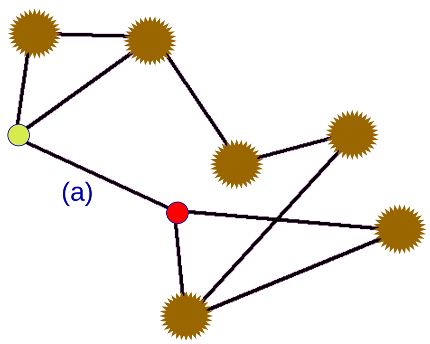
How do I prove that I know a solution *s* to a problem *P*?
I reveal the solution. . .

How can do it without revealing any information?
Zero-knowledge: simulator



Proof of Knowledge

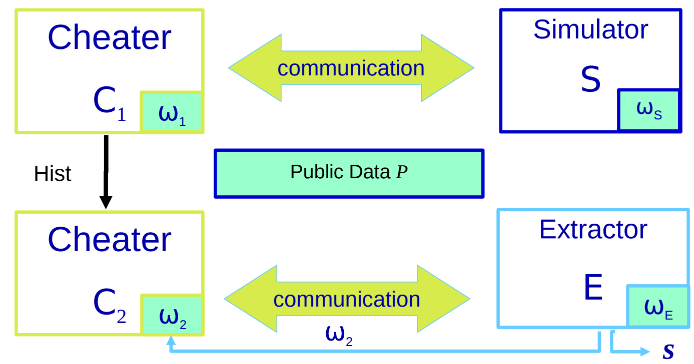
How do I prove that I know a 3-color covering,
without revealing any information?



I choose a random permutation on the colors
and I apply it to the vertices I mask the vertices
and send it to the verifier The verifier chooses an edge I open it
The verifier checks the validity: 2 different colors

Secure Multiple Proofs of Knowledge: Authentication

If there exists an efficient adversary,
then one can solve the underlying problem:



Schnorr Proofs

[Schnorr – Eurocrypt '89 - Crypto '89]

Zero-Knowledge Proof

- Setting: ($\mathbb{G} = \langle g \rangle$) of order q
  $\mathcal{P}$ knows x, such that $y = g^{-x}$
  and wants to prove it to $\mathcal{V}$
- $\mathcal{P}$ chooses $K \stackrel{R}{\leftarrow} \mathbb{Z}_q^\star$
  sets and sends $r = g^K$
- $\mathcal{V}$ chooses $h \stackrel{R}{\leftarrow} \{0,1\}^k$
  and sends it to $\mathcal{P}$
- $\mathcal{P}$ computes and sends
  $s = K + xh \bmod q$
- $\mathcal{V}$ checks whether $r \stackrel{?}{=} g^s y^h$

Signature

- ($\mathbb{G} = \langle g \rangle$) of order q
  $\mathcal{H}$: $\{0,1\}^\star \to \mathbb{Z}_q$
- Key Generation $\to (y, x)$
  private key $x \in \mathbb{Z}_q^\star$
  public key $y = g^{-x}$
- Signature of $m \to (r, h, s)$
  $K \stackrel{R}{\leftarrow} \mathbb{Z}_q^\star$ $r = g^K$
  $h = \mathcal{H}(m, r)$ and
  $s = K + xh \bmod q$
- Verification of (m, r, s)
  compute $h = \mathcal{H}(m, r)$
  and check $r \stackrel{?}{=} g^s y^h$

Generic Zero-Knowledge Proofs

Zero-Knowledge Proof

- Proof of knowledge of x,
  such that $\mathcal{R}(x, y)$
- $\mathcal{P}$ builds a commitment r
  and sends it to $\mathcal{V}$
- $\mathcal{V}$ chooses a challenge
  $h \stackrel{R}{\leftarrow} \{0,1\}^k$ for $\mathcal{P}$
- $\mathcal{P}$ computes and sends
  the answer s
- $\mathcal{V}$ checks (r, h, s)

Signature

$\mathcal{H}$ viewed as a random oracle

- Key Generation $\to (y, x)$
  private: x public: y
- Signature of $m \to (r, h, s)$
  Commitment r
  Challenge $h = \mathcal{H}(m, r)$
  Answer s
- Verification of (m, r, s)
  compute $h = \mathcal{H}(m, r)$
  and check (r, h, s)

Σ Protocols

Zero-Knowledge Proof

- Proof of knowledge of x
- $\mathcal{P}$ sends a commitment r
- $\mathcal{V}$ sends a challenge h
- $\mathcal{P}$ sends the answer s
- $\mathcal{V}$ checks (r, h, s)

Signature

- Key Generation $\rightarrow (y, x)$
- Signature of $m \rightarrow (r, h, s)$
  Commitment r
  Challenge $h = \mathcal{H}(m, r)$
  Answer s
- Verification of (m, r, s)
  compute $h = \mathcal{H}(m, r)$
  and check (r, h, s)

Special soundness

If one can answer to two different challenges $h \neq h'$: s and s'
for a unique commitment r, one can extract x

Outline

Splitting Lemma

Idea

When a subset A is "large" in a product space $X \times Y$,
it has many "large" sections.

The Splitting Lemma

Let $A \subset X \times Y$ such that $\Pr[(x, y) \in A] \geq \varepsilon$. For any $\alpha < \varepsilon$, define

$$B_\alpha = \left\{ (x, y) \in X \times Y \mid \Pr_{y' \in Y}[(x, y') \in A] \geq \varepsilon - \alpha \right\}, \qquad \text{then}$$

(*i*) $\Pr[B_\alpha] \geq \alpha$

(*ii*) $\forall (x, y) \in B_\alpha, \Pr_{y' \in Y}[(x, y') \in A] \geq \varepsilon - \alpha$.

(*iii*) $\Pr[B_\alpha \mid A] \geq \alpha/\varepsilon$.

Splitting Lemma – Proof

(*i*) we argue by contradiction, using the notation $\bar{B}$ for the complement of B in $X \times Y$. Assume that $\Pr[B_\alpha] < \alpha$. Then,

$$\varepsilon \leq \Pr[B] \cdot \Pr[A \mid B] + \Pr[\bar{B}] \cdot \Pr[A \mid \bar{B}] < \alpha \cdot 1 + 1 \cdot (\varepsilon - \alpha) = \varepsilon.$$

(*ii*) straightforward.

(*iii*) using Bayes' law:

$$\begin{aligned} \Pr[B \mid A] &= 1 - \Pr[\bar{B} \mid A] \\ &= 1 - \Pr[A \mid \bar{B}] \cdot \Pr[\bar{B}] / \Pr[A] \geq 1 - (\varepsilon - \alpha)/\varepsilon = \alpha/\varepsilon. \end{aligned}$$

Forking Lemma

[Pointcheval-Stern – Eurocrypt '96]

Theorem (The Forking Lemma)

Let $(\mathcal{K}, \mathcal{S}, \mathcal{V})$ be a digital signature scheme with security parameter k, with a signature as above, of the form (m, r, h, s), where $h = \mathcal{H}(m, r)$ and s depends on r and h only.
Let $\mathcal{A}$ be a probabilistic polynomial time Turing machine whose input only consists of public data and which can ask q_H queries to the random oracle, with $q_H > 0$.
We assume that, within the time bound T, $\mathcal{A}$ produces, with probability $\varepsilon \geq 7q_H/2^k$, a valid signature (m, r, h, s).
Then, within time $T' \leq 16 q_H T/\varepsilon$, and with probability $\varepsilon' \geq 1/9$, a replay of this machine outputs two valid signatures (m, r, h, s) and (m, r, h', s') such that $h \neq h'$.

Forking Lemma – Proof

- $\mathcal{A}$ is a PPTM with random tape ω.
- During the attack, $\mathcal{A}$ asks a polynomial number of queries to $\mathcal{H}$.
- We may assume that these questions are distinct:
  - $\mathcal{Q}_1, \ldots, \mathcal{Q}_{q_H}$ are the q_H distinct questions
  - and let $H = (h_1, \ldots, h_{q_H})$ be the list of the q_H answers of $\mathcal{H}$.

  Note: a random choice of $\mathcal{H}$ = a random choice of H.
- For a random choice of $(\omega, \mathcal{H})$, with probability ε, $\mathcal{A}$ outputs a valid signature (m, r, h, s).
- Since $\mathcal{H}$ is a random oracle, the probability for h to be equal to $\mathcal{H}(m, r)$ is less than $1/2^k$, unless it has been asked during the attack.

Accordingly, we define $Ind_{\mathcal{H}}(\omega)$ to be the index of this question:
$(m, r) = \mathcal{Q}_{Ind_{\mathcal{H}}(\omega)}$ $\quad$ ($Ind_{\mathcal{H}}(\omega) = \infty$ if the question is never asked).

Forking Lemma – Proof

We then define the sets

$$\begin{aligned} \mathcal{S} &= \{(\omega, \mathcal{H}) \mid \mathcal{A}^{\mathcal{H}}(\omega) \text{ succeeds } \& \ Ind_{\mathcal{H}}(\omega) \neq \infty\}, \\ \mathcal{S}_i &= \{(\omega, \mathcal{H}) \mid \mathcal{A}^{\mathcal{H}}(\omega) \text{ succeeds } \& \ Ind_{\mathcal{H}}(\omega) = i\} \quad i \in \{1, \ldots, q_H\}. \end{aligned}$$

Note: the set $\{\mathcal{S}_i\}$ is a partition of $\mathcal{S}$.

$$\nu = \Pr[\mathcal{S}] \geq \varepsilon - 1/2^k.$$

Since $\varepsilon \geq 7q_H/2^k \geq 7/2^k$, then

$$\nu \geq 6\varepsilon/7.$$

Forking Lemma – Proof

Let I be the set consisting of the most likely indices i,

$$I = \{i \mid \Pr[\mathcal{S}_i \mid \mathcal{S}] \geq 1/2q_H\}.$$

Lemma

$$\Pr[Ind_{\mathcal{H}}(\omega) \in I \mid \mathcal{S}] \geq \frac{1}{2}.$$

By definition of $\mathcal{S}_i$,

$$\Pr[Ind_{\mathcal{H}}(\omega) \in I \mid \mathcal{S}] = \sum_{i \in I} \Pr[\mathcal{S}_i \mid \mathcal{S}] = 1 - \sum_{i \notin I} \Pr[\mathcal{S}_i \mid \mathcal{S}].$$

Since the complement of I contains fewer than q_H elements,

$$\sum_{i \notin I} \Pr[\mathcal{S}_i \mid \mathcal{S}] \leq q_H \times 1/2q_H \leq 1/2.$$

Forking Lemma – Proof

- We run $2/\varepsilon$ times $\mathcal{A}$, with independent random ω and random $\mathcal{H}$. Since $\nu = \Pr[\mathcal{S}] \geq 6\varepsilon/7$, with probability greater than $1 - (1-\nu)^{2/\varepsilon} \geq 4/5$, we get at least one pair $(\omega, \mathcal{H})$ in $\mathcal{S}$.
- We apply the Splitting Lemma, with $\varepsilon = \nu/2q_h$ and $\alpha = \varepsilon/2$, for $i \in I$. We denote by $\mathcal{H}_{|i}$ the restriction of $\mathcal{H}$ to queries of index $< i$.
  Since $\Pr[\mathcal{S}_i] \geq \nu/2q_H$, there exists a subset Ω_i such that,

$$\forall(\omega,\mathcal{H}) \in \Omega_i, \quad \Pr_{\mathcal{H}'}[(\omega,\mathcal{H}') \in \mathcal{S}_i \,|\, \mathcal{H}'_{|i} = \mathcal{H}_{|i}] \geq \frac{\nu}{4q_H}$$

$$\Pr[\Omega_i \,|\, \mathcal{S}_i] \geq \frac{1}{2}.$$

Forking Lemma – Proof

Since all the subsets $\mathcal{S}_i$ are disjoint,

$$\begin{aligned}\Pr_{\omega,\mathcal{H}}[(\exists i \in I)\ (\omega,\mathcal{H}) \in \Omega_i \cap \mathcal{S}_i \,|\, \mathcal{S}] &= \Pr\left[\bigcup_{i\in I}(\Omega_i \cap \mathcal{S}_i) \,|\, \mathcal{S}\right] = \sum_{i \in I} \Pr[\Omega_i \cap \mathcal{S}_i \,|\, \mathcal{S}] \\ &= \sum_{i\in I} \Pr[\Omega_i \,|\, \mathcal{S}_i] \cdot \Pr[\mathcal{S}_i \,|\, \mathcal{S}] \geq \left(\sum_{i\in I} \Pr[\mathcal{S}_i \,|\, \mathcal{S}]\right)/2 \geq \frac{1}{4}.\end{aligned}$$

We let β denote the index $Ind_{\mathcal{H}}(\omega)$ of to the successful pair.
With prob. at least $1/4$, $\beta \in I$ and $(\omega,\mathcal{H}) \in \mathcal{S}_\beta \cap \Omega_\beta$.
With prob. greater than $4/5 \times 1/4 = 1/5$, the $2/\varepsilon$ attacks provided a successful pair $(\omega,\mathcal{H})$, with $\beta = Ind_{\mathcal{H}}(\omega) \in I$ and $(\omega,\mathcal{H}) \in \mathcal{S}_\beta$.

Forking Lemma – Proof

We know that $\Pr_{\mathcal{H}'}[(\omega,\mathcal{H}') \in \mathcal{S}_\beta \,|\, \mathcal{H}'_{|\beta} = \mathcal{H}_{|\beta}] \geq \nu/4q_H$. Then

$$\begin{aligned}&\Pr_{\mathcal{H}'}[(\omega,\mathcal{H}') \in \mathcal{S}_\beta \text{ and } h_\beta \neq h'_\beta \,|\, \mathcal{H}'_{|\beta} = \mathcal{H}_{|\beta}] \\ &\geq \Pr_{\mathcal{H}'}[(\omega,\mathcal{H}') \in \mathcal{S}_\beta \,|\, \mathcal{H}'_{|\beta} = \mathcal{H}_{|\beta}] - \Pr_{\mathcal{H}'}[h'_\beta = h_\beta] \geq \nu/4q_H - 1/2^k,\end{aligned}$$

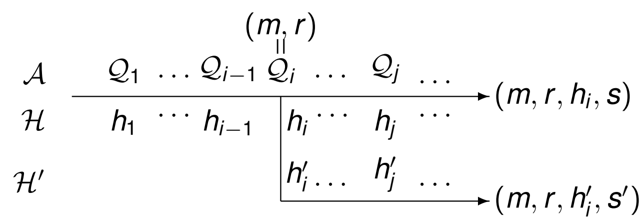
where $h_\beta = \mathcal{H}(\mathcal{Q}_\beta)$ and $h'_\beta = \mathcal{H}'(\mathcal{Q}_\beta)$.

Using the assumption that $\varepsilon \geq 7q_H/2^k$, the above prob. is $\geq \varepsilon/14q_H$.

We replay the attack $14q_H/\varepsilon$ times with a new random oracle $\mathcal{H}'$ such that $\mathcal{H}'_{|\beta} = \mathcal{H}_{|\beta}$, and get another success with probability greater than

$$1 - (1 - \varepsilon/14q_H)^{14q_H/\varepsilon} \geq 3/5.$$

Forking Lemma – Proof



Finally, after less than $2/\varepsilon + 14q_H/\varepsilon$ repetitions of the attack, with probability greater than $1/5 \times 3/5 \geq 1/9$, we have obtained two signatures (m, r, h, s) and (m, r, h', s'), both valid w.r.t. their specific random oracle $\mathcal{H}$ or $\mathcal{H}'$:

$$\mathcal{Q}_\beta = (m, r) \text{ and } h = \mathcal{H}(\mathcal{Q}_\beta) \neq \mathcal{H}'(\mathcal{Q}_\beta) = h'.$$

Chosen-Message Attacks

In order to answer signing queries, one simply uses the simulator of the zero-knowledge proof: (r, h, s), and we set $\mathcal{H}(m, r) \leftarrow h$.
The random oracle programming may fail, but with negligible probability.

Outline

Conclusion

Two generic methodologies for signatures

- hash and invert
- the Forking Lemma

Both in the random-oracle model

- Cramer-Shoup: based on the flexible RSA problem
- Based on Pairings
- etc