Blockwise Reduction and Security Estimates

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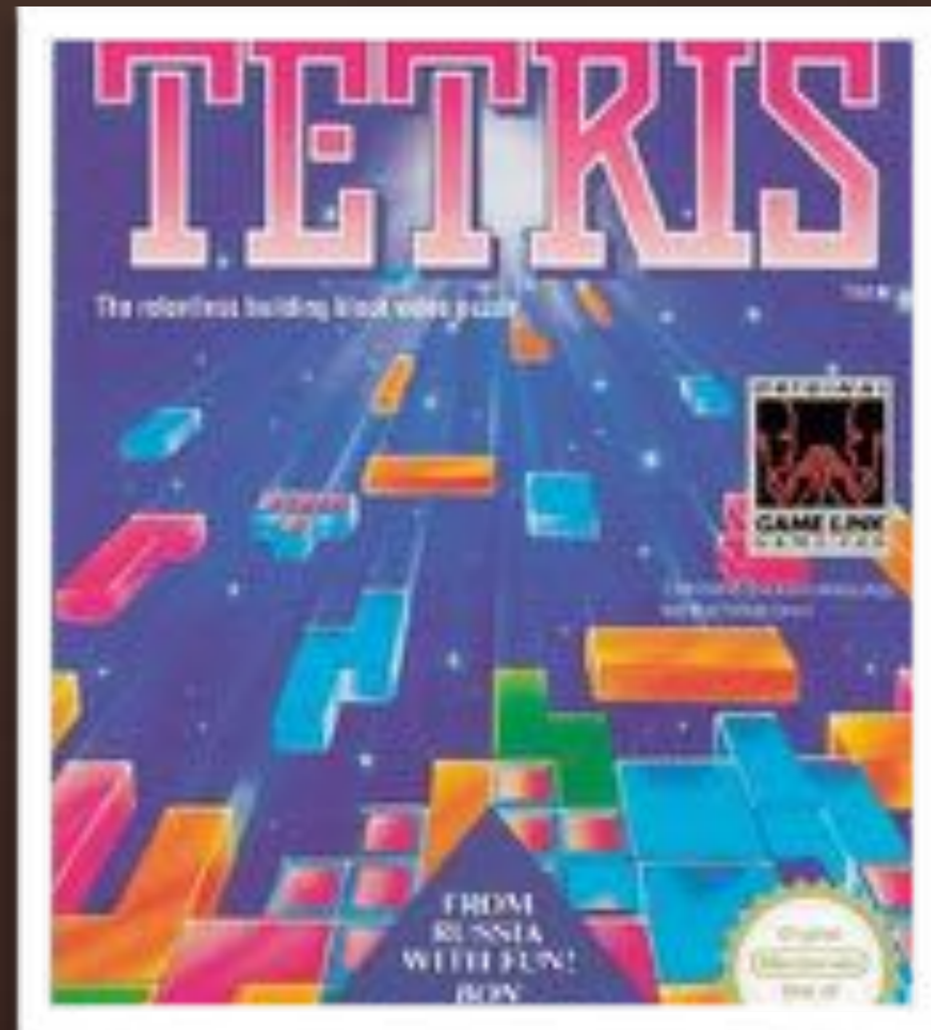
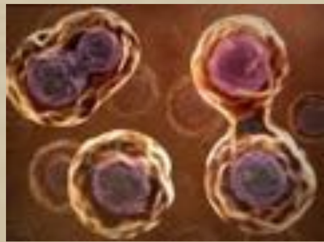
SVP Algorithms

- Poly-time approximation algorithms.
  - The LLL algorithm [LLL82].
  - Block generalizations by [Schnorr87,GHKN06,GamaN08,MiWa16,ALNS20].

- Exponential exact algorithms.
  - Poly-space enumeration [Pohst81,Kannan83,ScEu94]
  - Exp-space sieving [AKS01,MV10].

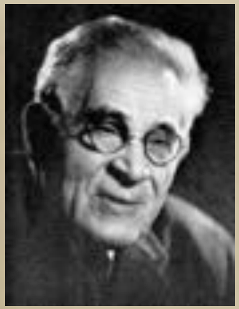
Blockwise Algorithms





Divide and Conquer

- LLL is based on a local reduction in dim 2.
- Blockwise algorithms find shorter vectors than LLL by using an exact SVP-subroutine in low dim k called the blocksize.
  - This subroutine can be done using $2^{O(k)}$ poly-time operations [AKS01,MV10,ADRS15], which is poly in d if k=log d.



Mathematical Analogy

- If we show the existence of very short lattice vectors in dim k, can we prove the existence of very short lattice vectors in dim d > k?
- [Mordell1944]'s inequality generalizes Hermite's inequality:

$$\sqrt{\gamma_d} \leq \sqrt{\gamma_k}^{(d-1)/(k-1)}$$

$$\lambda_1(L) \leq \sqrt{\gamma_k}^{(d-1)/(k-1)} \mathrm{vol}(L)^{1/d}$$

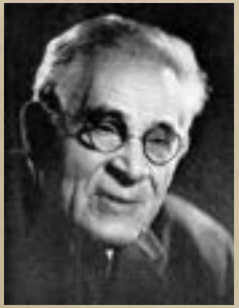
Approximation Algorithms for SVP

- Related to upper bounds on Hermite's constant, i.e. proving the existence of short lattice vectors.
- [LLL82] corresponds to [Hermite1850]'s inequality.

$$\lambda_1(L) \leq \sqrt{\gamma_2}^{d-1} \mathrm{vol}(L)^{1/d} = \left(\frac{4}{3}\right)^{(d-1)/4} \mathrm{vol}(L)^{1/d}$$

- Blockwise algorithms [Schnorr87, GHKN06, GN08,MW16,ALNS20] are related to [Mordell1944]'s inequality.

$$\lambda_1(L) \leq \sqrt{\gamma_k}^{(d-1)/(k-1)} \mathrm{vol}(L)^{1/d}$$



Mordell's Inequality (1944)

- Hermite's inequality is the k=2 particular case of Mordell's inequality:

$$\gamma_d \le \gamma_k^{(d-1)/(k-1)} \text{ if } 2 \le k \le d$$

- All known proofs of Mordell's inequality are based on duality.

Mordell's Proof

- For Hermite's inequality, the lattice rank was decreased by considering the quotient L mod b_1.
- Duality provides another way to reduce dimensions:
  - If L is a d-rank lattice and $v \in L^{\times}$ is non-zero, then $L \cap v^{\top}$ is a (d-1)-rank sublattice.

More Details

- See Aggarwal, Li, N, Stephens-Davidowitz: Slide Reduction, Revisited – Filling the Gaps in SVP Approximation. In CRYPTO 2020.
- https://arxiv.org/abs/1908.03724

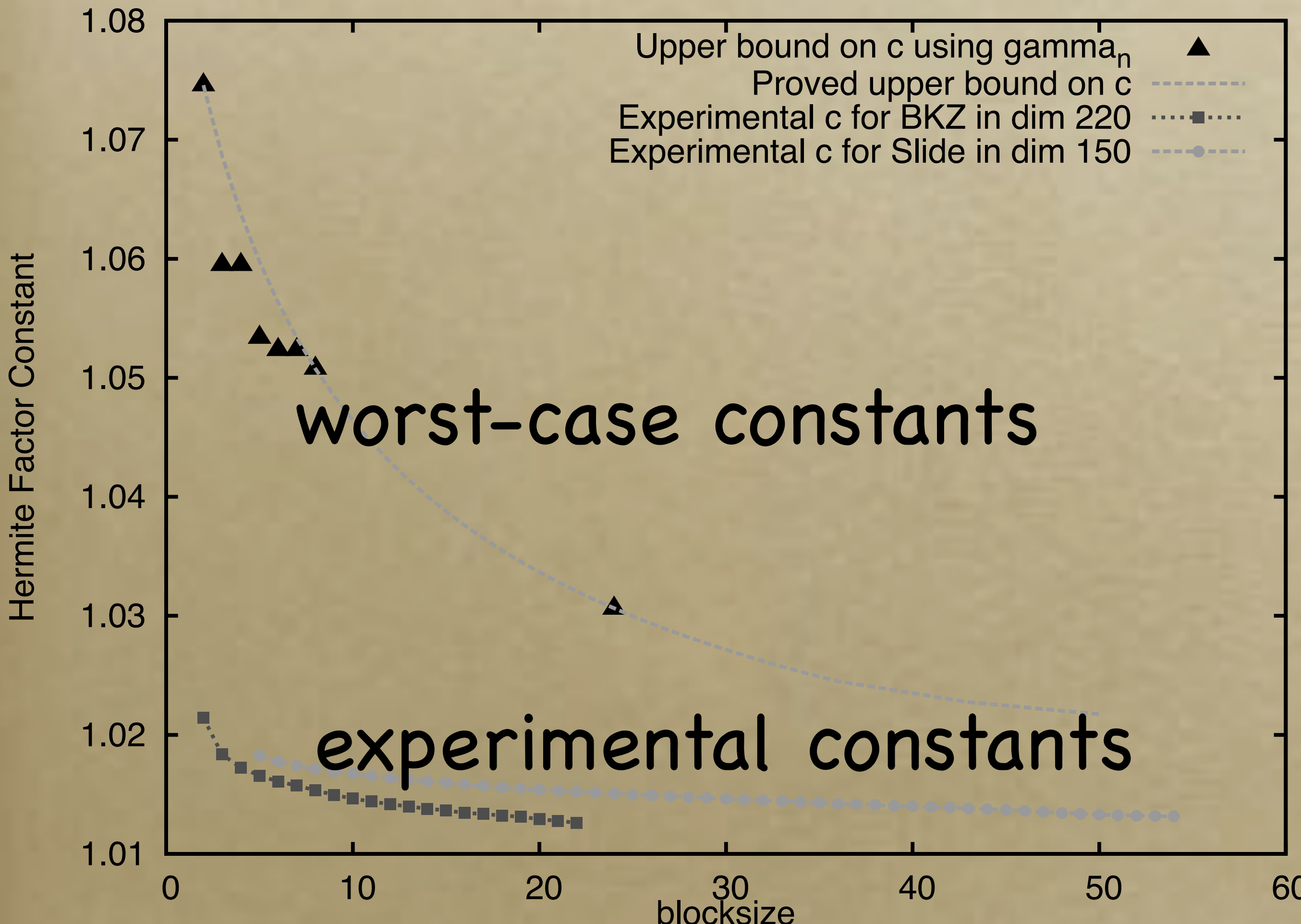
Random Lattices and Average-Case Behaviour

Average-Case Behaviour

- Experimentally [MiWa16], not many differences between blockwise algorithms, despite different theoretical bounds.
- The old BKZ algorithm [ScEu94,CN11] is widely used in lattice record computations.

BKZ Issues: Output Quality

- Theoretical worst-case bounds >> Practice.



c^d vs c'^d with $c > c'$

Same phenomenon as LLL

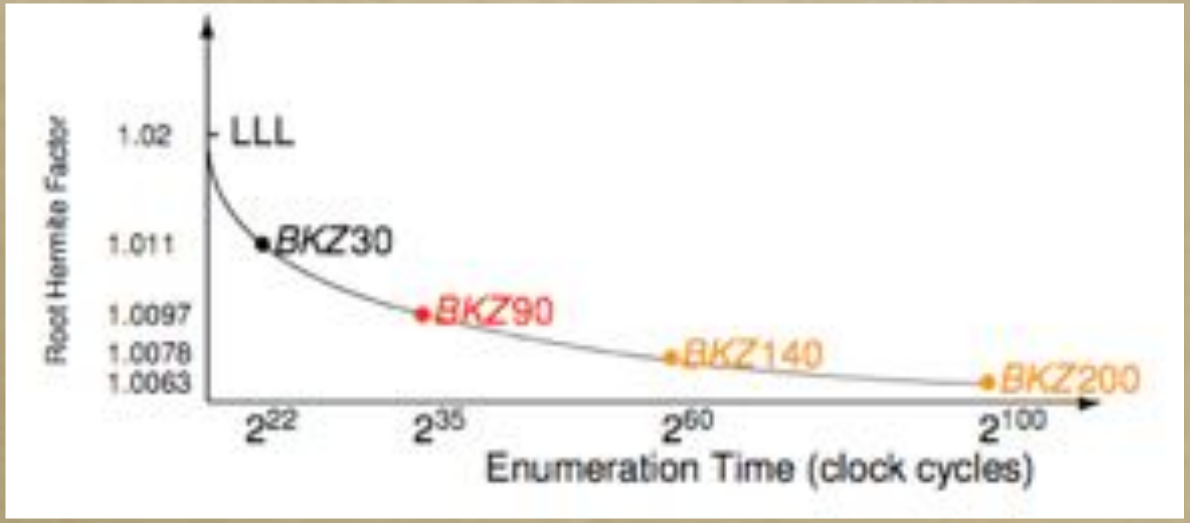
Predicting BKZ [CN11,BSW18]

- Predicts the approx behaviour of high-blocksize BKZ (k≥50), using an efficient simulation algorithm: the minimum of most k-rank blocks seems to behave like random lattices.

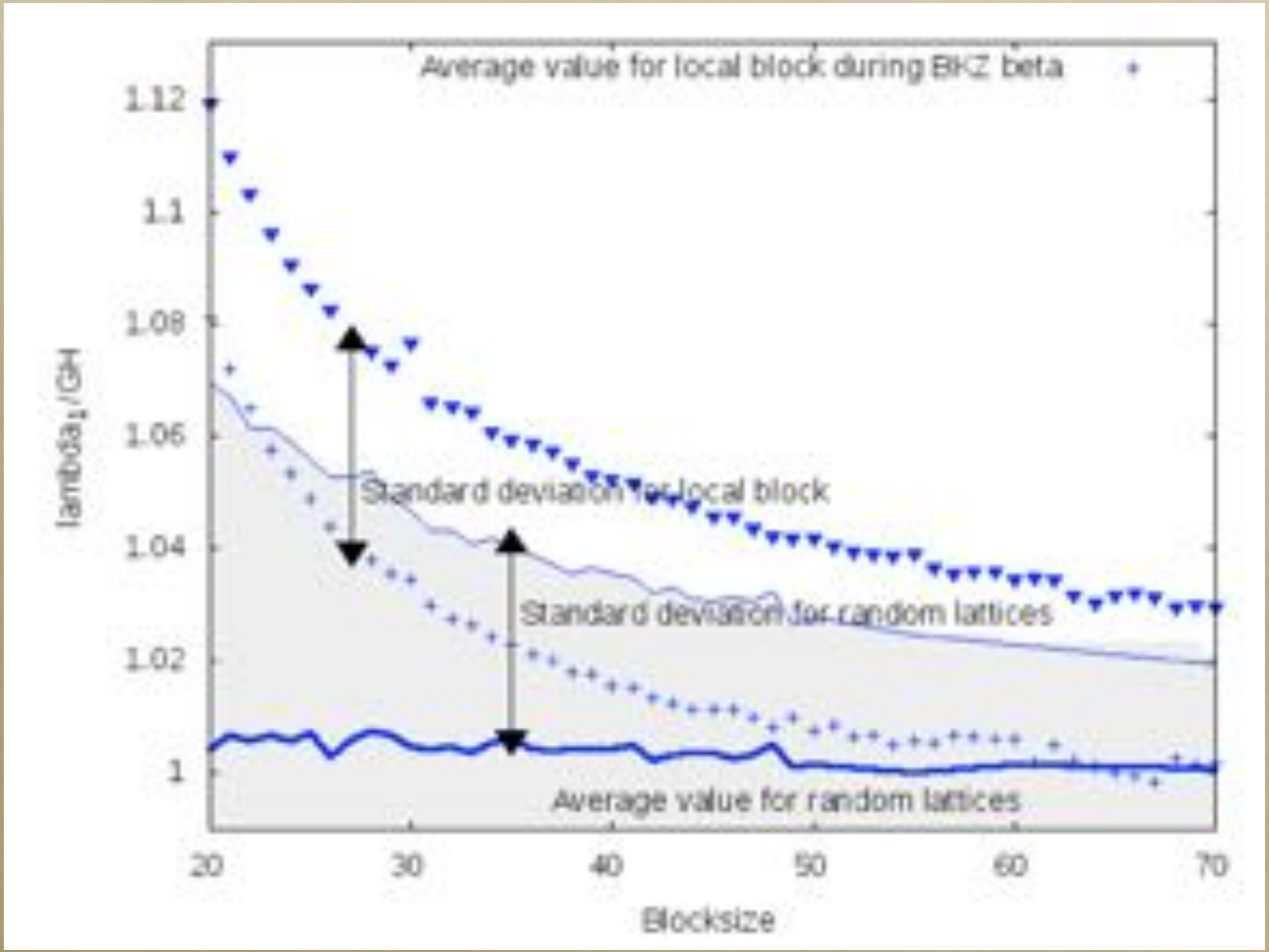
$$\sqrt{\gamma_k}^{(d-1)/(k-1)}$$

becomes roughly

$$\mathrm{GaussHeurist(k)}^{(d-1)/(k-1)}$$



Blocks vs Random Lattices



Security Estimates

Security Estimates

- Somewhat independent of security proofs
- Identify the best attack based on the state-of-the art
  - Find as many attacks as possible
  - Identify the ``best" one
  - Select keysizes/parameters accordingly

Selecting Keysizes

- [LenstraVerheul00] suggested to:
  - Model the performances of the best algorithm known, based on record benchmarks.
  - Add a security margin by speculating on:
    - Hardware improvements: Moore's law, etc.
    - Algorithmic improvements

Not So 

- A hardness assumption typically asks that no algorithm running in time $\leq T$ can solve a random instance with probability $\geq \varepsilon$.

- A complexity analysis typically says that an algorithm runs in time $\leq T'$ for all instances (of given size).

What is Needed

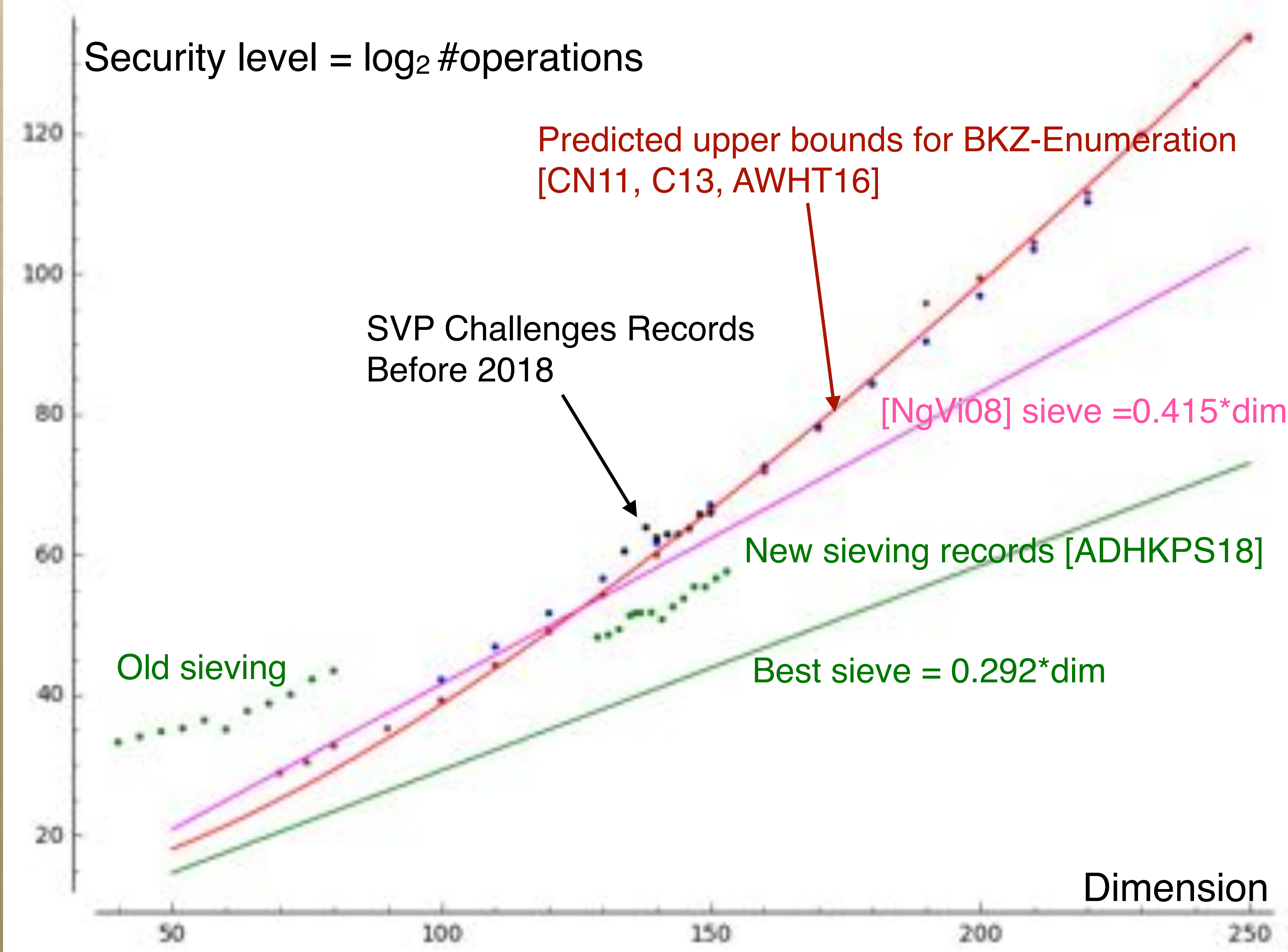
- A lower bound on the running time of the algorithm.
- Or more information on the distribution of the running time: expectation and variance.
- Typically not done in cryptanalysis.

NIST submissions

- Lattice-based submissions to NIST rely on a script to assess the security level: it does not fully reflect various uncertainties.

https://estimate-all-the-lwe-ntru-schemes.github.io/docs/

- The script says the best attack runs the SVP subroutine in some blocksize:
  - Estimate the cost of the SVP subroutine.
  - Estimate the number of calls.



Security level = log2 #operations
Predicted upper bounds for BKZ-Enumeration
[CN11, C13, AWHT16]
SVP Challenges Records
Before 2018
[NgVi08] sieve =0.415*dim
New sieving records [ADHKPS18]
Old sieving
Best sieve = 0.292*dim
Dimension
120
100
80
60
40
20
50
100
150
200
250

Open problem

- Efficient algorithms to approximate SVP within a polynomial factor, possibly quantum or subexponential.