Lattice Exercises

Notation:

— $\langle \vec{u}, \vec{v} \rangle$ is the standard Euclidean inner product of $\mathbb{R}^n$, that is $\langle \vec{u}, \vec{v} \rangle = \sum_{i=1}^{n} u_i v_i$.

— The Euclidean norm: $\| \vec{u} \|_2 = \langle \vec{u}, \vec{u} \rangle$.

— $\text{span}(\cdot)$ denotes the subspace generated by the vectors or the set inside the parentheses. It is the smallest subspace containing the vectors or the set inside the parentheses.

— $B_r(\vec{v}) = \{ \vec{w} \in \mathbb{R}^n, \| \vec{v} - \vec{w} \| < r \}$ is the open ball of $\mathbb{R}^n$ of center $\vec{v}$ and radius $r$.

1. Properties of lattices. (*)

Let $L$ be a discrete subgroup of $\mathbb{R}^n$. Show that:

1. There exists $r > 0$ s.t. for all $\vec{v} \in L$, $L \cap B_r(\vec{v}) = \{ \vec{v} \}$.

2. Show that any convergent sequence of $L$ is stationary: in particular, $L$ is closed.

3. For all $r > 0$ and $\vec{v} \in \mathbb{R}^n$, $L \cap B_r(\vec{v})$ is finite.

4. $L$ is countable.

2. Discreteness of subgroups. (*)

Let $L$ be a subgroup of $\mathbb{R}^n$. Show that $L$ is discrete if and only if one of the following conditions holds:

1. $0$ is isolated in $L$, i.e. there exists $r > 0$ s.t. $L \cap B_r(\vec{0}) = \{ \vec{0} \}$.

2. There is no injective sequence of $L$ converging to zero.

3. Examples of lattices. (*)

1. Show that $\mathbb{Z}^n$ is a lattice.

2. Show that any subgroup of $\mathbb{Z}^n$ is a lattice.

3. Let $\vec{b}_1, \ldots, \vec{b}_d$ be vectors in $\mathbb{Z}^n$. Show that the set of all integral linear combinations $\mathcal{L}(\vec{b}_1, \ldots, \vec{b}_d) = \{ \sum_{i=1}^{d} x_i \vec{b}_i, x_i \in \mathbb{Z} \}$ is a lattice.

4. Let $\vec{b}_1, \ldots, \vec{b}_d$ be linearly independent vectors in $\mathbb{R}^n$. Show that $\mathcal{L}(\vec{b}_1, \ldots, \vec{b}_d)$ is a lattice.

4. Projection. (**)  

Let $L$ be a discrete subgroup of $\mathbb{R}^n$ and $\vec{v} \in L$ be non zero. Let $H$ be the hyperplane orthogonal to $\vec{v}$.

1. Show that $D = L \cap \text{span}(\vec{v})$ is a discrete subgroup of $\mathbb{R}^n$.

2. Show that the orthogonal projection $\Lambda$ of $L$ over $H$ is a discrete subgroup of $\mathbb{R}^n$.

3. Let $d$ be the dimension of $\text{span}(L)$. Show by induction over $d$ that there exists a free family $(\vec{b}_1, \ldots, \vec{b}_d)$ of $\mathbb{R}^n$ such that $L = \mathcal{L}(\vec{b}_1, \ldots, \vec{b}_d)$. Such a $(\vec{b}_1, \ldots, \vec{b}_d)$ is called a $\mathbb{Z}$-basis of $L$. 

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5. **Successive Minima.**

Let $L$ be a lattice of $\mathbb{R}^n$ of rank $d$. Show that:

1. For any $k \in \{1, ..., d\}$, there exists a unique $\lambda_k(L) > 0$ such that $\text{span}(\{L \cap B_{\lambda_k(L)}(0)\})$ has dimension $< k$ and there exist $k$ linearly independent vectors in $L$ of norm $\leq \lambda_k(L)$. This $\lambda_k(L)$ is called the $k$-th minimum of $L$.

2. There exist linearly independent lattice vectors $\vec{a}_1, ..., \vec{a}_d \in L$ such that $\|a_k\| = \lambda_i(L)$ for all $1 \leq k \leq d$.

3. Let $L$ be the set of all $(y_1, ..., y_d) \in \mathbb{Z}^d$ such that $\sum_{i=1}^d y_i$ is even. Show that $L$ is a lattice and that there exist linearly independent lattice vectors $\vec{a}_1, ..., \vec{a}_d \in L$ such that $\|a_i\| = \lambda_i(L)$ for all $i$ but surprisingly, $L \neq L(\vec{a}_1, ..., \vec{a}_4)$.

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6. **Bases and Volume.**

Let $B = (\vec{b}_1, ..., \vec{b}_n)$ be a basis of a lattice $L$. Let $U = (u_{i,j})_{1 \leq i,j \leq n}$ be an $n \times n$ matrix over $\mathbb{R}$. Let $\vec{c}_i = \sum_{j=1}^n u_{i,j} \vec{b}_j$ for $1 \leq i \leq n$. Show that:

1. $\vec{c}_1, ..., \vec{c}_n \in L$ iff all $u_{i,j} \in \mathbb{Z}$.

2. $(\vec{c}_1, ..., \vec{c}_n)$ is a basis of $L$ iff all $u_{i,j} \in \mathbb{Z}$ and $\det U = \pm 1$.

3. $\det_{1 \leq i,j \leq n}(\vec{b}_i, \vec{b}_j) > 0$. Thus, we may define $\text{vol}(B) = \sqrt{\det_{1 \leq i,j \leq n}(\vec{b}_i, \vec{b}_j)}$.

4. If $C = (\vec{c}_1, ..., \vec{c}_n)$ is a basis of $L$, then $\text{vol}(B) = \text{vol}(C)$. Thus, we may define the lattice volume $\text{vol}(L) = \text{vol}(B)$.

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7. **Duality.**

Let $L$ be a lattice of $\mathbb{R}^n$. Show that:

1. For any group morphism $f : L \to \mathbb{Z}$, there exists a unique $\vec{v} \in \text{span}(L)$ s.t. for all $\vec{w} \in L$, $f(\vec{w}) = \langle \vec{v}, \vec{w} \rangle$.

2. The set $L^\times$ of all $\vec{v} \in \text{span}(L)$ such that for all $\vec{w} \in L$, $\langle \vec{v}, \vec{w} \rangle \in \mathbb{Z}$ is a lattice, called the dual lattice of $L$.

3. The additive group of all group morphisms $f : L \to \mathbb{Z}$ is isomorphic to $L^\times$.

4. $\text{vol}(L)\text{vol}(L^\times) = 1$.

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8. **Bonus.**

Let $L$ be a lattice in $\mathbb{R}^n$, i.e. a discrete subgroup.

1. Let $v_n(r)$ denote the volume of $B_r(\vec{0})$ and $N(r)$ denote the cardinal of $L \cap B_r(0)$. Show that when $r$ grows to $\infty$, $N(r)/v_n(r)$ converges to $1/\text{vol}(L)$. Thus, $N(r)$ is asymptotically equivalent to $v_n(r)/\text{vol}(L)$.

2. Let $E$ be a $d$-dimensional subspace of $\mathbb{R}^n$. Let $F = E$ be the orthogonal supplement of $E$. We know that $E \cap L$ is a lattice. Show that the orthogonal projection of $L$ over $F$ is a lattice if and only if the rank of $E \cap L$ is equal to $d$. 

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\textbf{Page 2}