Hermite's Inequality and the LLL Algorithm

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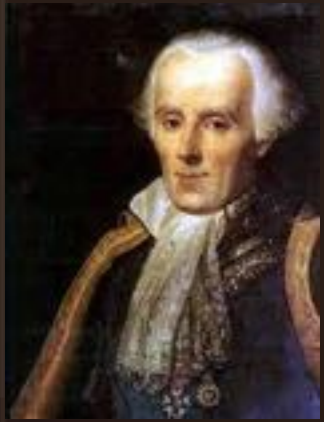
Inria — INVENTORS FOR THE DIGITAL WORLD

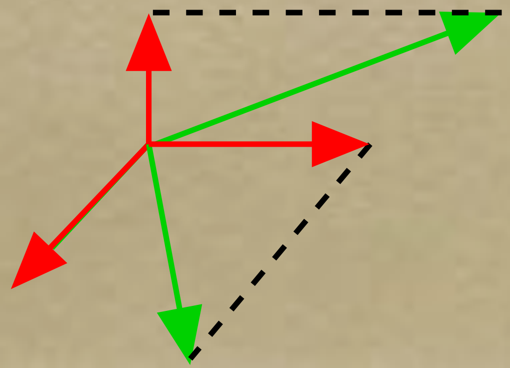
ENS — ÉCOLE NORMALE SUPÉRIEURE

Gram-Schmidt and Size-Reduction



Recall Gram-Schmidt

- Let $b_1,\ldots,b_n \in \mathbf{R}^m$.
- Its Gram-Schmidt Orthogonalization is $b_1^*,\ldots,b_n^* \in \mathbf{R}^m$ defined as:
  - $b_1^* = b_1$
  - For $2 \le i \le n$, b_i^* = projection of b_i over $\mathrm{span}(b_1,\ldots,b_{i-1})^\perp$



Linearly Independent Vectors

Formula

- Let $b_1,\ldots,b_n \in \mathbf{R}^m$ be linearly independent.
- Then all $b_j^* \neq 0$.
  - For $1 \leq j < i \leq n$, let $\mu_{i,j} = \dfrac{\langle \vec{b}_i, \vec{b}_j^* \rangle}{\|\vec{b}_j^*\|^2}$.
  - Then:

$$\vec{b}_1^\star = \vec{b}_1 \qquad \vec{b}_i^\star = \vec{b}_i - \sum_{j=1}^{i-1} \mu_{i,j} \vec{b}_j^\star$$

Induction Formulas

$$\|\vec{b}_i^*\|^2 = \|\vec{b}_i\|^2 - \sum_{j=1}^{i-1} \mu_{i,j}^2 \|\vec{b}_j^*\|^2$$

$$\mu_{i,j} = \frac{\langle \vec{b}_i, \vec{b}_j \rangle - \sum_{k=1}^{j-1} \mu_{j,k}\mu_{i,k} \|\vec{b}_k^*\|^2}{\|\vec{b}_j^*\|^2}$$

- This gives an algorithm, but not necessarily efficient: we want cheap operations on reasonably-sized numbers.

Efficient Computations

- We only deal with integers, so assume that $b_1,\ldots,b_n \in \mathbf{Z}^m$ and let $M=\max_{1\leq i\leq n}\|b_i\|$.
- Define the following integers:
  - $d_0=1$
  - $d_i=\mathrm{Gram}(b_1,\ldots,b_i) = \|b_1^*\|^2 \times \ldots \times \|b_i^*\|^2$ for $1\leq i\leq m$. Thus: $1\leq d_i\leq M^{2i}$
- Then $\mu_{i,j}$, $\|b_i^*\|^2 \in \mathbf{Q}$ and $b_j^* \in \mathbf{Q}^m$

Integral Gram-Schmidt

- Lemma: Let $b_1,\ldots,b_n \in \mathbf{Z}^m$ be linearly independent. Then for all $1 \le j < i \le n$:
  - $d_{i-1} b_i^* \in L(b_1,\ldots,b_i) \subseteq \mathbf{Z}^m$ with $\|d_{i-1} b_i^*\| \le M^{2i-1}$
  - $d_j \mu_{i,j} \in \mathbf{Z}$ with $|d_j \mu_{i,j}| \le M^{2j}$

Proof

- $B = \mu B^*$ for some lower-triangular matrix μ with unit diagonal: $B^* = \nu B$ where $\nu = \mu^{-1}$ is lower-triangular with unit diagonal.

$$\vec{b}_i^* = \vec{b}_i + \sum_{j=1}^{i-1} \nu_{i,j} \vec{b}_j$$

$$\Rightarrow \langle \vec{b}_i, \vec{b}_k \rangle = - \sum_{j=1}^{i-1} \nu_{i,j} \langle \vec{b}_j, \vec{b}_k \rangle, \text{if } k < i$$

- Thus, $\mathrm{Gram}(b_1,\ldots,b_{i-1})\, \nu_{i,j} \in \mathbf{Z}$ therefore $d_{i-1} b_i^* \in \mathbf{Z}^m$

Alternative Proof by Duality

- Let $L=L(b_1,\dots,b_i)$ and denote by $L^\times$ its dual lattice. Then $[L^\times:L]=\mathrm{covol}(L)^2=d_i$.
  - Note that: $b_i^*/||b_i^*||^2 \in L^\times$
  - Therefore $[L^\times:L]b_i^*/||b_i^*||^2 \in L$,

    i.e. $d_{i-1}b_i^* \in L(b_1,\dots,b_i) \subseteq \mathbf{Z}^m$.

Gram-Schmidt Algorithm

- Induction formulas can be rewritten with integers, giving an efficient algorithm.
- Let $\lambda_{i,j} = d_j \mu_{i,j} \in \mathbf{Z}$.

$$d_i = d_{i-1} \|\vec{b}_i\|^2 - \sum_{j=1}^{i-1} \frac{\lambda_{i,j}^2}{d_j d_{j-1}}$$

$$\lambda_{i,j} = d_{j-1} \langle \vec{b}_i, \vec{b}_j \rangle - \sum_{k=1}^{j-1} \frac{d_{j-1} \lambda_{j,k} \lambda_{i,k}}{d_k d_{k-1}}$$

- Could also derive b_i^*, but usually not needed

Recap

- If $b_1,\ldots,b_n \in \mathbf{Z}^m$ are linearly independent, we can compute efficiently all the integers $d_i = \mathrm{Gram}(b_1,\ldots,b_i) = \|b_1^*\|^2 \times \ldots \times \|b_i^*\|^2$ and $\lambda_{i,j} = d_j \mu_{i,j} = d_j \langle b_i, b_j^* \rangle / \|b_j^*\|^2$.

$$\vec{b}_i^* = \vec{b}_i - \sum_{j=1}^{i-1} \frac{\lambda_{i,j}}{d_j} \vec{b}_j^*$$

$$\|\vec{b}_i^*\|^2 = \frac{d_i}{d_{i-1}}$$

Application: Lattice Membership

- Let $b_1,\ldots,b_n \in \mathbf{Z}^m$ be linearly independent: let $L=L(b_1,\ldots,b_n)$.
- Given $t \in \mathbf{Z}^m$, decide if $t \in L$, and if so, find its integer coefficients in the decomposition $t=x_1b_1+\ldots+x_nb_n$.

Lattice Membership

- Let $b_1,\dots,b_n \in \mathbf{Z}^m$ be linearly independent.
- Assume that $t = x_1 b_1 + \dots + x_n b_n$.
- Then $\langle t, b_n^* \rangle = x_n \langle b_n, b_n^* \rangle = x_n \|b_n^*\|^2$
- Letting $b_{n+1} = t$, then $x_n = \mu_{n+1,n}$:
  - Derive x_n from Gram-Schmidt over $(b_1, \dots, b_n, t)$,
  - Repeat with $t - x_n b_n$ and $L(b_1,\dots,b_{n-1})$, etc.

Lattice Membership

- Let $b_1,\ldots,b_n \in \mathbf{Z}^m$ be linearly independent.
- Assume that $t=x_1b_1+\ldots+x_nb_n$.
- Then we can find efficiently x_n, $x_{n-1},\ldots$ $x_1 \in \mathbf{Z}$ using Gram-Schmidt.
- By checking if $t=x_1b_1+\ldots+x_nb_n$, we can decide if $t \in L$.
- Hence: we can decide lattice membership efficiently.

Application: Size-reduction

- Let $b_1,\ldots,b_d \in \mathbf{Z}^m$ be linearly independent.
- $B=(b_1,\ldots,b_d)$ is size-reduced if all $|\mu_{i,j}| \leq \frac{1}{2}$
- Th: There is an efficient algorithm to size-reduce B, without changing the Gram-Schmidt vectors.

Visualizing Size-reduction

- If we take an appropriate orthonormal basis, the matrix of the lattice basis becomes triangular.

$$\begin{pmatrix} \|\vec{b}_1^*\| & 0 & 0 & \dots & 0 \\ \mu_{2,1}\|\vec{b}_1^*\| & \|\vec{b}_2^*\| & 0 & \dots & 0 \\ \mu_{3,1}\|\vec{b}_1^*\| & \mu_{3,2}\|\vec{b}_2^*\| & \|\vec{b}_3^*\| & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \mu_{d,1}\|\vec{b}_1^*\| & \mu_{d,2}\|\vec{b}_2^*\| & \dots & \mu_{d,d-1}\|\vec{b}_{d-1}^*\| & \|\vec{b}_d^*\| \end{pmatrix}$$

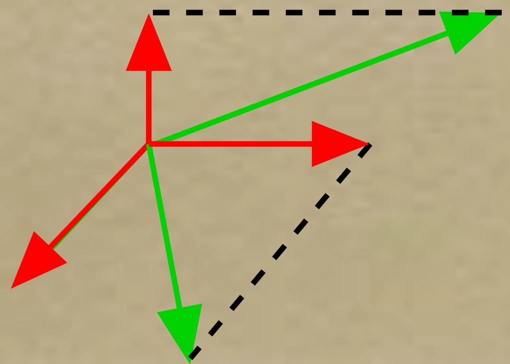
Size-reduction Algorithm

- For $i = 2$ to d
  - For $j = i-1$ downto 1
    - Size-reduce b_i with respect to b_j: make $|\mu_{i,j}| \leq 1/2$ by $b_i := b_i - \mathrm{round}(\mu_{i,j}) b_j$
    - Update all $\mu_{i,j'}$ for $j' \leq j$.
- The translation does not affect the previous $\mu_{i',j'}$ where $i' < i$, or $i'=i$ and $j'>j$.

Linearly Dependent Vectors

Reminder

- Let $b_1,\ldots,b_n \in \mathbf{R}^m$.
- Its Gram-Schmidt Orthogonalization is $b_1^*,\ldots,b_n^* \in \mathbf{R}^m$ defined as:
  - $b_1^* = b_1$
  - For $2 \leq i \leq n$, b_i^* = projection of b_i over $\mathrm{span}(b_1,\ldots,b_{i-1})^\perp$



Generalization

- Let $b_1,\ldots,b_n \in \mathbf{R}^m$ possibly linearly dependent.
- Then not all $b_j^* \neq 0$.
  - For $1 \le j < i \le n$, let $\mu_{i,j} = \frac{\langle \vec{b}_i, \vec{b}_j^* \rangle}{\|\vec{b}_j^*\|^2}$ if $b_j^* \neq 0$, and 0 otherwise.
  - Then we still have:

$$\vec{b}_1^\star = \vec{b}_1 \qquad \vec{b}_i^\star = \vec{b}_i - \sum_{j=1}^{i-1} \mu_{i,j} \vec{b}_j^\star$$

Induction Formulas

$$\|\vec{b}_i^*\|^2 = \|\vec{b}_i\|^2 - \sum_{j=1}^{i-1} \mu_{i,j}^2 \|\vec{b}_j^*\|^2$$

$$\mu_{i,j} = \frac{\langle \vec{b}_i, \vec{b}_j \rangle - \sum_{k=1}^{j-1} \mu_{j,k}\mu_{i,k} \|\vec{b}_k^*\|^2}{\|\vec{b}_j^*\|^2}$$

- If $b_j^*=0$, then we let $\mu_{i,j}=0$.

Efficient Computations

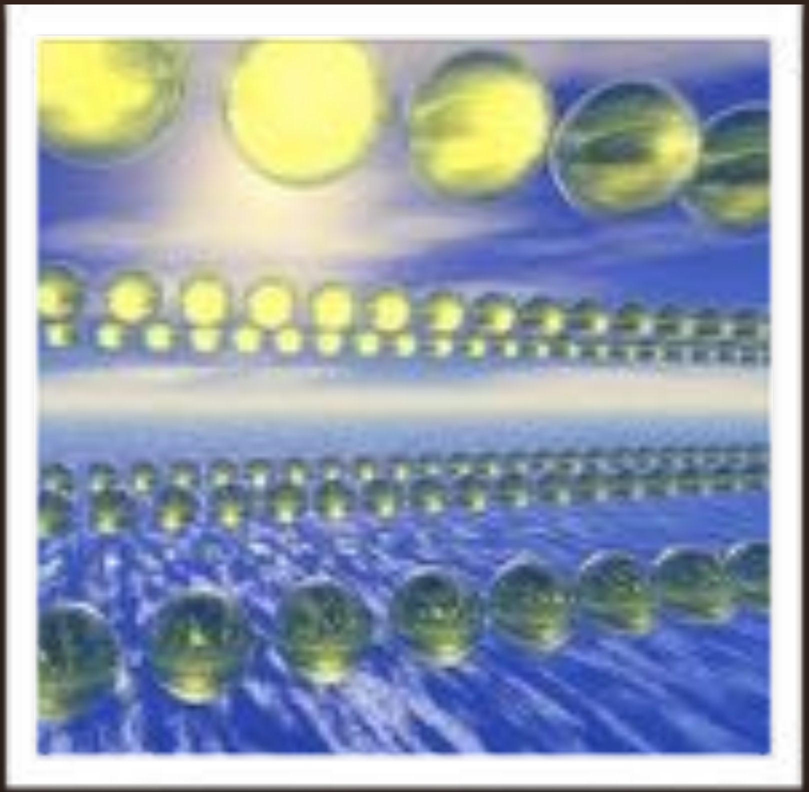
- We only deal with integers, so assume that $b_1,\ldots,b_n \in \mathbf{Z}^m$ and let $||B||=\max_{1\leq i\leq n}||b_i||$.
- Define the following integers:
  - $d_0=1$
  - $d_i=\mathrm{Gram}(b_j)$ over $1\leq j\leq i$, $b_j^* \neq 0$ = $\prod_{1\leq j\leq i}$non-zero $||b_j^*||^2$. Still: $1\leq d_i\leq ||B||^{2i}$
- Then $\mu_{i,j}$, $||b_i^*||^2 \in \mathbf{Q}$ and $b_j^* \in \mathbf{Q}^m$

Generalized Integral Gram-Schmidt

- Lemma: Let $b_1,\dots,b_n \in \mathbf{Z}^m$. Then for all $1 \le j < i \le n$:
  - $d_{i-1} b_i^* \in L(b_1,\dots,b_i) \subseteq \mathbf{Z}^m$ with $\|d_{i-1} b_i^*\| \le M^{2i-1}$
  - $d_j \mu_{i,j} \in \mathbf{Z}$ with $|d_j \mu_{i,j}| \le M^{2j}$

Recap

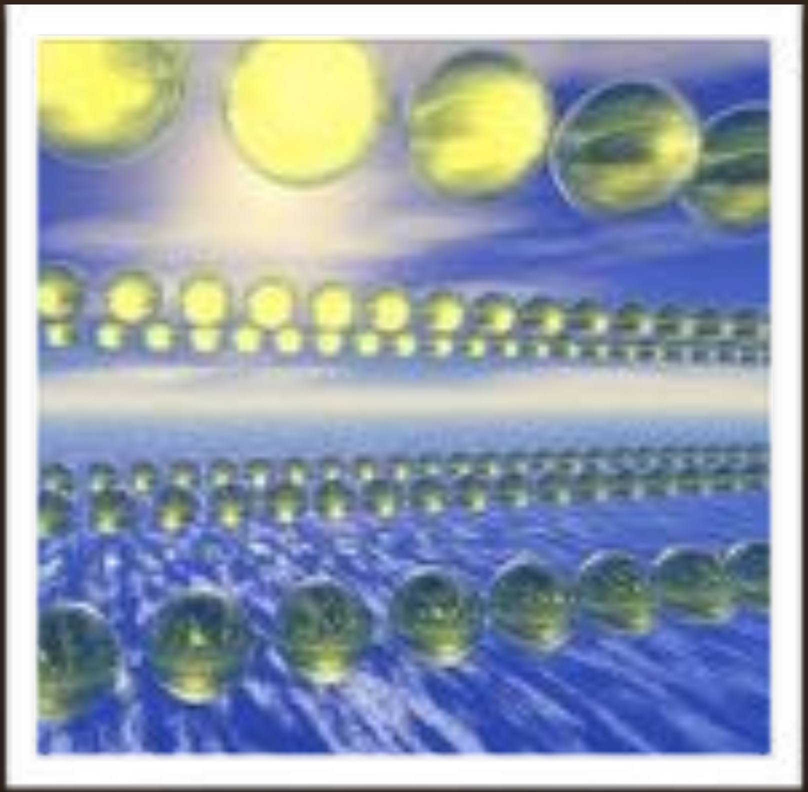
- If $b_1,\ldots,b_n \in \mathbf{Z}^m$, we can compute efficiently (polynomial time) all the generalized integers d_i and $\lambda_{i,j}=d_j\mu_{i,j}$ and decide which b_i^* are zero.



A Non-Trivial Lattice Algorithm

Euclid with Vectors

- If $b_1,\ldots,b_n \in \mathbf{Z}^m$, $L(b_1,\ldots,b_n)$ is a lattice: Find an efficient algorithm to find a lattice basis.
  - If n=2 and m=1, this is exactly the gcd problem, so we are trying to generalize Euclid's algorithm.



Overview on Lattice Algorithms

Insight

- The most classical problem is to prove the existence of short lattice vectors.
- All known upper bounds on Hermite's constant have an algorithmic analogue:
  - Hermite's inequality: the LLL algorithm.
  - Mordell's inequality: Blockwise generalizations of LLL.
  - Mordell's proof of Minkowski's inequality: worst-case to average-case reductions for SIS and sieve algorithms [BJN14,ADRS15]

SVP Algorithms

- Poly-time approximation algorithms.

  - The LLL algorithm [1982].
  - Block generalizations by [Schnorr1987], [GHKN06], [GamaN08], [MiWa16].
- Exponential exact algorithms.

  - Poly-space enumeration [Pohst1981,Kannan1983,ScEu1994]
  - Exp-space sieving [AKS01,MV10,ADRS15].

Hermite's Inequality and LLL

Hermite's Inequality

- Hermite proved in 1850:

$$\gamma_d \leq \gamma_2^{d-1} = \left(\frac{4}{3}\right)^{(d-1)/2}$$

- [LLL82] finds in polynomial time a non-zero lattice vector of norm $\leq (4/3+\varepsilon)^{(d-1)/4} \mathrm{vol}(L)^{1/d}$.

  It is an algorithmic version of Hermite's inequality.

Proof of Hermite's Inequality

- Induction over d: obvious for d=1.
- Let b_1 be a shortest vector of L, and π the projection over $b_1^{\perp}$.
- Let $\pi(b_2)$ be a shortest vector of $\pi(L)$.
- We can make sure by lifting that: $\|b_2\|^2 \leq \|\pi(b_2)\|^2 + \|b_1\|^2/4$ (size-reduction)
- On the other hand, $\|b_1\| \leq \|b_2\|$ and $vol(\pi(L)) = vol(L)/\|b_1\|$.

Hermite's Reduction

- Hermite proved the existence of bases such that:

$$|\mu_{i,j}| \le \frac{1}{2} \qquad \text{and} \qquad \frac{\|\vec{b}_i^\star\|^2}{\|\vec{b}_{i+1}^\star\|^2} \le \frac{4}{3}$$

- Such bases approximate SVP to an exp factor:

$$\|\vec{b}_1\| \le \left[(4/3)^{1/4}\right]^{d-1} \mathrm{vol}(L)^{1/d} \qquad \gamma_d \le (4/3)^{(d-1)/2}$$

$$\|\vec{b}_i\| \le \left[(4/3)^{1/2}\right]^{d-1} \lambda_i(L)$$

Graphically

- Condition 1 is over off-diagonal coeffs: size-reduction.
- Condition 2 is over diagonal coeffs.

$$
\left(\begin{array}{c|c|c|c|c}
\|\vec{b}_1^*\| & 0 & 0 & \dots & 0 \\
\mu_{2,1}\|\vec{b}_1^*\| & \|\vec{b}_2^*\| & 0 & \dots & 0 \\
\mu_{3,1}\|\vec{b}_1^*\| & \mu_{3,2}\|\vec{b}_2^*\| & \|\vec{b}_3^*\| & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\mu_{d,1}\|\vec{b}_1^*\| & \mu_{d,2}\|\vec{b}_2^*\| & \dots & \mu_{d,d-1}\|\vec{b}_{d-1}^*\| & \|\vec{b}_d^*\|
\end{array}\right)
$$

Question

- Is the proof constructive?
- Does it build a non-zero lattice vector satisfying Hermite's inequality:

$$\|\vec{b}_1\| \leq \left(\frac{4}{3}\right)^{(d-1)/4} \mathrm{vol}(L)^{1/d}$$

An Algorithmic Proof

- Let b_1 be a primitive vector of L, and π the projection over $b_1^{\perp}$.
- Find recursively $\pi(b_2) \in \pi(L)$ satisfying Hermite's inequality.
- Size-reduce so that $\|b_2\|^2 \leq \|\pi(b_2)\|^2 + \|b_1\|^2/4$
- If $\|b_2\| < \|b_1\|$, swap(b_1, b_2) and restart, otherwise stop.

An Algorithmic Proof

- This algorithm will terminate and output a non-zero lattice vector satisfying Hermite's inequality:

$$\|\vec{b}_1\| \leq \left(\frac{4}{3}\right)^{(d-1)/4} \mathrm{vol}(L)^{1/d}$$

- But it may not be efficient: LLL does better by strengthening the test $\|b_2\| < \|b_1\|$.

Computing Hermite reduction

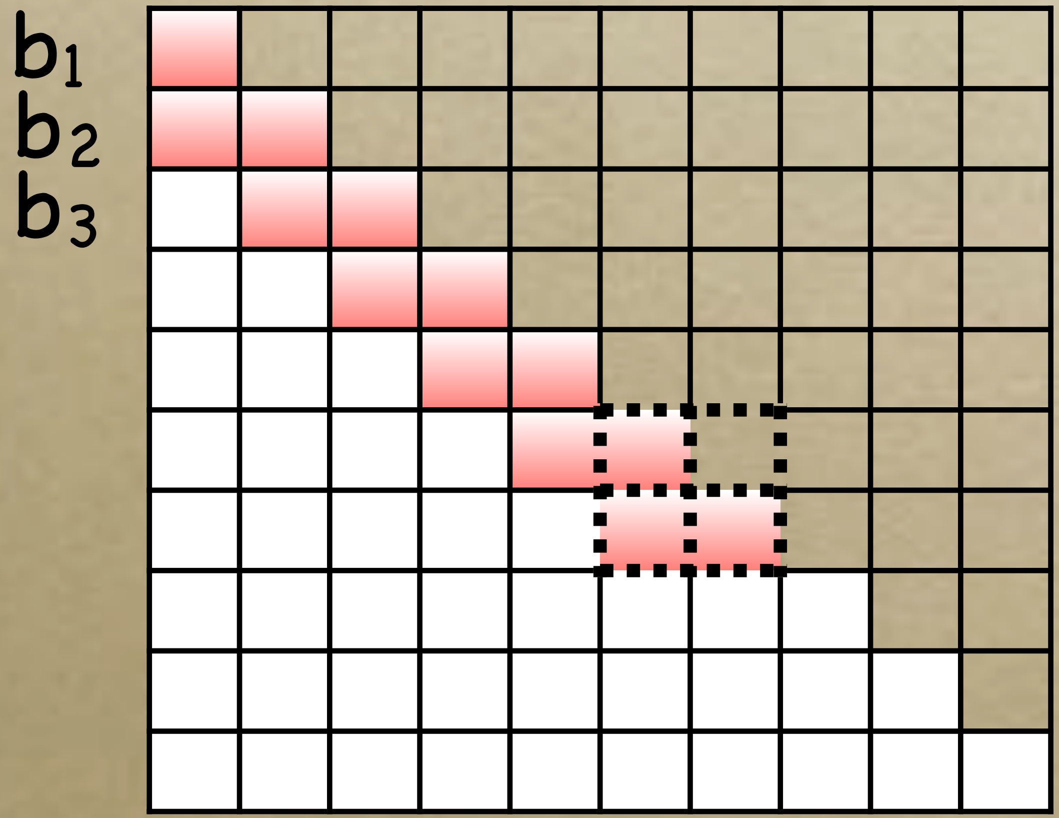
- Hermite proved the existence of bases s.t.:

$$|\mu_{i,j}| \leq \frac{1}{2} \quad \text{and} \quad \frac{\|\vec{b}_i^\star\|^2}{\|\vec{b}_{i+1}^\star\|^2} \leq \frac{4}{3}$$

- By relaxing the 4/3, [LLL1982] obtained a provably polynomial-time algorithm.

How LLL Works

- LLL is an elegant divide-and-conquer based on 2-dim reduction.



Lenstra-Lenstra-Lovász

$$\vec{b}_i^\star = \vec{b}_i - \sum_{j=1}^{i-1} \mu_{i,j} \vec{b}_j^\star \qquad \text{where } \mu_{i,j} = \frac{\langle \vec{b}_i, \vec{b}_j^\star \rangle}{\|\vec{b}_j^\star\|^2}$$

- A basis is LLL-reduced for ε >0 if and only if
  - it is size-reduced $|\mu_{i,j}| \leq \dfrac{1}{2}$
  - Lovász' conditions are satisfied

$$(1-\varepsilon)\|\vec{b}_{i-1}^\star\|^2 \leq \|\vec{b}_i^\star + \mu_{i,i-1}\vec{b}_{i-1}^\star\|^2$$

$$\Rightarrow \|\vec{b}_{i-1}^\star\|^2 \leq \left(\frac{4}{3} + \varepsilon'\right)\|\vec{b}_i^\star\|^2$$

Description of the LLL Algorithm

- While the basis is not LLL-reduced
  - Size-reduce the basis
  - If Lovász' condition does not hold for some pair $(i-1,i)$: swap b_{i-1} and b_i.

Recursive LLL

- Input: $(b_1,b_2,\ldots,b_d)$ basis of L and $\varepsilon > 0$.
- LLL-reduce $(\pi(b_2),\ldots,\pi(b_d))$ where π is the projection over $b_1^{\perp}$.
- Size-reduce so that $\|b_i\|^2 \leq \|\pi(b_i)\|^2 + \|b_1\|^2/4$
- If $\|b_2\| \leq (1-\varepsilon)\|b_1\|$, swap$(b_1, b_2)$ and restart, otherwise stop.

Evolution of Gram-Schmidt

- During LLL reduction:
  - $\text{Min}_i \ \|b^*_i\|$ never decreases.
  - $\text{Max}_i \ \|b^*_i\|$ never increases.
  - Each $\text{vol}(b_1,\ldots,b_i)$ never increases.
  - The only LLL operations that modify the b^*_i's are swaps.

Evolution of Gram-Schmidt

- We swap b_{i-1} and b_i whenever $(1-\varepsilon)\,||b^*_{i-1}||^2 > ||b^*_i + \mu_{i,i-1} b^*_{i-1}||^2$

- What happens to b^*_{i-1} and b^*_i?
  - $New(b^*_{i-1}) = b^*_i + \mu_{i,i-1} b^*_{i-1}$ has norm between $||b^*_i||$ and $\sqrt{(1-\varepsilon)}\,||b^*_{i-1}||$, hence $\geq\sqrt{(1-\varepsilon)}$ shorter.
  - $New(b^*_i)$ has norm between $||b^*_i||/\sqrt{(1-\varepsilon)}$ and $||b^*_{i-1}||$, hence $\geq 1/\sqrt{(1-\varepsilon)}$ longer.
  - $[new(||b^*_i||), new(||b^*_{i-1}||)] \subseteq [||b^*_i||, ||b^*_{i-1}||]$

Why LLL is polynomial

- Consider the quantity $P = \prod_{i=1}^{d} \|\vec{b}_i^*\|^{2(d-i+1)}$
- If the b_i's have integral coordinates, then P is a positive integer.
  - Size-reduction does not modify P.
  - But each swap of LLL makes P decrease by a factor <= 1- ε
- This implies that the number of swaps is polynomially bounded.

Remarks

- We described a simple version of LLL, which is not optimized for implementation.
- We did not fully prove that LLL is polynomial time, because we did not pay attention to the size of all temporary variables.

Recap of LLL

- The LLL algorithm finds in polynomial time a basis such that:

$$|\mu_{i,j}| \leq \frac{1}{2} \qquad \text{and} \qquad \frac{\|\vec{b}_i^\star\|^2}{\|\vec{b}_{i+1}^\star\|^2} \leq \frac{4}{3} + \varepsilon$$

- Such bases approximate SVP to an exp factor:

$$\|\vec{b}_1\| \leq \left[(4/3+\varepsilon)^{1/4}\right]^{d-1} \mathrm{vol}(L)^{1/d}$$

$$\gamma_d \leq (4/3)^{(d-1)/2}$$

$$\|\vec{b}_i\| \leq \left[(4/3+\varepsilon)^{1/2}\right]^{d-1} \lambda_i(L)$$

Take Away

- Hermite's inequality and LLL are based on two key ideas:
  - Projection
  - Lifting projected vectors aka size-reduction.

1773

1850

1982

LLL in Practice

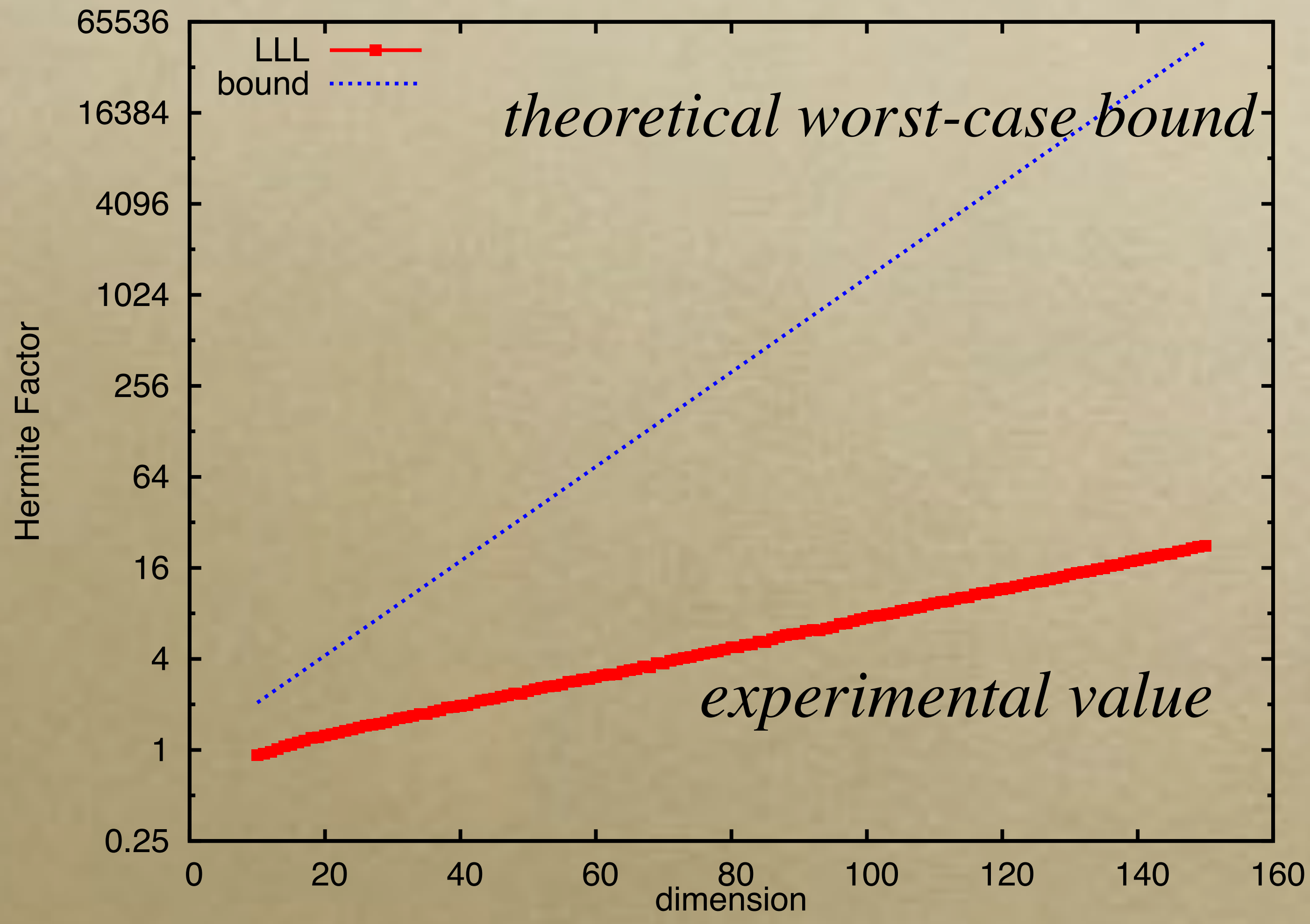
The Magic of LLL

- One of the main reasons behind the popularity of LLL is that it performs "much better" than what the worst-case bounds suggest, especially in low dimension.
- This is another example of worst-case vs. "average-case" and the difficulty of security estimates.

LLL: Theory vs Practice

- The approx factors $(4/3+\varepsilon)^{(d-1)/4}$ is tight in the worst case and for uniformly random LLL bases [KiVe16].
- Experimentally, $4/3+\varepsilon \approx 1.33$ can be replaced by a smaller constant ≈ 1.08, for any lattice, by randomizing the input basis.
- No good explanation for this phenomenon, and no known formula for the experimental constant $\approx$ 1.08.

Illustration



Log(Hermite Factor)

Random Bases

- There is no natural probability space over the infinite set of bases.
- Folklore: generate a « random » unimodular matrix and multiply by a fixed basis. But distribution not so good.
- Better method:
  - Generate say n+20 random long lattice points
  - Extract a basis, e.g. using LLL.

Random LLL

- Surprisingly, [KiVe16] showed that most LLL bases of a random lattice have a $\|b_1\|$ close to the worst case. Note: in fixed dimension, the number of LLL bases can be bounded, independently of the lattice.
- This means that LLL biases the output distribution: it is not the uniform distribution.

Open problems

- Take a random integer lattice L.
- Let B be the Hermite normal form of L, or a « random » basis from the discrete Gaussian distribution.
- Is is true that with overwhelming probability, after LLL-reducing B, $||b_1|| \leq c^{d-1} vol(L)^{1/d}$ for some $c<(4/3)^{1/4}$?
- Can we guess the distribution of $||b_1||$ and the running time?