LATTICES: MATHEMATICAL BACKGROUND

PHONG NGUYEN

http://www.di.ens.fr/~pnguyen

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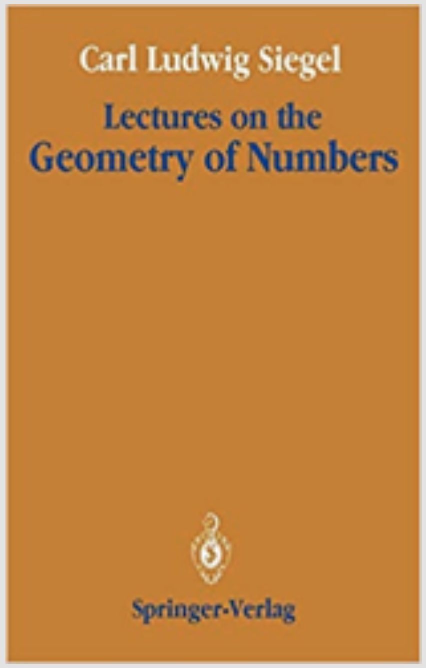
ADVICES

- Interaction: please ask questions during my talks; interruptions are welcome.

- Implementation: to understand an algorithm, it is helpful to implement it: sage, NTL, fplll, etc.

MATH REFERENCES

- Siegel's Lectures on the Geometry of Numbers (Springer).

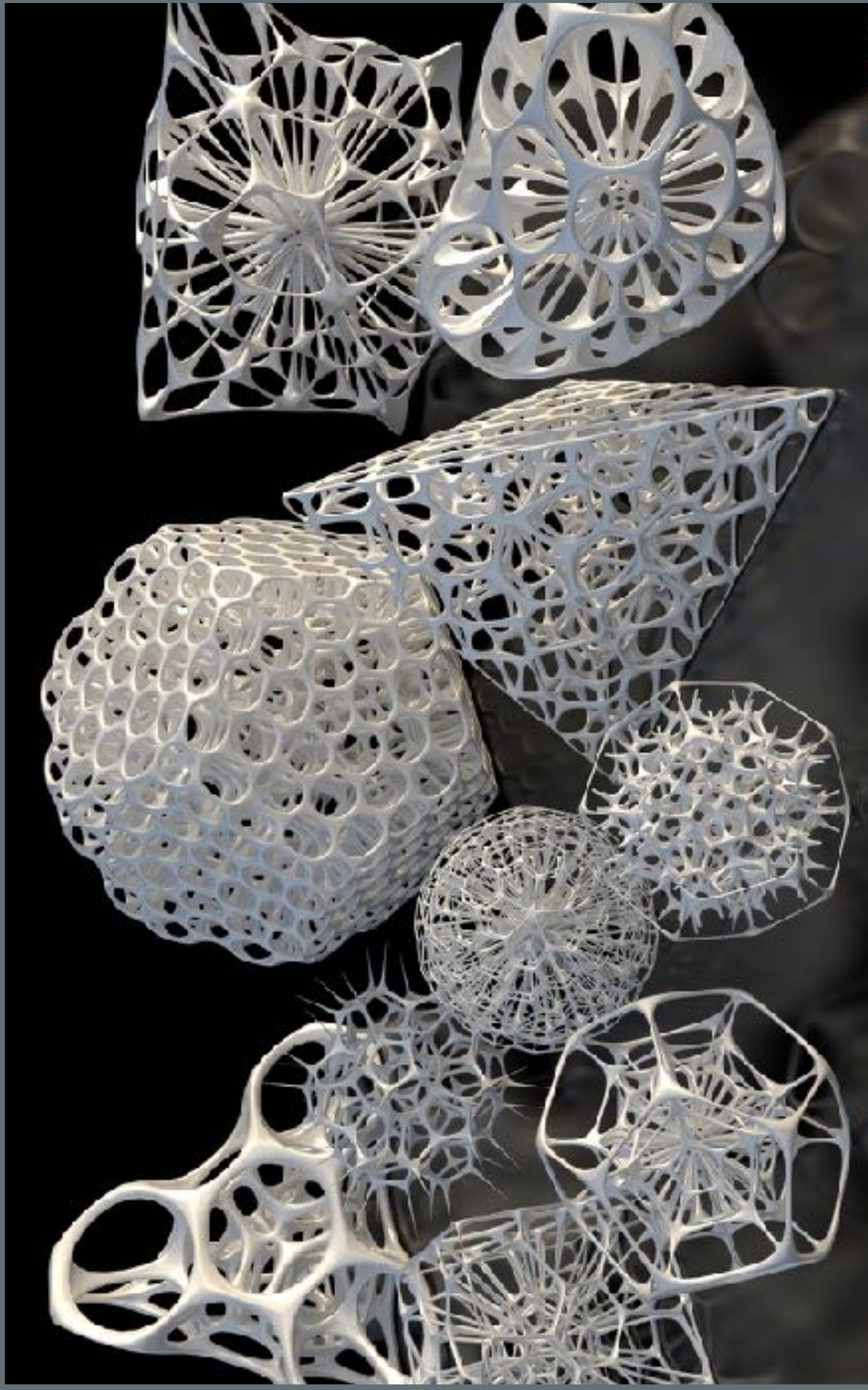


- Venkatesh's Stanford lecture notes on geometry of numbers

TODAY: LATTICES MATHEMATICAL BACKGROUND

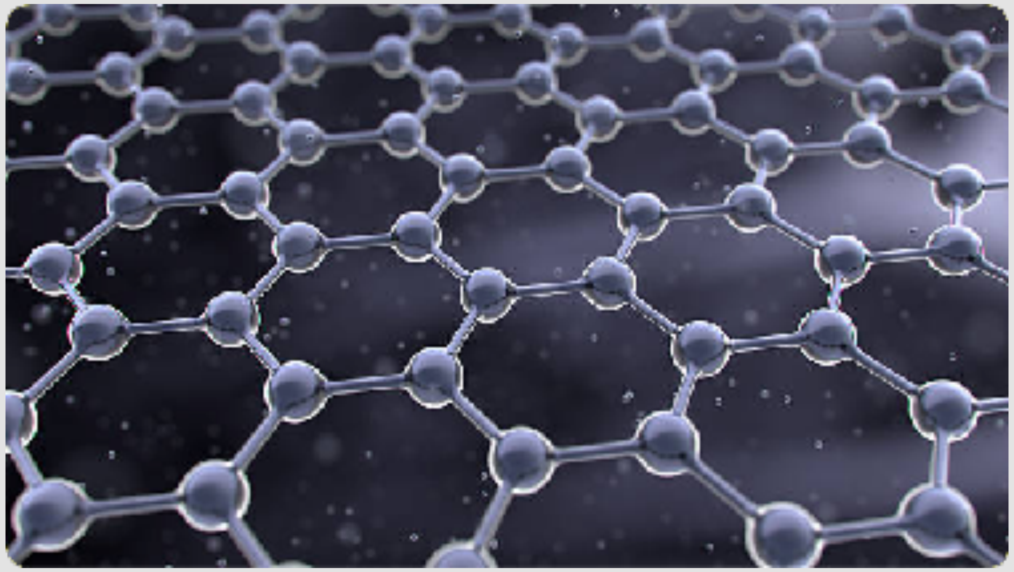
- What is a lattice?
- Characterization of lattices
- Counting lattice points

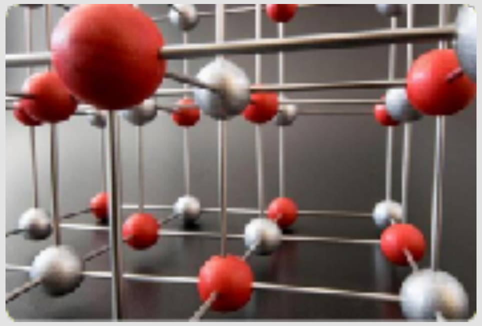
WHAT IS LATTICE?

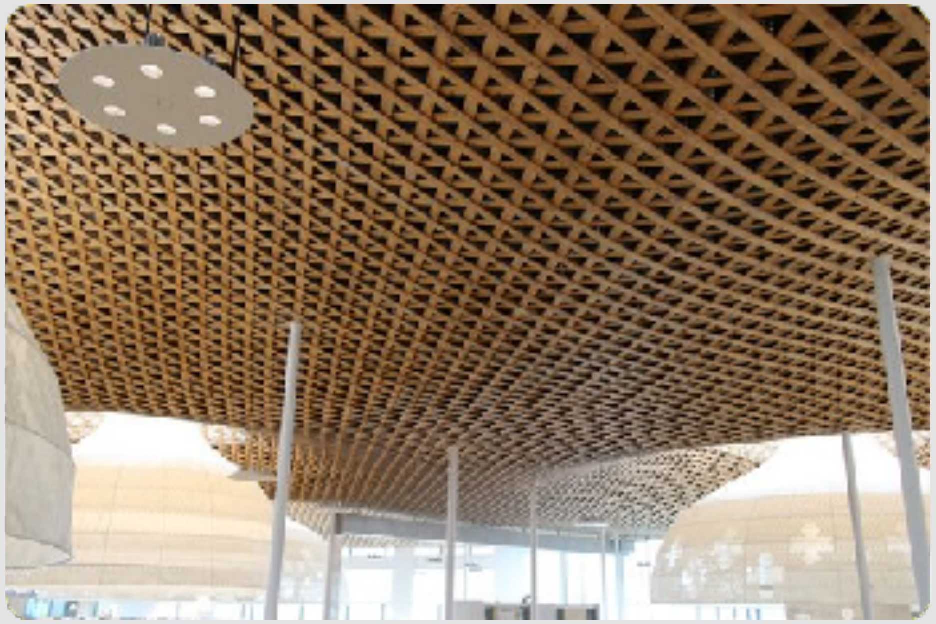


WHAT IS A LATTICE?

- An infinite arrangement of “regularly spaced” points







WHAT IS A LATTICE?

- A linear deformation of $\mathbf{Z}^n$.
  - Let B be a non-singular n x n matrix.
  - The lattice spanned by the basis B is $L = \mathbf{Z}^n B$.

2	0	0	0	0
0	2	0	0	0
0	0	2	0	0
0	0	0	2	0
1	1	1	1	1

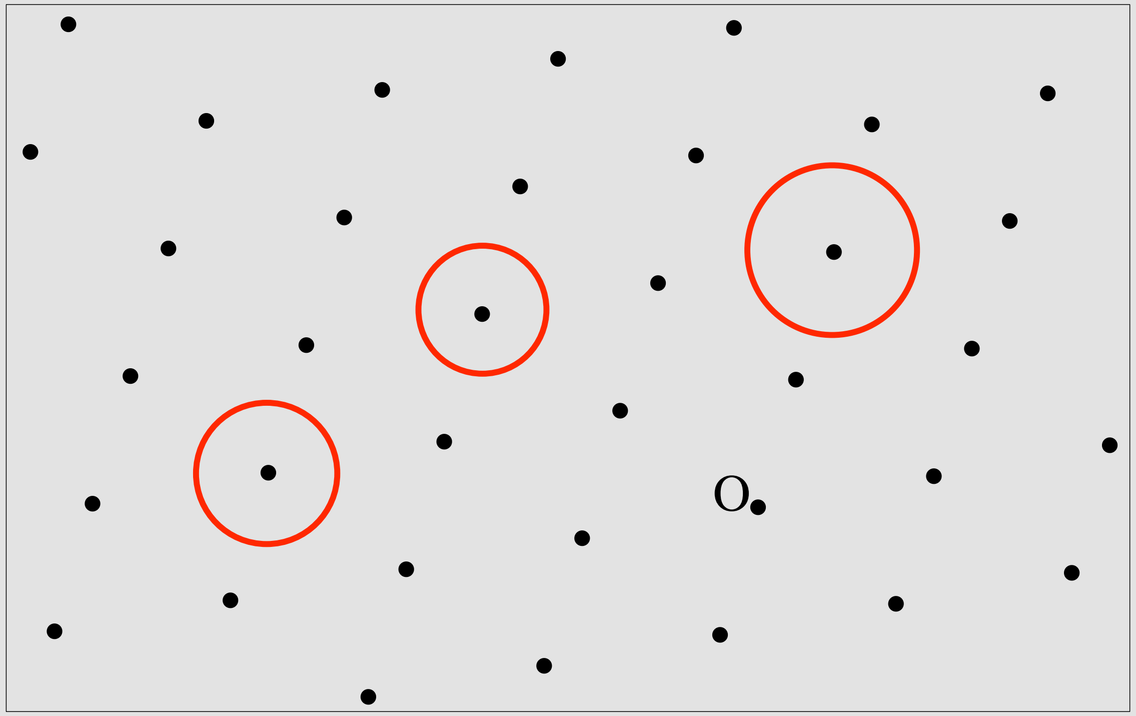
EUCLIDEAN LATTICES

- Consider $\mathbf{R}^n$ as a Euclidean space: let $\langle u,v \rangle$ be the dot product $||\overrightarrow{w}||$ be the norm.

- A lattice is a discrete subgroup L of $\mathbf{R}^n$:

  $\forall v \in L\ \exists r>0$ s.t. $L \cap \mathrm{Ball}(v,r)=\{v\}$

  $\dim(L)=\mathrm{rank}(L)=\dim(\mathrm{span}(L))$



- Ex: $\mathbf{Z}^n$ and its subgroups.

EXERCISES

- Show that for any lattice L of $\mathbf{R}^n$:
  - ➤ $\exists r>0$ s.t. $\forall v \in L$, $L \cap \mathrm{Ball}(v,r) = \{v\}$.
  - ➤ L is closed.
  - ➤ $\forall r>0$ and $x \in \mathbf{R}^n$, $L \cap \mathrm{Ball}(x,r)$ is finite. In particular, L has shortest non-zero vectors.
  - ➤ L is countable.

EXAMPLES

- Let L and L' be lattices in $\mathbf{R}^n$.

- Let E be a subspace of $\mathbf{R}^n$.
  - Is $E \cap L$ a lattice?

- Let E be a subspace of $\mathbf{R}^n$.
  - Is $L \cap L'$ a lattice?
  - Is $L \cup L'$ a lattice?

NOTATION: LINEAR COMBINATIONS

- For any vectors $b_1,\ldots,b_n \in \mathbf{R}^m$,

  let $L(b_1,\ldots,b_n)$ denote their $\mathbf{Z}$-span.

- $L(b_1,\ldots,b_n) = \mathbf{Z}b_1+\ldots+\mathbf{Z}b_n = \{\ x_1b_1+\ldots+x_nb_n \text{ where each } x_i\in\mathbf{Z}\ \}$

QUESTION 1

- If $b_1,\ldots,b_n \in \mathbf{Z}^m$, is $L(b_1,\ldots,b_n)$ a lattice?

ANSWER 1

- If $b_1,\ldots,b_n \in \mathbf{Z}^m$, is $L(b_1,\ldots,b_n)$ a lattice?

➤ Yes, because it is a subgroup of $\mathbf{Z}^m$, therefore a discrete subgroup of $\mathbf{R}^m$.

QUESTION 2

- If $b_1,\ldots,b_n \in \mathbf{Q}^m$, is $L(b_1,\ldots,b_n)$ a lattice?

ANSWER 2

- If $b_1,\ldots,b_n \in \mathbf{Q}^m$, is $L(b_1,\ldots,b_n)$ a lattice?

➤ Yes, because it is can be reduced to the previous question.

QUESTION 3

- If $b_1,\ldots,b_n \in \mathbf{R}^m$, is $L(b_1,\ldots,b_n)$ a lattice?

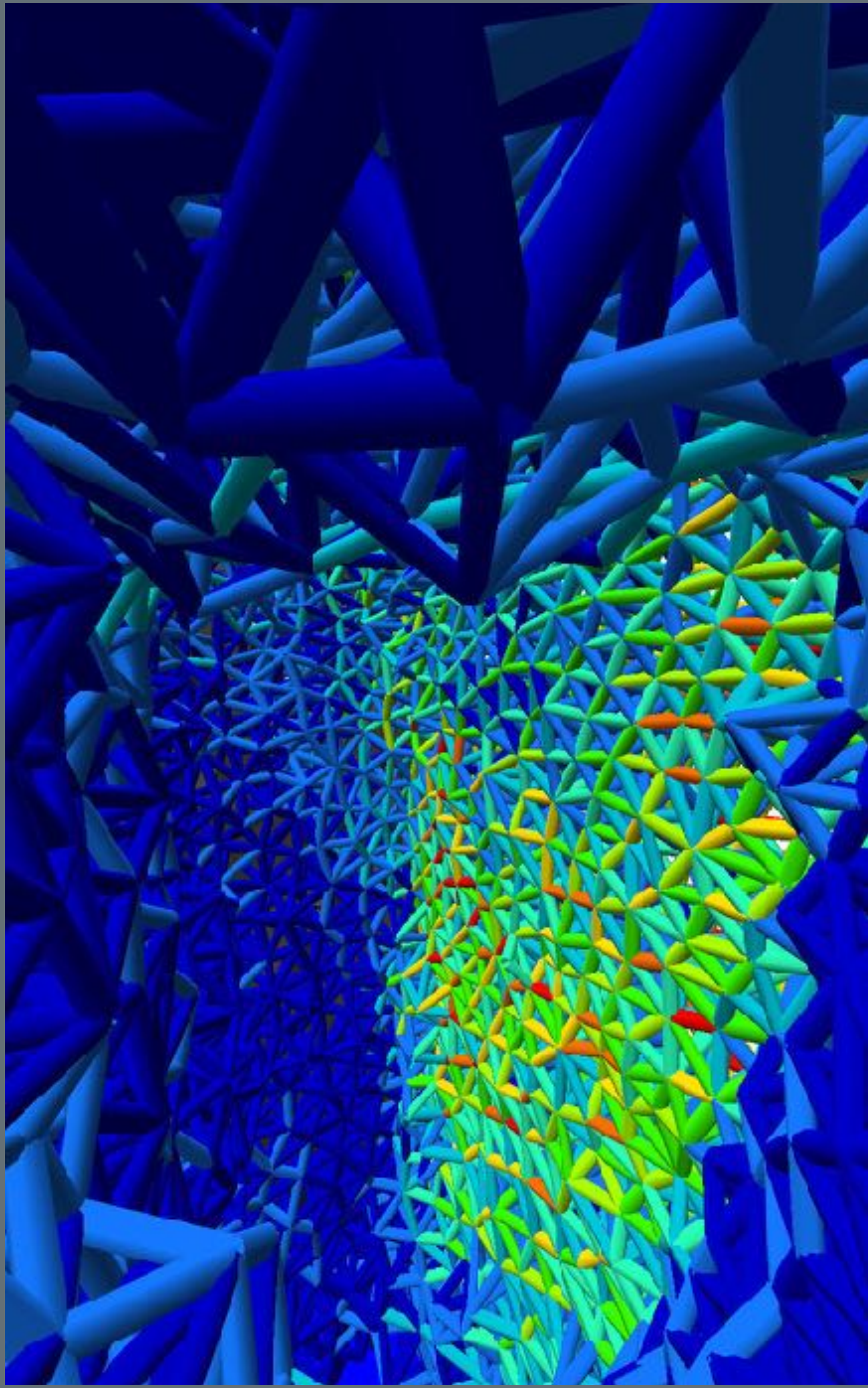
ANSWER 3

- If $b_1,\dots,b_n \in \mathbf{R}^m$, is $L(b_1,\dots,b_n)$ a lattice?

➤ Not necessarily, even for $n=2$ and $m=1$: $L(1,\sqrt{2})$ is not a lattice because it is dense in $\mathbf{R}$.

➤ Yet, $L(1,\sqrt{2})$ can also be "viewed" as a lattice: it is the ring of integers of $\mathbf{Q}(\sqrt{2})$.

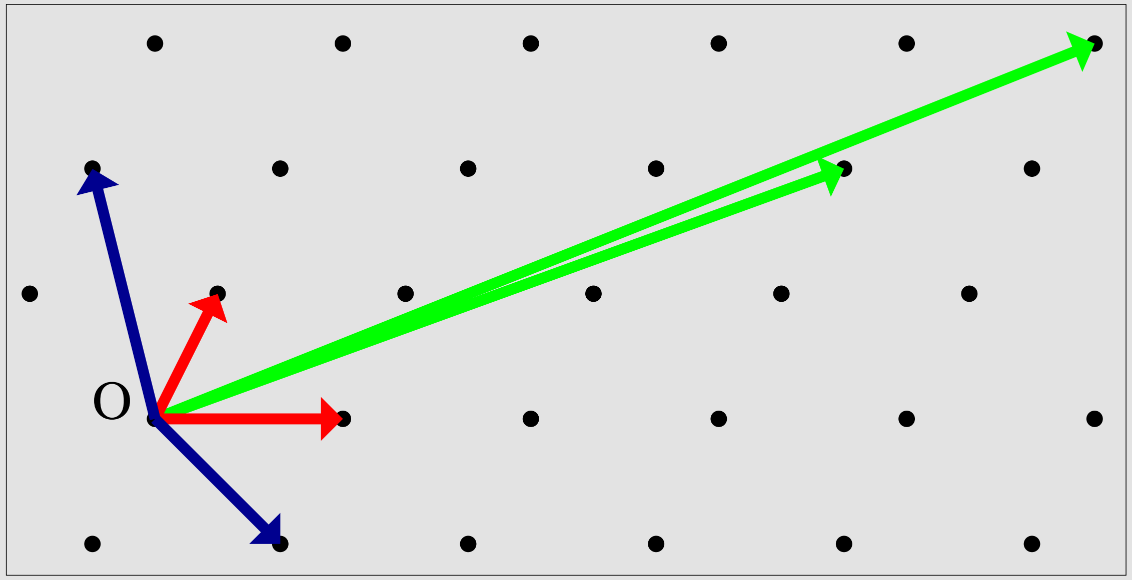
CHARACTERIZATION OF LATTICES



A NON-TRIVIAL LATTICE THEOREM

STATEMENT

- Th: Let $L \subseteq \mathbf{R}^m$ be non-empty. There is equivalence between:
  - L is a lattice.
  - $\exists b_1,\ldots,b_n \in \mathbf{R}^m$ linearly independent s.t. $L = L(b_1,\ldots,b_n)$: such $(b_1,\ldots,b_n)$ is called a basis of L and dim(L) :=n.



PROVING "⇐"

- Let $b_1, b_2, \ldots b_n$ be linearly independent in $\mathbf{R}^m$.

- Consider an injective sequence (v_i) of $L = L(b_1, \ldots, b_n)$ converging to 0.
  - $\exists! m_i \in \mathbf{Z}^n$ s.t. $v_i = m_i B$ where $B = (b_1, \ldots, b_n)$.
  - Then $m_i = (v_i B^t)(BB^t)^{-1}$ converges to 0 but $\mathbf{Z}^n$ is discrete: the m_i's must become 0.

- Thus L is discrete.

PROVING “⇒” BY INDUCTION

- Induction over n = dim(span(L)).

- If n=1:
  - Let b∈L be a shortest non-zero vector of L.
  - Then span(L)=**R**b and L=**Z**b.

PROVING "$\Rightarrow$" BY INDUCTION

- Now, assume that dim(span(L))=n.

- Let $b \in L$ non-zero.
  - Then $\mathrm{span}(b) \cap L = \mathbf{Z} b_1$ because it is a 1-dim lattice.

- Let π be the projection over $b_1^{\perp}$.
  - Claim 1: $\pi(L)$ is a lattice whose dim is n-1.
  - Claim 2: If $(\pi(b_2), \ldots, \pi(b_n))$ is a basis of $\pi(L)$, then $(b_1, \ldots, b_n)$ is a basis of L.

PROOF OF CLAIM 1

- Note: the projection of a lattice may not be a lattice! Why?
- $\mathrm{span}(b)\cap L=\mathbf{Z}b_1$ and π is the projection over $b_1^{\perp}$.
- Consider an injective sequence $\pi(v_i)$ converging to 0, where $v_i\in L$.
- $|\langle v_i,b_1\rangle|\leq\|b_1\|^2/2$ by lifting: $v_i=v_i-\lfloor\langle v_i,b_1\rangle/\langle b_1,b_1\rangle\rceil b_1$
- Then the v_i's are bounded: contradiction! Why?

PROOF OF CLAIM 2

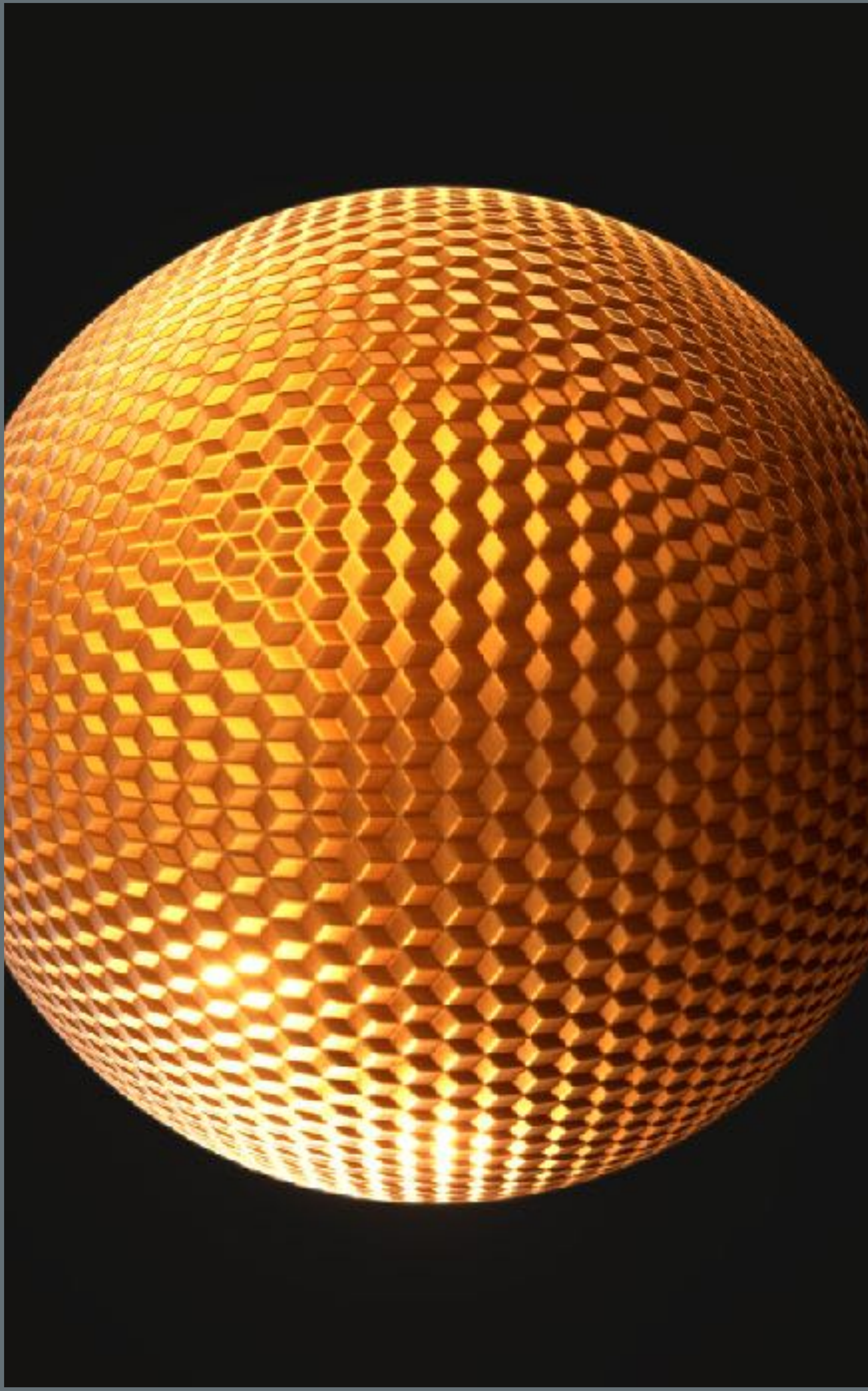
- Let $(\pi(b_2),..., \pi(b_n))$ is a basis of $\pi(L)$.
- Let $v \in L$.
  - Then $\pi(v)=x_2\pi(b_2)+\ldots+x_n\pi(b_n)$ for some $x_i \in \mathbf{Z}$.
  - And $v-(x_2b_2+\ldots+x_nb_n) \in L \cap \mathrm{span}(b_1)=\mathbf{Z}b_1$
  - So $v=x_1b_1+\ldots+x_nb_n$ where $x_i \in \mathbf{Z}$.
- Hence: $L = L(b_1,\ldots,b_n)$.

TAKE AWAY

- Key ideas:
  - Projecting a lattice to decrease its dimension: quotient with a sublattice.
  - Lifting short projections into short lattice vectors.
  - $v = \pi(v) + (v-\pi(v))$

v	$=$	$\pi(v)$	$+$	$(v-\pi(v))$
$\in$lattice		**Projection$\in$subspace$^\perp$** **$\in$lattice**		**Lift$\in$subspace**

COUNTING LATTICE POINTS



CO-VOLUMES

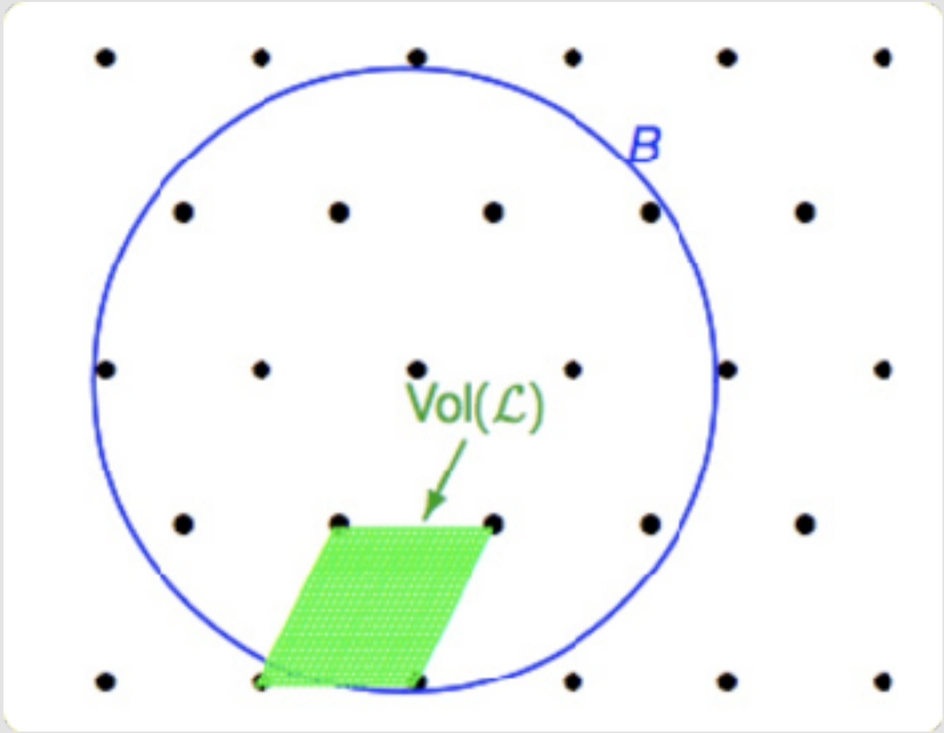
- $\mathrm{Gram}(b_1,\ldots,b_n) = \det(\langle b_i,b_j\rangle)_{1\leq i,j\leq n} \geq 0$.
- Th: If B is a basis of a lattice L, then Gram(B) only depends on L: $\sqrt{\mathrm{Gram}(B)}$ is called the (co-)volume of L.
- Ex: $\mathrm{vol}(\mathbf{Z}^n)=1$.

  $\mathrm{vol}(L)\mathrm{vol}(L^x)=1$.

THE GAUSSIAN HEURISTIC

- The volume measures the density of lattice points.
- For "nice" full-rank lattices L, and "nice" measurable sets C of $\mathbf{R}^n$:

$$\mathrm{Card}(L \cap C) \approx \frac{\mathrm{vol}(C)}{\mathrm{vol}(L)}$$



VOLUME OF THE BALL

The n-dimensional volume of a Euclidean ball of radius R in n-dimensional Euclidean space is

$$V_n(R) = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)} R^n,$$

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x}\,\mathrm{d}x$$

The unit-volume ball has radius $\sim \sqrt{\dfrac{n}{2\pi e}}$

VALIDITY OF THE GAUSSIAN HEURISTIC

- Fails for $L=\mathbf{Z}^n$, and $C=\text{Ball}(0,\sqrt{(n/10)})$.
- Easy to prove for asymptotically large balls: $1/\text{vol}(L) = \lim_{r\to\infty}$ (number of lattice points of norm $\leq r$)$/\text{vol}(\text{Ball}(0,r))$

SHORT LATTICE VECTORS

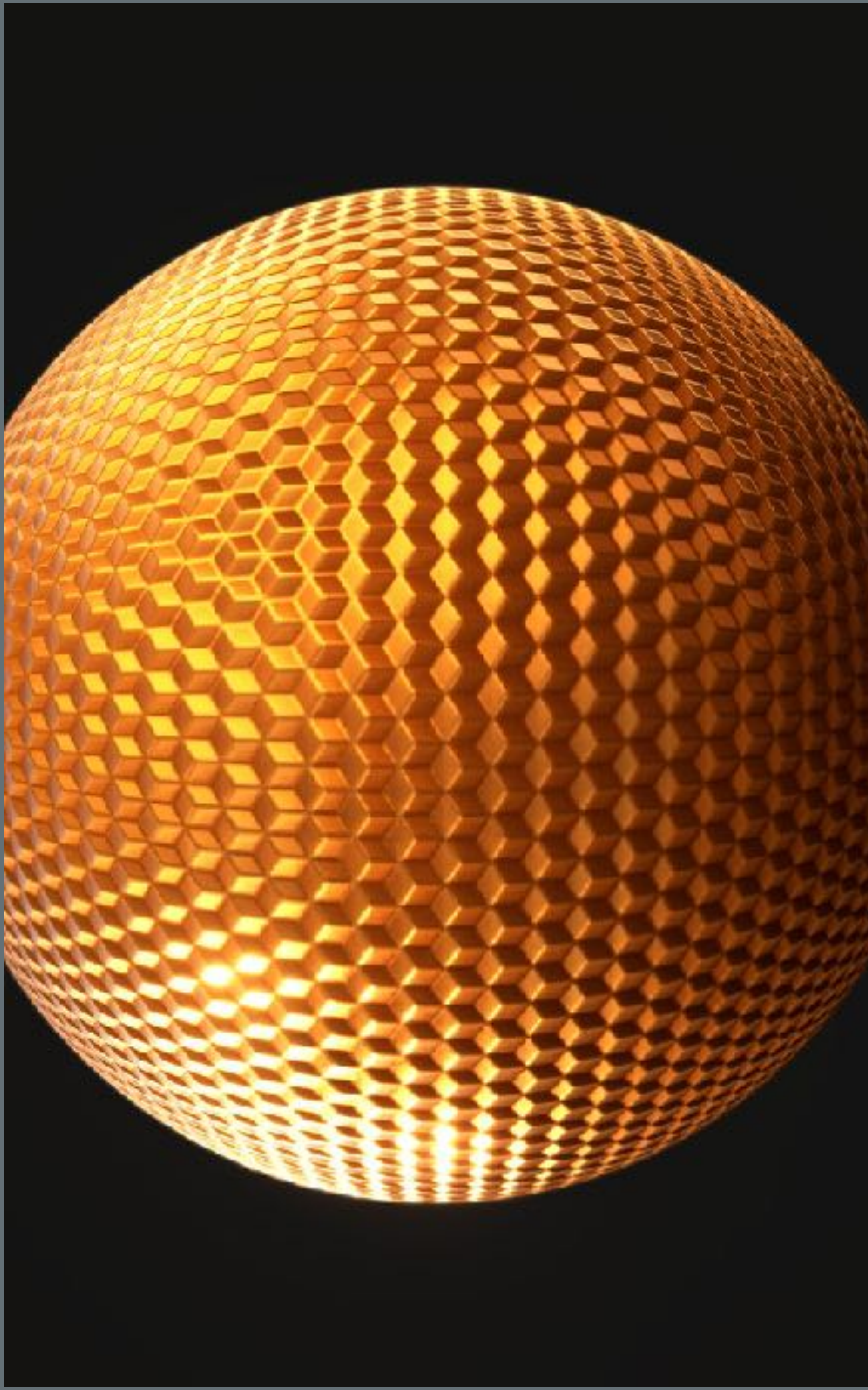
- Th: Any d-dim lattice L has exponentially many vectors of norm ≤

$$O\left(\sqrt{d}\right)\mathrm{vol}(L)^{1/d}$$

- Th: In a random d-dim lattice L, all non-zero vectors have norm ≥

$$\Omega\left(\sqrt{d}\right)\mathrm{vol}(L)^{1/d}$$

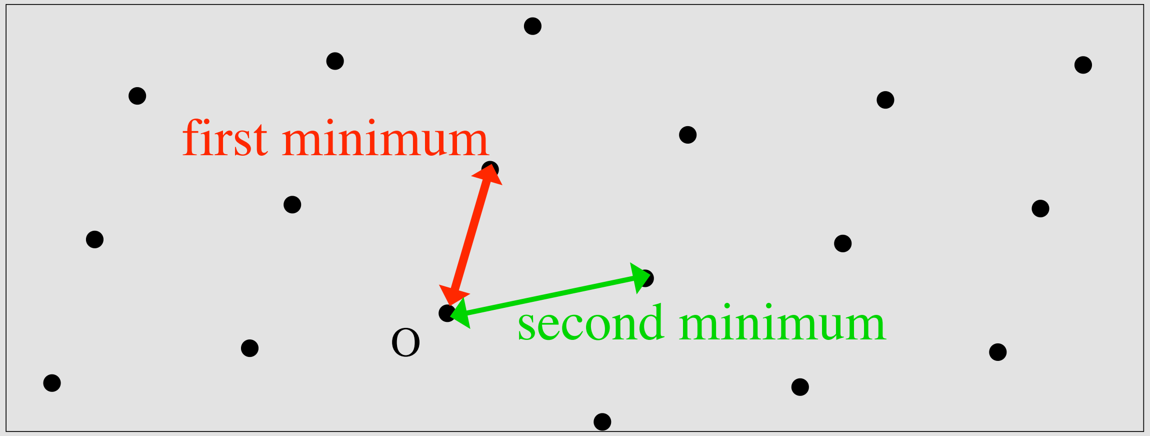
SHORT LATTICE VECTORS



THE FIRST MINIMUM

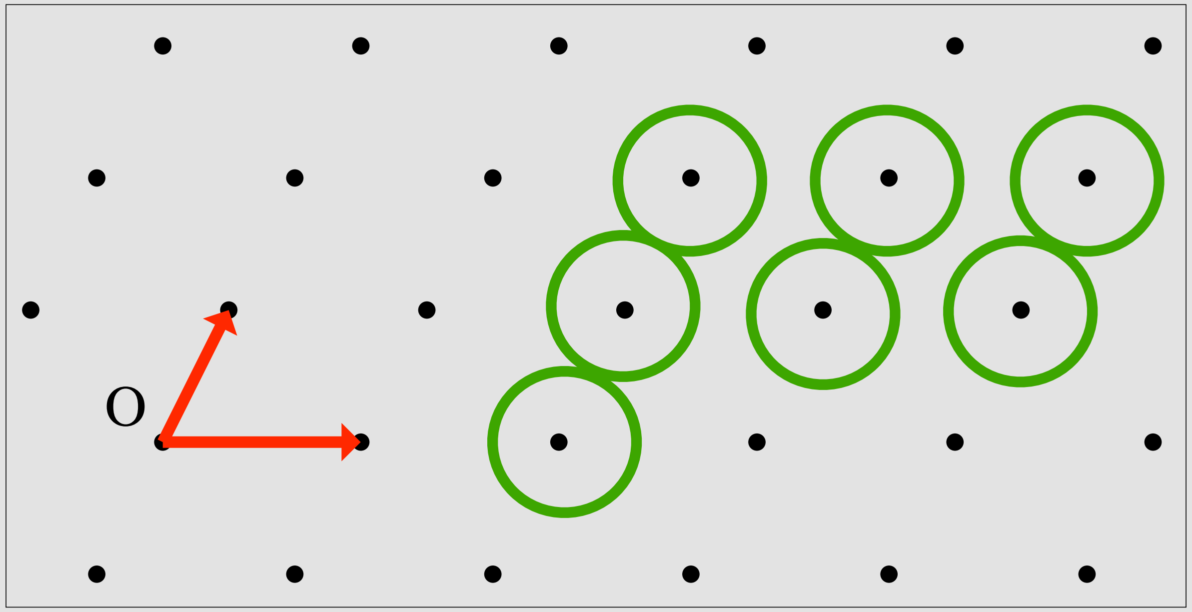
- The intersection of a lattice with any bounded set is finite.
- In a lattice L, there are non-zero vectors of minimal norm: this is the first minimum $\lambda_1(L)$ or the minimum distance..

- Ex: $\lambda_1(\mathbf{Z}^n)=1$.



LATTICE PACKINGS

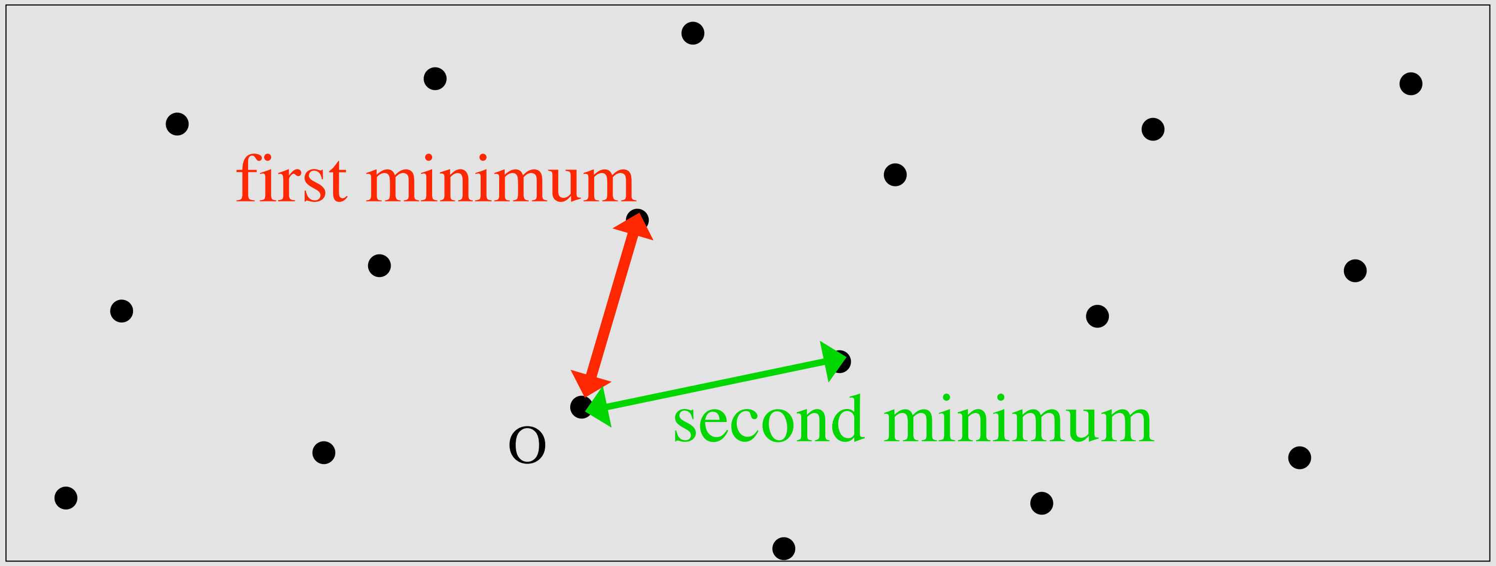
- Every lattice defines a sphere packing:



- The diameter of spheres is the first minimum of the lattice: the shortest norm of a non-zero lattice vector.

MINKOWSKI'S MINIMA

- Denoted by: $$\lambda_1(L), \ldots, \lambda_d(L)$$
- The k-th minimum is the radius of the smallest (centered) ball containing k linearly independent lattice vectors.



- There exist linearly independent lattice vectors $c_1,\ldots,c_d$ such that $\|\vec{c}_i\| = \lambda_i(L)$ for each $1 \le i \le d$.

HERMITE'S CONSTANT

HERMITE'S CONSTANT (1850)

- This is the "worst-case" for short lattice vectors.
- Hermite showed the existence of:

$$\sqrt{\gamma_d} = \max_L \frac{\lambda_1(L)}{\mathrm{vol}(L)^{1/d}}$$

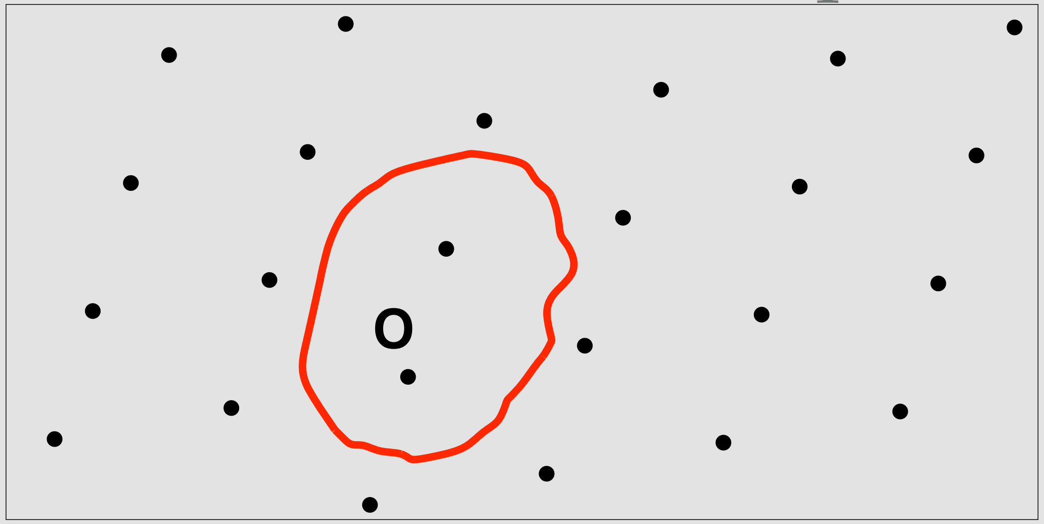
- Here, $\lambda_1(L)$ is the minimal norm of a non-zero lattice vector.
- Hermite's constant is asymptotically linear:

$$\Omega(n) \leq \gamma_n \leq O(n)$$

- The exact value of the constant is only known up to dim 8, and in dim 24.

THE EXISTENCE OF SHORT LATTICE VECTORS

- Hermite proved in 1850: $\gamma_d \leq \left(\frac{4}{3}\right)^{(d-1)/2}$
- Minkowski's theorem implies: $\gamma_d \leq d$

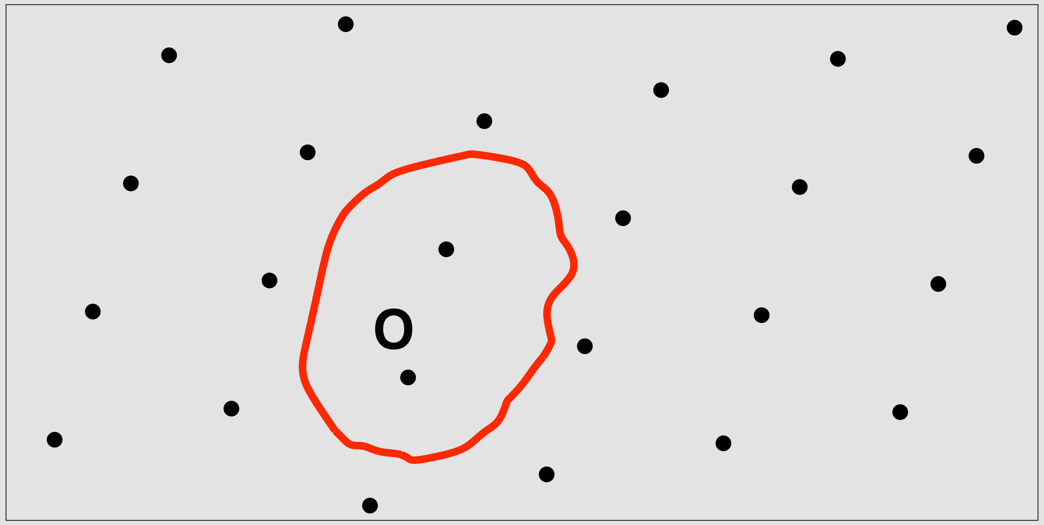


- Thus, any lattice contains a non-zero vector of norm

$$\leq \sqrt{d}\mathrm{vol}(L)^{1/d}$$

MINKOWSKI'S CONVEX BODY THEOREM (1896)

- Let L be a full-rank lattice of R^n. Let C be a measurable subset of R^n, convex, symmetric, and of measure > 2^nvol(L).
- Then C contains at least a non-zero point of L.



- The volume bound is optimal in the worst case.
- If C is furthermore compact, the > can be replaced by ≥.

APPLICATION TO A BALL

- Let C be the n-dim ball of radius r. Then its volume is r^n multiplied by:

$$v_n = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(1+\frac{n}{2}\right)} \sim \left(\frac{2e\pi}{n}\right)^{\frac{n}{2}} \frac{1}{\sqrt{\pi n}}$$

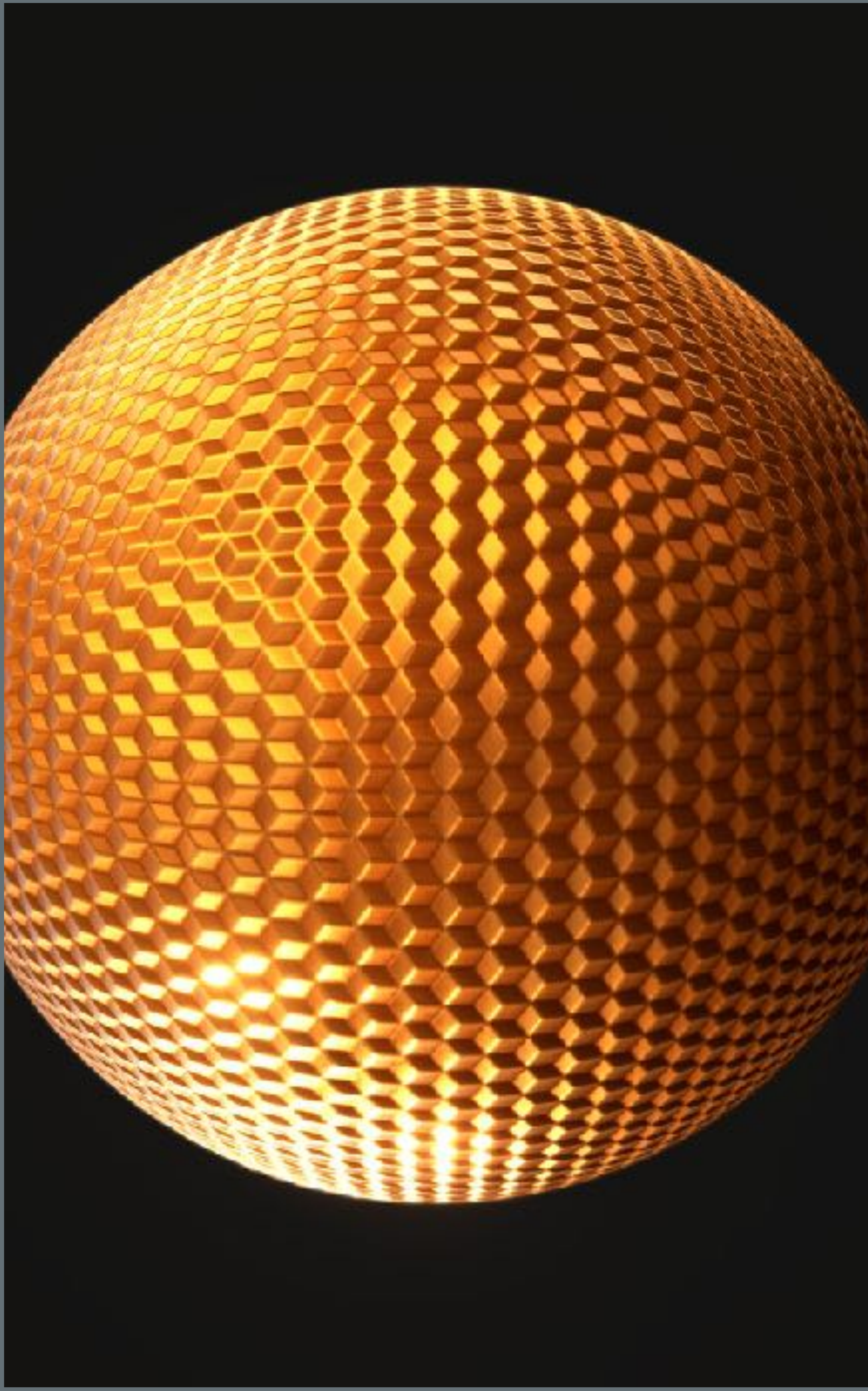
- To apply Minkowski's theorem, one can take:

$$r = \frac{2}{(v_n)^{\frac{1}{n}}} vol(L)^{\frac{1}{n}}$$

PROVING MINKOWSKI'S THEOREM

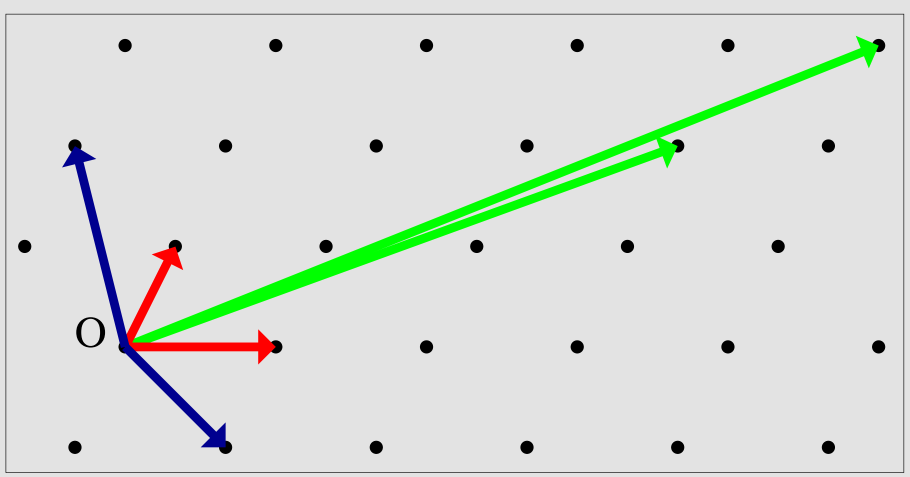
- Blichfeldt's lemma:
  - Let L be a full-rank lattice of R^n.
  - Let F be a measurable subset of R^n, of measure > vol(L).
  - Then F contains at least two distinct vectors whose difference is in L.
- Take F=C/2 to prove Minkowski.

LATTICE REDUCTION



LATTICE REDUCTION

- Euclidean spaces have orthogonal bases.
- Lattices have **reduced bases** whose vectors are short and **nearly-orthogonal**.



non-reduced

35184372088891	0
8497214565171	1

reduced

-3219347	2033901
-5233012	-7622957

MINIMA AND BASES

- As soon as d≥4, a free family reaching the minima is not necessarily a basis. Ex: the sublattice of $\mathbf{Z}^4$ formed by all vectors whose sum of coordinates is even.

Basis

1	1	0	0
1	0	-1	0
0	0	1	1
0	0	1	-1

Not a basis

1	1	0	0
1	-1	0	0
0	0	1	1
0	0	1	-1

MINIMAL BASES

- As soon as d≥5, there may not exist a basis reaching all the minima.
- Ex: this lattice whose minima are all equal to 2.

2	0	0	0	0
0	2	0	0	0
0	0	2	0	0
0	0	0	2	0
1	1	1	1	1

REDUCED BASES

- There is no basis which is “naturally” shorter than all others, as soon as $d \geq 5$.
- But the first minimum can always be extended to a basis.
- A **reduced basis** is a basis close to the minima. There are many notions of reduction.