HARD LATTICE PROBLEMS

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REMARK ON THE HNF (1851)

13.

Sur l'introduction des variables continues dans la théorie des nombres.

(Par Mr. *C. Hermite*, examinateur d'admission à l'école polytechnique, à Paris.)

La définition précédente peut être simplifiée, en observant que toute forme Φ a une équivalente, dans laquelle le système:

$$\begin{array}{cccccc} m_1 & p_1 & \ldots & r_1 & s_1 \\ m_2 & p_2 & \ldots & r_2 & s_2 \\ \vdots & \vdots & & \vdots & \vdots \\ m_n & p_n & \ldots & r_n & s_n \end{array}$$

dont le déterminant a p' valeurs Δ, est remplacé par le suivant:

$$\begin{array}{ccccc} \delta & g & h & \ldots & l \\ 0 & \delta_1 & h' & \ldots & l' \\ 0 & 0 & \delta_2 & \ldots & l'' \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & \delta_{n-1} \end{array}$$

Les nombres entiers, désignés par les lettres, $g, h, \ldots l$, sont positifs et vérifient tous les conditions

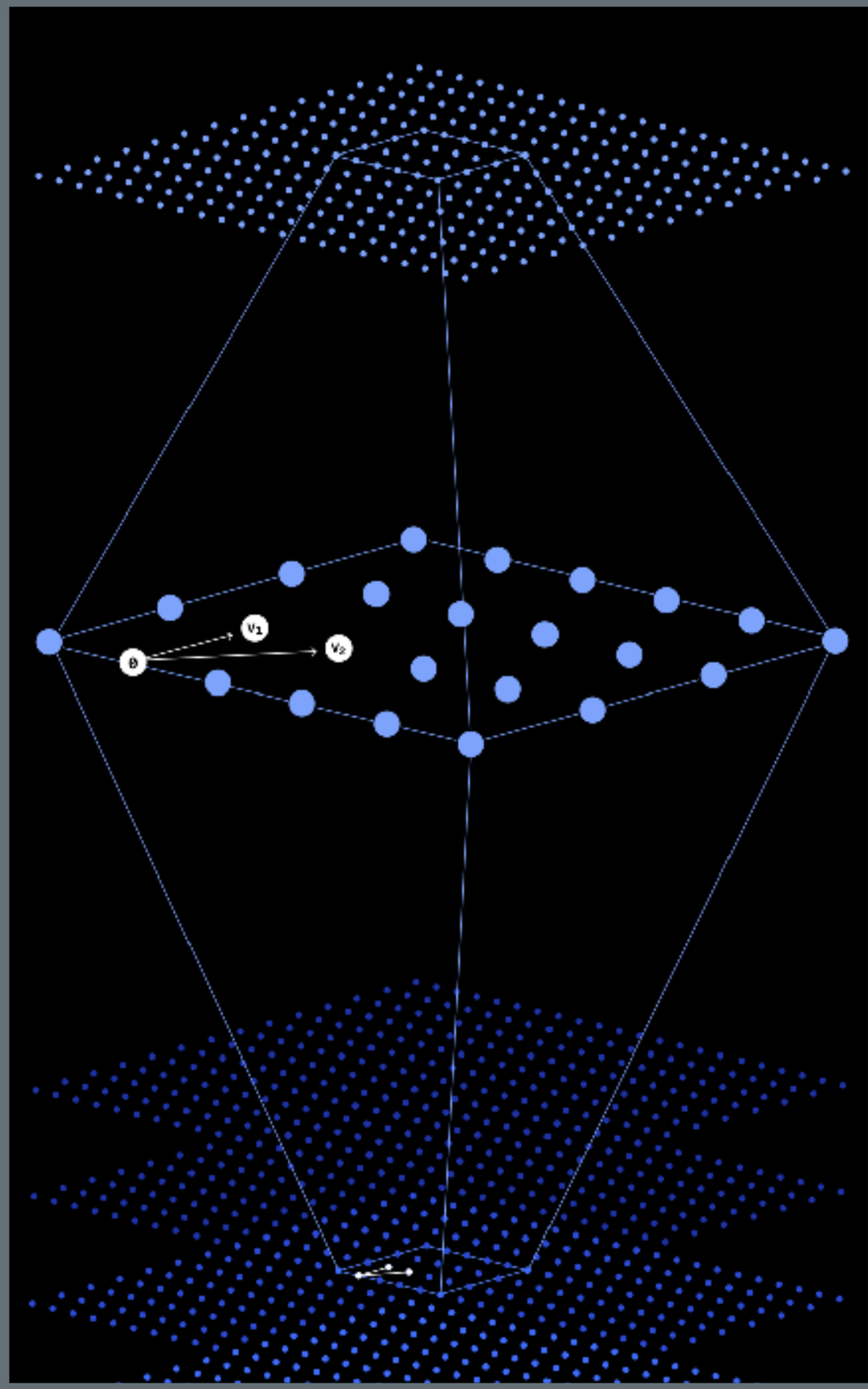
$$g < \delta_1, \quad h < \delta_2, \quad \ldots \quad l < \delta_{n-1},$$

et on a toujours

$$\delta . \delta_1 . \delta_2 \ldots . \delta_{n-1} = \Delta.$$

Ainsi pour chaque valeur de Δ, on voit qu'il n'existe jamais qu'un nombre fini d'expressions Φ, distinctes. Mais je m'occuperai tout d'abord des formes binaires, qui offrent dans des circonstances analytiques plus simples, l'application des mêmes principes.

LATTICE PROBLEMS

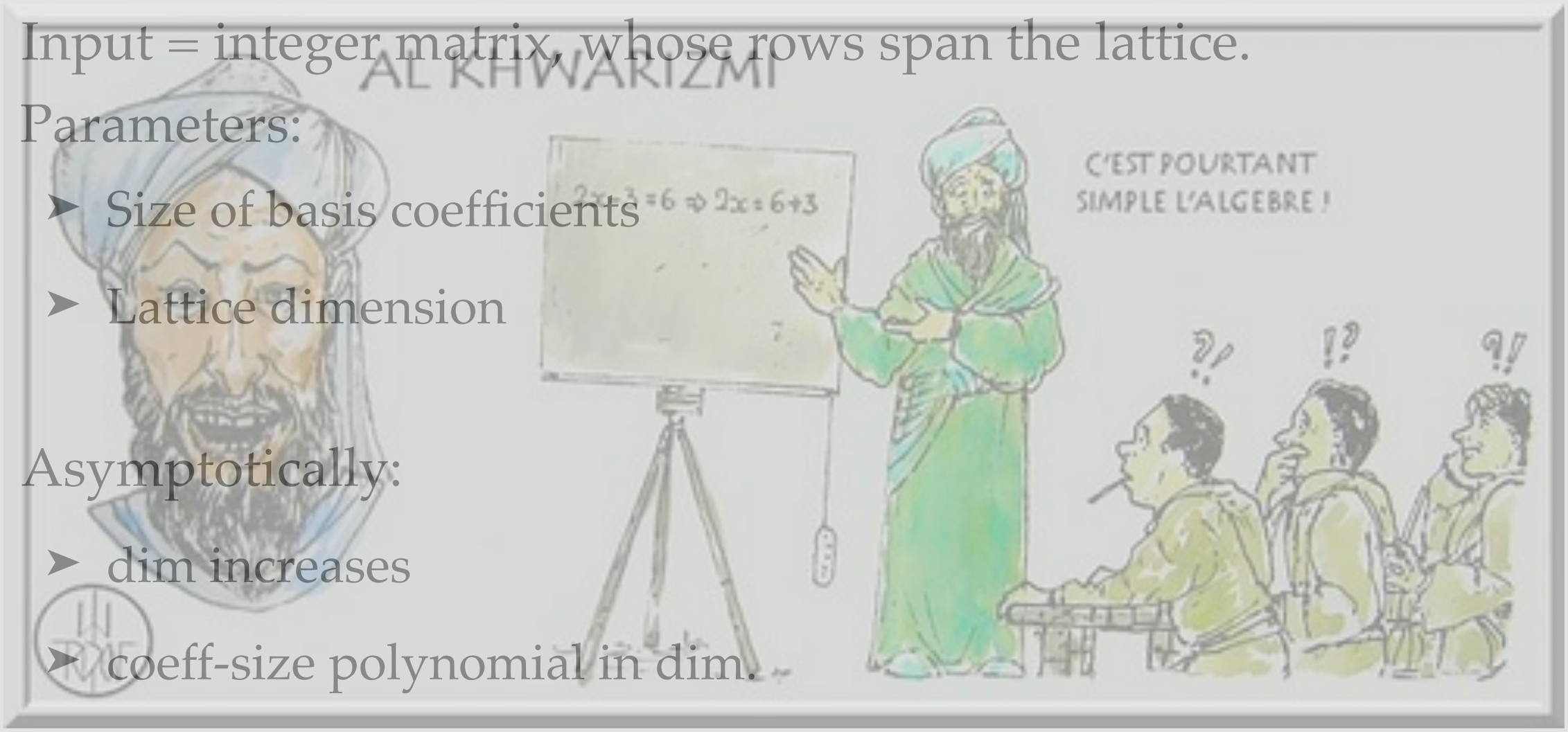


LATTICE ALGORITHMS

- Input = integer matrix, whose rows span the lattice.

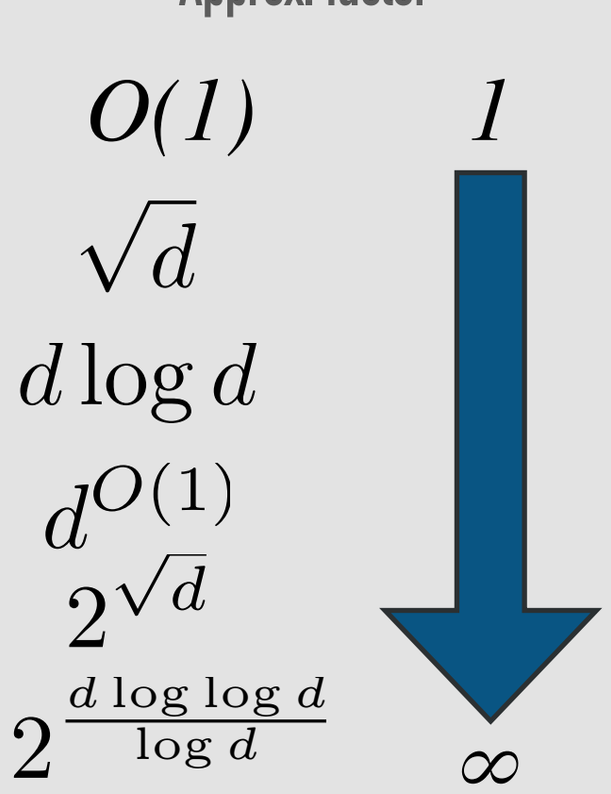
  Parameters:
  - Size of basis coefficients
  - Lattice dimension

- Asymptotically:
  - dim increases
  - coeff-size polynomial in dim.



HARD LATTICE PROBLEMS

- Since 1996, lattices are very trendy in classical and quantum complexity theory.
- Depending on the dimension d:
  - NP-hardness
  - non NP-hardness (NP∩co-NP)
  - worst-case / average-case reduction
  - cryptography
  - subexp-time algorithms
  - poly-time algorithms



HARD LATTICE PROBLEMS

- Input: a lattice L and an n-dim ball C.
- Output: decide if L∩C is non-trivial, and find a point when applicable. Easy if $L=\mathbf{Z}^n$.
- Two settings
  - ➤ Approx: L∩C has many points.

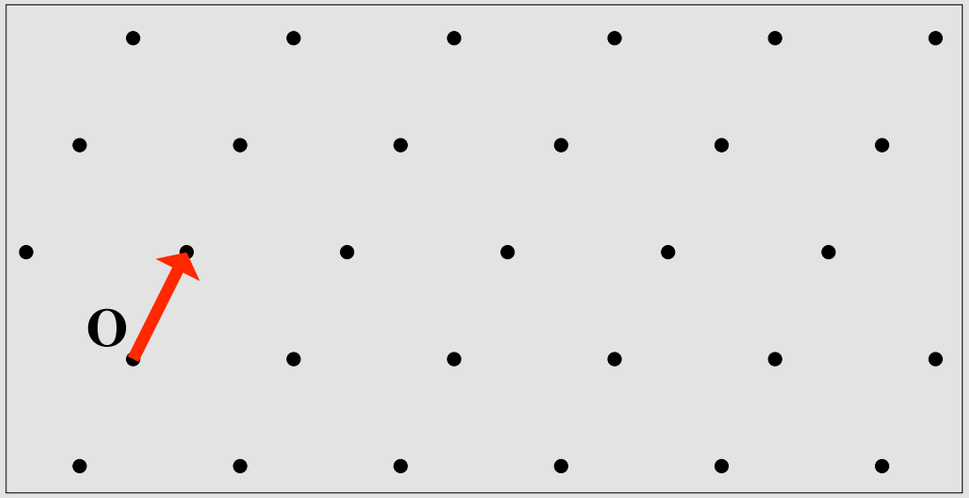
    Ex: SIS and ISIS.
  - ➤ Unique: only one non-trivial point.

    Ex: BDD.

THE SHORTEST VECTOR PROBLEM

THE SHORTEST VECTOR PROBLEM (SVP)

- Input: a basis of a d-dim lattice L.
- Output: nonzero $v \in L$ minimizing $||v||$ i.e. $||v|| = \lambda_1(L)$



2	0	0	0	0
0	2	0	0	0
0	0	2	0	0
0	0	0	2	0
1	1	1	1	1

THE SHORTEST VECTOR PROBLEM (SVP): DECISIONAL VARIANT

- Input: a basis of a d-dim lattice L and a rational number r
- Output: is there v∈L such that $||v|| \leq r$, i.e. is $\lambda_1(L) \leq r$?

THE SHORTEST VECTOR PROBLEM (SVP): OPTIMIZATION VARIANT

- Input: a basis of a d-dim lattice L and a rational number r
- Output: the integer $\lambda_1(L)^2$

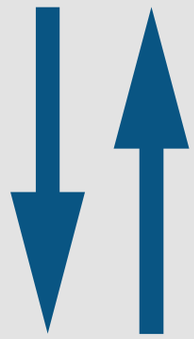
EXERCISE

- Show that given an oracle for Decisional-SVP, one can solve Optimization-SVP and SVP in polynomial time.
- Lagrange's algorithm shows how to solve rank-2 SVP in polynomial time.

RELAXING SVP

- Input: a basis of a d-dim lattice L.
- Output: nonzero $v \in L$ such that:

➤ Approximate-SVP: $||v|| \leq f(d)\,\lambda_1(L)$ [relative]



➤ Hermite-SVP: $||v|| \leq g(d)\,\mathrm{vol}(L)^{1/d}$ [absolute]

GAP-SVP

- Input: a basis of a d-dim lattice L and rational number r and gap γ
- Output: decide if $\lambda_1(L) \leq r$ or $\lambda_1(L) \geq \gamma r$ (we are promised to be in either situation)

LATTICE CHALLENGES

TU DARMSTADT LATTICE CHALLENGE

SVP CHALLENGE

HALL OF FAME

WILMINGTON, MA (PRWEB) FEBRUARY 05, 2015

Security Innovation is pleased to launch the NTRU Challenge today, February 5th, 2015. The NTRU Challenge will increase the understanding of the shortest vector problem in NTRU lattices while encouraging and stimulating further research into the security analysis of NTRU-based cryptosystems. The NTRU Challenge has been designed to provide additional information to users of NTRU public-key cryptosystems to aid in their selection of suitable key lengths for a desired level of security.

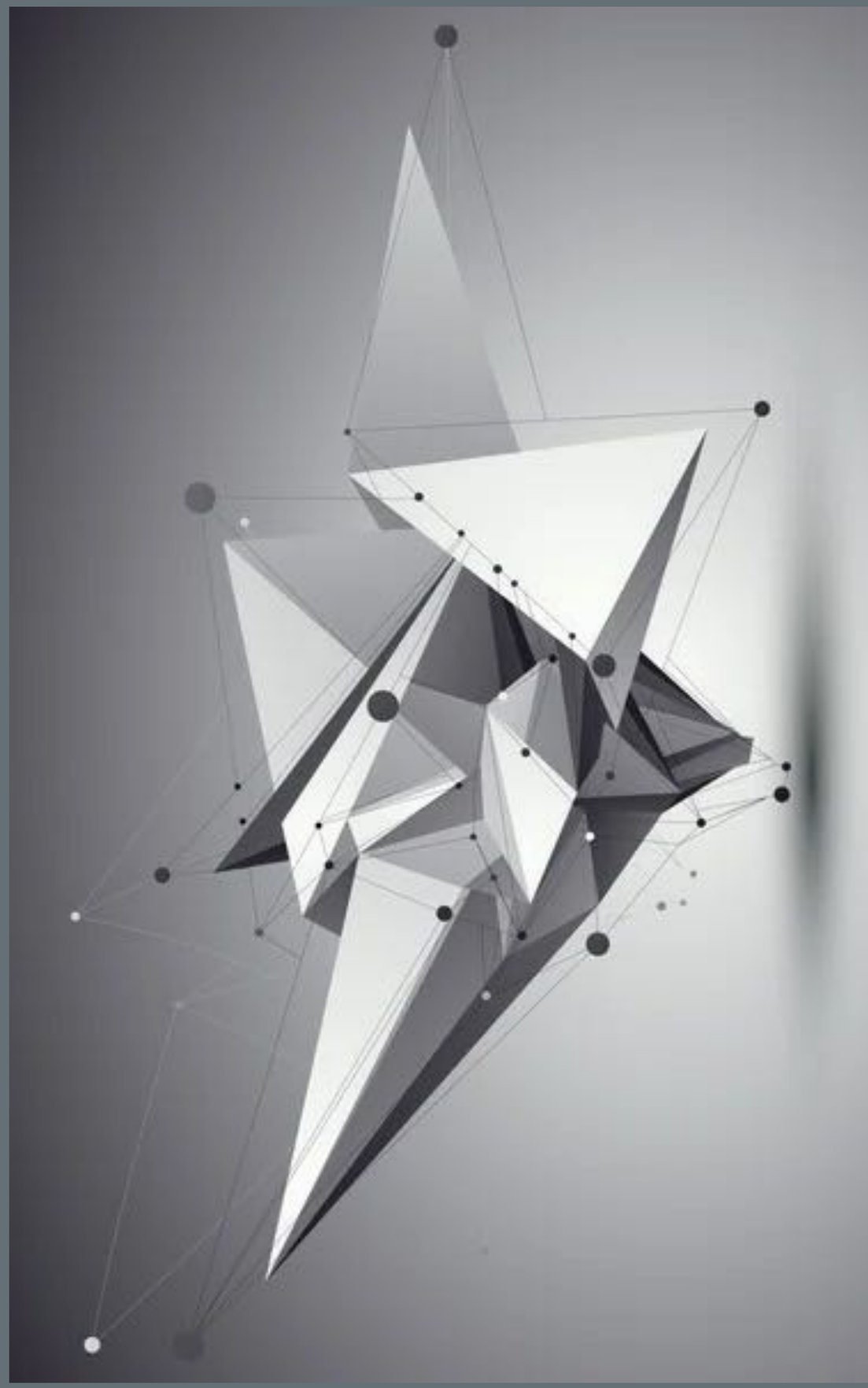
Access the challenge here: http://www.SecurityInnovation.com/NTRUChallenge

The Challenge asks participants to compute the NTRU private keys from the given list of public keys and associated system parameters. This is the type of problem faced by hackers who wish to defeat an NTRU-based cryptosystem. The Challenge consists of several individual NTRU challenges, targeted at different security levels, some of which can be solved in a day, some in a few months and some which are considered to be computationally intractable.

The prize for the first correct solution for the 11 lower security level challenges will be $1,000 each, with $5,000 being awarded per solution for the 16 higher security level challenges. Additionally, participants who arrive at innovative and unique solutions may be chosen for induction into the NTRU Hall of Fame, an award which includes an all-expenses paid trip to a major cryptographic conference.

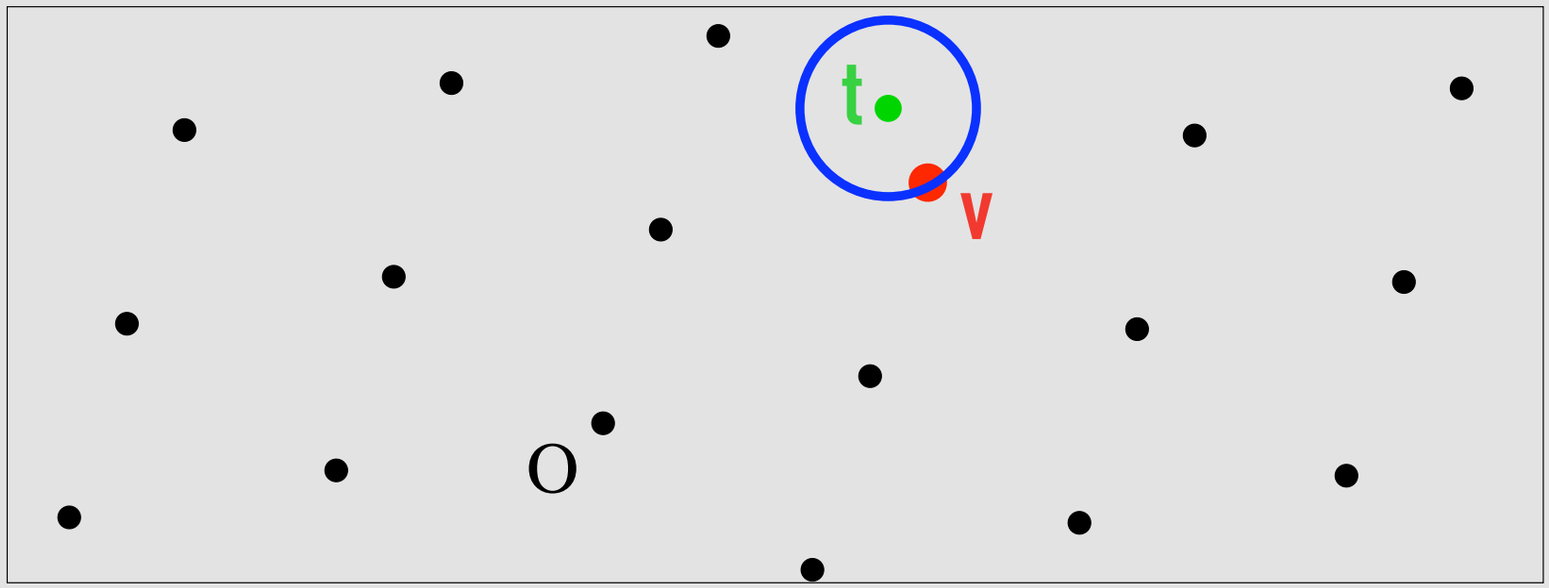
...m. ...te	Approx. Factor
20-12	1.04690
19-20	1.04906
18-18	1.00803
18-30	1.02102
18-3	1.03313

THE CLOSEST VECTOR PROBLEM



THE CLOSEST VECTOR PROBLEM (CVP)

- Input: a basis of a lattice L of dim d, and a target vector t.
- Output: v∈L minimizing ||v-t||.



- BDD (bounded distance decoding): special case when t is very close to L.

EXERCISES

- Solve CVP for rank-1 lattices.
- CVP is NP-hard.
- SVP is not harder than CVP: if one can solve exact CVP, one can solve exact SVP.

COVERING RADIUS

- $\mu(L)$ is the largest distance dist(x,L) where x runs over span(L): why is it reached for some x?
  - ➤ Show that $\mu(L) \geq \lambda_1(L)/2$ and even $\mu(L) \geq \lambda_n(L)/2$.
  - ➤ Show that $\mu(L) \leq \Sigma_i \lambda_i(L)/2$

A FEW REDUCTIONS

DECISION-CVP IS NP-COMPLETE

- Subset sum: given integers $a_1, \ldots, a_n$ and s, decide if $s = \sum_{i=1}^{n} x_i a_i$ where $x_i \in \{0,1\}$.
- Consider L spanned by $\begin{pmatrix} a_1 & 2 & 0 & \ldots & 0 \\ a_2 & 0 & 2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ a_n & 0 & \ldots & 0 & 2 \end{pmatrix}$ and $\vec{t} = (s \quad 1 \quad \ldots \quad 1)$
- Ask if $\text{dist}(\vec{t}, L) \leq \sqrt{n}$

SVP IS NOT HARDER THAN CVP

- Input: a basis $(\vec{b}_1, \ldots, \vec{b}_n)$ of L
- Output: $\sum_{i=1}^{n} x_i \vec{b}_i$ of minimal non-zero norm
- For any solution, one of the $x_i's$ is odd: $x_i = 2y_i + 1$
- Then $\sum_{j \neq i} x_j \vec{b}_j + 2y_i \vec{b}_i \in L(\vec{b}_1, \ldots, \vec{b}_{i-1}, 2\vec{b}_i, \vec{b}_{i+1}, \ldots, \vec{b}_n)$ (sublattice of L) is a closest vector to $-\vec{b}_i$
- By calling n times a CVP-oracle, we solve SVP.

SEARCH-TO-DECISION CVP

- Input: a basis $(\vec{b}_1, \ldots, \vec{b}_n)$ of L and a target $\vec{t}$
- Output: $\sum_{i=1}^{n} x_i \vec{b}_i$ minimizing $\| \sum_{i=1}^{n} x_i \vec{b}_i - \vec{t} \|$
- We can compute $\text{dist}(\vec{t'}, L')$ for any $(\vec{t'}, L')$.
- Let the sublattice $L_i = L(\vec{b}_1, \ldots, \vec{b}_{i-1}, 2\vec{b}_i, \vec{b}_{i+1}, \ldots, \vec{b}_n)$
- Check if $\text{dist}(\vec{t}, L) = \text{dist}(\vec{t}, L_1)$
  - Yes: There is a solution s.t. x_1 is even. We replace L by L_1.
  - No: all solutions have odd x_1, so we want to minimize $\|x_1(2\vec{b}_1) + \sum_{i=2}^{n} x_i \vec{b}_i - (\vec{t} + \vec{b}_1)\|$, so we replace $(\vec{t}, L)$ by $(\vec{t} + \vec{b}_1, L_1)$
- By iterating, we obtain the full x_1 of one solution: we know we have removed all the bits of x_1 if $\text{dist}(\vec{t}, L) = \text{dist}(\vec{t}, L(\vec{b}_2, \ldots, \vec{b}_n))$
- Now, replace $\vec{t}$ by $\vec{t} - x_1 \vec{b}_1$ and L by $L(\vec{b}_2, \ldots, \vec{b}_n)$: this decreased the rank, and we can iterate.

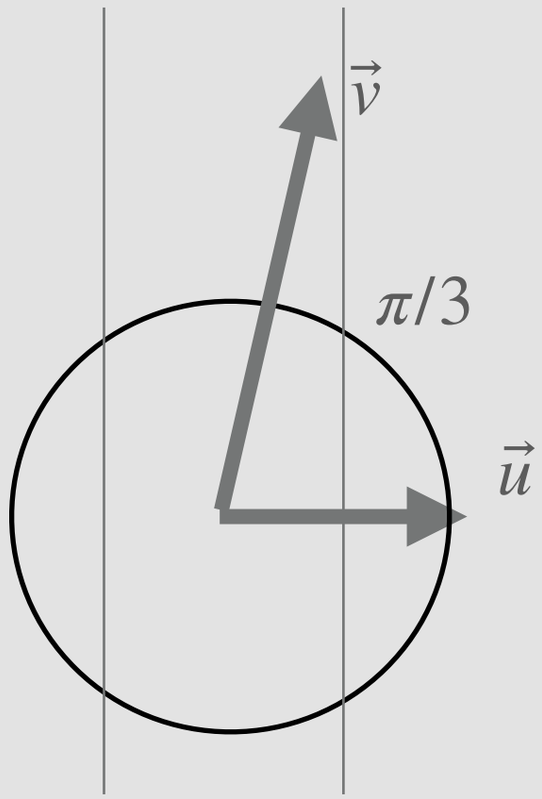
DIMENSION TWO

LAGRANGE'S REDUCTION (1773)

1. Lagrange reduction. (⋆⋆)

Let L be a two-rank lattice. A basis $(\vec{u}, \vec{v})$ of L is *Lagrange-reduced* if $\|\vec{u}\| \leq \|\vec{v}\|$ and $|\langle \vec{u}, \vec{v} \rangle| \leq \|\vec{u}\|^2/2$. Show that :

1. *If* $(\vec{u}, \vec{v})$ *is reduced, then* $\|\vec{u}\| = \lambda_1(L) \leq (4/3)^{1/4}\mathrm{vol}(L)^{1/2}$ *and* $\|\vec{v}\| = \lambda_2(L)$.
2. *There exists a reduced basis* $(\vec{u}, \vec{v})$ *of* L.
3. *There exists a lattice* L *such that* $\lambda_1(L) = (4/3)^{1/4}\mathrm{vol}(L)^{1/2}$.



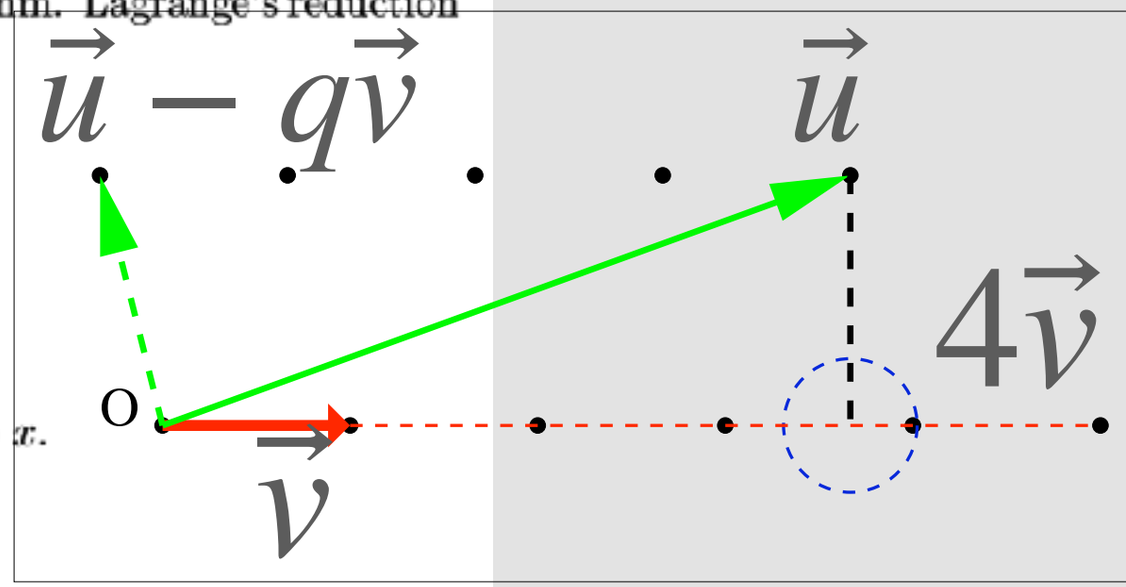
LAGRANGE'S ALGORITHM

2. Lagrange's Algorithm. (⋆⋆)

In 1773, Lagrange published the following two-dimensional reduction algorithm. Lagrange's reduction algorithm.

Input: a basis $(\vec{u}, \vec{v})$ of a two-rank lattice L.
Output: a Lagrange-reduced basis of L.

1: **if** $\|\vec{u}\| < \|\vec{v}\|$ **then**
2: swap $\vec{u}$ and $\vec{v}$
3: **end if**
4: **repeat**
5: $\vec{r} \longleftarrow \vec{u} - q\vec{v}$ where $q = \left\lfloor \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{v}\|^2} \right\rceil$ and $\lfloor x \rceil$ denotes an integer closest to x.
6: $\vec{u} \longleftarrow \vec{v}$
7: $\vec{v} \longleftarrow \vec{r}$
8: **until** $\|\vec{u}\| \leq \|\vec{v}\|$
9: Output $(\vec{u}, \vec{v})$.



1. *Consider Line 5 of Algorithm : show that this choice of $q \in \mathbb{Z}$ minimizes $\|\vec{u} - q\vec{v}\|$.*
2. *Show that Lagrange's algorithm terminates,* i.e. *that the repeat/until loop is not infinite, and that the output basis is Lagrange-reduced.*
3. *Consider the integer q of Step 5. Show that :*
   — *if $q = 0$, then this must be the last iteration of the loop.*
   — *if $|q| = 1$, then this must be either the first or last iteration of the loop.*
4. *Show that the number τ of iterations of the repeat/until loop is bounded by : $\tau = O(1 + \log B - \log \lambda_1(L))$ where B denotes the maximal Euclidean norm of the input basis vectors $\vec{u}$ and $\vec{v}$.*
5. *Show that when $L \subseteq \mathbb{Z}^n$, the bit-complexity of Lagrange's algorithm is polynomial in $\log B$.*

KEY ARGUMENTS

- If q=0, then the basis is reduced by definition.
- If it is not the last iteration then $\|\vec{u} - q\vec{v}\| < \|\vec{v}\| < \|\vec{u}\|$.
  - If $|q|=1$, then $\|\vec{u} - q\vec{v}\| = \|\vec{v} - q\vec{u}\|$ so $\|\vec{v} - q\vec{u}\| < \|\vec{v}\|$ which means that $\vec{v}$ could be shortened by $\vec{u}$. This can only happens at the first iteration.
- Except maybe the first and last iteration, we always have $|q| \geq 2$.
  - Then $\mu = \dfrac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{v}\|^2}$ satisfies $|\mu| \geq 3/2$.
  - But $\mu\vec{v}$ is the projection of $\vec{u}$ over span($\vec{v}$) so $\|\vec{u}\|^2 \geq \|\mu\vec{v}\|^2 \geq \dfrac{9}{4}\|\vec{v}\|^2$ therefore $\|\vec{v}\|^2 \leq \dfrac{4}{9}\|\vec{u}\|^2$: the norms decrease geometrically!

RANDOM INSTANCES

RANDOM INSTANCES

- Which distributions of integer lattices (SVP and CVP) and target (CVP/BDD)?

- A full-rank integer lattice $L \subseteq \mathbf{Z}^m$ defines a finite Abelian group $\mathbf{Z}^m/L$. Two different lattices can define the same quotient.

- Can we fix a quotient, then generate a random lattice with that quotient? Is it hard to find short vectors in such a random lattice?

THE SIS PROBLEM (1996): SMALL INTEGER SOLUTIONS

Miklós Ajtai

- Let $(G,+)$ be a finite Abelian group: $G=(\mathbf{Z}/q\mathbf{Z})^n$ in [A
  View G as a $\mathbf{Z}$-module.
- Pick $g_1,\ldots,g_m$ uniformly at random from G.
- Goal: Find short $(x_1,\ldots,x_m)\in\mathbf{Z}^m$ s.t. $\Sigma_i\, x_i\, g_i = 0$,
  e.g. $\|x\| \le \sqrt{m}(\#G)^{1/m}$.

➤ This is essentially finding a short vector in a (uniform) random lattice of $L_m(G) = \{$ lattices $L\subseteq\mathbf{Z}^m$ s.t. $\mathbf{Z}^m/L \sim G\}$.

EX: CYCLIC G

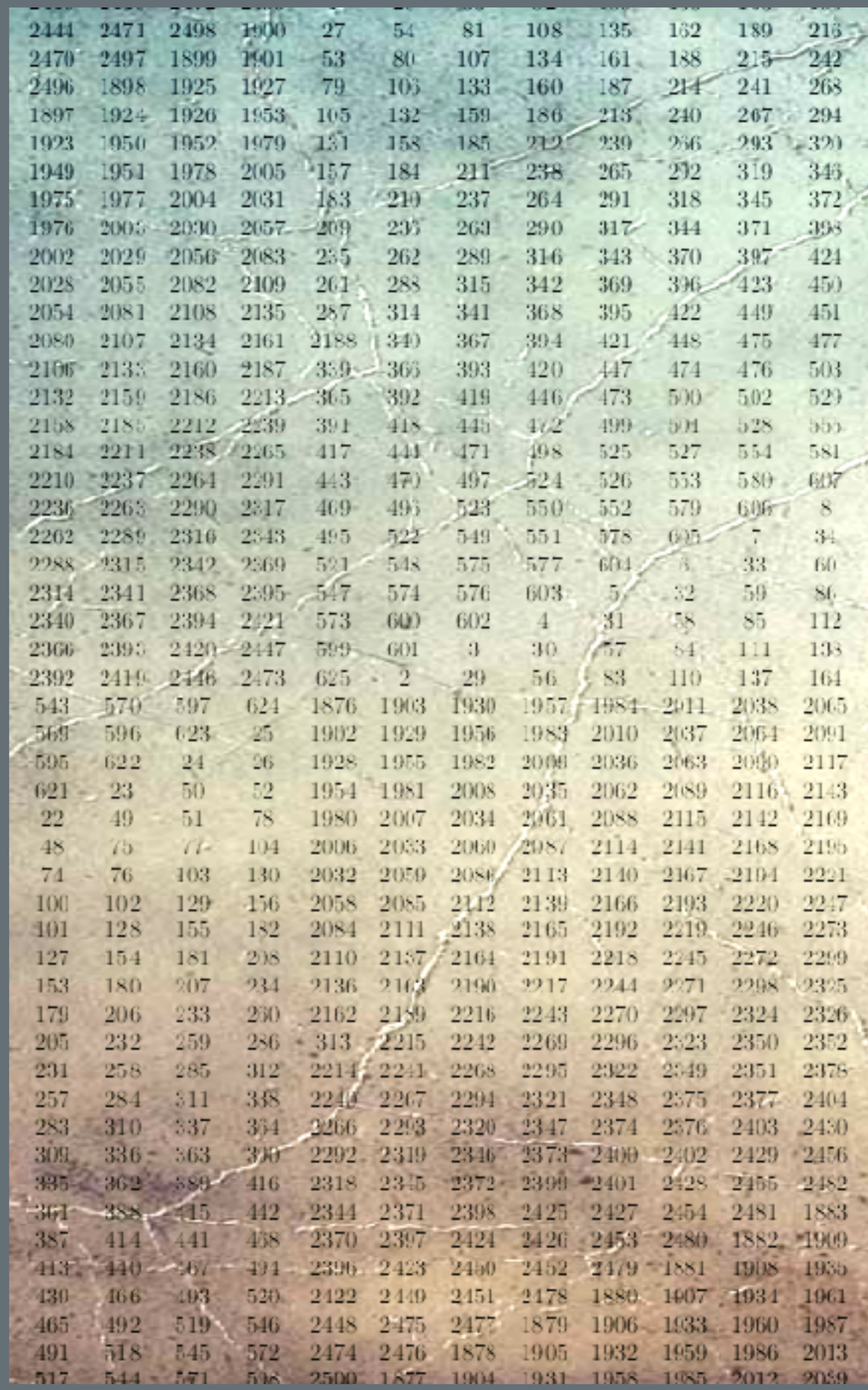
- Let $G = \mathbf{Z}/q\mathbf{Z}$

- Pick $g_1,\ldots,g_m$ uniformly at random mod q.

- Goal: Find short $x=(x_1,\ldots,x_m)\in\mathbf{Z}^m$ s.t. $\Sigma_i\, x_i\, g_i \equiv 0 \pmod q$.

➤ This is finding a short lattice vector for random lattices L such that $\mathbf{Z}^m/L \sim \mathbf{Z}/q\mathbf{Z}$.

SIS AND LWE


2444 2471 2498 1900 27 54 81 108 135 162 189 216
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2496 1898 1925 1927 79 106 133 160 187 214 241 268
1897 1924 1926 1953 105 132 159 186 213 240 267 294
1923 1950 1952 1979 131 158 185 212 239 266 293 320
1949 1951 1978 2005 157 184 211 238 265 292 319 346
1975 1977 2004 2031 183 210 237 264 291 318 345 372
1976 2003 2030 2057 209 235 263 290 317 344 371 398
2002 2029 2056 2083 235 262 289 316 343 370 397 424
2028 2055 2082 2109 261 288 315 342 369 396 423 450
2054 2081 2108 2135 287 314 341 368 395 422 449 451
2080 2107 2134 2161 2188 340 367 394 421 448 475 477
2106 2133 2160 2187 339 366 393 420 447 474 476 503
2132 2159 2186 2213 365 392 419 446 473 500 502 529
2158 2185 2212 2239 391 418 445 472 499 501 528 555
2184 2211 2238 2265 417 444 471 498 525 527 554 581
2210 2237 2264 2291 443 470 497 524 526 553 580 607
2236 2263 2290 2317 469 496 523 550 552 579 606 8
2262 2289 2316 2343 495 522 549 551 578 605 7 34
2288 2315 2342 2369 521 548 575 577 604 6 33 60
2314 2341 2368 2395 547 574 576 603 5 32 59 86
2340 2367 2394 2421 573 600 602 4 31 58 85 112
2366 2393 2420 2447 599 601 3 30 57 84 111 138
2392 2419 2446 2473 625 2 29 56 83 110 137 164
543 570 597 624 1876 1903 1930 1957 1984 2011 2038 2065
569 596 623 25 1902 1929 1956 1983 2010 2037 2064 2091
595 622 24 26 1928 1955 1982 2009 2036 2063 2090 2117
621 23 50 52 1954 1981 2008 2035 2062 2089 2116 2143
22 49 51 78 1980 2007 2034 2061 2088 2115 2142 2169
48 75 77 104 2006 2033 2060 2087 2114 2141 2168 2195
74 76 103 130 2032 2059 2086 2113 2140 2167 2194 2221
100 102 129 156 2058 2085 2112 2139 2166 2193 2220 2247
101 128 155 182 2084 2111 2138 2165 2192 2219 2246 2273
127 154 181 208 2110 2137 2164 2191 2218 2245 2272 2299
153 180 207 234 2136 2163 2190 2217 2244 2271 2298 2325
179 206 233 260 2162 2189 2216 2243 2270 2297 2324 2326
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231 258 285 312 2214 2241 2268 2295 2322 2349 2351 2378
257 284 311 338 2240 2267 2294 2321 2348 2375 2377 2404
283 310 337 364 2266 2293 2320 2347 2374 2376 2403 2430
309 336 363 390 2292 2319 2346 2373 2400 2402 2429 2456
335 362 389 416 2318 2345 2372 2399 2401 2428 2455 2482
361 388 415 442 2344 2371 2398 2425 2427 2454 2481 1883
387 414 441 468 2370 2397 2424 2426 2453 2480 1882 1909
413 440 467 494 2396 2423 2450 2452 2479 1881 1908 1935
439 466 493 520 2422 2449 2451 2478 1880 1907 1934 1961
465 492 519 546 2448 2475 2477 1879 1906 1933 1960 1987
491 518 545 572 2474 2476 1878 1905 1932 1959 1986 2013
517 544 571 598 2500 1877 1904 1931 1958 1985 2012 2039

WORST-CASE TO AVERAGE-CASE REDUCTION

- [Ajtai96]: If one can efficiently solve SIS for $G=(\mathbf{Z}/q_n\mathbf{Z})^n$ on the average, then one can efficiently find short vectors in every n-dim lattice.

- [GINX16]: This can be generalized to any sequence (G_n) of finite abelian groups, provided that $\#G_n$ is sufficiently large $\geq n^{\Omega(\max(n,\mathrm{rank}(G)))}$ and m too. Ex: $(\mathbf{Z}/2\mathbf{Z})^n$ is not.

THE SIS ONE-WAY FUNCTION

- Let $(G,+)$ be a finite Abelian group
- Pick $g_1,\ldots,g_m$ uniformly at random from G.
- Let f: short $(x_1,\ldots,x_m)\in\mathbf{Z}^m \longmapsto \Sigma_i\, x_i\, g_i \in G$.
  - f is many-to-one.
  - Given $h = \Sigma_i\, y_i\, g_i \in G$, finding a short $(x_1,\ldots,x_m)\in\mathbf{Z}^m$ s.t. $h=\Sigma_i\, x_i\, g_i \in G$ is as hard as SIS.

DUALITY

- Remember the SIS lattice:
  - $g_1,\ldots,g_m$ in some finite Abelian group $(G,+)$
  - $L=\{x=(x_1,\ldots,x_m)\in\mathbf{Z}^m \text{ s.t. } \Sigma_i\, x_i\, g_i = 0\}$

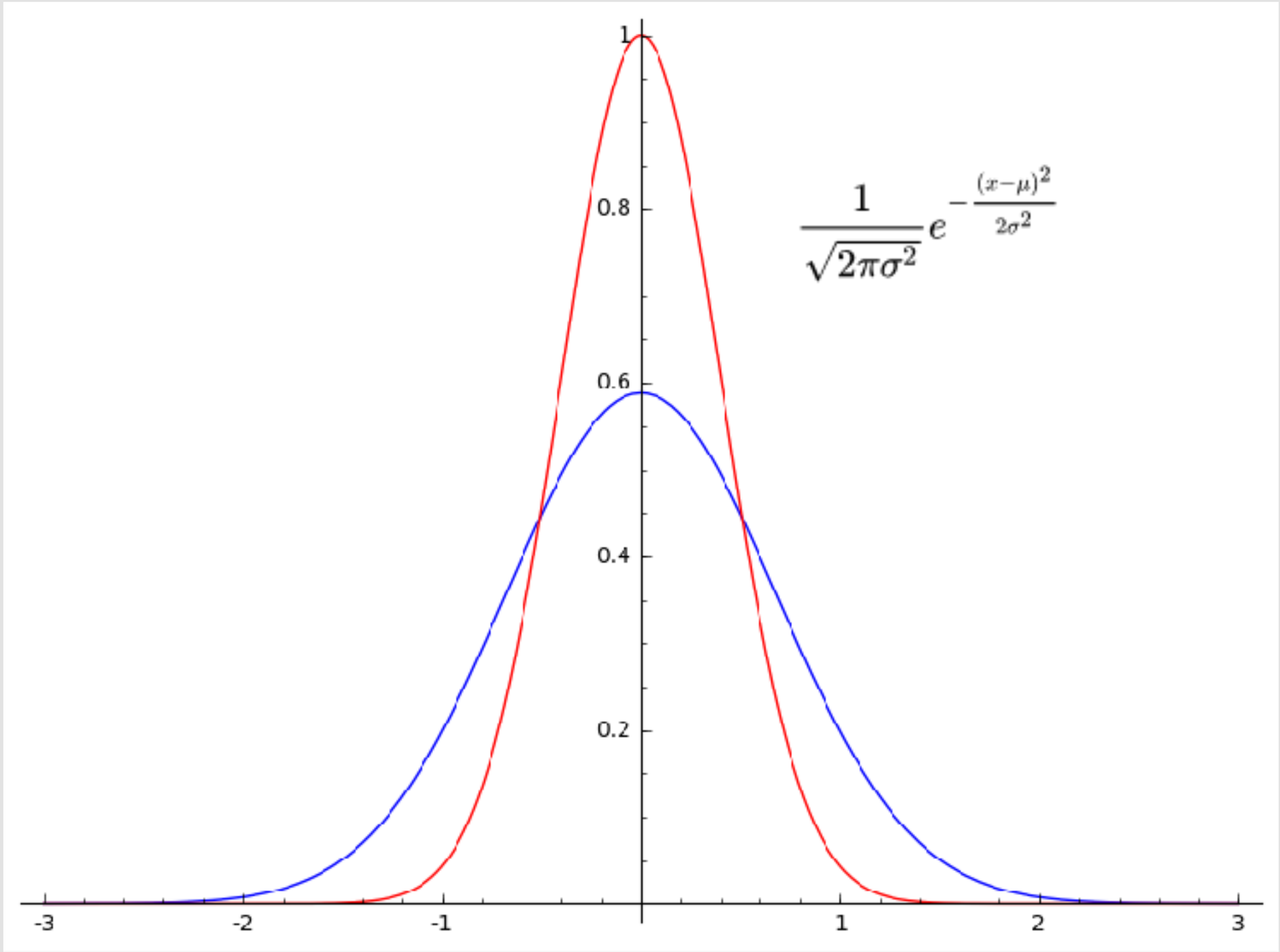
- The dual lattice of L is related to the dual group G^x of (additive) characters of G: morphisms from G to $\mathbf{T}=\mathbf{R}/\mathbf{Z}$
  - $L^x=\{(y_1,\ldots,y_m)\in\mathbf{R}^m \text{ s.t. for some } s\in G^x, \text{ for all i } y_i\equiv s(g_i) \pmod 1\}$

THE LWE PROBLEM: LEARNING (A CHARACTER) WITH ERRORS

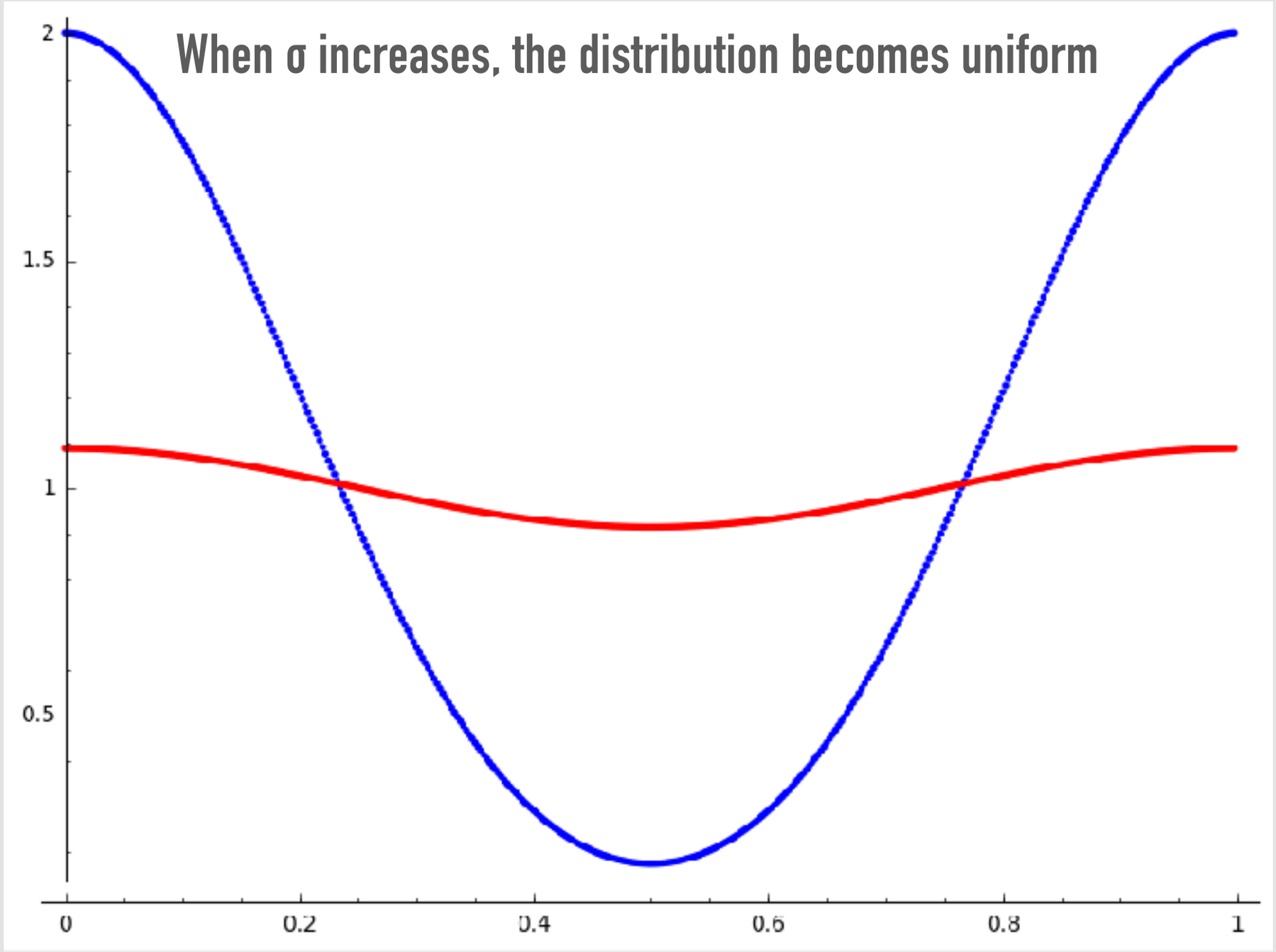
Oded Regev

- Let (G,+) be any finite Abelian group e.g. G=(Z/[illegible]05].
- Pick $g_1,\ldots,g_m$ uniformly at random from G.
- Pick a random character s in G^x.
- Goal: recover s given $g_1,\ldots,g_m$ and noisy approximations of $s(g_1),\ldots, s(g_m)$. Ex: Gaussian noise.

GAUSSIAN NOISE OVER R



GAUSSIAN NOISE OVER R/Z



EX: CYCLIC G

- Let $G = \mathbf{Z}/q\mathbf{Z}$
- Pick $g_1,\ldots,g_m$ uniformly at random mod q.
- Goal: recover $s \in \mathbf{Z}$ given $g_1,\ldots,g_m$ and randomized approximations of $sg_1 \bmod q,\ldots, sg_m \bmod q$.

➤ This is exactly a randomized variant of Boneh-Venkatesan's Hidden Number Problem from CRYPTO '96.

HARDNESS OF LWE

- [Regev05]: If one can efficiently solve LWE for $G=(\mathbf{Z}/q_n\mathbf{Z})^n$ on the average, then one can quantum-efficiently find short vectors in every n-dim lattice.
- [GINX16]: This can be generalized to any sequence (G_n) of finite abelian groups, provided that $\#G_n$ is sufficiently large.