

LATTICE EXERCISES

Notation :

- $\langle \vec{u}, \vec{v} \rangle$ is the standard Euclidean inner product of $\mathbb{R}^n$, that is $\langle \vec{u}, \vec{v} \rangle = \sum_{i=1}^n u_i v_i$.
- The Euclidean norm : $\|\vec{u}\|^2 = \langle \vec{u}, \vec{u} \rangle$.
- $\text{span}()$ denotes the subspace generated by the vectors or the set inside the parentheses. It is the smallest subspace containing the vectors or the set inside the parentheses.
- $\mathcal{B}_r(\vec{v}) = \{\vec{w} \in \mathbb{R}^n, \|\vec{v} - \vec{w}\| < r\}$ is the open ball of $\mathbb{R}^n$ of center $\vec{v}$ and radius r .

1. Properties of lattices.

(★)

Let L be a discrete subgroup of $\mathbb{R}^n$. Show that :

1. There exists $r > 0$ s.t. for all $\vec{v} \in L$, $L \cap \mathcal{B}_r(\vec{v}) = \{\vec{v}\}$.
 2. Show that any convergent sequence of L is stationary : If $(v_n)_{n \in \mathbb{N}}$ has a limit, where each $v_n \in L$, then there exists N such that $v_n = v_N$ for all $n \geq N$.
 3. Deduce that L is closed : If $(v_n)_{n \in \mathbb{N}}$ has a limit, where each $v_n \in L$, then the limit is also in L .
 4. For all $r > 0$ and $\vec{v} \in \mathbb{R}^n$, $L \cap \mathcal{B}_r(\vec{v})$ is finite.
 5. L is countable.
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2. Discreteness of subgroups.

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Let L be a subgroup of $\mathbb{R}^n$. Show that L is discrete if and only if one of the following conditions holds :

1. 0 is isolated in L , i.e. there exists $r > 0$ s.t. $L \cap \mathcal{B}_r(\vec{0}) = \{\vec{0}\}$.
 2. There is no injective sequence of L converging to zero.
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3. Examples of lattices.

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1. Show that $\mathbb{Z}^n$ is a lattice.
 2. Show that any subgroup of $\mathbb{Z}^n$ is a lattice.
 3. Let $\vec{b}_1, \dots, \vec{b}_d$ be vectors in $\mathbb{Z}^n$. Show that the set of all integral linear combinations $\mathcal{L}(\vec{b}_1, \dots, \vec{b}_d) = \{\sum_{i=1}^d x_i \vec{b}_i, x_i \in \mathbb{Z}\}$ is a lattice.
 4. Let $\vec{b}_1, \dots, \vec{b}_d$ be linearly independent vectors in $\mathbb{R}^n$. Show that $\mathcal{L}(\vec{b}_1, \dots, \vec{b}_d)$ is a lattice.
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4. Projection.

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Let L be a discrete subgroup of $\mathbb{R}^n$ and $\vec{v} \in L$ be non zero. Let H be the hyperplane orthogonal to $\vec{v}$.

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1. Show that $D = L \cap \text{span}(\vec{v})$ is a discrete subgroup of $\mathbb{R}^n$.
 2. Show that the orthogonal projection Λ of L over H is a discrete subgroup of $\mathbb{R}^n$.
 3. Let d be the dimension of $\text{span}(L)$. Show by induction over d that there exists a free family $(\vec{b}_1, \dots, \vec{b}_d)$ of $\mathbb{R}^n$ such that $L = \mathcal{L}(\vec{b}_1, \dots, \vec{b}_d)$. Such a $(\vec{b}_1, \dots, \vec{b}_d)$ is called a $\mathbb{Z}$ -basis of L .
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5. Successive Minima.

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Let L be a lattice of $\mathbb{R}^n$ of rank d . Show that :

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1. For any $k \in \{1, \dots, d\}$, there exists a unique $\lambda_k(L) > 0$ such that $\text{span}(L \cap \mathcal{B}_{\lambda_k(L)}(\vec{0}))$ has dimension $< k$ and there exist k linearly independent vectors in L of norm $\leq \lambda_k(L)$. This $\lambda_k(L)$ is called the k -th minimum of L .
 2. There exist linearly independent lattice vectors $\vec{a}_1, \dots, \vec{a}_d \in L$ such that $\|\vec{a}_k\| = \lambda_k(L)$ for all $1 \leq k \leq d$.
 3. Let L be the set of all $(y_1, \dots, y_4) \in \mathbb{Z}^4$ such that $\sum_{i=1}^4 y_i$ is even. Show that L is a lattice and that there exist linearly independent lattice vectors $\vec{a}_1, \dots, \vec{a}_4 \in L$ such that $\|\vec{a}_i\| = \lambda_i(L)$ for all i but surprisingly, $L \neq \mathcal{L}(\vec{a}_1, \dots, \vec{a}_4)$.
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6. Bases and Volume.

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Let $B = (\vec{b}_1, \dots, \vec{b}_n)$ be a basis of a lattice L . Let $U = (u_{i,j})_{1 \leq i,j \leq n}$ be an $n \times n$ matrix over $\mathbb{R}$. Let $\vec{c}_i = \sum_{j=1}^n u_{i,j} \vec{b}_j$ for $1 \leq i \leq n$. Show that :

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1. $\vec{c}_1, \dots, \vec{c}_n \in L$ iff all $u_{i,j} \in \mathbb{Z}$.
 2. $(\vec{c}_1, \dots, \vec{c}_n)$ is a basis of L iff all $u_{i,j} \in \mathbb{Z}$ and $\det U = \pm 1$.
 3. $\det_{1 \leq i,j \leq n} \langle \vec{b}_i, \vec{b}_j \rangle > 0$. Thus, we may define $\text{vol}(B) = \sqrt{\det_{1 \leq i,j \leq n} \langle \vec{b}_i, \vec{b}_j \rangle}$.
 4. If $C = (\vec{c}_1, \dots, \vec{c}_n)$ is a basis of L , then $\text{vol}(B) = \text{vol}(C)$. Thus, we may define the lattice volume $\text{vol}(L) = \text{vol}(B)$.
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7. Duality.

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Let L be a lattice of $\mathbb{R}^n$. Show that :

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1. For any group morphism $f : L \rightarrow \mathbb{Z}$, there exists a unique $\vec{v} \in \text{span}(L)$ s.t. for all $\vec{w} \in L$, $f(\vec{w}) = \langle \vec{v}, \vec{w} \rangle$.
 2. The set $L^\times$ of all $\vec{v} \in \text{span}(L)$ such that for all $\vec{w} \in L$, $\langle \vec{v}, \vec{w} \rangle \in \mathbb{Z}$ is a lattice, called the dual lattice of L .
 3. The additive group of all group morphisms $f : L \rightarrow \mathbb{Z}$ is isomorphic to $L^\times$.
 4. $\text{vol}(L)\text{vol}(L^\times) = 1$.
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8. Classical Lattices.

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Let A_n be the set of all $(x_1, \dots, x_{n+1}) \in \mathbb{Z}^{n+1}$ such that $\sum_{i=1}^{n+1} x_i = 0$. Let D_n be the set of all $(x_1, \dots, x_n) \in \mathbb{Z}^n$ such that $\sum_{i=1}^n x_i$ is even.

1. Show that A_n and D_n are lattices of rank n .
 2. Give bases of A_n and D_n .
 3. What is the co-volume of A_n and D_n ?
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9. Bonus.

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Let L be a full-rank lattice in $\mathbb{R}^n$.

1. Let $v_n(r)$ denote the volume of $\mathcal{B}_r(\vec{0})$ and $N(r)$ denote the cardinal of $L \cap \mathcal{B}_r(0)$. Show that when r grows to ∞ , $N(r)/v_n(r)$ converges to $1/\text{vol}(L)$. Thus, $N(r)$ is asymptotically equivalent to $v_n(r)/\text{vol}(L)$.
 2. Let E be a d -dimensional subspace of $\mathbb{R}^n$. Let $F = E^\perp$ be the orthogonal supplement of E . We know that $E \cap L$ is a lattice. Show that the orthogonal projection of L over F is a lattice if and only if the rank of $E \cap L$ is equal to d .
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