Graphs Algorithms 2-29-1
Solutions to (some) Exercises
(Correction)

Exercice 11 :

Show that graphs G of treewidth at most k with $k \geq 1$ have strictly less than $k|V(G)|$ edges.

> ▷ **Correction :** As seen in the course in any graph of tree width at most k there exists a vertex of degree at most k. Remove it and apply induction

Exercice 14 :

Prove that if G contains (as a subgraph) a complete bipartite graph with parts A and B, then in every tree decomposition there exists a bag that contains A or a bag that contains B.

> ▷ **Correction :** Consider an optimal tree-decomposition (T, W) of G. For a vertex x of G, denote as usual by T_x the subtree of T induced by the nodes of T whose corresponding bag contains x.
> By Helly's property if no bag entirely contains A, there exists $(a, a') \in A^2$ such that the subtrees T_a and $T_{a'}$ do not intersect. Therefore there exists an edge tt' of T such that T_a and $T_{a'}$ live in distinct connected component of $T \setminus \{tt'\}$.
> Now consider any vertex $b \in B$. Since it is adjacent to a and a', T_b should intersect both T_a and $T_{a'}$ and therefore contain t and t'. Since this is true for any b, we get that the bag W_t (and in fact also $W_{t'}$) entirely contains B.

> ▷ **Correction :** (Alternate solution) Consider an optimal tree-decomposition (T, W) of G and Delete all vertices but the vertices of the complete bipartite. Any bag that is not a leaf must be a cutset, but the only cutsets in a complete bipartite graph are one of the two sides.

Exercice 15 :

Prove that if x and y are two vertices that are joined by $k + 1$ internally vertex disjoints paths, then in every tree decomposition of G of width at most k, there exists a bag containing both x and y.

> ▷ **Correction :** Consider an optimal decomposition where no bag is included in another (see Lecture). In such a decomposition for any edge tt' the intersection $W_t \cap W_{t'}$ is a separator of G of size at most $tw(G)$.
> If there is no bag containing x and y, we thus would get a separator of size k between x and y which is not possible if there exists $k + 1$ internally vertex disjoints paths between the two.

Exercice 16 :

Prove that if G is $K_{2,3}$-minor-free then $tw(G) \leq 3$.

▷ **Correction :** Note that if the graph admits a separator $S = \{a, b\}$ of size 2, then we can do as in the proof of the fact that K_4-minor free graphs have treewidth at most 2. Either ab is an edge and we can decompose by induction the two sides and glue them by a clique sum operation. Or ab is not an edge, and we can prove that adding the edge does not change the fact that both parts are still $K_{2,3}$ minor- free and apply again the clique sum argument.

So we can assume the graph is 3 connected. If there exists two non adjacent vertices x and y, by Menger's Theorem there exists 3 vertex disjoint paths linking them, and thus a $K_{2,3}$ minor, which is not possible. So G is a clique. But K_5 contains $K_{2,3}$ (as a subgraph). So G is a clique on at most 4 vertices and thus has treewidth at most 3.