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INFORMATIQUE

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Excluding slightly more than a cycle

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Contents

Introduction en français)

L'objet principal de cette thèse est l'étude de classes de graphes héréditaires. Une classe de graphes est héréditaire si et seulement si, pour tout graphe G dans la classe, tout graphe H obtenu en supprimant des sommets de G est dans la classe (par contre, si on enlève des arrêtes de G, il est possible que l'on sorte de la classe). Autrement dit tout sous-graphe induit de G est dans la classe. Il est facile de voir que toute classe héréditaire peut-être définie comme l'ensemble des graphes qui ne contiennent pas une certaine liste de graphes en tant que sous-graphes induits.

La relation "être un sous-graphe induit de" est trés naturelle en mathématique, elle correspond à la notion de "sous-structure" que l'on rencontre partout en mathématique. Dans les années 1960, Gabriel Dirac à ouvert ce domaines de la théorie des graphes avec son travail sur les graphes triangulés qui sont les graphes ne contenant pas de trous en tant que sous-graphe induit (un *trou* est un cycle sans cordes de longueur au moins quatre) . A la même période, Claude Berge proposait deux conjectures sur les graphes parfaits qui ont été le point de départ de beaucoup de travaux trés profonds, jusqu'à la résolution en 2002 de la conjecture forte des graphes parfaits par Chudnovsky, Robertson, Seymour et Thomas.

Dans les années 1980 commença ce qu'on appelle le "graph minor project", développé par Robertson et Seymour, qui décrit une théorie aussi profonde que générale sur les classes de graphes définies par interdiction de mineurs (à la place de sous-graphes induits). Il est naturel de se demander si il existe une théorie analogue qui décrirait le monde des classes définies par interdiction de sous-graphes induits. Jusqu'à maintenant, les classes de graphes definies par interdiction de sous-graphes induits parraissent trop compliquées et bordéliques pour les décrire dans une théorie générale. Malgré tout, certaines conjectures comme la conjecture d'Erdős-Hajnal suggèrent qu'une telle théorie pourraient exister.

Dans ce document, la majorité des résultats sont sur des classes de graphes particulières. La plupart d'entre elles sont définies en interdisant certaines configurations de Truemper. Les configurations de Truemper sont quatre familles de graphes qui, toutes ensembles, peuvent être vues comme une subtile généralisation des trous. Elles jouent un rôle central dans le monde de la théorie structurelle des graphes. Plus précisément, elles sont une structure clé pour comprendre les classes de graphes définies par interdiction de trous de parité donnée, comme dans les graphes équilibrés ("balanced graphs"), les graphes sans trou pairs ou impairs et les graphes parfaits. Avant de donner le plan du document, nous allons définir une première famille de graphes qui apparaissent un peu partout dans la thèse: les k-roues. Une *k-roue* est un graphe formé par un cycle sans corde et un sommet, extérieur au cycle, qui a au moins k voisins dans le cycle.

Plan du document

Chapitre 1: Définitions de bases
On rappelle dans ce chapitre les définitions de bases de la théorie des graphes. On y donne aussi un apperçu bref et informel de la théorie de la complexité.

Chapter 2: Configurations de Truemper
Dans ce chapitre nous introduisons les configurations de Truemper, nous expliquons leurs relations avec les trous de parités données et leurs rôles dans la théorie structurelle des graphes. On y explique aussi le concept de *théorème de décomposition* et quelques techniques pour l'utiliser. Les méthodes expliquées dans ce chapitre sont beaucoup utilisées tout au long du document, nous les illustrons avec l'exemple simple des graphes triangulés.

Chapitre 3: La technique du moplex
On décrit, grâce à l'algorithm LexBFS, des propriétés structurelles de différentes classes de graphes, pouvant toutes être définies en interdisant certaines configurations de Truemper en tant que sous-graphes induits. Les résultats présentés dans ce chapitre sont le fruit d'un tavail en collaboration avec Pierre Charbit, Nicolas Trotignon et Kristina Vušković (soumis à Discrete Mathematics [3]).

Chapitre 4: Classes définies par contrainte sur la connectivité
Ca chapitre traite majoritairement de la classe bien connue des graphes minimallement 2-connexes. On commence par agrandir la classe pour la rendre héréditaire, puis, en appliquant les métodes décrites dans le chapitre 2, nous donnons de nouvelles preuves de résultats connus sur les graphes minimallement 2-connexe. Ensuite, nous expliquons pourquoi nous ne pouvons pas mener le même genre de travaux sur les graphes critiquement 2-connexe et nous montrons le lien qu'entretiennent les graphes critiquement 2-connexe avec les graphes qui ne contiennent pas de 2-roues en tant que sous-graphes induits.

Chapitre 5: Interdire les 2-roues en tant que sous-graphes induits
On étudie la classe des graphes qui ne contiennent pas de 2-roues en tant que sous-graphes induits. On donne un charactérisation structurelle complète de cette classe de laquelle nous obtenons un algorithme polynomial pour détecter les 2-roues induites et une solution au problème de l'arrête-coloration pour cette classe. Les Chapitres 4 et 5 sont le fruit d'un travail commun avec Marko Radovanović, Nicolas Trotignon et Kristina Vušković qui est publié au *SIAM Journal on Discrete Mathematics* [7] .

Chapitre 6: Les graphes équillibrables
Ce chapitre est consacré à la preuve d'une conjecture de Conforti et Rao sur les graphes équilibrés. La preuve repose sur un théorème de décomposition des graphes équilibrés prouvé par Conforti, Cornuéjols, Kapoor et Vušković en 2001 et sur une nouvelle utilisation des décompositions extrêmes. Ces travaux ont étés réalisé en collaboration avec Marko Radovanović, Nicolas Trotignon, Théophile Trunck et Kristina Vušković et sont acceptés dans *Journal of Graph Theory* [6]

Chapitre 7: Exclure les k-wheel en tant que sous-graphes
On étudie la classe des graphes qui ne contiennent pas de k-roues en tant que sous-graphes. Le principal résultat est que les graphes sans 4-roues sont 4-colorables. On montre aussi quelques propriétés les graphes sans 3-roues. Les résultats concernant les graphes sans 3-roues ont été obtenus en collaboration avec Frédéric Havet et Nicolas Trotignon [4], le résultat sur les graphes sans 4-roues est le travail de l'auteur seul et est accepté dans *Journal of Graph Theory* [1].

Chapitre 8: Exclure les cycles avec un nombre fixé de cordes
On démontre que la classe de graphes qui ne contiennent pas de cycles avec exactement k-cordes est χ-bornée pour $k = 2$ et $k = 3$. Ce travaille a été mené avec Nicolas Bousquet et est soumis dans Discrete Mathematics [2].

On peut observer que, dans chaque chapitre, nous étudions des classes de graphes définies en interdisant un cycle plus un sommet en dehors du cycle qui a des voisins dans le cycle, ou un chemin le liant au cycle etc... Il est bien connu qu'un graphe sans cycles est une fôret, les résultats présentés dans ce document peuvent donc être vus comme des généralisation de ce fait!

Introduction

This thesis is concerned with classes of graphs defined by forbidding induced subgraphs. The "induced subgraph" relation is mathematically very natural, it corresponds in graphs to the notion of substructure that is everywhere in mathematics. We may first observe that any class of graphs closed under taking induced subgraphs can be defined by forbidding a list of induced subgraphs. In the 1960's, Gabriel Dirac started the field with pioneer work on chordal graphs, that are graphs that do not contain holes (a *hole* is a chordless cycle of length at least four) as induced subgraph. At the same period, Claude Berge proposed his two conjectures on perfect graphs that has been the starting point of a very rich theory, until the resolution of the strong perfect graph conjecture by Chudnovsky, Robertson, Seymour and Thomas in 2002.

In the 1980's started the Graph Minor Project, developed by Robertson and Seymour, that describes a very deep and general theory about any class of graphs closed under taking minor (instead of induced subgraphs). It is natural to wonder whether there exists such a general theory for classes closed under taking induced subgraphs. Until now, classes closed under taking induced subgraphs seem to be too messy to hold in a general theory, but some conjectures like the Erdős-Hajnal's Conjecture suggest that there might be some features shared by every class of graphs defined by forbidding induced subgraphs.

In this document, most of the theorems are concerned with particular classes of graphs. Most of them are defined by forbidding some Truemper configurations as induced subgraphs. Truemper configurations are formed by four families of graphs that, all together, can be seen as a slight generalization of holes. Truemper configurations, that first appeared for polyhedral reasons, play a special role in the world of structural graph theory. More precisely, they appear as a key structure to understand some classes of graphs defined by forbidding holes of prescribed parity such as balanced graphs, even and odd-hole free graphs and perfect graphs. Before we give an outline of the thesis, let us define a family of graphs that are very present in the document: the k-wheels. A *k-wheel* is a graph formed by a chordless cycle and a vertex, outside the cycle, that has at least k neighbors in the cycle.

Outline of the document

Chapter 1: Basic definitions

We recall some definitions and notation of graph theory that we use all along this document as well as a brief and informal overview of complexity theory.

Chapter 2: Truemper Configurations

We introduce Truemper configurations and explain their links with holes of prescribed parity as well as their role in structural graph theory. We also introduce the notion of *decomposition theorem* and several techniques to get algorithmic and structural properties from such a theorem. The methods explained in this chapter are used in several proofs of this document, we illustrate them with the very simple example of chordal graphs.

Chapter 3: The moplex technique
We use the algorithm LexBFS to get structural properties and fast algorithms for several classes of graphs defined by forbidding induced subgraphs, all of them being Truemper configurations. Results of this chapter is a joint work with Pierre Charbit, Nicolas Trotignon and Kristina Vušković submitted to Discrete Mathematics [3].

Chapter 4: Constraint on connectivity
This chapter is mainly dedicated to the well known class of minimally 2-connected graphs. After enlarging the class, we show how to get new easy proofs of several known theorems on minimally 2-connected graphs using methods described in Chapter 2. Then, we explain why the same kind of work cannot be lead on the class of critically 2-connected graphs and we point out a link between graphs that do not contain 2-wheels as induced subgraphs and critically 2-connected graphs.

Chapter 5: 2-wheel-free graphs
We study the class of graphs that do not contain 2-wheels as induced subgraphs. We give a complete structural characterization for this class and, as an application, we describe a polynomial-time algorithm to recognize them as well as a solution to the edge-color problem. Chapters 4 and 5 come from a joint work with Marko Radovanović, Nicolas Trotignon and Kristina Vušković published in *SIAM Journal on Discrete Mathematics* [7] .

Chapter 6: Balanceable graphs
We prove a conjecture of Conforti and Rao on linear balanceable graphs. The proof leans on a very deep decomposition theorem for balanceable graphs proved by Conforti, Cornuéjols, Kapoor and Vušković in 2001, and some new idea on extreme decompositions. It comes from a joint work with Marko Radovanović, Nicolas Trotignon, Théophile Trunck and Kristina Vušković accepted in *Journal of Graph Theory* [6]

Chapter 7: Excluding k-wheels as subgraphs
This chapter is concerned with classes of graphs defined by forbidding k-wheels as subgraphs. The main result of this chapter is the proof that graphs with no 4-wheels as subgraphs are 4-colourable. We also give some new properties of some subclasses of the class of graphs that do not contain 3-wheels as subgraphs. Results concerned with 3-wheels come from a joint unpublished work with Frédéric Havet and Nicolas Trotignon [4], the result on 4-wheels is proved by the author of this document alone and is accepted in *Journal of Graph Theory*.

Chapter 8: Excluding cycles with a fixed number of chords
We give some χ-boundedness results about classes of graph that do not contain cycles with exactly k chords for $k = 2$ and $k = 3$. It is a joint work with Nicolas Bousquet submitted to Discrete Mathematics [2].

Observe that, in each chapter, we exclude a cycle plus a vertex outside the cycle that has neighbors in the cycle, or a path linking two vertices of the cycle etc etc... It is well-known that a graph with no cycle is a forest, so every result of this document can be seen as a generalization of this simple fact!

Chapter 1

Definitions and preliminaries

In this chapter we give some definitions and notation that we use all along the document. Most of them follow from classical text-book such as [14] and [42].

1.1 Graphs, paths, cycles and classical invariants

If V is a set and k a positive integer, we note $\binom{V}{k}$ the set of subsets of exactly k elements of V. A graph is a pair $G = (V, E)$ of finite sets such that E is a subset of $\binom{V}{2}$. The elements of V are the *vertices* of G, the element of E are its *edges*. For notational simplicity, we write uv for the unordered pair $\{u, v\}$. The vertex set of a graph G is referred to as $V(G)$, its edge set as $E(G)$. We refer to the number of vertices of a graph as the *order* of the graph. The graph of order 0 is called the *empty graph*.

Let G be a graph, u and v two distinct vertices of G and A a subset of $V(G)$. If uv is an edge of G, then u and v are the *extremities* of the edge uv and we say that u is *adjacent* to v, or that u *sees* v, or that u is a *neighbor* of v. If $uv \notin E(G)$, we say that u is *non-adjacent* to v, or that u is a *non-neighbor* of v. We denote by $N(v)$ the *neighborhood* of v, that is the set of neighbors of v in G and $N[v] = N(v) \cup \{v\}$ the *closed neighborhood* of v. The degree of v in G, denoted by $d(v)$ is the number of neighbors of v in G. We denote by $N(A)$ the set of vertices of $V(G) \setminus A$ that see at least one vertex in A, and $N[A] = N(A) \cup A$. If $v \notin A$, we denote by $N_A(v) = N(v) \cap A$ and $d_A(v) = |N(v) \cap A|$.

A *path* P is a sequence of distinct vertices $x_1x_2 \dots x_k$, $k \geq 1$, such that x_ix_{i+1} is an edge for all $1 \leq i < k$. Edges x_ix_{i+1}, for $1 \leq i < k$, are called the *edges of* P. Vertices x_1 and x_k are the *endvertices* of P, and $x_2 \dots x_{k-1}$ is the *interior* of P. P is referred to as a p_1p_k*-path*. For $1 \leq i \leq j \leq k$, we write $x_iPx_j := x_i \dots x_j$, $\mathring{P} := x_2 \dots x_{k-1}$, $\mathring{x}_jP\mathring{x}_i := x_{j+1} \dots x_{i-1}$. Two paths P_1 and P_2 that share their endvertices are said to be *internally disjoint* if their interior is disjoint.

A cycle C is a sequence of vertices $p_1p_2 \dots p_kp_1$, $k \geq 3$, such that $p_1 \dots p_k$ is a path and p_1p_k is an edge. Edges p_ip_{i+1}, for $1 \leq i < k$, and edge p_1p_k are called the *edges of* C. Let Q be a path or a cycle. The *length* of Q is the number of its edges. An edge $e = uv$ is a *chord* of Q if $u, v \in V(Q)$, but uv is not an edge of Q. A path or a cycle Q in a graph G is *chordless* if no edge of G is a chord of Q. A chordless cycle of length at least 4 is called a *hole*.

Let G be a graph. The graph G is called a *clique* if $E(G) = \binom{V}{2}$. The clique on 3 vertices is called a *triangle* and we denote by K_n the clique on n vertices. If A is a subset of $V(G)$ such that

$E(G[A]) = \emptyset$, we say that A is a *stable set*. We denote by $\omega(G)$ the size of the largest clique of G and by $\alpha(G)$ the size of the largest stable set in G. We call *k-coloration* of G any partition of $V(G)$ into k sets $A_1, \ldots, A_k$ such that A_i is a stable set for $1 \leq i \leq k$. We call *chromatic number* of G the smallest integer k such that G admits a k-coloration. The chromatic number is denoted by $\chi(G)$. The largest degree of G is denoted by $\Delta(G)$ and the smallest by $\delta(G)$.

A hereditary class of graphs is *χ-bounded* (see [53]) if for some function f, every graph G in the class satisfies $\chi(G) \leq f(\omega(G))$.

A graph G is said to be a *complete k-partite graph* if $V(G)$ can be partitioned into k non-empty subsets $A_1, \ldots, A_k$ such that, for $i = 1, \ldots, k$, A_i is a stable set and, for any $\{i, j\} \subseteq \{1, \ldots, k\}$, there are all possible edges between A_i and A_j. G is denoted by $K_{a_1,\ldots,a_k}$ where $a_i = |A_i|$ for $i = 1, \ldots, k$. If $k = 2$ then G is said to be a *complete bipartite graph* and if $k = 3$, G is said to be a *complete tripartite graph*. The graph $K_{1,1,2}$ is called a *diamond*.

The complement of a graph G, denoted $\overline{G}$ is defined by $V(\overline{G}) = V(G)$ and $E(\overline{G}) = \{uv : u, v \in V(G) \text{ and } uv \notin E(G)\}$.

1.2 Subgraphs and induced subgraphs

Let G and F be two graphs. We say that G and F are isomorphic if there exists a bijection $\phi : V(G) \to V(F)$ such that $uv \in E(G) \Leftrightarrow \phi(x)\phi(y) \in E(F)$ for all u, v in $V(G)$. We do not distinguish between isomorphic graphs and write $G = F$ is G and F are isomorphic. If $V(F) \subseteq V(G)$ and $E(F) \subseteq E(G)$, then F is a *subgraph* of G.

If A is a subset of $V(G)$, we denote by $G[A]$ the graph that has A as vertex set and $\binom{A}{2} \cap E(G)$ as edge set. We say that $G[A]$ is the *subgraph of G induced by A*. If there exists $A \subseteq V(G)$ such that $G[A]$ is isomorphic to a graph H, we say that H is *an induced subgraph of G*. If H is a subgraph (resp. an induced subgraph) of G, we say that G *contains* (or *admits*) *H as a subgraph* (resp. *as an induced subgraph*).

Saying that G is *F-free* can take two different meanings in this document. Either it means that that G does not contain F as a subgraph, or that G does not contain F as an induced subgraph. We clearly indicate at the beginning of each chapter which definition is used. If $\mathcal{F}$ is a class of graphs, we say that G is *$\mathcal{F}$-free* if for any graph $F \in \mathcal{F}$, G is F-free.

A class of graph $\mathcal{C}$ is *hereditary* if for any graph G in $\mathcal{C}$, every induced subgraph H of G belongs to $\mathcal{C}$. It is clear that a class of graphs defined by forbidding subgraphs or induced subgraphs is hereditary.

1.3 Connectivity

A non-empty graph is *connected* if any two of its vertices is linked by a path in G. A maximal connected subgraph of G is a *component* of G. A set $S \subseteq V(G)$ is a *cutset* of G if $G \setminus S$ is not connected. It is a *minimal cutset* if no proper subset of S is a cutset. It is a *k-cutset* if $|S| = k$. A 1-cutset is called a *cutvertex*. A graph is said to be *k-connected* if it has at least $k+1$ vertices and, for any set $S \subseteq V(G)$ such that $|S| \leq k-1$, S is not a cutset of G. The greatest integer k such that G is k-connected is the *connectivity* $\kappa(G)$ of G. Note that $\kappa(G) = 0$ if and only if G is not connected or is K_1, and $\kappa(K_n) = n - 1$. We end this section with the most famous theorem about connectivity that is used in several proofs of this document.

Theorem 1.1 (Menger Theorem, see [14]) *If a graph G is a k-connected graph, then for any vertices x, y of G, there exist k internally vertex-disjoint xy-paths.*

1.4 Algorithm and complexity

Definitions given here are informal, for a precise treatment see [50].

By *complexity* of an algorithm, we mean the number of basic computational steps required for its execution. This number clearly depends on the size and the nature of the input. In case of graphs, it depends on the number of vertices and the number of edges of the input graph. In this document, in all complexity analysis of algorithms, n stands for the number of vertices of the input graph and m for the number of its edges.

If the complexity of an algorithm is bounded above by a polynomial in n and m, we say it is a *polynomial-time* algorithm. Such an algorithm is further said to be *linear-time* if the polynomial is a linear function. The class of problems solvable in polynomial-time is denoted by $\mathcal{P}$.

A *decision problem* is a question whose answer is "yes" or "no". Such a problem belongs to $\mathcal{P}$ if there exists a polynomial-time algorithm that solves any instance of this problem. It belongs to the class $\mathcal{NP}$ if, given any instance of the problem whose answer is "yes", there is a certificate validating this fact which can be checked in polynomial-time. Analogously, a decision problem belongs to the class *co*-$\mathcal{NP}$ if, given any instance of the problem whose answer is "no", there is a certificate validating this fact which can be check in polynomial-time.

It is clear that $\mathcal{P}$ is included in both $\mathcal{NP}$ and co-$\mathcal{NP}$. The following conjectures are certainly two of the deepest conjectures in computer science.

Conjecture 1.2 $\mathcal{P} \neq \mathcal{NP}$

Conjecture 1.3 $\mathcal{P} = \mathcal{NP} \cap$ *co*-$\mathcal{NP}$

There exists a class of $\mathcal{NP}$ problems (whose definitions is ommited here) called the $\mathcal{NP}$-*complete*($\mathcal{NP}$-*c* for short) problems. A theorem of Cook states that, if there exists a polynomial-time algorithm for a problem in $\mathcal{NP}$-c then there exists a polynomial time algorithm for any problem in $\mathcal{NP}$. Here is an example of an $\mathcal{NP}$-complete problem:

Problem 1.4 (Clique)

Input: *A graph G and an integer k.*

Output: *Is there a clique of size k in G.*

Complexity: $\mathcal{NP}$-*complete [39].*

Chapter 2

Truemper configurations and decomposition method

In this chapter:

- **If *G* and H are graphs, then we say that G is *H-free* if G does not contain H as an <u>induced</u> subgraph.**

- **K_4 is not a wheel.**

2.1 Truemper configurations

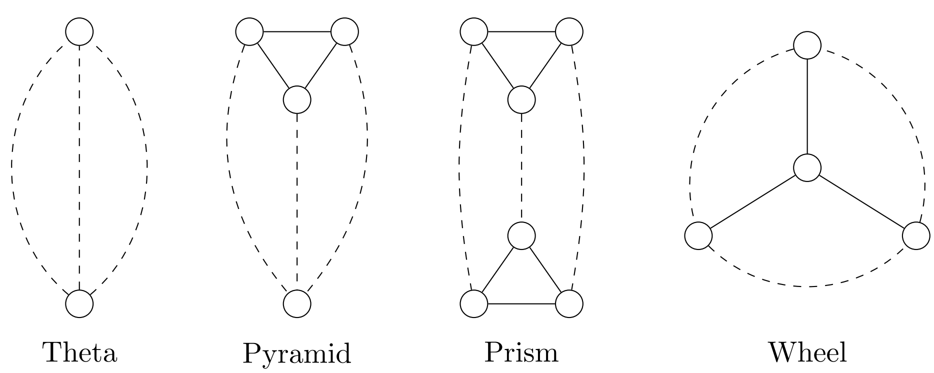


Figure 2.1: The Truemper configurations

Here are some definitions of special kinds of graphs.

- **Theta**

  A *theta* is a graph made of three internally disjoint paths $P_1 = a \dots b$, $P_2 = a \dots b$, $P_3 = a \dots b$ of length at least 2 and such that no edges exist between the paths except the three edges incident to a and the three edges incident to b.

- **Pyramid**

  A *pyramid* is a graph made of three paths $P_1 = a \dots b_1$, $P_2 = a \dots b_2$, $P_3 = a \dots b_3$ of length at least 1, two of which have length at least 2, vertex-disjoint except at a, and such that $b_1b_2b_3$ is a triangle and no edges exist between the paths except these of the triangle and the three edges incident to a.

- **Prism**

  A *prism* is a graph made of three vertex-disjoint paths $P_1 = a_1 \dots b_1$, $P_2 = a_2 \dots b_2$, $P_3 = a_3 \dots b_3$ of length at least 1, such that $a_1a_2a_3$ and $b_1b_2b_3$ are triangles and no edges exist between the paths except these of the two triangles.

- **Wheel**

  A *wheel* is a graph formed by a hole C called the *rim* together with a vertex v, called the *center* that has at least three neighbors in the hole. Such a wheel is denoted (v, C).

  Note that sometimes K_4 is also considered as a wheel. In this document, it is specified at the beginning of each chapter whether K_4 is considered as a wheel or not. In this chapter K_4 is not a wheel.

A *Truemper configuration* is a graph isomorphic to a prism, a pyramid, a theta or a wheel. Theta, pyramid and prism are sometimes called *3-path configurations*. It is important to observe that all Truemper configurations contain a hole.

In 1983 Truemper [94] gave a characterization of graphs whose edges can be labeled 0 or 1 in such a way that all chordless cycles have prescribed parity. The characterization states that this can be done for a graph G if and only if it can be done for all induced subgraphs of G isomorphic to a *Truemper configuration* or a K_4 (it is formally stated below). Truemper was originally motivated by the problem of obtaining a co-$\mathcal{NP}$ characterization of balanceable matrices (or balanceable graphs), a class that have important polyhedral properties. See Chapter 6 for more details about balanceable graphs.

We say that a graph F is *signed* if an assignment of weight is given to its edges. If G is a signed graph, the *weight* of a subgraph H of G is the sum of the weights assigned to the edges of H and is denoted by $v(H)$.

We now state formally Truemper's theorem:

Theorem 2.1 (Truemper [94]) *Let β be a $\{0, 1\}$ vector whose entries are in one-to-one correspondence with the chordless cycles of a graph G. Then there exists an assignment of $0, 1$ weights to the edges of G such that for every chordless cycle C of G, $v(C)$ is congruent to β_C mod 2 if and only if there exists such an assignment for every induced subgraph G' of G isomorphic to K_4 or to a Truemper configuration.*

This theorem shows that Truemper configurations are closely related to chordless cycle with given parity. Let us emphasize this relationship. An *even-hole* (resp. an *odd-hole*) is a hole of even (resp. odd) length. The structure of even-hole-free graphs and odd-hole-free graphs have been heavily studied and are of importance in the world of structural graph theory. Here, we just point out an obvious link that they have with Truemper configurations. For this purpose, let us show two very easy properties of Truemper configurations.

First, we need to define special kinds of wheels. A *sector* of a wheel is a subpath of length at least 1 of the rim whose ends are adjacent to the center and whose internal vertices are not. Observe that the sectors of a wheel edge-wise partition the edges of its rim. A wheel is *even* if it has an even number of sectors, or equivalently if it has an even number of spokes. A wheel is a *t-odd wheel* if it has an odd number of sectors of length 1 or, equivalently, if it contains an odd number of triangles.

Property 2.2 *Thetas, prisms and even-wheels contain even-holes.*

PROOF — Thetas and prisms both contain three vertex disjoint paths, so at least two of them have same parity, say P_1 and P_2. Then $G[V(P_1) \cup V(P_2)]$ is an even-hole.

Let (v, C) be an even wheel, and let $P_1, \dots, P_{2k}$ be the sectors of this wheel. If P_i is of odd length for $i = 1, \dots, 2k$, then C is an even-hole. So one of the P_i's, say P_1 is of even length and then $V(P_1) \cup \{v\}$ is an even-hole. □

We now give an analogue for odd-hole, we omit the straightforward proof.

Property 2.3 *Pyramids and t-odd-wheels contain odd-holes.*

These properties lead to the following two classes, that respectively generalize even-hole-free and odd-hole-free graphs, and capture some of their features. A graph is *odd-signable* if there exists an assignment of $0, 1$ weights to its edges that makes every chordless cycle of odd weight. A graph is *even-signable* if there exists an assignment of $0, 1$ weights to its edges that makes every triangle of odd weight and every chordless cycle of even weight.

Theorem 2.1, when applied to odd-signable and even-signable graphs, gives the following characterizations of these classes.

Theorem 2.4 *[30] A graph is* odd-signable *if and only if it is (theta, prism, even-wheel)-free.*

PROOF — Let G be an odd-signable graph. Let us first show that thetas, prisms and even-wheels are not odd-signable. Assume that G contains a theta and let x and y be the two vertices of degree three of this theta and P_1, P_2 and P_3 the three disjoint xy-paths. Put $C_1 = P_1 \cup P_2$, $C_2 = P_2 \cup P_3$ and $C_3 = P_3 \cup P_1$. By definition of a theta, C_1, C_2 and C_3 are holes. So $v(C_1) \equiv v(C_2) \equiv v(C_3) \equiv 1\,[2]$. However, $v(C_1) + v(C_2) + v(C_3) = 2(v(P_1) + v(P_2) + v(P_3)) \equiv 0\,[2]$, a contradiction.

Assume now that G admits a prism. Let $x_1x_2x_3$ and $y_1y_2y_3$ be the two triangles of the prism and, for $i = 1, 2, 3$, let P_i be the x_iy_i-path of the prism. Put $C_1 = x_1P_1y_1y_2P_2x_2x_1$, $C_2 = x_2P_2y_2y_3P_3x_3x_2$ and $C_3 = x_3P_3y_3y_1P_1x_1x_3$. By definition of a prism, C_1, C_2 and C_3 are holes. So $v(C_1) \equiv v(C_2) \equiv v(C_3) \equiv v(x_1x_2x_3) \equiv v(y_1y_2y_3) \equiv 1\,[2]$. However, $v(C_1) + v(C_2) + v(C_3) = 2(v(P_1) + v(P_2) + v(P_3)) + v(x_1x_2x_3) + v(y_1y_2y_3) \equiv 0\,[2]$, a contradiction.

Assume now that G contains an even-wheel (x, C). Let $x_1, \dots, x_{2k}$ be the neighbors of x in C, and name P_i for $i = 1, \dots, 2k$ the sectors with extremities x_i and x_{i+1} (subscripts are taken mod 2). Put, for $i = 1, \dots, 2k$, $C_i = x_iP_ix_{i+1}xx_i$. By definition of a wheel, C and C_i (for

$i = 1, \ldots, 2k$) are chordless cycles. So, $v(C) = 1\,[2]$ and $\sum_{i=1}^{i=2k} v(C_i) = 0\,[2]$. But, $\sum_{i=1}^{2k} v(C_i) = \sum_{i=1}^{2k} v(P_i) + 2\sum_{i=1}^{2k} v(xx_i) \equiv v(C)$, a contradiction.

Now, we need to prove that (theta, prism, even-wheel)-free are odd signable. By Theorem 2.1, we only need to check that K_4, pyramids and wheels that are not even-wheels are odd signable. For K_4, give the weight 1 to each edge of any triangle of K_4. For the pyramids, give the weight 1 to each edge of the triangle and 0 to every other edge. A wheel that is not an even-wheel has an odd number of sectors. So, for the wheels that are not even, give the weight 1 to exactly one edge in each sector and 0 to every other edge. It is easy to check that these signing give to every chordless cycle an odd weight. □

Next Theorem is an analogous of the previous one for even-signable graphs, we omit the proof that is very similar to the proof of the previous theorem.

Theorem 2.5 *[30] A graph is* even-signable *if and only if it is (pyramid, t-odd-wheel)-free.*

Let us now define another very important class of graphs related to Truemper configurations. A graph G is said to be *perfect* if, for every induced subgraph H of G, $\omega(H) = \chi(H)$. Odd-holes (and thus every graph containing odd-holes as induced subgraphs) are examples of *imperfect* graphs. An *antihole* is the complement of a hole. Odd-antiholes are also easily seen as being imperfect. Graphs that contain neither odd-hole nor odd anti-hole as induced subgraphs are said to be *Berge*.

In a celebrated paper, Chudnovsky, Robertson, Seymour and Thomas proved the following theorem, known as the Strong Perfect Graph Theorem (SPGT) and conjectured by Berge [8].

Theorem 2.6 (Chudnovsky, Robertson, Seymour and Thomas[23]) *A graph is perfect if and only if it is Berge.*

In other words, perfect graphs and (odd-hole, odd-antihole)-free graphs are the same. Note that, by Property 2.3, we know that perfect graphs do not contain pyramids nor t-odd-holes as induced subgraphs. Actually, as we will see in the next subsection, Truemper configurations play a special role in the proof of the SPGT.

2.2 Decomposition theorems

2.2.1 What is it?

We call a *decomposition theorem* for a class $\mathcal{C}$ a theorem with the following form.

Theorem 2.7 (Decomposition theorem) *For every graph G in $\mathcal{C}$, either G is "basic", or it admits a cutset S for $S \in \mathcal{S}$.*

Depending on what we want to prove about $\mathcal{C}$, "basic" graphs and the set of cutsets $\mathcal{S}$ need to have adequate properties.

Let us give a very simple example of a decomposition theorem. A graph is *chordal* (or *triangulated*) if it is hole-free. Next theorem is a decomposition theorem for chordal graphs, the basic class is the class containing all cliques (it is clearly a subclass of chordal graphs) and only one cutset is needed, the clique cutset. A cutset S of a graph G is a *clique cutset* if $G[S]$ induces a clique.

Theorem 2.8 (Dirac [44]) *If G is a chordal graph then either it is a clique or it admits a clique cutset.*

PROOF — Suppose that G is not a clique. Let S be a minimal vertex-cutset of G, and let C_1 and C_2 be two connected components of $G \setminus S$. Suppose that $G[S]$ is not a clique. So S contains two non-adjacent vertices u and v. Since S is minimal, both u and v have a neighbor in both C_1 and C_2. Hence, for $i = 1, 2$, there exists a chordless uv-path P_i whose interior vertices are in C_i. Then $P_1 \cup P_2$ induces a hole, a contradiction. So S is a clique-cutset of G. □

Decomposition theorems for even-hole-free graphs [41] and odd-hole-free graphs [31] exist but are too complicated to be stated here. In chapter 6 a decomposition theorem for *balanceable graphs* [34], that is also an example of a deep decomposition theorem, is described in details and heavily used to prove new theorems about balanceable graphs.

The SPGT has been proved through a decomposition theorem for Berge graphs. Again, this decomposition theorem is too complicated to be stated here, but let us say a very quick word on the proof itself that involves some Truemper configurations. We already observed that Berge graphs do not admit pyramids as induced subgraphs. This little fact is used very often to provide a contradiction when manipulating Berge graphs. Also, a long part of the proof is dedicated to the study of Berge graphs that admits a prism as an induced subgraphs. Finally, a very long part is devoted to Berge graphs that contain certain kinds of wheels as induced subgraphs. This last fact is a hint that wheel-free perfect graphs might have a way more simple structure than perfect graphs. We will come back to wheel-free graphs in Section 2.3 and, more generally, all along this document. To have a nice insight of the proof of the SPGT and the used of Truemper configurations in it, the survey of Nicolas Trotignon [90] is a good reading.

2.2.2 Decomposition trees and recognition problems

In this subsection, we explain the concept of *decomposition trees* that are based on decomposition theorems. Decomposition trees are useful to find polynomial-time algorithms for optimization problems such as coloring a graph, finding the biggest clique or the biggest stable set in a graph, that are $\mathcal{NP}$-hard in general, but become (sometime) polynomially solvable when some subgraphs are excluded. Here, we emphasize on the *recognition problem.* The *recognition problem for a class of graphs* $\mathcal{C}$ is: given a graph G, decide whether G belongs to $\mathcal{C}$ or not.

The removal of a cutset S from a graph G breaks G into at least two connected components. From these connected components, one can construct *blocks of decomposition* by possibly adding some vertices and edges. For a special type of cutset, we speak about his *associated blocks of decomposition.* A cutset S is said to be $\mathcal{C}$*-preserving* if it satisfies the following: G belongs to $\mathcal{C}$ if and only if all the blocks of decomposition belong to $\mathcal{C}$.

In an ideal situation, a recognition algorithm for a class $\mathcal{C}$ using a decomposition theorem has the following form. It decomposes the input graph G along $\mathcal{C}$-preserving decompositions into undecomposable graphs (note that it does not happen very often that every cutset used in a decomposition theorem is class-preserving). Since the decomposition is $\mathcal{C}$-preserving, we only need to check whether these undecomposable graphs belong to $\mathcal{C}$ which, according to Theorem 2.7, reduces to check if they are “basic”.

The decomposition can be represented by a *decomposition tree* where the root is the graph G and, for every non-leaf vertex H, the children of H are the blocks of decomposition of H. Leaves correspond to undecomposable graphs.

So, in order to get a polynomial-time recognition algorithm based on a decomposition tree, we need to be able to construct the decomposition tree in polynomial-time (which implies that we can find the cutsets in polynomial-time and that the size of the tree is polynomial) and to decide whether a "basic" graph belongs to $\mathcal{C}$ or not in polynomial-time (i.e. the recognition problem for the basic class is in P).

Let us illustrate this method with the example of chordal graphs. We already got a decomposition theorem for this class (Theorem 2.8), we now need to define blocks of decomposition associated with clique cutsets. Let S be a clique cutset of a graph G, and let $C_1, \dots, C_k$ be the connected components of $G \setminus S$. The *blocks of decomposition w.r.t.* S are the graphs $G_i = G[C_i \cup S]$ for $i = 1, \dots, k$.

Let us now prove that clique cutsets preserve being chordal.

Theorem 2.9 *Let G be a graph and S a clique cutset. G is chordal if and only if all the blocks of decomposition w.r.t. S are chordal.*

PROOF — Let $C_1, \dots, C_k$ be the connected components of $G \setminus S$ and $G_1, \dots, G_k$ the corresponding blocks of decomposition. Since all the blocks of decomposition are induced subgraphs of G, if G is chordal, then all the blocks are.

Suppose now that all the blocks $G_1, \dots, G_k$ are chordal and G contains a hole H. Since H cannot be contained in a block, it must contain some vertices of at least two connected components of $G \setminus S$. Consequently H contains at least two vertices of S that are not consecutive in H, so H has a chord, a contradiction. □

Observe that Theorems 2.8 and 2.9 give us a complete *structure theorem* for chordal graphs, i.e. they show that all chordal graphs can be built starting from cliques, gluing them together along cliques (the reverse operation of clique cutset) and that all graphs built this way are chordal.

We now show how to turn these theorems into a recognition algorithm. We construct a decomposition tree T as follows: the root of T is our input graph G; for every internal vertex G' of T, the children of G' are the blocks of decomposition of G' w.r.t. some clique cutset; and the leaves of T are graphs that have no clique cutset. An $\mathcal{O}(nm)$ algorithm is given in [100] to find a clique cutset, and a simple counting argument shows that number of vertices of T is bounded by $\mathcal{O}(n^2)$, giving an $\mathcal{O}(n^3m)$ algorithm for constructing T. Now, by Theorems 2.8 and 2.9, the input graph is chordal if and only if all the leaves of the decomposition tree are cliques. So, recognizing chordal graphs can be done in the same time as building the decomposition tree: build the decomposition tree, check if every leaves are cliques, if they are then the input graph is chordal, otherwise it is not. We will see in the next subsection that this is not the fastest algorithm to recognize chordal graphs.

It is, most of the time, very hard to turn a decomposition theorem into a polynomial-time recognition algorithm. Some examples where this is possible are balanceable graphs [35], even-hole-free graphs [36] (or [19] for a faster one) and perfect graphs [22] (note that in [22] two recognition algorithms are given, one based on the decomposition theorem that runs in $\mathcal{O}(n^{18})$-time and another one, based on a more direct method, that run in $\mathcal{O}(n^9)$-time). On the other hand, the question for odd-hole-free graphs is still open.

2.2.3 Extreme decompositions

Let G be a graph and *gizmo* be a special type of a cutset with an associated block of decomposition. A gizmo cutset S of G is an *extreme gizmo cutset* if one of the block of decomposition w.r.t. S does not admit any gizmo cutsets. Let us illustrate this by an example. Blocks of decomposition associated with clique cutsets are the same as in the previous subsection.

Theorem 2.10 *If a graph G admits a clique cutset, then it admits an extreme clique cutset.*

PROOF — Let S be a clique cutset of H such that, among all clique cutsets of G, a connected component C of $G \setminus S$ is the smallest possible. Let $G' = G[C \cup S]$ be the block of decomposition that contains C. Assume that G' has a clique cutset S'. Since S is a clique, there is a component C' of $G' \setminus S'$ that is disjoint from S. Moreover, it is clear that $S' \cap C \neq \emptyset$ and thus $|V(C')| < |V(C)|$. Hence S' is a clique cutset of G that contradicts our choice of S and C. Thus G' does not admit clique cutsets and hence S is an extreme clique cutset. □

As an application, let us show how one can speed up the recognition algorithm for chordal graph using extreme clique cutset. Suppose that S is an extreme clique cutset and G_i an *extreme block* associated with S (i.e. a block that does not admit clique cutsets). In order to build what we call an *extreme decomposition tree*, we construct only two blocks of decomposition: $G_B = G_i = G[C_i \cup S]$ and $G_A = G \setminus C_i$. The *extreme decomposition tree* T is constructed as follows: the root is the input graph G; for every internal vertex G' of T, the children of G' are the blocks of decomposition G'_A and G'_B of G' w.r.t. an extreme clique cutset; the leaves of T are the graphs with no clique cutset. Note that every G'_B is a leaf, so T is a binary tree such that every internal vertex has a child that is a leaf.

In [86], it is shown that an *extreme tree decomposition* w.r.t. clique cutsets can be constructed in time $\mathcal{O}(nm)$ and thus one can decide if a graph is chordal in time $\mathcal{O}(nm)$. Once again, it is not the fastest known recognition algorithm for chordal graphs. In Section 3.2.1, an algorithm to recognize them in linear time is described.

It is quite rare that special type of cutsets admit extreme decomposition in any graph like clique cutsets. k-cutsets with adequate blocks of decomposition also admit extreme decomposition in any graphs, see Chapter 7 for more details. More generally, extreme decompositions are used very often in this document, most of the time to prove some local structural properties for several classes of graphs. In Chapter 6 a new method is explained that makes the use of extreme decomposition for the so-called *star cutset* (notoriously difficult to use) possible in certain cases.

2.3 Wheel-free graphs

In this section, we survey known results around wheel-free graphs, several of them are proved in this document.

As we already mentioned, a very long part of the proof of the SPGT is devoted to Berge graphs that contain kinds of wheels, which suggest that wheel-free perfect graphs and more generally wheel-free graphs should have interesting structural properties. Understanding their structure might shed a new light on the work that have been done on even-hole-free graphs, odd-hole-free graphs or perfect graphs.

If, as we will soon see, several subclasses of wheel-free graphs are well-understood, very few results are known on wheel-free graphs in general, let us state them all here.

The only structural known property of wheel-free graph is the following. The original proof, due to Chudnovky is by induction, a different proof is given in Subsection 3.2.4.

Theorem 2.11 (Chudnovsky [20]) *Every non-empty wheel-free graph contains a vertex whose neighborhood is a disjoint union of cliques.*

As a corollary, we also give the following result that can be seen as an extension of a famous result: a chordal graph G has at most n maximal cliques.

Corollary 2.12 *A wheel-free graph G has at most m maximal cliques.*

PROOF — Induction on m. By Theorem 2.11, consider a vertex v of degree d whose neighborhood is a disjoint union of cliques. By the induction hypothesis, $G \setminus \{v\}$ has at most $m - d$ maximal cliques, and because of its neighborhhood, v is in at most d maximal cliques. □

It is also proved in Subsection 3.2.4 that there exists an algorithm in time $\mathcal{O}(mn)$ to find a largest clique in a wheel-free graph.

Subclasses of wheel-free graphs

Many proper subclasses of wheel-free graphs have been studied, we now list them. A *k-wheel* is a wheel with at least k spokes. So wheels and 3-wheels are the same.

- The class of 2-wheel-free graphs is clearly a subclass of wheel-free graphs. Its structure is precisely described in Chapter 5.

- Say that a graph is *unichord-free* if it does not contain a cycle with a unique chord as an induced subgraph. The class of unichord-free graphs is a subclass of wheel-free graphs because every wheel contains a cycle with a unique chord as an induced subgraph. The class of unichord-free graphs have a complete structural description [92] that implies that it is χ-bounded and that coloring problem, clique number problem and recognition problem are polynomially solvable in this class. See Chapter 8 for more results about classes of graphs defined by forbidding cycles with a fixed number of chords.

- The class of graphs that do not contain wheels as subgraphs has been studied in [87], [88] and [4]. Several structural properties and extremal results for this class as well as some results on graphs that do not contain a k-wheel ($k \geq 4$) as a subgraph are described in details in Chapter 7.

- The class of graphs that do not contain K_4 nor induced subdivision of wheels is clearly a subclass of wheel-free graphs. Here again, this subclass of wheel-free graphs has a complete structural description with consequences that graphs in this class are 3-colorable and the recognition problem is polynomially solvable [66]. Observe also that forbiding K_4 and induced subdivision of wheels is equivalent with forbidding wheels and induced subdivision of K_4.

Chromatic number

A nice question concerning the chromatic number of wheel-free graphs is the following:

Question 2.13 *Is the class of wheel-free graphs χ-bounded?*

In order to answer this question, the following has been proved. A *square* is a hole of size four.

Theorem 2.14 (Bousquet, Thomassé [15]) *The class of (triangle, square, wheel)-free has bounded chromatic number.*

We also propose the following conjecture that would generalize the previous theorem.

Conjecture 2.15 *Let H be a fixed complete bipartite graph. The class of (triangle, H, wheel)-free graphs has bounded chromatic number.*

Chapter 3

The moplex technique

In this chapter:

- **If *G* and H are graphs, then we say that G is *H-free* if G does not contain H as an <u>induced</u> subgraph.**
- **K_4 is not a wheel.**

Most of the results presented in this chapter come from a joint work with P. Charbit, N. Trotignon and K. Vušković submitted to Discrete Mathematics [3].

LexBFS is an algorithm due to Rose, Tarjan and Lueker [79] that computes in linear time an ordering of the vertices of an input graph (such an ordering is called a LexBFS order). Berry and Bordat [11] proved that for every graph, the last vertex in a LexBFS order is part of what they call a moplex, a set of vertices with strong structural properties. Maffray, Trotignon and Vušković [71] defined a property of graphs (called property ($\star$)) that implies the existence of vertex elimination orderings with structural properties. The goal of this chapter is to show how these three works lead to short proofs of some structural results, and fast algorithms for maximum weighted clique problem for several classes defined by forbidding induced subgraphs. Surprisingly, all of these forbidden subgraphs are Truemper configurations.

Here is the plan of this chapter. In the first section, we present the tools around LexBFS algorithm and the notion of a moplex. In the second section, we apply the tools on every graph on two and three vertices, then we explain every consequence it has. In the third section, we show how our tools can be used to speed-up some algorithms for classes related to even-hole-free graphs and Berge Graphs. In the fourth section, the class of universally-signable graphs (to be defined later), that we meet in Section 3.2 is studied from a more structural point of view.

3.1 LexBFS, moplexes and property ($\star$)

An order $\prec$ of the vertices of a graph G is a *LexBFS order* if and only if it satisfies the following property: for all vertices a, b, c of G such that $c \prec b \prec a$, $ca \in E(G)$ and $cb \notin E(G)$ there exists a vertex d in G such that $d \prec c$, $db \in E$ and $da \notin E$ (see Figure 3.1). This is not the original definition, but it is proved to be equivalent to it by Brandstädt, Dragan and Nicolai [16].

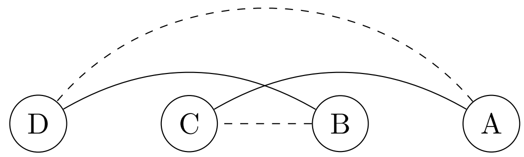


Figure 3.1: A solid line denote an edge and a dashed line a non-edge

Theorem 3.1 (Rose, Tarjan and Lueker [79]) *There exists an $O(n+m)$-time algorithm that outputs a LexBFS order of an input graph.*

A *module* (sometimes called *homogeneous set*) in a graph G is a set A of vertices that share the same external neighborhoods, i.e. $\forall a, b \in A, N(a) \setminus A = N(b) \setminus A$. A is a *clique module* if A is a clique (i.e. all pairs of vertices of A are adjacent) and a module. Recall that a cutset S is a *minimal cutset* if no proper subset of S is a cutset.

A *moplex* of a graph G is a set $S \subseteq V(G)$ such that S is a clique, S is a module and $N(S)$ is a minimal cutset. Note that in [11] a slightly different definition is given: S is a moplex if S is maximal with respect to the property of being a clique and a module, and $N(S)$ is a minimal cutset. It is easy to see that the two definitions are equivalent by the following observation.

Lemma 3.2 *Let S be a moplex of a graph G and $S \subseteq S' \subseteq V(G)$. If S' is a clique and a module of G, then $S' = S$.*

PROOF — Let $X = V(G) \setminus N[S]$. Assume there exists x in $S' \setminus S$. Since S' is a clique, we have $S' \subseteq N[S]$, so $x \in N(S)$. Since $N(S)$ is a minimal cutset, x has a neighbor in X. But this contradicts the fact that S' is a module since no vertex of S has a neighbor in X. □

Theorem 3.3 (Berry and Bordat [11]) *If G is a graph that is not a clique and $(v_1, \dots, v_n)$ is a LexBFS order of G, then v_n is contained in a moplex of G.*

A vertex v of a graph F is *F-universal* if v is adjacent to all vertices of $F \setminus v$. To prove the existence of a vertex with a particular neighborhood, the following graph property was introduced in [71]. We slightly rephrased it here for convenience.

Definition 3.4 *A graph G satisfies property $(\star)$ w.r.t. a class of graphs $\mathcal{F}$ when for every $x \in V(G)$ and every connected component C of $G \setminus N[x]$, if $F \in \mathcal{F}$ is contained in $G[N(x)]$, then there exists a vertex of F that is not F-universal and that has no neighbor in C.*

The next theorem makes the link between satisfying property $(\star)$ and being in a moplex.

Theorem 3.5 *Let G be a graph that satisfies property $(\star)$ for a class of graphs $\mathcal{F}$. Let x be a vertex of G contained in a moplex. Then $G[N(x)]$ is $\mathcal{F}$-free.*

PROOF — Assume G satisfies property $(\star)$ w.r.t. $\mathcal{F}$, and let x be a vertex that is contained in a moplex S of G. Suppose that some $F \in \mathcal{F}$ is contained in $G[N(x)]$. Since S is a moplex, $N[x] = N[S]$ and for some connected component C of $G \setminus N[S]$, every vertex of $N(S)$ has a neighbor in C. By property ($\star$), there is a vertex y in F that is not universal for F that has no neighbor in C. Since y is not universal for F, it follows that $y \in N(S)$, a contradiction. □

An ordering $(v_1, \dots, v_n)$ of the vertices of a graph G is an *$\mathcal{F}$-elimination ordering* if for every $i = 1, \dots, n$, $N_{G[v_1,\dots,v_i]}(v_i)$ is $\mathcal{F}$-free. The following theorem sums up all the results of this subsection.

Theorem 3.6 *Let $\mathcal{F}$ be a class of graphs that contains no cliques and let $\mathcal{C}$ be a hereditary class of graphs such that every graph in $\mathcal{C}$ has property ($\star$) w.r.t. $\mathcal{F}$. Then there exists a linear time algorithm whose input is any graph G and whose output is an ordering of the vertices of G such that if $G \in \mathcal{C}$, then the ordering is an $\mathcal{F}$-elimination ordering.*

PROOF — By Theorem 3.1, we compute a LexBFS order of G, say $(v_1, \dots, v_n)$. Observe that from the definition of LexBFS, for every $1 \leq i \leq n$, the order $(v_1, \dots, v_i)$ is a LexBFS order of $G[v_1, \dots, v_i]$. If $G[v_1, \dots, v_i]$ is a clique, then $N_{G[v_1,\dots,v_i]}(v_i)$ is clearly $\mathcal{F}$-free, since $\mathcal{F}$ contains no clique. Otherwise, by Theorem 3.3, v_i is in a moplex of $G[v_1, \dots, v_i]$. By Theorem 3.5, $N_{G[v_1,\dots,v_i]}(v_i)$ is $\mathcal{F}$-free again. This proves that $(v_1, \dots, v_n)$ is an $\mathcal{F}$-elimination ordering of G. □

Theorem 3.6 is extensively used in the next section to obtain structural properties with algorithmic consequences for several classes of graphs.

In [71], the existence of a moplex in a graph was proved by an ordering different from LexBFS. Finding a moplex as in Theorem 3.6 will allow us to speed up the algorithm from [84] for finding a maximum weighted clique in an even-hole-free graph, as explained in Subsection 3.3.

3.2 Systematic applications

Here is how we use the tools presented in the previous section. Fix a graph F that is not a clique. The goal is to find a class of graph $\mathcal{H}$ such that every $\mathcal{H}$-free graph satisfies property ($\star$) w.r.t. F. Here is how we proceed. Observe that a graph G does not satisfy property ($\star$) w.r.t. a graph F if there exists a vertex $v \in V(G)$ such that F is an induced subgraph of $N(v)$ and, for some component of $G \setminus N[v]$, every vertex of F that is not F-universal has a neighbor in C. So, we need to describe every graph that can be built from a vertex (here v) adjacent to every vertex of F and a component C such that each non F-universal vertex in F has a neighbor in C.

By this way we obtain a class of graphs defined by forbidding induced subgraphs for which, by Theorem 3.6, we can obtain an F-elimination ordering in linear time.

In this section, we apply this method for every non-clique graphs on two and three vertices. This gives a bunch of theorems, all of the same kind, and with very short proofs. Most of them are new, but some have been known for a long time.

3.2.1 Graphs on two vertices and chordal graphs

There is a unique non-clique graph on two vertices that is the independent graph on two vertices, denoted by S_2. This leads to the class of chordal graphs that we already analysed in the previous chapter.

Lemma 3.7 *If G is a chordal graph, then G has property ($\star$) w.r.t. S_2.*

PROOF — Suppose not. Then for some $x \in V(G)$ and some connected component C of $G \setminus N[x]$, $G[N(x)]$ contains an induced subgraph F isomorphic to S_2, and every vertex of F has a neighbor in C. This clearly implies that G contains a hole, a contradiction. □

Now, by Theorem 3.6, every chordal graph has a vertex whose neighborhood is S_2-free, and being S_2-free means being a clique. In particular, we prove that every chordal graph contains a vertex whose neighborhood is a clique, which implies easily the decomposition theorem for chordal graphs (Theorem 2.8). More precisely, Theorem 3.6 says that any LexBFS order on the vertices of a chordal graph is an S_2-elimination order. This is a celebrated result of Rose, Tarjan and Lueker [79] that can be easily turned into a recognition algorithm for chordal graphs thanks to the following theorem.

Theorem 3.8 (Rose, Tarjan and Lueker [79]) *Let G be a graph and $\{v_1, \dots, v_n\}$ a LexBFS ordering of its vertices. Then G is chordal if and only if $\{v_1, \dots, v_n\}$ is an S_2-elimination ordering of G.*

PROOF — We have already seen that if G is chordal then any LexBFS order of G is an S_2-elimination order.

Suppose that $\{v_1, \dots, v_n\}$ is an S_2-elimination ordering of G and that G is not chordal. Sot G contains a hole H as an induced subgraph. Let i be the largest integer such that $v_i \in V(H)$ and let v_j and v_k the two neighbors of v_i in H. So $N_{G[v_1,\dots,v_i]}(v_i)$ is S_2-free, i.e is a clique and thus $v_i v_k$ is an edge, a contradiction. □

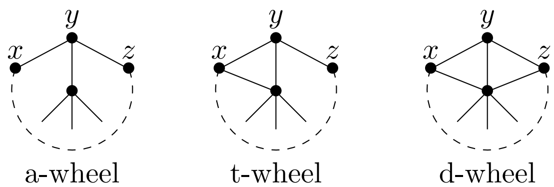
Now, here is a way to decide if a graph is chordal in linear time :

- Apply a LexBFS on G ($\mathcal{O}(n+m)$-time).
- Check if the obtained ordering is an S_2-elimination ordering (it is proved in [55] that in can be implemented in $\mathcal{O}(n+m)$-time). If it is then G is chordal, otherwise it is not.

3.2.2 Graphs on three vertices

We now apply the method for each non-clique graphs on three vertices. Up to isomophism, there are four graphs on three vertices, and three of them are not cliques: the independent graph on three vertices denoted by S_3, the path of length 2 denoted by P_3 and the complement of the path of length 2 denoted by $\overline{P_3}$ (note that $\overline{P_3}$ is not connected). Note that, once we have found the classes $\mathcal{H}$ for which $\mathcal{H}$-free graphs satisfy property ($\star$) for respectively S_3, P_3 and $\overline{P_3}$, it is easy to find the classes $\mathcal{H}'$ such that $\mathcal{H}'$-free graphs satisfy property ($\star$) for any combination of S_3, P_3 and $\overline{P_3}$.

Let us first specify several kinds of wheels.
A wheel is a *universal wheel* if the center is adjacent to all vertices of the rim.
A wheel is an *a-wheel* if for some consecutive vertices x, y, z of the rim, the center is adjacent to y and non-adjacent to x and z.
A wheel is a *triangle-wheel* (*t-wheel* for short) if for some consecutive vertices x, y, z of the rim, the center is adjacent to x and y, and non-adjacent to z.
A wheel is a *diamond-wheel* (*d-wheel* for short) if for some consecutive vertices x, y, z of the rim, the center is adjacent to x, y and z.
Observe that a wheel can be simultaneously an a-wheel, a t-wheel and a d-wheel. But every wheel is an a-wheel, a t-wheel or a d-wheel. Also, any d-wheel is either a t-wheel or a universal wheel.



The three next lemmas describe which induced subgraphs must contain a graph that does not satisfy property ($\star$) w.r.t. S_3, P_3 and $\overline{P_3}$ respectively. Observe that Truemper configurations appear everywhere!

Lemma 3.9 *Let G be a graph and $v \in V(G)$ be such that $G[N(v)]$ contains S_3. If for some component C of $G \setminus N[v]$ every vertex of the S_3 has a neighbor in C, then G contains a theta, a pyramid or an a-wheel.*

PROOF — Denote by x, y, z the three members of S_3. Let P be a chordless path from x to y with interior in C. Let Q be a chordless path from z to z', such that $V(Q) \setminus \{z\} \subseteq C$, z' has neighbors in the interior of P, and is of minimum length among such paths (possibly, $Q = z = z'$).

Suppose that at least one of x or y has neighbors in Q (this implies that Q has length at least 1). Call w the vertex of Q closest to z along Q, that has neighbors in $\{x, y\}$, and suppose up to symmetry that w is adjacent to y. Call w' the vertex of Q closest to z along Q that has neighbors in $P - y$. Call x' the neighbor of w' in P, closest to x along P. Now, $V(xPx') \cup V(zQw') \cup \{v, y\}$ induces a theta or a a-wheel centered at y.

Therefore, we may assume that none of x, y has a neighbor in Q. If z' has a unique neighbor in P, then $V(P) \cup V(Q) \cup \{v\}$ induces a theta. If z' has exactly two neighbors in P that are adjacent, then $V(P) \cup V(Q) \cup \{v\}$ induces a pyramid. Otherwise, $V(P) \cup V(Q) \cup \{v\}$ contains a theta. □

Lemma 3.10 *Let G be a graph and $v \in V(G)$ be such that $G[N(v)]$ contains a chordless path xyz. If for some component C of $G \setminus N[v]$, x and z have neighbors in C, then G contains a d-wheel.*

PROOF — Let P be a chordless path from x to z with interior in C. The graph induced by $V(P) \cup \{v, y\}$ is a d-wheel. □

Lemma 3.11 *Let G be a graph and $v \in V(G)$ be such that $G[N(v)]$ contains a $\overline{P_3}$. If for some component C of $G \setminus N[v]$ every vertex of the $\overline{P_3}$ has a neighbor in C, then G contains a prism, a pyramid, or a t-wheel.*

PROOF — Denote by x, y, z the vertices of $\overline{P_3}$ in such a way that xy is the only edge of $G[x, y, z]$. Let P be a path from x to y with interior in C whose unique chord is xy. Let Q be a chordless path from z to z', such that $V(Q) \setminus \{z\} \subseteq C$, z' has neighbors in the interior P, and is of minimum length among such paths (possibly, $Q = z = z'$).

Suppose that at least one of x or y has neighbors in Q. Call w the vertex of Q closest to z along Q, that has neighbors in $\{x, y\}$, and suppose up to symmetry that w is adjacent to y. Call w' the vertex of Q closest to z along Q that has neighbors in $P - y$. Call x' the neighbor of w' in P, closest to x along P. Now, $V(xPx') \cup V(zQw') \cup \{v, y\}$ induces a t-wheel centered at y.

Therefore, we may assume that none of x, y has a neighbor in Q. If z' has a unique neighbor in P, then $V(P) \cup V(Q) \cup \{v\}$ induces a pyramid or a t-wheel (when P has length 2). If z' has

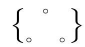
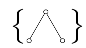
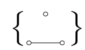
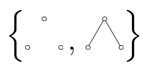
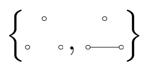
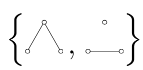
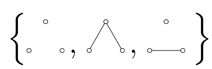
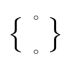
i	Class $\mathcal{C}_i$	$\mathcal{F}_i$	Neighborhood
1	no a-wheel, no theta, no pyramid		no stable set of size 3
2	no d-wheel		disjoint union of cliques
3	no t-wheel, no prism, no pyramid		complete multipartite
4	no a-wheel, no d-wheel, no theta, no pyramid		disjoint union of at most two cliques
5	no a-wheel, no t-wheel, no prism, no theta, no pyramid		stable sets of size at most 2 with all possible edges between them
6	no t-wheel, no d-wheel, no prism, no pyramid		clique or stable set
7	no Truemper configuration		clique or stable set of size 2
8	no hole		clique

Table 3.1: Eight classes of graphs

exactly two neighbors in P that are adjacent, then $V(P) \cup V(Q) \cup \{v\}$ induces a prism. Otherwise, $V(P) \cup V(Q) \cup \{v\}$ contains a pyramid. □

3.2.3 Sum-up

Table 3.1 describes every class that satisfies property ($\star$) w.r.t. any fixed combination of graphs on three vertices and the class that satisfy property ($\star$) w.r.t. S_2. So, it describes eight different classes of graphs $\mathcal{C}_1, \dots, \mathcal{C}_8$, all defined by excluding the induced subgraphs listed in the second column of the table. Every graph in $\mathcal{C}_i$ satisfies property ($\star$) w.r.t. the class $\mathcal{F}_i$ described in the third column. So, by Theorem 3.6, every graph in $\mathcal{C}_i$ contains a vertex v such that $N(v)$ is $\mathcal{F}_i$-free. The class of $\mathcal{F}_i$-free graphs is described in the last column.

The next theorem states formally what information gives the application of Theorem 3.6 on each C_i. What we need to prove it that, indeed, graphs in $\mathcal{C}_i$ satisfy property ($\star$) w.r.t. to $\mathcal{F}_i$.

These proofs being easy and similar for each i, we do not write them all.

Theorem 3.12 *There exists a linear time algorithm whose input is any graph G and whose output is an ordering of the vertices of G. Moreover, for $i = 1, \dots, 8$, if G is in $\mathcal{C}_i$, the ordering is an $\mathcal{F}_i$-elimination ordering of G (where $\mathcal{C}_i$ and $\mathcal{F}_i$ are the classes defined as in Table 3.1). In particular, every non-empty graph in $\mathcal{C}_i$ has a vertex whose neighborhood is $\mathcal{F}_i$-free.*

PROOF —
Class $\mathcal{C}_i$ is clearly hereditary. By Theorem 3.6, it suffices to prove that every graph in $\mathcal{C}_i$ satisfies property ($\star$) w.r.t. $\mathcal{F}_i$. For $i = 1, 2, 3$, the result holds by Lemmas 3.9, 3.10 and 3.11 respectively. We write the proof for $i = 6$, the proof is exactly the same for the other values. Suppose that $G \in \mathcal{C}_6$ does not have property ($\star$) w.r.t. $\mathcal{F}_6$. This means that for some $v \in V(G)$, some $F \in \mathcal{F}_6$ and some component C of $G \setminus N[x]$, every vertex of F that is not F-universal has a neighbor in C. If F is a P_3, then by Lemma 3.10, G contains a d-wheel, a contradition. If F is a $\overline{P_3}$, then by Lemma 3.11, G contains a prism, a pyramid, or a t-wheel, a contradition. □

Inclusions between our classes and several known classes are represented in Figure 3.2 (where a *cap* is cycle of length at least 5 with a unique chord joining two vertices at distance 2 on the cycle, a *d-hole* is a d-wheel such that the center has degree exactly 3, and the *claw* is $K_{1,3}$). Observe that a d-hole is also a t-wheel. Note also that the class $\mathcal{C}_7$ has already been studied in literature, see Section 3.4 for more details.

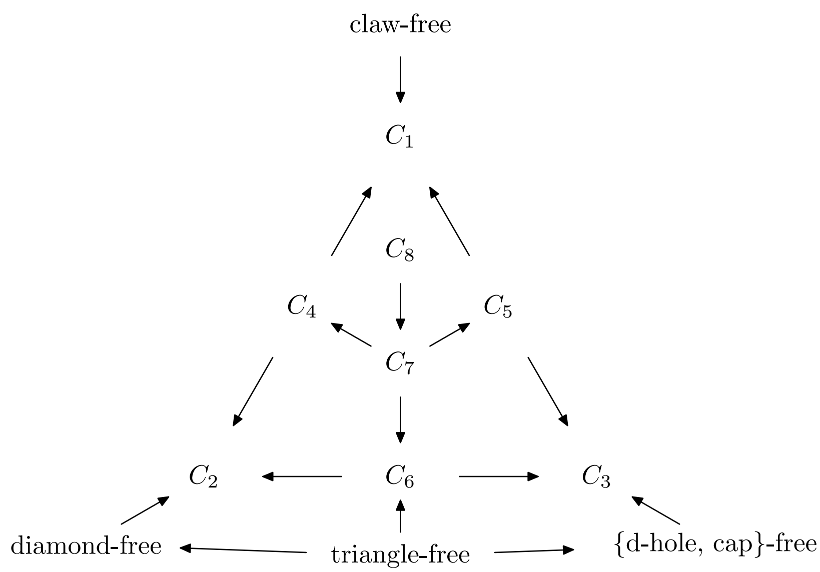


Figure 3.2: Inclusion for several classes of graphs. An arrow from A to B means "A is contained in B". Arrows arising from transitivity are not represented.

Of course, we can push further our analysis. For instance, we can study the 10 (up to isomorphism) non-clique-graphs on four vertices, and for each of them (say H), describe the structure that can be extracted from a vertex adjacent to all vertices of H, plus a component C such that each non-H-universal vertex of H has a neighbor in C. This kind of study would lead to infinitely many theorems and describing them would be rather uncomfortable and possibly purposeless. We decided to carry out the study for graphs on at most three vertices because it is really remarkable that it leads only to describe classes defined by excluding some Truemper configurations.

3.2.4 Consequences

i	χ-bounded	Max clique	Coloring
1	$f(x) = O(x^2/\log x)$	NP-hard [78]	NP-hard [54]
2	No [101]	$O(nm)$ [79]	NP-hard [69]
3	No [101]	$O(nm)$	NP-hard [69]
4	$f(x) = 2x - 1$	$O(n+m)$	?
5	$f(x) = 2x - 1$	$O(nm)$	?
6	No [101]	$O(n+m)$	NP-hard [69]
7	$f(x) = \max(3, x)$ [29]	$O(n+m)$	$O(n+m)$
8	$f(x) = x$ [44]	$O(n+m)$ [79]	$O(n+m)$ [79]

Table 3.2: Several properties of classes defined in Table 3.1

Table 3.2 describes several properties of the classes defined in Table 3.1. We indicate a reference for the properties that are already known. Before we analyze and prove each result of Table 3.2, observe that the class C_2 contains the class of wheel-free graph, and thus Table 3.2 contains the result announced in Section 2.3 stating that one can find a maximum clique in wheel-free graphs in time $\mathcal{O}(mn)$.

χ-boundedness

Let us analyze the column "χ-bounded" of Table 3.2. The column indicates whether the class C_i is χ-bounded, and if so, gives the smallest known function proving so. Classes $\mathcal{C}_2, \mathcal{C}_3$ and $\mathcal{C}_6$ are not χ-bounded because they contain all triangle-free graphs, and these may have arbitrarily large chromatic number as first shown by Zykov [101].

For classes $\mathcal{C}_1, \mathcal{C}_4$ and $\mathcal{C}_5$, we may rely on degeneracy. Say that a hereditary class of graphs is *ω-degenerate* if there exists a function g such that every non-empty graph in the class has a vertex of degree at most $g(\omega(G))$. It is easy to check that by the greedy coloring algorithm, if a hereditary class of graphs is ω-degenerate with a non-decreasing function g, then it is χ-bounded with function $g + 1$.

The function given for classes $\mathcal{C}_4$ and $\mathcal{C}_5$ follows from the fact that these classes are clearly ω-degenerate with function $g(x) = 2x - 2$.

For the class $\mathcal{C}_1$, we use Ramsey theory. Kim [60] proved that for some constant c, every graph on $ct^2/\log t$ vertices admits a stable set of size 3 or a clique of size t. Therefore, the vertex whose neighborhood is S_3-free in any graph in $\mathcal{C}_1$ proves that $\mathcal{C}_1$ is ω-degenerate with function $g(x) = O(x^2/\log x)$.

Observe that these results about χ-boundedness just improve bounds. Indeed, a theorem due to Kühn and Osthus [64] proves that theta-free graphs (and therefore graphs in $\mathcal{C}_1$, $\mathcal{C}_4$ and $\mathcal{C}_5$) are ω-degenerate, but their function is quite big.

Clique number

Let us now analyse the column "Max clique" of Table 3.2, that gives the best complexity of finding a maximum weighted clique in a graph of the corresponding class. By a result of Poljak [78], it is $\mathcal{NP}$-hard to compute a maximum stable set in a triangle-free graph. Rephrased in the complement, it is $\mathcal{NP}$-hard to compute a maximum clique in an S_3-free graph, and therefore in graphs from $\mathcal{C}_1$.

Finding a maximum weighted clique in $\mathcal{C}_2$ is easy as follows: for every vertex v, look for a maximum weighted clique in $N(v)$, and choose the best clique among these. This can be implemented by running n times the $O(n + m)$ algorithm of Rose, Tarjan and Lueker, because $N(v)$ is chordal for every v. In fact, this algorithm works in the larger class of universal-wheel-free graphs.

For $\mathcal{C}_4$, we need to be careful about the complexity analysis. Here is an algorithm that finds a maximum (weighted) clique in $G \in \mathcal{C}_4$. First by Theorem 3.12, we find in linear time an $\{S_3, P_3\}$-elimination ordering of G, say $(v_1, \dots, v_n)$. This means that in $G[\{v_1, \dots, v_i\}]$, $N(v_i)$ is a disjoint union of at most two cliques. We now show that, having this order, we can compute a maximum clique in time $O(m)$. We may assume that G is connected (otherwise we work on components separately), so $m \geq n - 1$. Suppose inductively that a maximum clique of $G[v_1, \dots, v_{n-1}]$ is found in time $O(m - d(v_n))$. We now take the vertices of $N(v_n)$ one by one. We give name x and label X to the first one, and check whether the next ones are adjacent to x. If so, we give them label X. If some are not adjacent to x, we give name y and label Y to the first one that we meet. The next vertices receive label X or Y according to their adacency to x or y. Note that exactly one of these adjacencies must occur, since $N(v_n)$ is the union of at most two cliques. At the end of this loops, the vertices with label X and Y form at most two cliques in $N(v_n)$. They are identified in time $O(d(v_n))$. So, we now know all the maximal cliques of $G[N[v_n]]$ and a maximum clique of $G[v_1, \dots, v_{n-1}]$. A maximum clique among these is a maximum clique of G. All this takes time $O(m - d(v_n)) + O(d(v_n)) = O(m)$. Observe that this algorithm relies on a constant time checking of the adjacency, so it needs the graph to be represented by an adjacency matrix. Therefore, the time complexity is $O(n + m)$, but the space complexity is $O(n^2)$. Observe also that this algorithm is not robust. If the input graph is not in $\mathcal{C}_4$, the output is a set of vertices, and if it is a clique, we cannot be sure that it has maximum weight.

For class $\mathcal{C}_6$, the algorithm is similar to the previous one. We have to find a maximum clique in $N(v_n)$ in time $O(d(v_n))$. It is easy to verify quickly whether the neighbohood of v_n is a clique or a stable set, and in both cases, it is immediate to find in time $O(d(v_n))$ a maximum weighted clique in it. We omit further details.

For $\mathcal{C}_3$ (that contains $\mathcal{C}_5$), the algorithm is similar to the previous one, except that we rely on a $\{\overline{P_3}\}$-elimination ordering of G instead of an $\{S_3, P_3\}$-elimination ordering. As a result, the neighborhood of the last vertex v induced a complete multipartite graph. We do not know how to find a maximum clique in $N(v)$ in time $O(d(v))$, so we do not know how to obtain a linear time algorithm. Instead, we look for a maximum clique in $N(v)$ in time $O(m)$, and therefore the overall complexity is $O(nm)$.

Chromatic number

Let us now analyse the column "Coloring" of Table 3.2, that gives the best complexity for coloring a graph of the corresponding class. Since the edge-coloring problem is $\mathcal{NP}$-hard [54], it follows that coloring line graphs is $\mathcal{NP}$-hard, and therefore, so is coloring claw-free graphs (that are all in $\mathcal{C}_1$).

Classes $\mathcal{C}_2$, $\mathcal{C}_3$ and $\mathcal{C}_6$ contain all triangle-free graphs, that are $\mathcal{NP}$-hard to color as proved by Preissmann and Maffray [69].

For $\mathcal{C}_7$, we first try to find a 2-coloring of the graph by the classical BFS algorithm. If it does not exist, we look for a $\max(3, \omega(G))$-coloring of the input graph G as follows. By Theorem 3.12 we obtain an $\{S_3, P_3, \overline{P_3}\}$-elimination ordering in linear time. As a result, the neighborhood of the last vertex of the ordering is a clique or has size 2. We remove the last vertex v, color recursively the remaining vertices, and give some available color to v. Note that the existence of the ordering we use is proved in [29], but how to obtain it in linear time was not known so far.

3.3 Even and odd-signable graphs

In the previous section, checking property $(\star)$ relied on an almost automatic method. Here, we describe two classes of graphs, Berge graphs and even-hole-free graphs, that are proved in existing papers to have property $(\star)$ w.r.t. particular classes in a less obvious way.

A *square* in a graph is a hole of length 4.

Theorem 3.13 (da Silva and Vušković [84]) *Square-free odd-signable graphs have property* $(\star)$ *w.r.t. holes.*

This theorem is used in [84] to obtain a robust $O(n^2m)$-time algorithm for computing a maximum weighted clique in a square-free odd-signable graph (and hence in an even-hole-free graph). We now show how to reduce this complexity to $O(nm)$.

Theorem 3.14 *There is an $O(nm)$ time algorithm whose input is a weighted graph G and whose output is a maximum weighted clique of G or a certificate proving that G is not square-free odd-signable.*

PROOF — Let $\mathcal{H}$ denote the class of all holes and consider the following algorithm. By Theorems 3.6 and 3.13 compute in linear time an ordering $(v_1, \dots, v_n)$ of vertices of G that is an $\mathcal{H}$-elimination ordering if G is a square-free odd-signable graph. We already proved that testing whether a graph is chordal can be done in linear time, and hence it can be checked in $O(nm)$ time whether $(v_1, \dots, v_n)$ is an $\mathcal{H}$-elimination ordering.

So, we may assume that $(v_1, \dots, v_n)$ is an $\mathcal{H}$-elimination ordering of G. We suppose inductively that a maximum weighted clique of $G[v_1, \dots, v_{n-1}]$ is found in time $O((n-1)m)$. A maximum weighted clique of $G[N[v_n]]$ can be found in time $O(m)$ by the algorithm of Rose, Tarjan and Lueker (see [79]) applied to $G[N(v)]$. So, we now know a maximum weighted clique of $G[N[v_n]]$ and a maximum weighted clique of $G[v_1, \dots, v_{n-1}]$. A maximum weighted clique among these is a maximum weighted clique of G. All this takes time $O((n-1)m) + O(m) = O(nm)$. □

We now turn our attention to Berge graphs (and their generalization to even-signable graphs). A *square-theta* is a theta that contains a square. A *long hole* is a hole of length greater or equal to 5.

Theorem 3.15 (Maffray, Trotignon and Vušković [71]) *(Square, theta)-free even-signable graphs have property $(\star)$ w.r.t. long holes.*

Based on Theorem 3.15 an $O(n^7)$ time algorithm is given in [71] for computing a maximum weighted clique in a square-theta-free Berge graph. It relies on a long-hole-elimination ordering. With the machinery presented here, we can obtain this ordering in linear time, but unfortunately, this does not improve the overall complexity of the maximum clique algorithm.

3.4 Universally signable graphs

We said that the class $\mathcal{C}_7$ had already been studied in [29]. This study was concerned with structural properties and our result permits to speed up some algorithms for problems already known to be polynomially solvable. Let us investigate a little bit deeply this class of graphs.

A graph G is *universally signable* if, for any $(0, 1)$ vector β whose entries are in one-to-one correspondence with the holes of G, there exists an assignment of $0, 1$ weights to the edges such that every hole H have weight congruent to β_H mod 2.

The following theorem follows easily from Theorem 2.1

Theorem 3.16 (Conforti, Cornuéjols, Kapoor, K. Vušković [29]) *A graph is universally signable if and only if it does not contain any Truemper configurations as induced subgraphs.*

So universally signable graphs is our class $\mathcal{C}_7$.

The following theorem is a decomposition theorem for universally signable graph. The first proof is in [29], a way more simple proof is given in [28], and the proof we give here is a slight simplification due to Diot and Trotignon [43].

Theorem 3.17 (Conforti, Cornuéjols, Kapoor and Vušković [29]) *Let G be a connected graph. If G is universally signable then either it is a hole, or it is a clique, or it admits a clique cutset.*

PROOF — Let G be a universally signable graph. By Theorem 2.8, we may assume that G is not chordal, so let H be a hole in G. We may assume that $G \setminus H$ is non-empty, so let C be a connected component of $G \setminus H$.

(1) If $c \notin V(H)$ then it has at most two neighbors in H. Moreover, if it has two neighbors then they are adjacent.

Let $c \in V(H)$ and suppose for contradiction that c has two non-adjacent h_1 and h_2 neighbors in H. If $N_H(c) = \{h_1, h_2\}$ then $V(H) \cup \{c\}$ induces a theta, a contradiction. So c has at least three neighbors in H and thus (c, H) is an induced wheel, a contradiction. This proves (1).

Now, suppose for a contradiction that G has no clique cuset. This implies that $N_H(C)$ contains two non-adjacent vertices h_1 and h_2. Let c_1 (resp. c_2) be a neighbor of h_1 (resp. h_2) in C. Observe that by (1), $c_1 \neq c_2$. Let $P = c_1 \dots c_2$ be a path of $G[C]$. Suppose that h_1, h_2, c_1, c_2 and P, are chosen subject to the minimality of P (so P is chordless, and has length at least 1 because $c_1 \neq c_2$).

Case 1: At least one internal vertex of P has a neighbor in H.
Hole H is edge-wise partioned into two paths $H_1 = h_1 \dots h_2$ and $H_2 = h_1 \dots h_2$. Observe first that vertices in $\mathring{P}$ cannot be adjacent to h_1 or h_2, otherwise some subpath of P contradicts the minimality of P. Moreover, if $N_H(\mathring{P})$ has a vertex in both $\mathring{H}_1$ and $\mathring{H}_2$, then some subpath of P contradicts the minimality of P. So, up to symmetry, we assume that if $N_H(\mathring{P}) \cap H_2 = \emptyset$. Now, at least one internal vertex of P, say c, is adjacent to an internal vertex of H_1, say h. It follows by

the minimality of P that H_1 has length 2, i.e. $H_1 = h_1 h h_2$. Observe that c has no neighbor x in H_2, for otherwise a shortest path from x to h with interior in P contradicts the minimality of P. Also, by (1), c has no neighbor in H_2. So, $(h, H_2 \cup P)$ is an induced wheel, a contradiction.

Case 2: No internal vertex of P has a neighbor in H.
If both c_1, c_2 have a unique neighbor in H, then $V(H) \cup V(P)$ induces a theta, a contradiction. So, by (1), up to symmetry c_1 has two adjacent neighbors in H. If c_2 has a unique neighbor in H, then $V(H) \cup V(P)$ induces a pyramid, a contradiction. So, by (1), c_2 has two adjacent neighbors in H. Now, if c_1 and c_2 have a common neighbor in H, $V(H) \cup V(P)$ induces a wheel, and otherwise $V(H) \cup V(P)$ induces a prism, a contradiction. □

We already noticed that, since every Truemper configuration contains a hole, any class defined by forbidding some Truemper configurations is a superclass of chordal graph.

The above theorem show that universally signable graphs are a "slight" generalization of chordal graph: by adding the class of holes as a basic class in the decomposition theorem for chordal graphs (Theorem 3.8), one get the decomposition theorem for universally signable graphs. So, it is easy to turn Theorem 3.17 into a recognition algorithm that run in time $\mathcal{O}(nm)$ (proceed exactly as in the recognition theorem for chordal graphs described in Subsection 2.2.3, with the slight difference that you need to check that the leaves of the decomposition tree are either cliques or holes).

One could wonder if, as for chordal graphs, it is possible to use LexBFS to recognize universally signable graphs. An analogue of Theorem 3.8 for universally signable graphs would state that, if G is a graph and $\{v_1, \dots, v_n\}$ is a LexBFS ordering, then G is universally signable if and only if $\{v_1, \dots, v_n\}$ is a $(S_3, P_3, \overline{P_3})$-elimination ordering (recall that $\{v_1, \dots, v_n\}$ is a $(S_3, P_3, \overline{P_3})$-elimination ordering if, for $i = 1, \dots, n$, $N_{G[v_1,\dots,v_i]}(v_i)$ is either a clique or a stable set of size two). Unfortunately, Figure 3.3 is a counter-example of this statement.

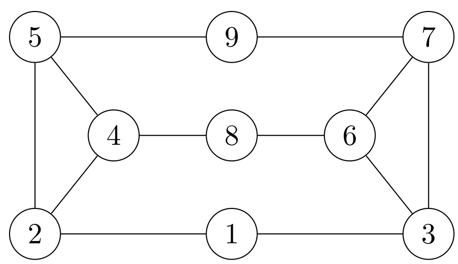


Figure 3.3: Numbers on the vertices of this prism correspond both to a LexBFS ordering and to a $(S_3, P_3, \overline{P_3})$-elimination ordering.

Next theorem states four different characterization of universally signable graphs. These characterizations, and more particularly (4), might be usable to force LexBFS to recognize universally signable graphs in linear time.

Theorem 3.18 *Let G be a connected graph. Following conditions are equivalent:*

1. *G is universally signable.*
2. *G does not contain any Truemper configurations as induced subgraphs.*
3. *For any induced subgraph H of G, either H is a hole, or is a clique, or admits a clique cutset.*

4. *For any induced subgraph H of G, either H admits a vertex whose neighborhood is a clique, or a vertex of degree 2 that is contained in a unique hole of H.*

PROOF — (1) ⇔ (2) ⇔ (3) come from Theorems 3.16 and 3.17.

Let us prove that (3) ⇒ (4). It suffices to prove that any graph G that satisfies (3) satisfies (4). If G is a hole or a clique then it satisfies (4). So G admits a clique cutset. Let S be an extreme clique cutset of G. So there exists a component C_1 of $G \setminus S$ such that $G[C_1 \cup S]$ does not admit clique cutsets. So either $G[C_1 \cup S]$ is a clique and C_1 contains a vertex whose neighborhood is a clique in G, or $G[C_1 \cup S]$ is a hole and then C_1 contains a vertex of degree 2 that is contained in a unique cycle of G, namely $G[C_1 \cup S]$.

We now now prove that (4) ⇒ (2). To prove it, it is enough to check that every vertex of degree 2 in a Truemper configuration is contained in at least two holes, and that Truemper configurations contain no vertex whose neighborhood is a clique, which is straightforward to check. □

3.5 Open questions

Corollary 2.12 suggests that a linear time algorithm for the maximum clique problem might exist for $\mathcal{C}_2$.

We are not aware of a polynomial-time coloring algorithm for graphs in $\mathcal{C}_4$ or $\mathcal{C}_5$, but it would be surprising to us that it exists.

Since class $\mathcal{C}_1$ generalizes claw-free graphs, it is natural to ask which of the properties of claw-free graphs it has, such as a structural description (see [26]), a polynomial-time algorithm for the maximum stable set (see [48]), approximation algorithms for the chromatic number (see [61]), a polynomial-time algorithm for the induced linkage problem (see [49]), and a polynomial χ-bounding function (see [53]).

We also wonder whether theta-free graphs are χ-bounded by a *polynomial* (quadratic?) function (recall that in [64], they are proved to be χ-bounded). Note that since there exist some graphs G satisfying $\alpha(G) = 2$, $\omega(G) = k$ and $|V(G)| = \mathcal{O}(t^2/log(t))$ (where c is a constant, see [60]), by the trivial following bound on the chromatic number: $\chi(G) \leq |V(G)|/\alpha(G)$, the best possible χ-bounded function for theta-free graphs is a $\mathcal{O}(t^2/log(t))$.

In [29], an $O(nm)$ time algorithm is described for the maximum weighted stable set problem in $\mathcal{C}_7$. Since the class is a simple generalization of chordal graphs, we wonder whether a linear time algorithm exists and, as we discussed in Section 3.4, whether a linear time recognition algorithm exists for universally signable graphs.

Chapter 4

Classes defined by constraints on connectivity

In this chapter:

- **If *G* and H are graphs, then we say that G is *H-free* if** G **does not contain H as an induced subgraph.**
- **K_4 is a wheel.**

Most of the results presented in this chapter come from a joint work with M. Radovanović, N. Trotignon and K. Vušković published in SIAM Journal on Discrete Mathematics [7].

A graph is *minimally k-connected* if it is k-connected and if the removal of any edge yields a graph of connectivity $k-1$. A graph is *critically k-connected* if it is k-connected and if the removal of any vertex yields a graph of connectivity $k-1$. Minimally and critically k-connected graphs were the object of much research, see [63] for a survey on this subject. Observe that the classes of critically k-connected graphs and minimally k-connected graphs are not hereditary classes (since graphs of connectivity at most $k-1$ are not in these classes) which, at first sight, discourage attempt to attack these classes with the decomposition method.

In the first section of this chapter we show how to enlarge the class of minimally 2-connected graphs in order to get a hereditary class of graphs. Moreover, this leads us to chordless graphs (a graph is *chordless* if all its cycles are chordless) that are particularly interesting to us since one can easily check that a line-graph is wheel-free if and only if it is the line graph of a chordless graph that contains no K_4 and that has maximum degree at most 3. We then apply the decomposition method on it and revisit several known theorems on minimally 2-connected graphs.

In the second subsection, we investigate if an analogue of the results of the first section can be find with critically 2-connected graphs instead of minimally 2-connected graphs and, despite a negative answer, we explain how it leads us to 2-wheel-free graphs (that are studied in the next chapter).

We close this introduction with a very famous Theorem of Dirac that will be usefull in this chapter and the next one.

Theorem 4.1 *If G is a 2-connected graph and u, v are two distinct vertices of G, then there exists a cycle passing through u and v.*

Note that a more general version of this result is given in Theorem 7.17.

4.1 Minimally 2-connected graphs

This section is divided into three subsections. In the first one we explain how to enlarge the class of minimally 2-connected graphs in order to get a hereditary class of graphs (namely chordless graphs). In the second subsection we present a simple decomposition theorem for chordless graphs. Finally, in the last subsection, we lean on the decomposition theorem to prove some results about chordless and minimally 2-connected graphs.

4.1.1 Enlarging minimally-2-connected graphs

Let $\mathcal{C}_1'$ be the class of *chordless graphs*, that are graphs whose cycles are all chordless (or in other words, the class of graphs that do not contain a cycle with a chord). Observe that a cycle with a chord is not minimally 2-connected, and in a sense is the "smallest" 2-connected graph that is not minimally 2-connected. Class $\mathcal{C}_1'$ was studied by Dirac [45] and Plummer [77] in the 1960's. As we said at the beginning of the section, it is the class we are going to study in place of minimally 2-connected graphs. Indeed, $\mathcal{C}_1'$ is a hereditary class that contains every minimally 2-connected graph (see Theorem 4.2).

One might think it is a strange choice. Indeed, there is a natural way to embed a class $\mathcal{C}$ into an hereditary class $\mathcal{C}'$: taking the closure of $\mathcal{C}$, that is the class $\mathcal{C}'$ of all subgraphs (or induced subgraphs according to the containment relation you are interested in) of graphs from $\mathcal{C}$. This way, one clearly obtains the smallest hereditary class containing $\mathcal{C}$. But, a chordless graph may fail to be a subgraph of some minimally 2-connected graph. For instance consider a triangle with a pending edge (the *paw*). It is chordless and it is easy to check that a 2-connected graph that contains a paw as a subgraph also contains a cycle with a chord and thus is not minimally 2-connected. Hence $\mathcal{C}_1'$ is a proper superclass of the class of subgraphs of minimally 2-connected graphs. We choose $\mathcal{C}_1'$ anyway, first because it is a very natural class of graphs and, as the next theorem suggests, it captures every structural property that minimally 2-connected graphs might have.

Next theorem show the close relationship minimally 2-connected graphs and chordless graph maintain.

Theorem 4.2 (Dirac [45], Plummer [77]) *A 2-connected graph is chordless if and only if it is minimally 2-connected.*

PROOF — Suppose first that G is a 2-connected chordless graph that is not minimally 2-connected. So it admits an edge $e = xy$ such that $G \setminus e$ is 2-connected. So, by Theorem 4.1 there exists a cycle C passing through x and y. So e is a chord of C, a contradiction.

Conversely, suppose that G is a minimally 2-connected graph and let uv be an edge of G. So, $G \setminus uv$ has connectivity 1 and therefore contains a cutvertex x. Since G is 2-connected, it follows that $(G \setminus uv) \setminus x$ has two connected components, one containing u, the other containing v. This implies that every cycle of G that contains u and v must go through uv, so uv cannot be a chord of any cycle of G. This proof can be repeated for all edges of G. It follows that G is chordless. □

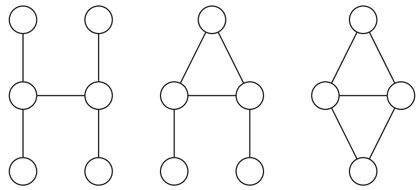


Figure 4.1: List of obstructions for $\mathcal{C}'_0$

4.1.2 Decomposition and structure of chordless graphs

Let $\mathcal{C}'_0$ be the class of graphs such that vertices of degree at least 3 induce an independent set.

Observe that class $\mathcal{C}'_0$ is closed under taking subgraph (the list of obstruction is given in Figure 4.1) and that $\mathcal{C}'_0 \subsetneq \mathcal{C}'_1$.

We are now going to show that $\mathcal{C}'_1$ admits a simple decomposition theorem with $\mathcal{C}'_0$ serving as a basic class (Theorem 4.3). It is actually a known theorem, it is implicitly proved in [92] and explicitly stated and proved in [66]. Anyway, the proof proposed here is much shorter and simpler than the previous ones. It follows a classical way to prove a decomposition theorem that consists to start with a graph that is not basic, thus it admits an obstruction to the basic class and, studying how the rest of the graph attaches on this obstruction, we find one of the specified decomposition.

We say that a graph admits a 0-cutset if it is disconnected.

A 2-cutset $\{a, b\}$ is a *S_2-cutset* if $ab \notin E(G)$. It is a *proper S_2-cutset* if $G \setminus \{a, b\}$ can be partitioned into two sets K' and K'' such that: there is no edge with one extremity in K' and the other one in K'' and $G[\{a, b\} \cup K']$ (resp. $G[\{a, b\} \cup K'']$) is not a chordless ab-path. We say that $(\{a, b\}, K', K'')$ is a *split* of a proper S_2-cutset.

Theorem 4.3 (Decomposition Theorem for $\mathcal{C}'_1$) *A graph in $\mathcal{C}'_1$ is either in $\mathcal{C}'_0$, or has a 0-cutset, a 1-cutset, or a proper S_2-cutset.*

PROOF — Let G be in $\mathcal{C}'_1 \setminus \mathcal{C}'_0$ and suppose that G has no 0-cutset and no 1-cutset, i.e. G is 2-connected. So, by Theorem 4.2, G is minimally 2-connected. Thus, there is an edge $e = uv$ such that u and v have both degree at least 3 and by Theorem 4.2, $G \setminus e$ is not 2-connected so, it has a 0-cutset or a 1-cutset.

If $G \setminus e$ is disconnected, then u (and v) would be a cutvertex of G. So $G \setminus e$ has a cutvertex w. Since w is not a cutvertex of G, the graph $(G \setminus e) \setminus w$ has exactly two connected components C_u and C_v, containing u and v respectively, and $V(G) = C_u \cup C_v \cup \{w\}$. Let $u' \notin \{v, w\}$ be a neighbor of u (u' exists since u has degree at least 3). So, $u' \in C_u$. In G, u is not a cutvertex, so there is a path P_u from u' to w whose interior is in C_u. Together with a path P_v from v to w with interior in C_v, P_u, uu' and e form a cycle, so $uw \notin E(G)$ for otherwise uw would be a chord of this cycle. Because of the degrees of u and v, $(\{u, w\}, C_u \setminus \{u\}, C_v)$ is a split of a proper S_2-cutset of G. □

Theorem below can easily be turned into a complete structural characterization of chordless graphs. Theorems 4.4 and 4.5 state this characterization. We do not include the straightforward proofs.

We first define blocks of decomposition for each cutset involved in Theorem 4.3.

If S is a 0-cutset or a 1-cutset of G and $G \setminus S$ admits k distinct connected components $C_1, \ldots, C_k$ (note that $k \geq 2$), we define k blocks of decomposition $G_1, \ldots, G_k$ as follows: for $i = 1, \ldots, k$,

$G_i = G[C_i \cup S]$. We say that $G_1, \ldots, G_k$ are the blocks of decomposition with respect to S.

Let $(\{a,b\}, X, Y)$ be a split of a proper S_2-cutset. The block of decomposition with respect to this split are graph G_X and G_Y defined as follows. Block G_X is the graph obtained from $G[X \cup \{a,b\}]$ by adding a marker vertex m' and the two marker edges am' and bm'. Block G_Y is the graph obtained from $G[Y \cup \{a,b\}]$ by adding a marker vertex m'' and the two marker edges am'' and bm''. Note that by definition of a proper S_2-cutset, $G[\{a,b\} \cup X]$ and $G[\{a,b\} \cup Y]$ are not chordless ab-path and thus G_X and G_Y are not chordless cycles.

Theorem 4.4 *Let S be a 0-cutset or a 1-cutset and let $G_1, \ldots, G_k$ be the blocks of decomposition with respect to S. Then $G \in \mathcal{C}'_1$ if and only if $G_i \in \mathcal{C}'_1$ for $i = 1, \ldots, k$.*

Theorem 4.5 *Let G be a 2-connected graph, let (S, X, Y) be a split of a proper 2-cutset and let G_X and G_Y the corresponding blocks of decomposition. Then $G \in \mathcal{C}'_1$ if and only if $G_X \in \mathcal{C}'_1$ and $G_Y \in \mathcal{C}'_1$.*

4.1.3 Applications of the decomposition theorem

In this subsection, we show how the decomposition theorem for chrordless graphs (Theorem 4.3) can be used to get different kind of properties on chordless graphs.

Dirac [45] and Plummer [77] independently showed that minimally 2-connected graphs have at least two vertices of degree at most 2 and chromatic number at most 3. We now show how Theorem 4.3 can be used to give simple proofs of these results for chordless graphs in general.

Theorem 4.6 *Every chordless graph on at least two vertices has at least two vertices of degree at most 2.*

PROOF — We prove the result by induction on the number of vertices. If $G \in \mathcal{C}'_0$ then clearly the statement holds. Let $G \in \mathcal{C}'_1 \setminus \mathcal{C}'_0$, and assume the statement holds for graphs with fewer than $|V(G)|$ vertices. Suppose G has a 0-cutset or 1-cutset S, and let $C_1, \ldots, C_k$ be the connected components of $G \setminus S$. For $i = 1, \ldots, k$, by induction applied to $G_i = G[V(C_i) \cup S]$, C_i contains a vertex of degree at most 2 in G_i. Note that such a vertex is of degree at most 2 in G as well, and hence G has at least two vertices of degree at most 2.

So we may assume that G is 2-connected, and hence by Theorem 4.3 G has a proper S_2-cutset with split $(\{a,b\}, X, Y)$. We now show that both X and Y contain a vertex of degree at most 2.

Let $(\{a',b'\}, X', Y')$ be a split of a proper S_2-cutset of G such that $X' \subseteq X$, and out of all such splits assume that $|X'|$ is smallest possible. We now show that both a' and b' have at least two neighbors in X', i.e. they are of degree at least 3 in G and G'_X. Since G is 2-connected both a' and b' have a neighbor in every connected component of $G \setminus \{a', b'\}$. In particular $G[Y' \cup \{a', b'\}]$ contains an $a'b'$-path Q and a' has a neighbor a_1 in X'. Suppose $N(a') \cap X' = \{a_1\}$. If $a_1 b'$ is not an edge, then (since $G[X' \cup \{a', b'\}]$ is not a chordless path), for some $X'' \subseteq X'$, $(\{a_1, b'\}, X'', V(G) \setminus (X'' \cup \{a_1, b'\})$ is a split of a proper S_2-cutset of G, contradicting our choice of $(\{a', b, \}, X', Y')$. So $a_1 b'$ is an edge. Then since $G[X' \cup \{a', b'\}]$ is not a chordless path, $X' \setminus \{a_1\}$ contains a vertex c. Since a_1 cannot be a cutvertex of G, there is a $b'c$-path in $G \setminus a_1$ whose interior vertices are in X'. Since b' cannot be a cutvertex of G, there is an $a_1 c$-path in $G \setminus b'$ whose interior vertices are in X'. Therefore $G[(X' \cup b')] \setminus a_1 b'$ contains an $a_1 b'$-path P. But then $V(P) \cup V(Q)$ induces a cycle with a chord (namely $a_1 b'$), a contradiction. Therefore, a' has at least two neighbors in X' and by symmetry so does b'.

Let m' be the marker vertex of $G_{X'}$. Note that $|V(G_{X'})| < |V(G)|$, and clearly since G is chordless so is $G_{X'}$. So by induction, $G_{X'}$ contain at least two vertices of degree 2. Therefore there is a vertex $t \in V(G_{X'}) \setminus \{m'\}$ that is of degree at most 2 in $G_{X'}$. Since both a' and b' have at least two neighbors in X', it follows that $t \in X'$, and hence t is of degree at most 2 in G as well. So X contains a vertex of degree at most 2, and by symmetry so does Y, and the result holds. □

Note that for proving the theorem below, it is essential that the class we work on is closed under taking subgraphs. Proofs of 3-colorability in [45] and [77] are more complicated because they consider only minimally 2-connected graphs, that are not closed under taking subgraphs which forbids using induction.

Corollary 4.7 *If G is a chordless graph then $\chi(G) \leq 3$.*

PROOF — Let G be a chordless graph and by Theorem 4.6 let x be a vertex of G of degree at most 2. Inductively color $G \setminus \{x\}$ with at most 3 colors. This coloring can be extended to a 3-coloring of G since x has at most two neighbors in G. □

Next lemma shows that if a chordless graph G admits a proper S_2-cutset, then it admits an extreme proper S_2-cutset (recall it is a proper S_2-cutset such that one of the block of decomposition does not contain proper S_2-cutset). The proof can be find in [67] but we reproduce it here for completeness.

Lemma 4.8 (Machado, de Figueiredo and Trotignon [67]) *Let G be a 2-connected chordless graph not in $\mathcal{C}_0'$. Let $(\{a,b\}, X, Y)$ be a split of a S_2-cutset of G such that $|X|$ is minimum among all possible such splits. Then G_X is in $\mathcal{C}_0'$. Moreover, a and b both have degree at least 3 in G and in G_X.*

PROOF — The proof that a and b are of degree at least 3 in G and in G_X is the same as the proof, in Theorem 4.6, that a' and b' are of degree at least 3 in G' and G'_X.

Let m be the marker vertex of G_X. It is easy to check that G_X is a 2-connected chordless graph. Suppose $G_X \notin \mathcal{C}_0'$. So, by Theorem 4.3, it admits a split of a proper 2-cutset, say $(\{u,v\}, X_1, X_2)$. Choose it such that both u and v have degree at least 3 (this is possible as explained in the beginning of the proof). Hence $m \notin \{u,v\}$. If $\{u,v\} = \{a,b\}$ then $(\{a,b\}, X_1, Y \cup X_2)$ is a split of a proper 2-cutset of G that contradicts the minimality of X. So $\{u,v\} \neq \{a,b\}$ and we may assume w.l.o.g. that $b \notin \{u,v\}$. Hence b and m are in the same connected component of $G_X \setminus \{u,v\}$, so may assume w.l.o.g. that $\{b,m\} \subseteq \{X_2\}$. Thus $(\{u,v\}, X_1, Y \cup X_2 \setminus \{m\})$ is a split of a proper 2-cutset in G that contradicts the minimality of X. □

We now show how Theorem 4.3 may be used to prove the main result in [77], that is a nice characterization of minimally 2-connected graphs.

Theorem 4.9 (Plummer [77]) *Let G be a 2-connected graph. Then G is minimally 2-connected (or equivalently chordless) if and only if either*

1. *G is a cycle; or*

2. *if S denotes the set of vertices of degree 2 in G, then there are at least two components in $G \setminus S$, each component of $G \setminus S$ is a tree and if C is any cycle in G and T is any component of $G \setminus S$, then $(V(C) \cap V(T), E(C) \cap E(T))$ is empty or connected.*

PROOF — Suppose first that G is minimally 2-connected. If G is in C'_0 then $G \setminus S$ contains only isolated vertices. Hence, either $G \setminus S$ is empty, in which case all vertices of G are of degree 2, meaning that G is a cycle; or $G \setminus S$ is not empty, in which case G contains at least two vertices of degree at least 3, and the second outcome holds.

So, by Theorem 4.3, we may assume that G admits a split of a proper S_2-cutset $(\{a, b\}, X, Y)$ and, by Lemma 4.8, we may assume that G_X is in $\mathcal{C}'_0$ and a, b have degree at least 3 in G_X. Note that from the definition of a proper S_2-cutset, none of G_X, G_Y is a cycle. Note also that, since $G_X \in \mathcal{C}'_0$, the set of vertices of degree at least 3 in G_X induces a stable set.

Let us first prove that each component of $G \setminus S$ is a tree. Let S_Y be the set of vertices of degree 2 in G_Y and let $T_1, \dots, T_k$ be the components of $G_Y \setminus S_Y$. By induction, $T_1, \dots, T_k$ are trees. If a or b is of degree 2 in G_Y and has a neighbor of degree at least 3 (it has at most one such neighbor) in T_i say ($i \in \{1, \dots, k\}$), then define T'_i to be the tree obtained by adding the pending vertex a (resp. b, resp. both a and b) to T_i. For all $j = 1, \dots, k$ such that T'_j is not defined above, we put $T'_j = T_j$. Now, if we remove the vertices of degree 2 of G, $T'_1, \dots, T'_k$ are connected components (here we use the fact that since G_X is in $\mathcal{C}'_0$, all neighbors of a or b in X have degree 2). The other components are the vertices of degree 3 from X. They are all trees because G_X is in $\mathcal{C}'_0$. That proves that each component of $G \setminus S$ is a tree.

It remains to prove that if C is any cycle in G and T is any component of $G \setminus S$, then $(V(C) \cap V(T), E(C) \cap E(T))$ is empty or connected. Let C be a cycle of G. There are three cases. Either $V(C) \subseteq X \cup \{a, b\}$, or $V(C) \subseteq Y \cup \{a, b\}$, or C is formed of a path P_X from a to b with interior in X and a path P_Y from a to b with interior in Y. In the first case, the trees intersected by C are all formed of one vertex, so outcome 2 holds. In the second case, C is also a cycle of G_Y. Let T be a tree of $G \setminus S$ such that $V(T) \subset Y \cup \{a, b\}$ (all the other trees of $G \setminus S$ are on 1 vertex). Note that $a \in V(C) \cap V(T)$ implies that a has degree at least 3 in G_Y and so T is also a tree of $G_Y \setminus S_Y$. Hence, $(V(C) \cap V(T), E(C) \cap E(T))$ is connected by the induction hypothesys applied to G_Y. In the third case, we consider the cycle C_Y formed by P_Y and the marker vertex of G_Y. We suppose that T has more than one vertex (otherwise T is indeed connected), so $V(T) \subseteq Y \cup \{a, b\}$. Note that if T contains a, then it must also contains some neighbor of a in Y (because marker vertices are of degree 2). This means that if a has degree 2 in G_Y and $a \in V(C) \cap V(T)$, then the neighbor a' of a in G_Y has degree at least 3 and is therefore in a tree of $G_Y \setminus S_Y$, so $a' \in V(C) \cap V(T)$. The same remark holds for b. Hence, $(V(C) \cap V(T), E(C) \cap E(T))$ is connected by the induction hypothesis applied to G_Y.

Suppose conversely that one of the outcomes 1, 2 is satisfied by some 2-connected graph G (here we reproduce the proof given by Plummer). If G is a cycle, then it is obviously minimally 2-connected. Otherwise, let $e = uv$ be an edge of G. It is enough to prove that $G \setminus e$ is not 2-connected. If u or v has degree 2 in G this holds obviously. Otherwise, u and v are in the same component T of $G \setminus S$. If $G \setminus e$ is 2-connected, then some cycle C of $G \setminus e$ goes through u and v, and $(V(C) \cap V(T), E(C) \cap E(T))$ is not connected nor empty because it contains u and v but not $e = uv$ (and removing any edge from a tree disconnects it), a contradiction to 2. □

Note that we do not use the existence of vertices of degree 2 to prove the theorem above. Hence, a new proof of their existence can be given: if G is 2-connected, then by Theorem 4.9, the vertices of degree 2 of G form a cutset of G. Thus, there must be at least two of them; otherwise, the existence of two vertices of degree at most 2 follows easily by induction.

The last theorem of this section is concerned with edge and total coloring of chordless graphs.

We do not reproduce the proof here because it is a bit long, but we write a short proof sketch that gives the flavor of it. Note that to prove this theorem, the only approach we are aware of is to use Theorem 4.3.

Theorem 4.10 (Machado, de Figueiredo and Trotignon [67]) *Let G be a chordless graph of maximum degree at least 3. Then G is $\Delta(G)$-edge colourable and $(\Delta(G)+1)$-total-colourable.*

SKETCH OF PROOF — The first step is to prove directly the result for graphs in $\mathcal{C}'_0$. Then, a minimal counter-example G belongs to $\mathcal{C}'_1 \setminus \mathcal{C}'_0$. So, by Theorem 4.3 it admits a split (S, X, Y) of a 0-cutset, a 1-cutset or a proper 2-cutset. Let G_X and G_Y be the two blocks of decomposition w.r.t this split. Note that by minimality of G, it is easy to check that results hold for G_X and G_Y. If S is a 0-cutset or a 1-cutset it is easy to recover directly an edge or a total coloring from a coloring of the blocks of decomposition. So S is a proper 2-cutset and thus, by Lemma 4.8 we may assume that $G_X \in \mathcal{C}'_0$. Taking advantage of the extreme simplicity of the graphs in $\mathcal{C}'_0$, it is possible to extend an edge or a total coloring of $G[Y]$ to G. □

We close this subsection by observing that there is another well studied hereditary class that properly contains the class $\mathcal{C}'_1$, namely the class of graphs that do not contain a cycle with a unique chord as an induced subgraph. In [92], a precise structural description of this class is given and used to obtain efficient recognition and coloring algorithms. Interestingly, it was proved by McKee [72] that these graphs can also be defined by constraints on connectivity: the graphs with no cycles with a unique chord are exactly the graphs such that all minimal cutsets are independent sets.

4.2 Critically 2-connected graphs

In this subsection we consider the class of critically 2-connected graphs, that were studied by Nebeský [75], and we investigate whether there exists an analogous sequence of theorems as in the previous subsection, starting with critically 2-connected graphs instead of minimally 2-connected graphs. To enlarge the class of minimally 2-connected graphs we chose the class that does not contain cycles with chords, and it was equivalent to forbid them as subgraphs or induced subgraphs. Thinking of it, it is clear that a similar way to enlarge the class of critically 2-connected graphs, is to forbid 2-wheels (indeed, 2-wheels are, in a sense, the smallest critically 2-connected graphs). Since forbidding 2-wheels as subgraphs or induced subgraphs is not equivalent, there are two way to find an analogue of chordless graphs, and we consider both.

Let $\mathcal{C}_1$ be the class of graphs defined by forbidding 2-wheels as subgraphs, and let $\mathcal{C}_2$ be the class of graphs defined by forbidding 2-wheels as induced subgraphs. The analogue of $\mathcal{C}'_0$ is naturally $\mathcal{C}_0$: the class of graphs with no vertices adjacent with at least two vertices of degree at least three. It is easy to check that $\mathcal{C}_0 \subsetneq \mathcal{C}_1 \subsetneq \mathcal{C}_2$.

Unfortunately, an exact analogue of Theorem 4.2 is hopeless. A critically 2-connected graph can contain anything as a subgraph: the only hereditary class that contains every critically 2-connected graphs is the class of all graphs. To see this, consider a graph G on vertices $\{v_1, \dots, v_n\}$. For all $i \in \{1, \dots, n\}$, add a vertex a_i adjacent to v_i and a vertex b_i adjacent to a_i. Add a vertex c adjacent to all b_i's. It is easy to see that the obtained graph is critically 2-connected, and contains G as a subgraph. So there cannot be a version of Theorem 4.2 with "critically" instead of "minimally", and particularly, a critically 2-connected graph may contain a 2-wheel (since it may contain anything). Also, any property of graphs closed under taking subgraphs, such as being k-colorable, is false for critically 2-connected graphs, unless it holds for all graphs. However, there is a sequence of theorems, proven in the next section, that mimics the sequence of results obtained on $\mathcal{C}'_0$ and $\mathcal{C}'_1$.

As we just said, an exact analogue of Theorem 4.2 cannot be interesting, but the following can be seen as a "semi"-analogue of it.

Lemma 4.11 *A graph G belongs in $\mathcal{C}_1$ if and only if for every subgraph H of G, either H has connectivity at most 1 or it is critically 2-connected.*

PROOF — A 2-wheel has connectivity 2 and is not critically 2-connected since removing the center yields a 2-connected graph. This proves the "if" part of the theorem. To prove the "only if" part, consider a graph G that contains no 2-wheel, and suppose for a contradiction that some subgraph H of G does not satisfy the requirement on connectivity that is to be proved. Hence H is 2-connected and not critically 2-connected. So by deleting a vertex v (that has degree at least 2 because of the connectivity of H), a 2-connected graph H' is obtained. Note that $|V(H)| \geq 4$. Therefore, by Theorem 1.1, H' contains a cycle, that together with v forms a 2-wheel of H, a contradiction. □

Before studying the structure of $\mathcal{C}_1$ and $\mathcal{C}_2$, let us see that an analogue of Theorem 4.6 can be proved directly (by a nice argument on the longest chordless path) for 2-wheel-free graphs. Nebeský [75] proved that every critically 2-connected graph contains a vertex of degree 2, but critically 2-connected graphs are not 3-colorable in general, since they may contain any 4-chromatic graph as a subgraph.

Theorem 4.12 *If $G \in \mathcal{C}_2$, then G has a vertex of degree at most 2 and G is 3-colorable.*

PROOF — Suppose that for every $v \in V(G)$, $d(v) \geq 3$. Let P be a longest chordless path in G, and x and y the endvertices of P. As $d(x) \geq 3$, x has at least two neighbors u and v not in P and u (resp. v) has a neighbor in $P \setminus x$, since otherwise $V(P) \cup \{u\}$ (resp. $V(P) \cup \{v\}$) would induce a longer path in G. We choose u_1 and v_1, neighbors of respectively u and v in $P \setminus x$ that are closest to x on P. W.l.o.g. let us assume that x, u_1, v_1 appear in this order on P. Then (u, xPv_1vx) is an induced 2-wheel of G, a contradiction. This proves that G has a vertex of degree at most 2. It follows by an easy induction that every graph from $\mathcal{C}_2$ is 3-colorable. □

Chapter 5

2-Wheel-free graphs

In this chapter:

- **If *G* and H are graphs, then we say that G is *H-free* if G does not contain H as an induced subgraph.**
- **K_4 is a wheel.**

All results presented in this chapter come from a joint work with M. Radovanović, N. Trotignon and K. Vušković published in SIAM Journal on Discrete Mathematics together with the results of the previous chapter [7].

Recall that a 2-wheel is a graph formed by a chordless cycle and a vertex, outside the cycle, that has at least two neighbors in the cycle. Recall also that, as in the previous chapter, $\mathcal{C}_1$ and $\mathcal{C}_2$ are respectively the class of graphs that do not admit 2-wheels as subgraphs and the class of graphs that do not admit 2-wheels as induced subgraphs.

We already observed that $\mathcal{C}_1 \subseteq \mathcal{C}_2$, and the main results of this section, even if they hold for $\mathcal{C}_1$, are interesting when concerning $\mathcal{C}_2$. We also already motivated the study of 2-wheel-free graphs in Section 2.3, emphasizing on the fact they were a subclass of wheel-free graphs. In this chapter, we study 2-wheel-free graphs using all techniques explained in Chapter 2.1. A decomposition theorem for this class is given in Section 5.1 and is turned into a complete structural characterization in Section 5.2. In Section 5.3, we give a polynomial-time recognition algorithm using a decomposition tree as explained in Section 2.2.2, as well as a $\mathcal{NP}$-completeness result concerned with the recognition problem for 4-wheel-free graphs. And finally, in Section 5.4, we use extremal cutsets to prove a local structural property of 2-wheel-free graphs that implies a polynomial-time algorithm for the edge-coloring problem.

Names of the classes studied in this chapter being a bit confusing, we recall there definitions here in order to facilitate the lecture.

- $\mathcal{C}_0$ is the class of graphs with no vertices adjacent with at least two vertices of degree at least 3.
- $\mathcal{C}_1$ is the class of graphs that do not contain 2-wheels as subgraphs.
- $\mathcal{C}_2$ is the class of graphs that do not contain 2-wheels as induced subgraphs.

5.1 Decomposition theorems

In this section we present decomposition theorems for graphs that do not contain 2-wheels as subgraphs ($\mathcal{C}_1$) and graphs that do not contain 2-wheels as induced subgraphs ($\mathcal{C}_2$).

We start to present the special cutsets we need to decompose these two classes.

Proper $\mathbf{K_2}$-cutset
A 2-cutset $\{a, b\}$ is a *K_2-cutset* if $ab \in E(G)$. It is a *proper K_2-cutset* if $G \setminus \{a, b\}$ can be partitioned into two non-empty sets K' and K'' such that there is no edge with one extremity in K' and the other one in K'' and no vertex of $K' \cup K''$ sees both a and b.

We say that $(\{a, b\}, K', K'')$ is a *split* of a proper K_2-cutset.

Proper $\mathbf{S_2}$-cutset
A 2-cutset $\{a, b\}$ is a *S_2-cutset* if $ab \notin E(G)$. It is a *proper S_2-cutset* if $G \setminus \{a, b\}$ can be partitioned into two sets K' and K'' such that there is no edge with one extremity in K' and the other one in K'' and $G[\{a, b\} \cup K']$ (resp. $G[\{a, b\} \cup K']$) is not a chordless ab-path.

We say that $(\{a, b\}, K', K'')$ is a *split* of a proper S_2-cutset.

Proper I-cutset
A set of vertices $S = \{u, v, w\}$ of a graph G is an *I-cutset* if $G[S]$ induced exactly one edge. It is a *proper I-cutset* if no vertex of $G \setminus \{u, v, w\}$ can be partitioned into two sets K' and K'' such that:

- There is no edge with one extremity in K' and the other one in K''
- For some component C' of $G[K']$, u, v and w all have a neighbor in C'.
- For some component C'' of $G[K'']$, u, v and w all have a neighbor in C''.
- Every vertex in $K' \cup K''$ is adjacent to at most one vertex in $\{u, v, w\}$.

We say that $(\{u, v, w\}, K', K''$ is a *split* of this I-cutset.

We now give some notation related with these cutsets that will be used all along this chapter.

If G is a 2-connected graph and $(\{a, b\}, K', K'')$ is a split of a proper S_2-cutset of G, then it is clear that $G[K' \cup \{a, b\}]$ (resp. $G[K'' \cup \{a, b\}]$) contains a chordless ab-path. For such a S_2-cutset, we denote this path by P'_{uv} (resp. P''_{uv}).

Similarly, if G is a 2-connected graph and $(\{u, v, w\}, K', K'')$ is the split of an I-cutset (where uv say is an edge), then $G[K' \cup \{u, v, w\}] \setminus uv$ (resp. $G[K'' \cup \{u, v, w\}] \setminus uv$) contains chordless uv-path, uw-path and vw-path. For such an I-cutset, we denote these paths by P'_{uv}, P'_{uw} and P'_{vw} (resp. P''_{uv}, P''_{uw} and P''_{vw}).

We start with a technical lemma on the structure of 2-connected graphs in $\mathcal{C}_1$ and $\mathcal{C}_2$ that do not admit K_2-cutset (note that since $\mathcal{C}_1 \subsetneq \mathcal{C}_2$, it is enough to prove it for $\mathcal{C}_2$). Recall that C_k is the chordless cycle on k vertices.

Lemma 5.1 *Let G be a 2-connected graph that does not have a K_2-cutset. If $G \in \mathcal{C}_2$ and it contains a C_k, for some $k \in \{3, 4, 5\}$, as an induced subgraph, then $G = C_k$.*

PROOF — Let $G \in \mathcal{C}_2$ and suppose that G contains a $C_k = x_1x_2 \dots x_kx_1$ as an induced subgraph, for some $k \in \{3, 4, 5\}$. Assume $G \neq C_k$ and that G has no 1-cutset nor K_2-cutset. Let K be a connected component of $G \setminus C_k$.

If a vertex $x \in K$ is adjacent to more than one vertex of C_k, then (x, C_k) is a 2-wheel of G. So a vertex of K can have at most one neighbor in C_k. Since G has no 1-cutset nor K_2-cutset, $|N(K) \cap V(C_k)| \geq 2$, and if $|N(K) \cap V(C_k)| = 2$, then the two vertices of $N(K) \cap V(C_k)$ are nonadjacent.

Suppose $k = 3$, and let P be a minimal path of K such that its endvertices are adjacent to different vertices of C_k. Then $V(P) \cup V(C_k)$ induces a 2-wheel. Therefore $k \in \{4, 5\}$, and hence $N(K) \cap V(C_k)$ contains nonadjacent vertices. Let P be a minimal path of K such that its endvertices are adjacent to nonadjacent vertices of C_k. We may assume w.l.o.g. that the endvertices of P are adjacent to x_1 and x_3. By the choice of P, we may assume w.l.o.g. that vertices of $V(C_k) \backslash \{x_1, x_2, x_3\}$ have no neighbors in P. But then $V(C_k) \cup V(P)$ induces a 2-wheel. □

Next Lemma is a direct application of the above one.

Lemma 5.2 *Let G be a 2-connected graph. If G has a K_2-cutset that is not proper, then $G \notin \mathcal{C}_2$.*

PROOF — If G has a K_2-cutset that is not proper then it contains a triangle and the result holds by Lemma 5.1 as $G = C_3$ and thereby has no K_2-cutset: contradiction. □

Theorem 5.3 (Decomposition Theorem for C_1) *A graph in $\mathcal{C}_1$ is either in $\mathcal{C}_0$ or it has a 0-cutset, a 1-cutset, a proper K_2-cutset or a proper S_2-cutset.*

PROOF — Let G be a 2-connected graph in $\mathcal{C}_1 \setminus \mathcal{C}_0$. So G contains a vertex w that has two neighbors u and v that are both of degree at least 3. By Lemma 5.1, $uv \notin E(G)$. If no vertex of $G \setminus w$ is a cutvertex separating u from v, then by Theorem 4.1, there is a cycle of $G \setminus w$ going through u and v, so that in G, w is the center of a 2-wheel, a contradiction. Hence there is such a cutvertex w'. So, in $G \setminus \{w, w'\}$, there are distinct components C_u and C_v containing u and v respectively, and possibly other components whose union is denoted by C. But then $(\{w, w'\}, C \cup C_u, C_v)$ is a split of either a S_2-cutset of G (when $ww' \notin E(G)$), which is proper because of the degrees of u and v, or a split of a K_2-cutset (when $ww' \in E(G)$), which is proper by Lemma 5.2. □

To prove the decomposition theorem for $\mathcal{C}_2$ we need the following lemma.

Lemma 5.4 *Let G be a 2-connected graph that has no K_2-cutset. If an I-cutset of G is not proper, then $G \notin \mathcal{C}_2$.*

PROOF — Let $(\{u, v, w\}, K', K'')$ be a split of an I-cutset such that uv is an edge. Suppose that $x \in G \setminus \{u, v, w\}$ has at least two neighbors in $\{u, v, w\}$. W.l.o.g. $x \in K'$ and by Lemma 5.1 w.l.o.g. $N(x) \cap \{u, v, w\} = \{u, w\}$. If $x \notin V(P'_{uw})$ then $(x, uP'_{uw}wP''_{uw}u)$ is a 2-wheel, and hence $G \notin \mathcal{C}_2$. So we may assume that $x \in V(P'_{uw})$ and that $P'_{uw} = uxw$. Let C' be the connected component of $G[K']$ that contains x, and let P be the shortest xv-path in $G[C \cup \{v\}]$. If w does not have a neighbor in $P \setminus \{x, v\}$, then $(u, vxwP''_{wv}v)$ is a 2-wheel with center u, and hence $G \notin \mathcal{C}_2$. So we may assume that w has a neighbor in $P \setminus \{x, v\}$. If u does not have a neighbor in $P \setminus \{x, v\}$, then $(w, xuvPx)$ is a 2-wheel, and hence $G \notin \mathcal{C}_2$. So we may assume that u also has a neighbor in $P \setminus \{x, v\}$. Let w_1 (resp. u_1) the neighbor of w (resp. of u) on P that is the nearest of x. If x, w_1, u_1 appear in this order along P or if $u_1 = w_1$, then $(w, uxPu_1u)$ is a 2-wheel. So (x, u_1, w_1) appear in this order long P and $(u, wxPw_1w)$ is a 2-wheel, a contradiction. □

Next theorem is the decomposition for $\mathcal{C}_2$.

Theorem 5.5 (Decomposition Theorem for C_2) *A graph in $\mathcal{C}_2$ is either in $\mathcal{C}_1$ or it has a proper I-cutset.*

PROOF — Let G be a graph in $\mathcal{C}_2 \setminus \mathcal{C}_1$. So G contains a 2-wheel as a subgraph. Let (x, C) be a 2-wheel of G whose rim has the fewest number of chords. Note that C must have at least one chord.

(1) Let $y'y''$ be a chord of C, and P_1 and P_2 the two $y'y''$-subpaths of C. If a vertex $u \in V(G) \setminus V(C)$ has more than one neighbor on C, then it has exactly two neighbors on C, one in the interior of P_1, and the other in the interior of P_2.

Let $u \in V(G) \setminus V(C)$ and suppose that u has at least two neighbors on C. If u has at least two neighbors on P_i, for some $i \in \{1, 2\}$, then $(u, yy'P_iy)$ is a 2-wheel that contradicts our choice of (x, C)). This proves (1).

By (1), x has exactly two neighbors x' and x'' on C.

(2) If $u \in V(G) \setminus (V(C) \cup \{x\})$ then u has at most one neighbor on C.

Assume not. Then (1), u has exactly two neighbors u' and u'' on C. Let P_1 and P_2 be the two $u'u''$-subpaths of C. Note that since C has a chord, by (1) that chord has one endvertex in the interior of P_1 and the other in the interior of P_2. In particular, neither P_1 nor P_2 is an edge. If $\{x', x''\} \subset V(P_i)$, for some $i \in \{1, 2\}$, then the graph induced by $V(P_i) \cup \{u, x\}$ contains a 2-wheel with center x that contradicts our choice of (x, C). So w.l.o.g. x' is contained in the interior of P_1 and x'' in the interior of P_2. Let $y'y''$ be a chord of C. Then by (1) we may assume that vertices u', x', y', u'', x'', y'' are all distinct and appear in this order when traversing C clockwise. If $u'y''$ is an edge then the graph induced by $V(P_1) \cup \{u, y''\}$ contains a 2-wheel with center y'' that contradicts our choice of (x, C). So $u'y''$ is not an edge, and by symmetry neither is $u''y'$. Let P_1' (respectively P_2') be the $u'y'$-subpath (respectively $u''y''$-subpath) of C that contains x' (respectively x''). Then the graph induced by $V(P_1') \cup V(P_2') \cup \{u, x\}$ contains a 2-wheel with center x that contradicts our choice of (x, C). This proves (2).

Let $y'y''$ be a chord of C. By (1), vertices x', y', x'', y'' are all distinct and w.l.o.g. appear in this order when traversing C clockwise. Let P' (respectively P'') be the $y'y''$-subpath of C that contains x' (respectively x'').

(3) C cannot have a chord $z'z''$ such that $z' \in V(P') \setminus \{y', y''\}$ and $z'' \in V(P'') \setminus \{y', y''\}$.

Assume it does. W.l.o.g. z' is on the $x'y'$-subpath of P'. Then, by (1), z'' is on the $x''y''$-subpath of P''. Let C' be the cycle obtained by following P' from z' to y'', going along edge $y''y'$, following P'' from y' to z'', and going along edge $z''z'$. Since C' cannot have fewer chords than C (by the choice of (x, C)), it follows that both $z'y'$ and $z''y''$ are edges. But then $G[V(P') \cup \{z''\}]$ contains a 2-wheel with center y', that contradicts our choice of (x, C). This proves (3).

We now prove that $S = \{y', y'', x\}$ is an I-cutset. By (3), S is a cutset of $G[V(C) \cup \{x\}]$. Assume it is not a cutset of G that separates the vertices of C, and let $P = p_1p_2 \dots p_k$ be a shortest path in $G \setminus S$ whose one endvertex has a neighbor in $P' \setminus \{y', y''\}$ and the other has a neighbor in $P'' \setminus \{y', y''\}$. W.l.o.g. p_1 has a neighbor in $P' \setminus \{y', y''\}$, and p_k in $P'' \setminus \{y', y''\}$. By (2) and definition of P: P is a chordless path, for some $u \in V(P') \setminus \{y', y''\}$ and $v \in V(P'') \setminus \{y', y''\}$, $N(p_1) \cap V(C) = \{u\}$ and $N(p_k) \cap V(C) = \{v\}$, and the only vertices of (x, C) that may have a neighbor in the interior of P are x, y' and y''.

Let $P_{uy''}$ (respectively $P_{y''v}$) be the uy''-subpath (respectively $y''v$-subpath) of C that does not contain y'. Let $P_{uy'}$ (respectively $P_{y'v}$) be the uy'-subpath (respectively $y'v$-subpath) of C that does not contain y''.

(4) No vertex of $\{y', y''\}$ has a neighbor in P.

First suppose that both y' and y'' have a neighbor in P. Let p_i (respectively p_j) be the vertex of P with smallest index adjacent to y' (respectively y''). W.l.o.g. $i \leq j$. Let Q be a chordless path from u to y'' in $G[V(P_{uy''})]$. Then $V(Q) \cup \{p_1, p_2, \ldots, p_j, y'\}$ induces in G a 2-wheel with center y', a contradiction.

So we may assume w.l.o.g that y'' does not have a neighbor in P. Suppose y' does. Let Q be the uv-subpath of C that contains y''. Let Q' be a chordless uv-path in $G[V(Q)]$. By (3), Q' contains y''. But then $G[V(Q') \cup V(P) \cup \{y'\}]$ is a 2-wheel with center y', a contradiction. This proves (4).

By symmetry it suffices to consider the following two cases.

Case 1: $x' \in V(P_{uy''})$ and $x'' \in V(P_{y''v})$.
Let C' be the cycle that consists of $P_{uy''}$, $P_{y''v}$ and P. Then by (4), (x, C') is a 2-wheel that contradicts our choice of (x, C).

Case 2: $x' \in V(P_{uy'})$ and $x'' \in V(P_{y''v})$.
Suppose $y'v$ is an edge. Let C' be the cycle that consists of $P_{uy''}$, $P_{y''v}$ and P. Then by (4), (C', y') is a 2-wheel that contradicts our choice of (x, C). So $y'v$ is not an edge, and by symmetry neither is uy''. Now let C' be the cycle that consists of $P_{uy'}$, $y'y''$, $P_{y''v}$ and P. Then by (4), (x, C') is a 2-wheel that contradicts our choice of (x, C). □

Note that in the above Theorem $\mathcal{C}_1$ is used as a basic class for $\mathcal{C}_2$. Here is another version of the decomposition theorem where $\mathcal{C}_0$ is used as a basic class and some cutsets are added.

Theorem 5.6 *A graph in $\mathcal{C}_2$ is either in $\mathcal{C}_0$ or it has a 0-cutset, a 1-cutset, a proper K_2-cutset, a proper S_2-cutset or a proper I-cutset.*

PROOF — The result holds by Theorem 5.5 and Theorem 5.3. □

5.2 Structural characterization

In [7], a complete structural characterization of graphs in $\mathcal{C}_1$ and $\mathcal{C}_2$ is given. Here, we only prove what is needed to build a decomposition based recognition algorithm for $\mathcal{C}_2$ (Section 5.3) and for the result about edge-coloring presented in Section 5.4.

We first define blocks of decomposition w.r.t. different cutsets.

If G has a 0-cutset, i.e. it is disconnected, then its *blocks of decomposition* are the connected components of G. If G has a 1-cutset $\{u\}$ and $C_1, \ldots, C_k$ are the connected components of $G \setminus u$, then the *blocks of decomposition* w.r.t. this cutset are graphs $G_i = G[C_i \cup \{u\}]$, for $i = 1, \ldots, k$.

Let (S, K', K'') be a split of a proper K_2-cutset of G. The *blocks of decomposition* of G with respect to this split are graphs $G' = G[S \cup K']$ and $G'' = G[S \cup K'']$.

Let $(\{u, v, w\}, K', K'')$ be a split of a proper I-cutset of a graph G, and assume uv is an edge. The *blocks of decomposition* of G w.r.t. this split are graphs G' and G'' defined as follows. Block G' is the graph obtained from $G[V(K') \cup \{u, v, w\}]$ by adding marker vertices u'_1, u'_2, v'_1 and

v_2' and marker edges uu_1', $u_1'u_2'$, $u_2'w$, vv_1', $v_1'v_2'$ and $v_2'w$. Block G'' is the graph obtained from $G[V(K'') \cup \{u, v, w\}]$ by adding marker vertices u_1'', u_2'', v_1'' and v_2'' and marker edges uu_1'', $u_1''u_2''$, $u_2''w$, vv_1'', $v_1''v_2''$ and $v_2''w$.

Theorem 5.7 *For 0-cutset, 1-cutsets and proper K_2-cutsets the following holds: G is in $\mathcal{C}_2$ if and only if all the blocks of decomposition are in $\mathcal{C}_2$.*

PROOF — Since a 2-wheel is 2-connected, the theorem obviously holds for 0-cutsets and 1-cutsets. Suppose that $(\{u, v\}, K', K'')$ is a split of a proper K_2-cutset of G, and let G' and G'' be the blocks of decomposition w.r.t. this split. Since G' and G'' are induced subgraphs of G, it follows that if $G \in \mathcal{C}_2$, then $G', G'' \in \mathcal{C}_2$.

Suppose now that G' and G'' are in $\mathcal{C}_2$. If G admits a 2-wheel (x, C) as an induced subgraph, then C being a chordless cycle, it must by included in G' or G''. Assume w.l.o.g. that C is included in G'. Now, since (x, C) is not a subgraph of G', $x \in V(K'')$ and thus the only way for x to be adjacent with at least two vertices of C is to be adjacent with both a and b, a contradiction with the fact that $\{a, b\}$ is a proper K_2-cutset. □

Theorem 5.8 *Let G be a 2-connected graph. Let $(\{u, v, w\}, K', K'')$ be a split of a proper I-cutset of G, and G' and G'' the corresponding blocks of decomposition. Then $G \in \mathcal{C}_2$ if and only if $G' \in \mathcal{C}_2$ and $G'' \in \mathcal{C}_2$. Moreover, G' and G'' are 2-connected.*

PROOF — First note that, since G is 2-connected, by construction of G' and G'', G' and G'' are 2-connected.

Let $G \in \mathcal{C}_2$ and assume w.l.o.g. that G' contains a 2-wheel (x, C) as an induced subgraph. Since x has at least two neighbors of degree at least 3, x is not a marker vertex. Since (x, C) cannot be contained in G, $V(C') \cap \{u_1', u_2', v_1', v_2'\} \neq \emptyset$. W.l.o.g. we may assume that $uu_1'u_2'w$ is a subpath of C. If $V(C) \cap \{v_1', v_2'\} \neq \emptyset$, then $C = uu_1'u_2'wv_1'v_2'u$, and it follows that x is adjacent to at least two vertices of $\{u, v, w\}$, contradicting the assumption that $\{u, v, w\}$ is a proper I-cutset. Therefore $V(C) \cap \{v_1', v_2'\} = \emptyset$. Let P' be the uw-subpath of C that does not contain u_1'. If $v \notin V(P')$ then $(x, uP'wP'_{wu}u)$ is 2-wheel in G, a contradiction. Hence $v \in V(P')$. If x has at least two neighbors in $P' \setminus \{u\}$, then $(x, vP'wP''_{wv}v)$ is a 2-wheel in G, a contradiction. So $N(x) \cap V(P') = \{u, a\}$, where a is a vertex of $P' \setminus \{u, v, w\}$. If v does not have a neighbor in $P''_{uw} \setminus \{u\}$, then $(x, uvP'wP''_{wu}u)$ is a 2-wheel in G, a contradiction. Hence v has a neighbor in $P''_{uw} \setminus \{u\}$. But then $(v, uxaP'wP'_{wu}u)$ is a 2-wheel in G, a contradiction.

To prove the converse assume that $G' \in \mathcal{C}_2$ and $G'' \in \mathcal{C}_2$, but that G contains as an induced subgraph a 2-wheel (x, C). Let us first assume that C is contained in G' or G'', w.l.o.g $V(C) \subset V(G')$. If $x \in K' \cup \{u, v, w\}$ then (x, C) is an induced subgraph of G', a contradiction. Otherwise $x \in K''$, and hence the two neighbors of x in C must be in $\{u, v, w\}$ which contradicts the assumption that $\{u, v, w\}$ is a proper I-cutset. So C must contain vertices from both K' and K'', and therefore it contains w and at least one vertex from the set $\{u, v\}$. W.l.o.g. we may assume that it contains u and that $x \in V(G')$. Let P be the uw-subpath of C contained in G'. If $x \neq v$, then $(x, uPwu_2'u_1'u)$ is a 2-wheel in G', a contradiction. So $x = v$, and therefore v is adjacent to a vertex y of C different from u. We may assume w.l.o.g. that $y \in G'$. Then the vertex set $(v, uPwu_2'u_1'u)$ is a 2-wheel in G', a contradiction. □

5.3 Recognition algorithms

Deciding whether a graph contains a 2-wheel as a subgraph can be done directly as follows: for every 3-vertex path xyz, check whether there are two internally disjoint xz-paths in $G\setminus y$. Checking whether there are two internally disjoint xz-paths can be done in $\mathcal{O}(n)$ time ([74], see also [82]) this leads to an $\mathcal{O}(n^4)$ recognition algorithm for class $\mathcal{C}_1$.

Recognizing whether a graph contains a 2-wheel as an induced subgraph is a more difficult problem, and we are not aware of any direct method for doing that. Observe that the above method would not work since checking whether there is a chordless cycle through two specified vertices of an input graph is $\mathcal{NP}$-complete [12].

In this section, we give an $\mathcal{NP}$-completeness result showing that the detection of "wheel-like" induced subgraph may be hard, then we describe a decomposition based recognition algorithm for $\mathcal{C}_2$.

5.3.1 Detecting 4-wheels

Recall that a k-wheel ($k \geq 2$) is a wheel with at least k spokes.

Theorem 5.9 *The problem whose instance is a graph G and whose question is "does G contain a 4-wheel as an induced subgraph?" is NP-complete.*

PROOF — Let H be a graph of maximum degree 3, with 2 non-adjacent vertices x and y of degree 2. Detecting an induced cycle through x and y in H is an NP-complete problem (see Theorem 2.7 in [65]). We now show how to reduce this problem to the detection of a 4-wheel. Let x' and x'' (resp. y' and y'') be the neighbors of x (resp. of y). Subdivide the edges xx', xx'', yy' and yy''. Call a, b,c, d the four vertices created by these subdivisions. Add a vertex v adjacent to a, b, c and d. Call G this new graph. Note that since H has maximum degree 3, v is the only vertex of degree at least 4 in G, so every 4-wheel of G must be centered at v. Hence, G contains a 4-wheel if and only if H contains an induced cycle through x and y. □

5.3.2 Recognition algorithm for $\mathcal{C}_2$

As we noticed at the begining of the section, it is easy to decide if a graph is in $\mathcal{C}_1$ in polynomial-time. So, to recognize $\mathcal{C}_2$, we can lean on the decomposition theorem of $\mathcal{C}_2$ that uses $\mathcal{C}_1$ as a basic class (Theorem 5.5).

The next lemma is essential to show that the decomposition tree obtained by decomposing along I-cutsets has polynomial size.

Lemma 5.10 *Let G be a 2-connected graph. Let $(\{u,v,w\},K',K'')$ be a split of a proper I-cutset of G, and G' and G'' the corresponding blocks of decomposition. If $|K'| \leq 4$ or $|K''| \leq 4$ then $G \notin \mathcal{C}_2$.*

PROOF — Assume w.l.o.g. that uv is an edge (the only one in $G[\{u,v,w\}]$). Assume by way of contradiction that $|K'| \leq 4$. Then P'_{uw} is of length at most 5. Suppose that v has a neighbor $x \in P'_{uw} \setminus \{u\}$. By definition of a proper I-cutset, vertices in K' see at most one vertex in $\{u,v,w\}$ and thus x is not a neighbor of w. So $uP'_{uw}xvu$ is a cycle of length at most 5 in G and thus, by Lemma 5.1, $G \notin \mathcal{C}_2$. So we may assume that v does not have a neighbor on $P'_{uw} \setminus \{u\}$. Since

$\{u, v, w\}$ is a proper I-cutset, P'_{uw} and P'_{vw} are both of length at least 3. Suppose that the interior vertices of P'_{uw} and P'_{vw} are disjoint. Then $P'_{uw} = ux_1x_2w$ and $P'_{vw} = vy_1y_2w$, and hence since x_1, x_2, y_1, y_2 all belong to the same connected component of $G \setminus \{u, v, w\}$, there must be an edge between a vertex of $\{x_1, x_2\}$ and a vertex of $\{y_1, y_2\}$. But then G admits a cycle of length at most 5 and by Lemma 5.1, $G \notin \mathcal{C}_2$. Finally we may assume w.l.o.g. that $P'_{uw} = ux_1x_2x_3w$ and $P'_{vw} = uy_1x_3w$ (else by Lemma 5.1, $G \notin \mathcal{C}_2$). But then either $(y_1, ux_1x_2x_3wP''_{wv}vu)$ is a 2-wheel (if u does not have a neighbor on $P''_{vw} \setminus \{v\}$) or $(u, vy_1x_3wP_{wv}v)$ is a 2-wheel (if u does have a neighbor on $P''_{vw} \setminus \{v\}$). □

Theorem 5.11 *There exists a polynomial time algorithm that decides whether an input graph is in $\mathcal{C}_2$.*

PROOF — Observe that finding a proper I-cutset in any input graph can be done in polynomial-time by checking for every triple $\{u, v, w\}$ that induces exactly one edge if it is a cutset or not.

We may assume that G has at least 12 vertices for otherwise, we proceed by brute force. Also we assume that G is 2-connected, for otherwise by the algorithm from [55] we compute its blocks (in the classical sense of 2-connectivity, see [14] for example) in linear time and run our algorithm for each block.

The algorithm has two steps.

Step 1 If G has no proper I-cutset, then by Theorem 5.5, G is in $\mathcal{C}_2$ if and only if G is in $\mathcal{C}_1$. So, we may rely on the recognition algorithm for $\mathcal{C}_1$ described at the beginning of the section.

Step 2 So, we may assume that G has a proper I-cutset, for which we compute the two blocks of decomposition G' and G''. By Theorem 5.5, G is in $\mathcal{C}_2$ if and only if G' and G'' are. By Theorem 5.8 and Lemma 5.10, we may assume that both G' and G'' are 2-connected and both have at least 12 vertices (otherwise G does not belong to $\mathcal{C}_2$). Now, test whether G' and G'' are in $\mathcal{C}_2$ by recursively calling our algorithm.

The only problem with the algorithm above is that there might be more than a polynomial number of recursive calls, so let us bound their number. (Note that the number of recursive calls corresponds to the number of vertices of the decomposition tree of G along I-cutset).

For any graph H, set $\varphi(H) = 2|V(H)| - 23$ and let $k(H)$ be the total number of recursive calls when running the algorithm for H. Observe that in Step 2 the algorithm is called only for graphs such that φ is at least 1. We claim that $k(G) \leq \varphi(G)$. If G is handled in Step 1, then $k(G) = 1$, so our claim is clear (because $|V(G)| \geq 12$). Otherwise, by the induction hypothesis and since $|V(G)| = |V(G_1)| + |V(G_2)| - 11$, we have

$$\begin{aligned} k(G) &= 1 + k(G_1) + k(G_2) \\ &\leq 1 + \varphi(G_1) + \varphi(G_2) \\ &= 2(|V(G_1)| + |V(G_2)| - 11) - 23 \\ &= \varphi(G). \end{aligned}$$

□

5.3.3 Conclusions and open questions

Based on Theorem 5.3, it is shown in [7] that one can decide if a graph is in $\mathcal{C}_1$ in $\mathcal{O}(nm)$-time and then, based on Theorem 5.6 (the theorem of decomposition for $\mathcal{C}_2$ using $\mathcal{C}_0$ as a basic class), one can decide if a graph belongs to $\mathcal{C}_2$ in $\mathcal{O}(n^2m^2)$-time. We decided not to enter in these complexity details because our main interest is not to get the fastest algorithm but to know which problem are in $\mathcal{P}$ and which are not.

The proof of Theorem 5.9 also implies that it is $\mathcal{NP}$-complete to decide if a graph admits a k-wheel as an induced subgraph for any $k \geq 4$. So the question for 3-wheel-free graph becomes of particular interest.

A 2-wheel that has exactly two spokes is called a *clock*. Detecting if a graph admits a clock as an induced subgraph is mentioned in [65] as an open problem (Section 3.3, the first of the 7 open problems).

Nicolas Trotignon also proposed the following conjecture:

Conjecture 5.12 (Trotignon [91]) *If G is a clock-free graph then, either it admits a clique-cutset, or a vertex of degree 2 or it is the cube.*

In order to prove this conjecture, the following result has been proven:

Theorem 5.13 (Aboulker, Li, Thomassé [5]) *If G is a clock-free graph with girth at least 9, then G admits either a clique-cutset or a vertex of degree 2. Moreover the class of clock-free graphs is χ-bounded.*

5.4 Flat edges and edge-coloring

A *flat edge* of a graph G is an edge both of whose endvertices are of degree 2. In this section we show that every 2-connected 2-wheel-free graph has a flat edge and use this property to edge-color it. The key of the proof is to show the existence of an extreme decomposition for proper S_2-cutset and proper I-cutset in graphs that do not admit K_2-cutsets (see Theorem 5.16). (recall that an extreme decomposition is a decomposition in which one of the blocks is basic.)

Lemma 5.14 *Let G be a 2-connected graph that does not have a K_2-cutset. Let $(\{u, v, w\}, K', K'')$ be a split of a proper I-cutset of G, and G' and G'' the corresponding blocks of decomposition. Then G' and G'' have no K_2-cutset.*

PROOF — W.l.o.g. uv is an edge and, by definition of an I-cutset, it is the only edge with both extremities in $\{u, v, w\}$. Assume by way of contradiction and w.l.o.g. that G' admits a K_2-cutset $S = \{a, b\}$. By Theorem 5.8, G' is 2-connected. Since G is 2-connected and has no K_2-cutset, every connected component of $G \setminus \{u, v, w\}$ must contain a neighbor of w and a neighbor of u or v. So, S does not contain any marker vertices and $S \neq \{u, v\}$. Moreover, since uv is the only edge with both extremities in $\{u, v, w\}$, $S \cap \{u, v, w\} \leq 1$. Then w.l.o.g. we may assume that $v \notin S$. Let C and D be two distinct connected components of $G' \setminus S$ such that $v \in C$. Then all the marker vertices and vertices of $\{u, w\} \setminus S$ are in C. Therefore $D \subseteq K'$, and hence S is a cutset of G (separating D from $G \setminus (D \cup S)$), a contradiction. □

We now need to define blocks of decomposition w.r.t. to proper S_2-cutset. Note that they are defined in a different way that in the previous chapter. Let $(\{u, v\}, K', K'')$ be a split of a proper

S_2-cutset of a graph G. The *blocks of decomposition* of G w.r.t. this split are graphs G' and G'' defined as follows. Block G' is the graph obtained from $G[V(K') \cup \{u, v\}]$ by adding marker vertices u', v' and marker edges uu', $u'v'$ and $v'v$. Block G'' is the graph obtained from $G[V(K'') \cup \{u, v\}]$ by adding marker vertices u'' and v'' and marker edges uu'', $u''v''$ and $v''v$.

Lemma 5.15 *Let G be a 2-connected graph that does not have a K_2-cutset. Let $(\{u, v\}, K', K'')$ be a split of a proper S_2-cutset of G, and G' and G'' the corresponding blocks of decomposition. Then G' and G'' are 2-connected, have no K_2-cutset and belong to $\mathcal{C}_2$.*

PROOF — Since G is 2-connected, by construction of G' and G'' it is clear that G' and G'' are also 2-connected.

Let us now prove that G' and G'' have no K_2-cutset. Suppose by way of contradiction and w.l.o.g. that G' admits a K_2-cutset $\{a, b\}$. Since G is 2-connected, any connected component of K' must contain a neighbor of u and v. Thus $\{a, b\} \cap \{u, v, u', v'\} = \emptyset$ and it follows that $\{u, v, u', v'\}$ are in the same connected component of $G \setminus \{a, b\}$. Let D be a connected component of $G \setminus \{a, b\}$ that does not contain $\{u, v, u', v'\}$. Then $\{a, b\}$ is a K_2-cutset of G (separating D from the rest of the graph), a contradiction. So G' (and by symmetry G'') does not admit K_2-cutset.

It now remains to show that G' and G'' belong to $\mathcal{C}_2$. Assume by way of contradiction and w.l.o.g. that G' admits a 2-wheel (x, C) as an induced subgraph. Since x has at least two neighbors of degree at least 3, x is not a marker vertex of G'. Since $G \in \mathcal{C}_2$, (x, C) is not contained in K' and thus we may assume that $uu'v'v$ is in C. Since G is 2-connected, $G[S \cup K'']$ contains an uv-path P. So, by replacing $uu'v'v$ by P, we obtain a 2-wheel in G, a contradiction. Therefore, $G' \in \mathcal{C}_2$. $\square$

Theorem 5.16 *Let $G \in \mathcal{C}_2 \setminus \mathcal{C}_0$ be a 2-connected graph that does not have a K_2-cutset. Then G has an I-cutset or a proper S_2-cutset S with split (S, K', K'') such that one of the blocks of decomposition, say G', belongs to $\mathcal{C}_0$. Furthermore, all vertices of S are of degree at least 3 in G'.*

PROOF — By Theorem 5.6, G has an I-cutset or a proper S_2-cutset. Let (S, K', K'') be a split of an I-cutset or a proper S_2-cutset of G such that among all such splits, $|K'|$ is minimized. Let G' be the block of decomposition that contains K'. If S is a proper S_2-cutset we let $S = \{u, v\}$, and if S is an I-cutset we let $S = \{u, v, w\}$ and assume that uv is an edge.

(1) G' is 2-connected, has no K_2-cutset and belongs to $\mathcal{C}_2$.

If S is an I-cutset, then by Theorem 5.8 G' is 2-connected and belongs to $\mathcal{C}_2$ and by Lemma 5.14 G' has no K_2-cutset. If S is a proper S_2-cutset, then the claim holds by Lemma 5.15. This proves (1).

(2) If S is a proper S_2-cutset, then both u and v have at least two neighbors in K'. In particular, all vertices of S have degree at least 3 in G'.

Suppose not and let u_1 be the unique neighbor of u in K'. If u_1v is an edge then $uu_1vv'u'u$ is a cycle of G' and thus, by Lemma 5.1, $G' = uu_1vv'u'u$, contradicting the assumption that S is a proper S_2-cutset of G. Then $(\{u_1, v\}, K' \setminus \{u_1\}, K'' \cup \{u\})$ is a split of a proper S_2-cutset of G that contradicts our choice of (S, K', K''). This proves (2).

Since it is clear that if S is an I-cutset, then every vertex of S is of degree at least 3 in G', it only remains to show that $G' \in \mathcal{C}_0$. Assume not. By (1) and Theorem 5.6, G' has an I-cutset or a proper S_2-cutset with split (C, C_1, C_2) (say). W.l.o.g. we may assume that (C, C_1, C_2) is chosen such that $|C_i|$, for some $i \in \{1, 2\}$, is minimized. Let M be the set of marker vertices of G' (if

S is a proper S_2-cutset $M = \{u', v'\}$ and if S is a proper I-cutset, $M = \{u'_1, u'_2, v'_1, v'_2\}$). If C is an I-cutset, then it is clear that every vertex in C is of degree at least 3 in G'. If C is a proper S_2-cutset, then by (2) (applied on C and G'), all vertices of C have degree at least 3 in G'. Hence, in both case, $C \cap M = \emptyset$. We now consider the following two cases.

Case 1: S is a proper S_2-cutset of G.
Since $C \cap M = \emptyset$, we may assume w.l.o.g. $M \subseteq C_2$. Thereby C_1 is a proper subset of K'. Then $(C, C_1, (C_2 \setminus M) \cup K'')$ is a split of an I-cutset or a proper S_2-cutset of G, contradicting our choice of (S, K', K'').

Case 2: S is an I-cutset of G.
First observe that, $|C \cap S| \leq 2$ (it is obvious in the case where C is a proper S_2-cutset and it is due to the fact that, by the choice of (S, K', K''), $G[K']$ is connected in the case where C is an I-cutset). If $|C \cap S| \leq 1$, then w.l.o.g. $(S \cup M) \setminus C \subseteq C_2$, and hence $(C, C_1, (C_2 \setminus M) \cup K'')$ is a split of an I-cutset or a proper S_2-cutset of G, contradicting our choice of (S, K', K'').

So we may assume from now on that $|C \cap S| = 2$. Suppose first that C is a proper S_2-cutset. So, w.l.o.g. $C = \{v, w\}$. Then, since each vertex of S has a neighbor in K', in particular u has a neighbor in K' and thus $K' \cup \{u, u'_1, u'_2\}$ are in the same connected component of $G' \setminus \{v, w\}$, say $K' \cup \{u, u'_1, u'_2\} \subseteq C_1$. So $C_2 = \{v'_1, v'_2\}$ contradicting the fact that C is a proper S_2-cutset. So we may assume that C is an I-cutset. If marker vertices u'_1, u'_2 are in C_1 and v'_1, v'_2 are in C_2 (which might be the case if $C \cap S = \{w, u\}$ *or* $\{w, v\}$), then $(C, C_1 \setminus \{u'_1, u'_2\}, C_2 \cup \{u'_1, u'_2\})$ is also a split of an I-cutset of G'. So we may assume that w.l.o.g. $(S \cup M) \setminus C \subseteq C_2$, and hence $(C, C_1, (C_2 \setminus M) \cup K'')$ is a split of an I-cutset or a proper S_2-cutset of G, contradicting our choice of (S, K', K''). □

Theorem 5.17 *If $G \in \mathcal{C}_2$ is 2-connected, then either G is an induced cycle or it has at least two flat edges that induce a matching.*

PROOF — We prove the result by induction on $|V(G)|$. It is true when $|V(G)| \leq 3$.

Case 1: $G \in \mathcal{C}_0$. Since G is 2-connected, all vertices have degree at least two. Let S be the vertices of G of degree 2, and $T = V(G) \setminus S$. If G is not an induced cycle, then $T \neq \emptyset$. It follows that the connected components of $G[S]$ are all paths of length at least 1, whose vertices are all of degree 2 in G. Let $u \in T$. By definition of $\mathcal{C}_0$, u can have at most one neighbor in T, and hence it has at least two neighbors in S, say u_1 and u_2. Since G is 2-connected, u_1 and u_2 cannot be in the same connected component of $G[S]$ (otherwise u is a cutvertex of G). Therefore, $G[S]$ has at least two connected components, and hence it has two flat edges that induce a matching. This completes the proof in Case 1.

Case 2: *G has a K_2-cutset.* Suppose $(\{a, b\}, C_1, C_2)$ is a split of a K_2-cutset of G, and let $G_1 = G[C_1 \cup \{a, b\}]$ and $G_2 = G[C_2 \cup \{a, b\}]$ be the corresponding blocks of decomposition. Note that by Lemma 5.2, $\{a, b\}$ is a proper K_2-cutset. For $i = 1, 2$, G_i is clearly 2-connected, and since G_i is a subgraph of G, $G_i \in \mathcal{C}_2$. Hence, by the induction hypothesis, G_i is either an induced cycle or it has at least two flat edges that induce a matching. Since $\{a, b\}$ is proper, G_i cannot be a triangle. Therefore, one of the flat edges of G_i must be completely contained in C_i, and hence it is flat in G as well. Hence, G has at least two flat edges (one in C_1, the other one in C_2) that induce a matching. This completes the proof in Case 2.

So, we may no assume that G has no K_2-cutset. Thus, by Theorem 5.16, G has an I-cutset or a proper S_2-cutset S with split (S, K', K'') such that the block of decomposition G' that contains K' belongs to $\mathcal{C}_0$, and all vertices of S have degree at least three in G'. This leads us to the following two cases.

Case 3: *S is an I-cutset.* Let C' be a connected component of $G[K']$ such that all vertices of S have a neighbor in C' and note that, since $G \in \mathcal{C}_0'$, all these neighbors must be of degree 2. Then C' must contain a vertex x that is of degree at least 3. Now by the same argument as in Case 1, there are at least two flat edges in C' that induce a matching. These edges are flat in G as well. This completes the proof in Case 3.

Case 4: *S is an S_2-cutset.* Let $S = \{u, v\}$ and let u_1, u_2 be two neighbors of u in K' Since $G' \in \mathcal{C}_0$, for $i = 1, 2$, u_i is of degree 2 and have a neighbor of degree 2, say u_i'. By Lemma 5.1, u_1u_1' and u_2u_2' are two flat edges that induce a matching in G that induce a mat This completes the proof in Case 4. □

An edge of a graph is *pending* if it contains at least one node of degree 1.

Corollary 5.18 *Every graph G in $\mathcal{C}_2$ with at least one edge contains an edge that is pending or flat.*

PROOF — We consider the classical decomposition of G into blocks, in the sense of 2-connectivity (see [14]). So, G has a block B that is either a pending edge of G, or a 2-connected graph containing at most one vertex x that has neighbors in $V(G) \setminus V(B)$. In the latter case, by Theorem 5.17, B is either a chordless cycle or it has two flat edges that induce a matching in B, and so at least one flat edge of B is non-incident to x, and is therefore a flat edge of G. □

An edge-coloring of G is a function $\pi : E \to C$ such that no two adjacent edges receive the same color $c \in C$. If $C = \{1, 2, \dots, k\}$, we say that π is a k-edge coloring. The chromatic index of G, denoted by $\chi'(G)$, is the least k for which G has a k-edge-coloring.

Vizing's theorem states that $\chi'(G) = \Delta(G)$ or $\chi'(G) = \Delta(G) + 1$, where $\Delta(G)$ is maximum degree of nodes in G. The edge-coloring problem or chromatic index problem is the problem of determining the chromatic index of a graph. The problem is NP-hard for several classes of graphs, and its complexity is unknown for several others. Here, we solve the edge-coloring problem for the class $\mathcal{C}_2$.

Theorem 5.19 *If G is a graph in $\mathcal{C}_2$ such that $\Delta(G) \geq 3$, then $\chi'(G) = \Delta(G)$.*

PROOF — Induction on $|E(G)|$. If $|E(G)| = 0$, the result clearly holds. By Corollary 5.18, G has an edge ab that is pending or flat. Note that $\mathcal{C}_2$ is not closed under removing edges in general, but it is closed under removing flat or pending edges. Set $G' = (V(G), E(G) \setminus \{ab\})$. If $\Delta(G') \geq 3$, then by the induction hypothesis, we can edge-color G' with $\Delta(G')$ colors. Otherwise, $\Delta(G') \leq 2$, so G' is 3-edge colorable. In either cases, G' is $\Delta(G)$-colorable. We can extend the edge-coloring of G' to an edge-coloring of G as follows. When ab is pending, by assigning a color to ab not used among the edges incident to ab, and when ab is flat by assigning to ab a color not used for the two edges adjacent to ab. □

Note that when $\Delta(G) \leq 2$, G is a disjoint union of cycles and paths, so χ' is easy to compute. The proof above is easy to transform into a polynomial time algorithm that outputs the coloring whose existence is proved.

Chapter 6

Balanced and balanceable graphs

In this chapter:

- **If *G* and H are graphs, then we say that G is *H-free* if** G **does not contain H as an** **induced** **subgraph.**
- $\mathbf{K_4}$ **is not a wheel.**

The work presented in this chapter comes from a joint work with Marko Radovanović, Nicolas Trotignon, Théophile Trunck and Kristina Vušković to appear in Journal of Graph Theory [6].

6.1 Introduction

Classes of balanced and balanceable graphs do not come from the world of graphs but from the world of combinatorial optimization. Let us say a word about their matrices' origin.

A $0,1$ matrix is *balanced* if for every square submatrix with two ones per row and per column, the number of ones is a multiple of four. This notion was introduced by Berge [9]. There is a natural way to associate $0,1$ matrices with bipartite graphs.

Given a $0,1$ matrix A, the *bipartite graph representation of* A is the bipartite graph having a vertex for every row in A, a vertex for every column in A, and an edge ij joining row i to column j if and only if the entry a_{ij} equals 1. We say that a bipartite graph G is balanced if it is the bipartite representation of a balanced matrix. The following property is an easy consequence of the definition of balanced graphs and give a characterization of balanced graphs by forbidding induced subgraphs.

Property 6.1 *A graph G is balanced if and only if the length of every chordless cycles of G is a multiple of four.*

In [94], Truemper extended the definition of $0,1$ balanced matrix to $0,\pm1$ matrices. A $0,\pm1$ matrix is *balanced* if for every square submatrix with two nonzero entries per row and column, the sum of the entries is a multiple of four. *Balanceable* graphs arose from this extension, let us explain how they are defined. A *signed graph* is a graph, together with an assignment of weights ±1 to its edges. A signed graph G is *balanced* if the length of every chordless cycle of G is a multiple of four. A graph is *balanceable* if there exists a signing of its edges i.e. an assignment of weights ±1 to its edges, such that the resulting signed graph is balanced. Since assigning weight $+1$ to each edge of a balanced graph give a balanced signed graph, the class of balanceable graphs is a superclass of balanced graphs. Note also that balanced and balanceable graphs are subclasses of bipartite graphs.

Another (more adequat in the context of structural graph theory) definition of balanceable come from the following theorem. Let G be a bipartite graph. An *odd theta* of G is a theta that connects two vertices that are on opposite sides of the bipartition of G. An *odd wheel* is a wheel that has an odd number of spokes.

Theorem 6.2 (Truemper [94]) *A bipartite graph is balanceable if and only if it does not contain odd wheels nor odd thetas as induced subgraphs.*

The following conjecture is the last unresolved conjecture about balanced graphs in Cornuéjols' book [40] (it is Conjecture 6.11).

Conjecture 6.3 (Conforti and Rao [38]) *Every balanced graph contains an edge that is not the unique chord of a cycle.*

It is interresting to observe that, as proved by the following property, this conjecture is equivalent to say that every balanced graph contains an edge whose removal leaves the graph balanced.

Property 6.4 *If G is a balanced graph and e is an edge of G, then $G \setminus e$ is balanced if and only if e is not the unique chord of a cycle.*

PROOF — Let G be a balanced graph and let $e = xy$ an edge of G. We first prove that if e the unique chord of a cycle C in G, then G is not balanced. Edges of C are edge-wise partitioned into two xy-path P_1 and P_2. So $P_1 \cup e$ and $P_2 \cup e$ are chordless cycle of G. So, since G is balanced, P_1 and P_2 are both of length 3 mod 4. Therefore $P_1 \cup P_2$, that is a chordless cycle in $G \setminus e$, is of length 2 mod 4 and thus $G \setminus e$ is not balanced.

Suppose now that $G \setminus e$ is not balanced. Then $G \setminus e$ contain a chordless cycle C such that the length of C is not a multiple 4. Since G is balanced, C is not a chordless cycle of G and thus e is the unique chord of C in G. □

Observe that in Conjecture 6.3, we cannot replace "balanced" by "balanceable". Indeed, in the graph R_{10}, that is the graph defined by the cycle $x_1x_2 \dots x_{10}x_1$ (of length 10) with chords x_ix_{i+5}, $1 \leq i \leq 5$ (see Figure 6.1), every edge is the unique chord of a cycle and it is balanceable: assign weight $+1$ to the edges of the cycle $x_1x_2 \dots x_{10}x_1$ and -1 to the chords. (Note that R_{10} is not balanced: $x_1x_2x_3x_4x_5x_6$ is a hole of length 6).

Anyway, Conjecture 6.3 generalises to balanceable graphs in the following way.

Conjecture 6.5 (Conforti, Cornuéjols and Vušković [32]) *In a balanceable graph either every edge belongs to some R_{10} or there is an edge that is not the unique chord of a cycle.*

Outline of the chapter

A graph is *linear balanceable* if it is balanceable and does not contain a square (i.e. a hole of length 4). A graph G is cubic if every vertex is of degree 3, and is *subcubic* if $\Delta(G) \leq 3$. In this chapter, we prove that conjecture 7.13 holds when restricted to linear balanceable graphs (see Corollary 6.14)

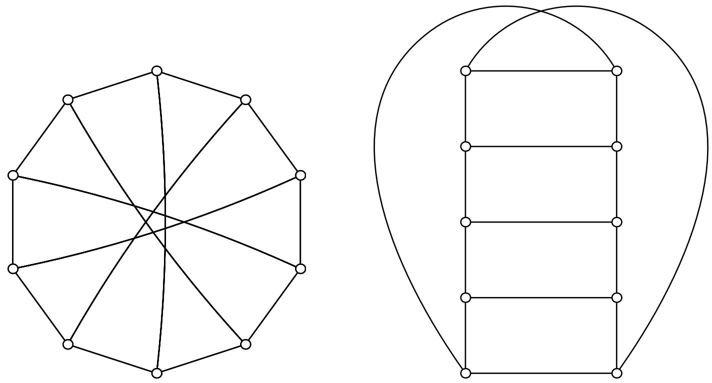


Figure 6.1: Two ways to draw the graph R_{10}.

and to subcubic balanceable graphs (see Corollary 6.18).
For the subcubic case, our proof relies on a result conjectured by Morris, Spiga and Webb [73], stating that every cubic balanced graph contains a pair of twins (see Corollary 6.17).

Our proofs are based on known decomposition theorems for the classes we consider, which we describe in Section 6.2. The decomposition theorems say that either the graph is ”basic”, or it has a 2-join, a 6-join or a star cutset. It is not straightforward to use this decomposition theorem to prove the desired result. In fact, the decomposition theorem for balanced graphs [33] has been known since the early 1990's, and still no one knows how to use it to prove the Conforti and Rao Conjecture. The key idea that makes things work for us, is the use of extreme decompositions, (recall it is a decompositions in which one of the blocks is basic). In Section 6.3 we prove that if star cutsets are excluded, then the graphs in our classes admit extreme decompositions. This is sufficient for the proof of the main result in the subcubic case in Section 6.5, since the induction hypothesis in this case goes through the star cutset nicely.

For the linear balanceable graphs, this is not the case. Here we cannot inductively get rid of star cutsets in a straightforward manner. Furthermore, it is not true that if a linear balanceable graph has a star cutset, then it has a star cutset one of whose blocks of decomposition does not have a star cutset (i.e. star cutsets do not admit extreme decomposition in linear balanceable graphs). To prove the main result for linear balanceable graphs (Section 6.4), we develop a new technique for finding an “extreme decomposition” with respect to star cutsets: we look for a minimally-sided double star cutset, and show that the corresponding block of decomposition does not have a star cutset.

6.2 Decomposition theorems

In this section we present the decomposition theorems for linear balanceable graphs and subcubic balanceable graphs. We first precisely describe the basic class and the different cutsets that we need for the decomposition theorems, then we state the decomposition theorem our proof leans on.

We now introduce the different cutsets that are needed to decompose balanceable graphs.

Star cutset
A connected graph G has a *star cutset* (x, R) if $R \subseteq N(x)$ and $\{x\} \cup R$ is a cutset of G. Note that

if $R = \emptyset$, then (x, R) is a cutvertex.

2-join
A graph G has a *2-join* (X_1, X_2) if $V(G)$ can be partitioned into sets X_1 and X_2 so that the following hold:

- For $i = 1, 2$, X_i contains disjoint nonempty sets A_i and B_i, such that every vertex of A_1 is adjacent to every vertex of A_2, every vertex of B_1 is adjacent to every vertex of B_2, and there are no other adjacencies between X_1 and X_2.

- For $i = 1, 2$, X_i contains at least one path from A_i to B_i, and if $|A_i| = |B_i| = 1$, then $G[X_i]$ is not a chordless path.

We say that $(X_1, X_2, A_1, A_2, B_1, B_2)$ is a *split* of this 2-join, and the sets A_1, A_2, B_1, B_2 are called the *special sets* of this 2-join (see Figure 6.2).

6-join
A graph G has a *6-join* (X_1, X_2) if $V(G)$ can be partitioned into sets X_1 and X_2 so that the following hold:

- X_1 (resp. X_2) contains disjoint nonempty sets A_1, A_3, A_5 (resp. A_2, A_4, A_6) such that, for every $i \in \{1, \dots, 6\}$, every vertex in A_i is adjacent to every vertex in $A_{i-1} \cup A_{i+1}$ (where subscripts are taken modulo 6), and these are the only adjacencies between X_1 and X_2.

- $|X_1| \geq 4$ and $|X_2| \geq 4$.

We say that $(X_1, X_2, A_1, A_2, A_3, A_4, A_5, A_6)$ is a *split* of this 6-join (see Figure 6.3).

We now define the unique basic class we need, called *sparse graphs.*

Sparse graphs
A bipartite graph is *sparse* if it admits a bipartition such that all the vertices in one side of the bipartition have degree at most 2. The class of sparse graphs will be the only basic class we use in our decomposition theorems.

We are now armed to state the decomposition theorems that are going to be used all along the chapter. This decomposition theorem can be recovered from the work in [33], [34] and [98] as explained in [6].

Theorem 6.6 ([6]) *Let G be a connected balanceable graph.*

- *If G is square-free, then G is sparse, or has a 2-join, a 6-join or a star cutset.*

- *If $\Delta(G) \leq 3$, then G is sparse or is R_{10}, or has a 2-join, a 6-join or a star cutset.*

6.3 Graphs with no star cutset

In this section, we investigate what are the properties that have 2-joins and 6-joins in balanceable graphs with no star cutset (see subsections 6.3.1 and 6.3.2). The main result of this section is the proof of the existence of extreme $\{2, 6\}$-join in linear balanceable graphs with no star cutset (see Subsection 6.3.3).

6.3.1 2-joins in graphs with no star cutset

We first define blocks of decomposition w.r.t. 2-join.

Let $(X_1, X_2, A_1, A_2, B_1, B_2)$ be a split of a 2-join of a graph G. The *blocks of decomposition* of G w.r.t. this 2-join are graphs G_1 and G_2 defined as follows. To obtain G_i, for $i = 1, 2$, we start from $G[X_i]$, and first add a vertex a_{3-i}, adjacent to all the vertices in A_i and no other vertex of X_i, and a vertex b_{3-i} adjacent to all the vertices in B_i and no other vertex of X_i. For $i = 1, 2$, let Q_{3-i} be a path in $G[X_{3-i}]$ with smallest number of edges connecting a vertex in A_{3-i} to a vertex in B_{3-i}. For $i = 1, 2$, add to G_i a *marker path* M_{3-i} connecting a_{3-i} and b_{3-i} with length $|E(M_{3-i})| \in \{4, 5\}$ having the same parity as Q_{3-i}.

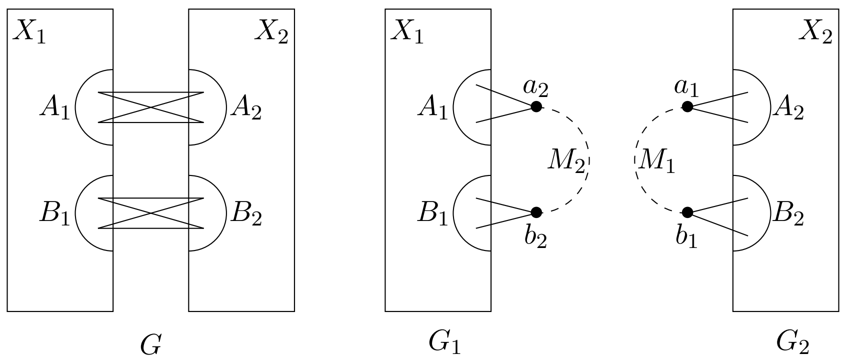


Figure 6.2: A graph G with a split of a 2-join $(X_1, X_2, A_1, B_1, A_2, B_2)$ and the associated blocks of decomposition G_1 and G_2.

The following lemma is proved in [93] (Lemma 3.2). It gives some easy property of 2-join that are practical when manipulating 2-joins.

Lemma 6.7 (Trotignon and Vušković [93]) *Let G be a graph that has no star cutset, and let $(X_1, X_2, A_1, A_2, B_1, B_2)$ be a split of a 2-join of G. Then for $i = 1, 2$, the following hold:*

(i) Every component of $G[X_i]$ meets both A_i and B_i.

(ii) Every $u \in X_i$ has a neighbor in X_i.

(iii) Every vertex of A_i has a non-neighbor in B_i.

(iv) Every vertex of B_i has a non-neighbor in A_i.

(v) $|X_i| \geq 4$.

Theorem 6.8 (Conforti, Cornuéjols, Kapoor, Vušković [35]) *Let G be a bipartite graph with no star cutset. Let (X_1, X_2) be a 2-join of G, and let G_1 and G_2 be the corresponding blocks of decomposition. Then the following hold:*

(i) If G is balanceable, then G_1 and G_2 are balanceable.

(ii) G_1 and G_2 have no star cutset.

(iii) If G has no 6-join, then G_1 and G_2 have no 6-join.

A 2-join (X_1, X_2) of G is a *minimally-sided 2-join* if for some $i \in \{1, 2\}$ the following holds: for every 2-join (X'_1, X'_2) of G, neither $X'_1 \subsetneq X_i$ nor $X'_2 \subsetneq X_i$. In this case X_i is a *minimal side* of this minimally-sided 2-join. Next lemma says that graphs with no star cutset admit extreme 2-join.

Lemma 6.9 (Trotignon and Vušković [93]) *Let G be a bipartite graph with no star cutset. Let $(X_1, X_2, A_1, A_2, B_1, B_2)$ be a split of a minimally-sided 2-join of G with X_1 being a minimal side, and let G_1 and G_2 be the corresponding blocks of decomposition. Then the following hold:*

1. *$|A_1| \geq 2$, $|B_1| \geq 2$, and in particular all the vertices of $A_2 \cup B_2$ are of degree at least 3.*

2. *G_1 has no 2-join.*

6.3.2 6-joins in graphs with no star cutset

We first define blocks of decomposition w.r.t. 6-join.

Let $(X_1, X_2, A_1, \dots, A_6)$ be a split of a 6-join of a graph G. The *blocks of decomposition* of G by this 6-join are graphs G_1 and G_2 defined as follows. For $i = 1, \dots, 6$ let a_i be any vertex of A_i. Then $G_1 = G[X_1 \cup \{a_2, a_4, a_6\}]$ and $G_2 = G[X_2 \cup \{a_1, a_3, a_5\}]$. vertices a_2, a_4, a_6 (resp. a_1, a_3, a_5) are called the *marker vertices* of G_1 (resp. G_2).

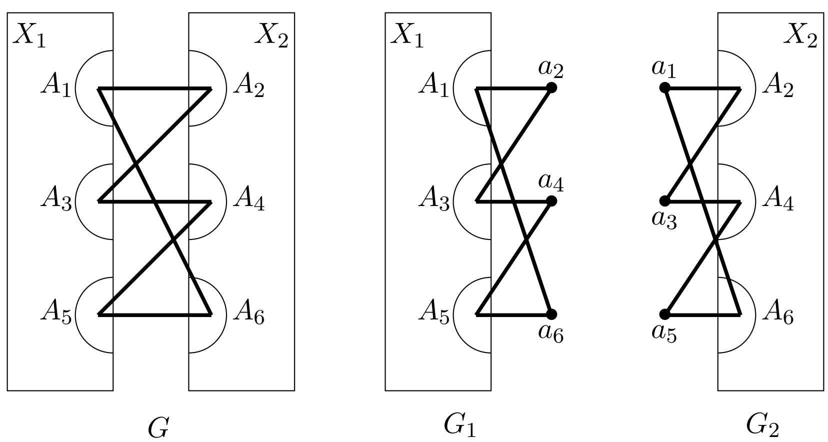


Figure 6.3: A graph G with a split of a 6-join $(X_1, X_2, A_1, A_2, A_3, A_4, A_5, A_6)$ and the associated blocks of decomposition G_1 and G_2.

Lemma 6.10 *Let G be a bipartite graph with no star cutset. Let $(X_1, X_2, A_1, \dots, A_6)$ be a split of a 6-join of G, and G_1 and G_2 the corresponding blocks of decomposition. Then the following hold:*

(i) $X_1 \setminus (A_1 \cup A_3 \cup A_5) \neq \emptyset$ and $X_2 \setminus (A_2 \cup A_4 \cup A_6) \neq \emptyset$.

(ii) If C is a connected component of $G[X_1 \setminus (A_1 \cup A_3 \cup A_5)]$ (resp. $G[X_2 \setminus (A_2 \cup A_4 \cup A_6)]$), then a vertex of A_i, for every $i = 1, 3, 5$ (resp. $i = 2, 4, 6$) has a neighbor in C.

(iii) If G is square-free or $\Delta(G) \leq 3$, then $|A_i| = 1$ for every $i \in \{1, \dots, 6\}$, and in particular every vertex of $\cup_{i=1}^{6} A_i$ is of degree at least 3 in G.

(iv) If G is balanceable, then so are G_1 and G_2.

(v) If G is square-free, then G_1 and G_2 do not have star cutsets.

PROOF — Note that G is bipartite so there are no edges in $A_1 \cup A_3 \cup A_5$ nor in $A_2 \cup A_4 \cup A_6$.

Suppose that $X_1 \setminus (A_1 \cup A_3 \cup A_5) = \emptyset$. By definition of a 6-join, $|X_1| \geq 4$. So, we may assume w.l.o.g. that $|A_1| \geq 2$. Hence for a vertex $a_1 \in A_1$, $\{a_1\} \cup A_2 \cup A_6$ is a star cutset of G, a contradiction. Therefore by symmetry (i) holds.

Let C be a connected component of $G[X_1 \setminus (A_1 \cup A_3 \cup A_5)]$ and suppose that no vertex of A_1 has a neighbor in C. Then for a vertex $a_4 \in A_4$, $\{a_4\} \cup A_3 \cup A_5$ is a star cutset of G separating C from the rest of the graph, a contradiction. Therefore by symmetry, (ii) holds.

If G is square-free then clearly $|A_i| = 1$ for every $i \in \{1, \dots, 6\}$. Suppose $\Delta(G) \leq 3$ and $|A_1| \geq 2$. Then, any vertex in A_2 have at least 3 neighbors in $A_1 \cup A_3$ and thus have no neighbors in X_2, but by (i), $X_2 \setminus (A_2 \cup A_4 \cup A_6) \neq \emptyset$ and thus by (ii), some vertex of A_2 must have a neighbor in X_2, a contradiction. Therefore, (iii) holds.

Since G_1 and G_2 are induced subgraphs of G, (iv) holds.

To prove (v) assume G is square-free and w.l.o.g. G_1 has a star cutset (x, R). Let a_2, a_4, a_6 be the marker vertices of G_1. By (ii), $x \notin \{a_2, a_4, a_6\}$. If $x \in A_1$, then $(x, R \cup A_2 \cup A_6)$ is a star cutset of G, a contradiction. Therefore by symmetry, $x \in X_1 \setminus (A_1 \cup A_3 \cup A_5)$. Since G is square-free R may contain vertices from at most one of the sets A_1, A_3, A_5, and hence a_2, a_4, a_6 are all contained in the same connected component of $G_1 \setminus (\{x\} \cup R)$. It follows that (x, R) is also a star cutset of G, a contradiction. Therefore (v) holds. □

We observe that property (v) above is not true in general for balanceable graphs. On the other hand, it is true for subcubic balanceable graphs. Since we use a different technique to prove the main result for subcubic balanceable graphs than the one we use for linear balanceable graphs, we do not need this result.

6.3.3 Extreme {2,6}-join

A partition (X_1, X_2) of $V(G)$ is a *$\{2,6\}$-join* if it is a 2-join or a 6-join of G. It is a *minimally-sided $\{2,6\}$-join* if for some $i \in \{1,2\}$ the following holds: for every $\{2,6\}$-join (X'_1, X'_2) of G, neither $X'_1 \subsetneq X_i$ nor $X'_2 \subsetneq X_i$. In this case X_i is a *minimal side* of this minimally-sided $\{2,6\}$-join. An *extreme $\{2,6\}$-join* is a $\{2,6\}$-join such that one of the block of decomposition does not admit $\{2,6\}$-joins.

Note that the next result holds for square-free bipartite graphs in general (no need to ask for them to be balanceable).

Lemma 6.11 *Let G be a square-free bipartite graph. Let (X_1, X_2) be a minimally-sided $\{2,6\}$-join of G, with X_1 being a minimal side. If G has no star cutset, then the block of decomposition G_1 has no $\{2,6\}$-join i.e. (X_1, X_2) is an extreme $\{2,6\}$-join.*

PROOF — Assume the contrary, and let (X'_1, X'_2) be a $\{2,6\}$-join of G_1. We now consider the following cases.

Case 1: (X_1, X_2) is a 2-join of G.
Let M_2 be the marker path of G_1. By Theorem 6.8 (ii) and Lemma 6.9 (ii), G_1 has no 2-join,

and thus (X'_1, X'_2) is a 6-join of G_1, say with split $(X'_1, X'_2, A'_1, \ldots, A'_6)$. By Lemma 6.10 (iii), every vertex in $\cup_{i=1}^6 A'_i$ is of degree at least 3 in G_1. Therefore, we may assume w.l.o.g. that $V(M_2) \subseteq X'_2$. If $V(M_2) \subseteq X'_2 \setminus (A'_2 \cup A'_4 \cup A'_6)$, then clearly $(X'_1, (X'_2 \setminus V(M_2)) \cup X_2)$ is a 6-join of G that contradicts the choice of (X_1, X_2). So $V(M_2) \cap (A'_2 \cup A'_4 \cup A'_6) \neq \emptyset$, and since vertices in $A'_2 \cup A'_4 \cup A'_6$ are of degree at least 3, $V(M_2) \cap (A'_2 \cup A'_4 \cup A'_6) \subseteq \{a_2, b_2\}$. Moreover, since a_2 and b_2 do not have common neighbors, $V(M_2) \cap (A'_2 \cup A'_4 \cup A'_6) \neq \{a_2, b_2\}$. So, we may assume w.l.o.g. that $V(M_2) \cap (A'_4 \cup A'_6) = \emptyset$. But then $(X'_1, (X'_2 \setminus V(M_2)) \cup X_2, A'_1, A_2, A'_3, A'_4, A'_5, A'_6)$ is a split of a 6-join of G that contradicts the choice of (X_1, X_2).

Case 2: (X_1, X_2) is a 6-join of G.
Let $(X_1, X_2, A_1, \ldots, A_6)$ be the split of this 6-join, and let a_2, a_4, a_6 be the marker vertices of G_1. We now consider the following two cases.

Case 2.1: (X'_1, X'_2) is a 6-join of G_1.
Let $(X'_1, X'_2, A'_1, \ldots, A'_6)$ be the split of this 6-join. By Lemma 6.10 (iii) every vertex in $\cup_{i=1}^6 A'_i$ is of degree at least 3 in G_1 and thus we may assume w.l.o.g. that $\{a_2, a_4, a_6\} \subseteq X'_2 \setminus (A'_2 \cup A'_4 \cup A'_6)$. But then $(X'_1, X'_2 \cup X_2)$ is a 6-join of G that contradicts the choice of (X_1, X_2).

Case 2.2: (X'_1, X'_2) is a 2-join of G_1.
Let $(X'_1, X'_2, A'_1, A'_2, B'_1, B'_2)$ be the split of this 2-join. By Lemma 6.10 (iii), let $A_1 = \{a_1\}$, $A_3 = \{a_3\}$ and $A_5 = \{a_5\}$, and let H be the 6-hole induced by $\{a_1, \ldots, a_6\}$. First suppose that both $X'_1 \setminus (A'_1 \cup B'_1)$ and $X'_2 \setminus (A'_2 \cup B'_2)$ contain a vertex of H. Then w.l.o.g. we may assume that $a_2 \in X'_2 \setminus (A'_2 \cup B'_2)$, $a_4 \in B'_1$ and $a_6 \in A'_1$. Since vertices a_2, a_4 and a_6 are all of degree 2 in G_1, it follows that $A'_2 = \{a_1\}$ and $B'_2 = \{a_3\}$, and hence by Lemma 6.10 (iii) $(a_2, \{a_1, a_3\})$ is a star cutset of G, a contradiction.

So we may assume w.l.o.g. that $(X'_2 \setminus (A'_2 \cup B'_2)) \cap V(H) = \emptyset$. By Lemma 6.7 (ii) and since a_2, a_4, a_6 are all of degree 2 in G_1, it follows that in fact w.l.o.g. we may assume that $V(H) \cap X'_2 \subseteq A'_2$. By Lemma 6.7 (ii), every vertex of A'_2 has a neighbor in X'_2, and hence (since a_2, a_4, a_6 are all of degree 2 in G_1) $\{a_2, a_4, a_6\} \subseteq X'_1$. But then $(X'_1 \cup X_2, X'_2)$ is a 2-join of G that contradicts the choice of (X_1, X_2). □

6.4 Linear balanceable graphs

In this section, we first show how looking at a minimally sided double star cutset leads to a block of decomposition with no star cutset. Then we prove the main result of this chapter on linear balanceable graphs (Theorem 6.13).

A *double star cutset* S of a graph G is a cutset of G such that S contains two adjacent vertices u and v such that every vertex of S is adjacent to at least one of u or v. Note that a star cutset is either a double star cutset or a cut vertex. If $U = (N(u) \cap S) \setminus \{v\}$ and $V = (N(v) \cap S) \setminus \{u\}$, then this double star cutset is denoted by (u, v, U, V). Note that if G is a square-free bipartite graph, $U \cup V$ induces a stable set and $U \cap V = \emptyset$.

Let S be a double star cutset and let C_i, for $i = 1, 2$, be a partition of the vertex set $V(G) \setminus S$, such that there are no edges between vertices of C_1 and C_2. Then $G_i = G[S \cup V(C_i)]$, $i = 1, 2$, are *blocks of decomposition* with respect to this double star cutset.

A double star cutset of a 2-connected graph G with blocks of decompositions G_1 and G_2 is a *minimally-sided double star cutset* if for some $i \in \{1,2\}$ the following holds: for every double star cutset of G with blocks of decompositions G'_1 and G'_2 neither $V(G'_1) \subsetneq V(G_i)$ nor $V(G'_2) \subsetneq V(G_i)$. In this case G_i is a *minimal side* of this minimally-sided double star cutset.

Lemma 6.12 *Let G be a 2-connected square-free bipartite graph that has a star cutset. Let G_i, for some $i \in \{1,2\}$ be a minimal side of a minimally-sided double star cutset of G. Then G_i does not have a star cutset.*

PROOF — Let (u,v,U,V) be a minimally-sided double star cutset, let G_1 be its minimal side, and let $S = \{u,v\} \cup U \cup V$. Observe that every vertex of $U \cup V$ has a neighbor in $G_1 \setminus S$. In particular, G_1 is 2-connected. Let us assume by way of contradiction that (x,R) is a star cutset of G_1. Since G_1 is 2-connected, $R \neq \emptyset$.

Case 1: $x \notin S$.

Since G is square-free and bipartite, x has at most one neighbor in S. If $R \cap \{u,v\} = \emptyset$, then vertices of $S \setminus R$ are in the same connected component of $G_1 \setminus (\{x\} \cup R)$, and therefore $(x,y,R\setminus\{y\},\emptyset)$, for a vertex $y \in R$, is a double star cutset of G that contradicts the minimality of G_1. So w.l.o.g. $u \in R$. Let C be a connected component of $G_1 \setminus (\{x\} \cup R)$ that does not contain a vertex of $\{v\} \cup V$. If $V(C) \setminus U \neq \emptyset$, then $(x,u,R\setminus\{u\},U)$ is a double star cutset of G that contradicts the minimality of G_1. So $V(C) \setminus U = \emptyset$. But then some vertex $u' \in U$ is of degree 1 in G_1 (since G_1 is square-free and bipartite), contradicting the fact that G_1 is 2-connected.

Case 2: $x \in S$.

First, let us assume that $x \in \{u,v\}$, say $x = u$. Since G is square-free and bipartite and every vertex of $U \cup V$ is of degree at least 2 in G_1, every connected component of $G_1 \setminus (\{x\} \cup R)$ that contains a vertex from U or a vertex from V contains a vertex from $G_1 \setminus S$. Therefore, $(x,v,(U \cup R)\setminus\{v\},V)$ is a double star cutset of G that contradicts the minimality of G_1. So, $x \in U \cup V$, and w.l.o.g. we may assume that $x \in U$. Then the vertices of $\{v\} \cup V$ are all contained in the same connected component of $G_1 \setminus (\{x\} \cup R)$. Again, since G is square-free and bipartite, every connected component of $G_1 \setminus (\{x\} \cup R)$ that contains a vertex from U contains a vertex from $G_1 \setminus S$. Therefore, $(x,u,R\setminus\{u\},U\setminus\{x\})$ is a double star cutset of G that contradicts the minimality of G_1. □

Our main result about linear balanceable graphs is the following.

Theorem 6.13 *If G is a linear balanceable graph on at least two vertices, then G contains at least two vertices of degree at most 2.*

PROOF — We prove the theorem by induction on $|V(G)|$. If $|V(G)| = 2$, then the theorem trivially holds. So, let G be a linear balanceable graph such that $|V(G)| > 2$. We may assume that G is connected, else we are done by induction.

Let u be a cut vertex of G, and let $\{C_1, C_2\}$ be a partition of $V(G) \setminus \{u\}$, such that there are no edges between vertices of C_1 and C_2. Then, by induction applied to graphs $G[C_i \cup \{u\}]$ for $i = 1,2$, there is a vertex $c_i \in C_i \setminus \{u\}$, for $i = 1,2$, that is of degree at most 2 in $G[C_i \cup \{u\}]$. But then c_1 and c_2 are also of degree at most 2 in G. So, we may assume that G is 2-connected.

Now suppose that G admits a star cutset. By Lemma 6.12, there is a double star cutset (u,v,U,V) of G, such that a block of decomposition w.r.t. this cutset, say G', has no star cutset.

Let $S = \{u, v\} \cup U \cup V$ and note that all vertices from U and V have a neighbor in $G' \setminus S$. By Theorem 6.6 G' is sparse or has a $\{2, 6\}$-join.

Case 1: G' is sparse.
Let (X, Y) be a bipartition of G' such that all vertices of Y are of degree 2. Vertices u and v are adjacent, so we may assume w.l.o.g. that $\{v\} \cup U \subseteq Y$ and $\{u\} \cup V \subseteq X$. In particular, $|V| \leq 1$.

Suppose first $|V| = 1$ and put $V = \{v'\}$. All the neighbors of v' in $G' \setminus S$ are of degree 2 in G' and in G, so we may assume that v' has a unique neighbor w in $G' \setminus S$. Let w' be the unique neighbor of w in $G' \setminus v'$. Since G' is square-free and bipartite, $w' \in V(G') \setminus S$. If w' is of degree 2 in G' (and hence in G), then w' and w are the desired two vertices. So we may assume that w' has at least three neighbors in G'. But then, since G' is square-free and bipartite, w' must have a neighbor $w'' \in V(G') \setminus (S \cup \{w\})$, and hence w and w'' are the desired two vertices.

Now suppose that $V = \emptyset$ and let v' be the neighbor of v in $V(G') \setminus S$. Since G is square-free and bipartite, v' has no neighbors in $U \cup \{u\}$. So, either $\deg_{G'}(v') \geq 3$, in which case v' has at least two neighbors in $V(G') \setminus S$ of degree 2 in G', and hence in G, or $\deg_{G'}(v') = 2$, in which case v' and the neighbor of v' in $V(G') \setminus S$ are both of degree 2 in G', and hence in G. Therefore G has at least two vertices of degree 2.

Case 2: G' has a $\{2, 6\}$-join.
Let (X'_1, X'_2) be a $\{2, 6\}$-join of G'. W.l.o.g. we may assume that $|X'_1 \cap \{u, v\}| \leq 1$. Let (X_1, X_2) be a minimally-sided $\{2, 6\}$-join of G' such that $X_1 \subseteq X'_1$, and let G_1 be the corresponding block of decomposition. Clearly G_1 is square-free and $|X_1 \cap \{u, v\}| \leq 1$. By Theorem 6.8 (in case (X'_1, X'_2) is a 2-join) or Lemma 6.10 (in case (X'_1, X'_2) is a 6-join), G_1 is linear balanceable and has no star cutset. By Lemma 6.11, G_1 has no $\{2, 6\}$-join, and hence by Theorem 6.6, G_1 is sparse. We now consider the following two cases.

Case 2.1: (X_1, X_2) is a 6-join of G'.
Let $(X_1, X_2, A_1, \dots, A_6)$ be the split of this 6-join. By Lemma 6.10, $A_1 = \{a_1\}$, $A_3 = \{a_3\}$, $A_5 = \{a_5\}$, and all these vertices are of degree at least 3 in G_1. Since G_1 is square-free, vertices a_1, a_3, a_5 do not have common neighbors in X_1. Since $|X_1 \cap \{u, v\}| \leq 1$, we may assume w.l.o.g. that $(X_1 \setminus \{a_1\}) \cap \{u, v\} = \emptyset$. Let a'_3 (resp. a'_5) be a neighbor of a_3 (resp. a_5) in X_1. Then $a'_3 \neq a'_5$ and $\{a'_3, a'_5\} \cap S = \emptyset$. Since G_1 is sparse, a'_3 and a'_5 are of degree 2 in G_1, and hence in G'. Since $\{a'_3, a'_5\} \cap S = \emptyset$, they are also of degree 2 in G.

Case 2.2: (X_1, X_2) is a 2-join of G'.
Let $(X_1, X_2, A_1, A_2, B_1, B_2)$ be the split of this 2-join, and let M_2 be the marker path of G_1. By Lemma 6.9, $|A_1| \geq 2$, $|B_1| \geq 2$ and the ends of M_2 are of degree at least 3 in G_1. Since G_1 is sparse, it follows that the vertices of $A_1 \cup B_1$ are all of degree 2 in G_1, and on the same side of bipartition of G_1, and hence of G' as well. In particular, it is not possible that both u and v are in $A_2 \cup B_2$. Since G' is square-free and bipartite, it follows that $|A_2| = |B_2| = 1$, and hence the vertices of $A_1 \cup B_1$ are of degree 2 in G'. Since $|X_1 \cap \{u, v\}| \leq 1$, w.l.o.g. $B_1 \cap S = \emptyset$, and hence the vertices of B_1 are also of degree 2 in G.

So, we may assume that G does not admit a star cutset. Thus, by Theorem 6.6 G is sparse or has a $\{2, 6\}$-join. So the theorem holds by the same proof as in Cases 1 and 2 above. □

Corollary 6.14 *Let G be a linear balanceable graph that has at least one edge. Then there is an edge of G that is not the unique chord of a cycle.*

PROOF — Follows immediately from Theorem 6.13 since an edge incident to a degree 2 vertex cannot be the unique chord of a cycle. □

6.5 Subcubic balanceable graphs

A *branch vertex* is a vertex of degree at least 3. A *branch* is a path connecting two branch vertices and containing no other branch vertices. Two branches are *non incident* if the sets of ends of the corresponding paths are disjoint. Note that a 2-connected graph that is not a cycle is edgewise partitioned into its branches. A pair of non-adjacent vertices (u, v) of G is a *pair of twins* if $N(u) = N(v)$ and $|N(u)| \geq 3$. Note that a cubic bipartite graph has a pair of twins if and only if it contains a $K_{2,3}$ as a subgraph. Note that R_{10} does not have a pair of twins.

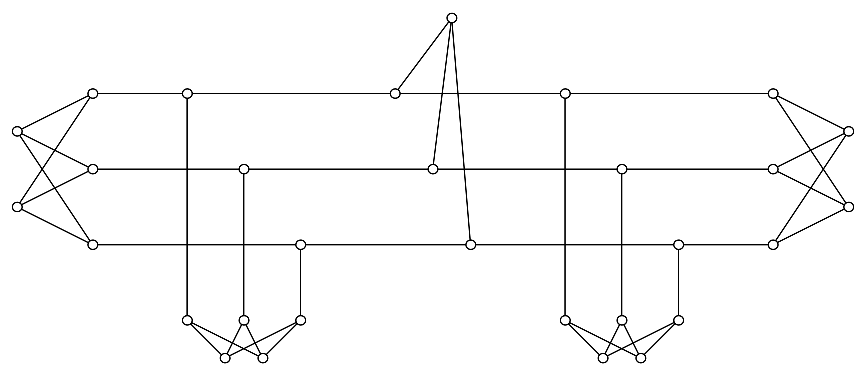


Figure 6.4: A cubic balanceable graph with 4 pairs of twins.

Our main result on subcubic balanceable graphs is the following theorem.

Theorem 6.15 *Let G be a 2-connected balanceable bipartite graph with $\Delta(G) \leq 3$. If G is not equal to R_{10} and has at least three branch vertices, then one of the following holds:*

(i) G has two vertices of degree 2 that are in non incident branches.

(ii) G has a pair of twins and a vertex of degree 2.

(iii) G has two disjoint pairs of twins.

In the previous theorem, if G has at least three branch vertices, then it has in fact at least four branch vertices (because 2-connected graphs have no vertex of degree 1).

The following lemma settles the case in which G does not admit a star cutset nor a 6-join. We treat this case separately because it does not need induction.

Lemma 6.16 *Let G be a 2-connected balanceable bipartite graph with $\Delta(G) \leq 3$, that is not equal to R_{10} and has at least three branch vertices. If G does not have a star cutset nor a 6-join, then G has two vertices of degree 2 that are in non incident branches.*

PROOF — By Theorem 6.6, G is either sparse or has a 2-join, so we consider the following two cases. Note that every vertex of G is of degree at least 2.

Case 1: G is sparse.
Since G is sparse, no two branch vertices are adjacent, and hence every branch of G contains a vertex of degree 2. Let a, b, c be distinct vertices of degree 3, such that there is a branch from a to b. There are three branches in G with end c. If one of the other ends of these branches is not a or b, the proof is complete. So we may assume w.l.o.g. that we have two branches between a and c and one branch between b and c. But then there is a branch from b with an end not in $\{a, c\}$, and hence the result follows.

Case 2: G has a 2-join.
Let $(X_1, X_2, A_1, A_2, B_1, B_2)$ be a split of a minimally-sided 2-join of G with X_1 being a minimal side. Let G_1 be the corresponding block of decomposition. By Theorem 6.8, G_1 is balanceable and it does not have a star cutset nor a 6-join. By Lemma 6.9 G_1 has no 2-join, $|A_1| = |B_1| = 2$ (because $\Delta(G) \leq 3$), and all vertices of $A_2 \cup B_2$ are of degree 3. So by Theorem 6.6 G_1 is sparse.

Claim: *$X_1 \setminus (A_1 \cup B_1)$ contains a vertex of degree 2.*
Proof of Claim: Assume not. Let (X, Y) be a bipartition of G_1 such that all vertices of X are of degree 2. Let $a_2 \dots b_2$ be the marker path of G_1, with a_2 complete to A_1 and b_2 complete to B_1. Then a_2 and b_2 are in Y and hence $A_1 \cup B_1 \subseteq X$. In particular, there are no edges in $G[A_1 \cup B_1]$. So by Lemma 6.7 (ii), $X_1 \setminus (A_1 \cup B_1)$ is not empty. Since we assumed that $X_1 \setminus (A_1 \cup B_1)$ contains no vertex of degree 2, and G_1 has no vertex of degree 1 (because G is 2-connected), every vertex in $X_1 \setminus (A_1 \cup B_1)$ is of degree 3. Moreover it implies that $X_1 \setminus (A_1 \cup B_1) \subseteq Y$ and thus $X_1 \setminus (A_1 \cup B_1)$ is a stable set. So for every $u \in X_1 \setminus (A_1 \cup B_1)$, $N(u) \subseteq A_1 \cup B_1$. But then since $|A_1 \cup B_1| = 4$, $|N(u)| = 3$ and the fact that each vertex of $A_1 \cup B_1$ is of degree 2 in G_1, we have a contradiction. This completes the proof of the claim.

By the claim let $c_1 \in X_1 \setminus (A_1 \cup B_1)$ be of degree 2 (in G_1, and hence in G as well). Let $(X'_1, X'_2, A'_1, A'_2, B'_1, B'_2)$ be a split of a minimally-sided 2-join of G with X'_2 being a minimal side and $X'_2 \subseteq X_2$. Then, as before, $|A'_2| = |B'_2| = 2$, and hence all the vertices of $A'_1 \cup B'_1$ are of degree 3. By the claim, there is a vertex $c_2 \in X'_2 \setminus (A'_2 \cup B'_2)$ that is of degree 2 in G.

Since $|A_1| = |B_1| = |A'_2| = |B'_2| = 2$, we see that no branch of G may overlap the three following sets: $A_1 \cup B_1$, $X_1 \setminus (A_1 \cup B_1)$ and $A'_2 \cup B'_2$ (resp. $A'_2 \cup B'_2$, $X'_2 \setminus (A'_2 \cup B'_2)$ and $A_1 \cup B_1$). It follows that c_1 and c_2 are in non incident branches. □

Proof of Theorem 6.15: We proceed by induction on $|V(G)|$. If $|V(G)| = 1$, then the theorem is vacuously true. By Theorem 6.6 and Lemma 6.16, we may assume that G has a star cutset or a 6-join.

Proof when G has a star cutset.
Let (x, R) be a star cutset of G such that $|R|$ is minimum. Since G is 2-connected, $|R| \geq 1$, and by the choice of (x, R) and since G is subcubic, every vertex of R has neighbors in every connected component of $G \setminus (\{x\} \cup R)$, every vertex of R is of degree 3. Moreover, $G \setminus (\{x\} \cup R)$ has exactly two connected components (because of the degree of the vetices in R), say C_1 and C_2. Let G_i be the block of decomposition w.r.t. this cutset that contains C_i, for $i = 1, 2$ (recall that $G_i = G[C_i \cup R \cup \{x\})$. Note that every vertex of R is of degree 2 in G_i. Note also that both G_1, G_2 are 2-connected.

Claim: *If x is of degree 2 in G_i, for some $i \in \{1, 2\}$, then C_i contains a vertex u of degree 2, or a pair of twins. Furthermore, if G_i has at least two branch vertices, then u can be chosen so that x and u are not in the same branch of G_i.*

Proof of Claim: If G_i has no branch vertices, then C_i contains a vertex of degree 2. If G_i has exactly two branch vertices, both are in C_i. Since these vertices can have at most one branch of length 1 connecting them, there must be a branch between them that is fully contained in C_i and is of length at least 2, and therefore there is a vertex of degree 2 in C_i that is not in the same branch as x. If G_i has at least 3 branch vertices, then, by the induction hypothesis, C_i contains a vertex of degree 2 that is not in the same branch as x, or C_i contains a pair of twins. This completes the proof of Claim.

We now consider the following cases.

Case 1: $|R| = 1$.
Note that since G is 2-connected, x has a neighbor in both C_1 and C_2, and in particular, x is of degree 2 in both G_1 and G_2. Since G has at least three branch vertices, at least one of G_1 or G_2 has at least two branch vertices, so, by Claim applied for $i = 1$ and $i = 2$, G satisfies the theorem.

Case 2: $|R| = 2$.
Let $R = \{y_1, y_2\}$. Suppose first that $\deg(x) = 2$. Then at least one of G_1 or G_2 has at least two branch vertices (since neither can have exactly one), w.l.o.g. say G_1 does. By Claim applied to G_1, either G_1 admits a pair of twins or there is a vertex u of degree 2 in C_1 that is not in the same branch of G_1 as x. If G_1 admits a pair of twins then, since we assumed that x was of degree 2 in G, outcome (ii) of the theorem holds. So we may assume there is a vertex u of degree 2 in C_1 that is not in the same branch of G_1 as x. Since y_1 and y_2 have degree 3 in G, x and u are degree 2 vertices of G that are contained in non incident branches of G, a contradiction.

So we may assume that $\deg(x) = 3$, and w.l.o.g. x has a neighbor in C_1 and does not in C_2. If G_1 has exactly two branch vertices and they are adjacent, then for a shortest path P from y_1 to y_2 in $G_2 \setminus \{x\}$, the set $V(G_1) \cup V(P)$ induces an odd wheel with centre x, contradicting Theorem 6.2. So, if G_1 has exactly two branch vertices, then there is a vertex of degree 2 in G_1 in a branch that does not contain y_1 nor y_2, and therefore, by Claim applied to G_2 (x is of degree 2 in G_2 so we can apply the claim), G satisfies the theorem, a contradiction. So G_1 must have at least three branch vertices, and hence by induction hypothesis, G_1 has a pair of twins or a vertex of degree 2 in a branch that has both of its ends in C_1. But then by Claim applied to G_2, G satisfies the theorem.

Case 3: $|R| = 3$.
Let $R = \{y_1, y_2, y_3\}$. First, let us suppose that both G_1 and G_2 have exactly two branch vertices, and that v_i is a branch vertex of G_i different from x, for $i = 1, 2$. If G_i, for $i = 1, 2$, does not have a vertex of degree 2 other than y_j, for $j = 1, 2, 3$, then G is a $K_{3,3}$, and hence it satisfies (iii) of the theorem. So, we may assume that there is a vertex of degree 2 (in G) in a branch of G_1 containing y_1. If y_2v_2 or y_3v_2 is not an edge, then G satisfies (i) of the theorem, so we may assume that y_2v_2 and y_3v_2 are edges. If y_1v_2 is also an edge, then x and v_2 form a pair of twins, and therefore G satisfies (ii) of the theorem. When y_1v_2 is not an edge, then by symmetry v_1y_2 and v_1y_3 are edges. But then y_2 and y_3 form a pair of twins, and therefore G satisfies (ii) of the theorem.

Observe that if G_i has at least three branch vertices, then, by induction hypothesis, there is a vertex u_i of degree 2 in a branch of G_i not having x as its end, or G_i has a pair of twins that does not contain x (since G_i has at least three branch vertices). So if both G_1 and G_2 have at least three branch vertices, then the theorem holds. Therefore we may assume that G_1 has at least three and G_2 exactly two branch vertices. If G_2 has a vertex2 u_2 of degree 2 not in $\{y_1, y_2, y_3\}$, then G satisfies (i) or (ii) of the theorem. So we may assume that the only vertices of G_2 of degree 2 are y_1, y_2 and y_3, and therefore x and the other branch vertex of G_2 form a pair of twins, hence G satisfies (ii) or (iii). This completes the proof when G has a star cutset.

Proof when G has a 6-join.

We may assume that G has no star cutset. In particular, G does not contain a pair of twins (for if u, v is a pair of twins of G, since G has at least three branch vertices, $V(G) \setminus (N(u) \cup \{u, v\}) \neq \emptyset$, and hence $N(u) \cup \{u\}$ is a star cutset). Let $(X_1, X_2, A_1, A_2, A_3, A_4, A_5, A_6)$ be a split of a 6-join of G and let $A = \cup_{i=1}^{6} A_i$. By Lemma 6.10 (iii), $|A_i| = 1$ for every $i \in \{1, \ldots, 6\}$ and all vertices of A are of degree 3 in G. It follows that both blocks of decomposition G_1 and G_2 have at least three branch vertices. By the choice of G, each of them has a vertex of degree 2 not in A, and hence G satisfies (i) of the theorem. This completes the proof. □

As a consequence of Theorem 7.9 we have the following corollary, a special case of which was conjectured in [73].

Corollary 6.17 *If G is a cubic balanceable graph that is not R_{10}, then G has a pair of twins none of whose neighbors is a cut vertex of G.*

PROOF — Inductively we may assume that G is connected. If G has no cutvertex then the result holds by Theorem 6.15. So we may assume that G has a cutvertex. Let G' be an end block of G (so G' is 2-connected). Then G' has exactly one vertex of degree 2 say x (that is a cutvertex of G), and all the other vertices of degree 3. Let G'' be the graph obtained from G' by subdividing twice an edge incident to x. Clearly G'' is 2-connected, balanceable and not equal to R_{10}. Note that G'' has exactly one branch of length greater than 1. By Theorem 6.15 G'' has a pair of twins $\{u_1, u_2\}$. Since x has only one neighbor of degree 3, $x \notin N(u_1) = N(u_2)$ and thus $\{u_1, u_2\}$ is the desired pair of twins of G. □

As was noticed in [73] (for the special case of cubic balanced graphs), Corollary 6.17 implies the following.

Corollary 6.18 *Let G be a cubic balanceable graph. Then the following hold:*

(i) G has girth four.

(ii) If $G \neq R_{10}$ then G contains an edge that is not the unique chord of a cycle.

(iii) G is not planar.

PROOF — It is easy to see that if $G = R_{10}$ then (i) and (iii) hold. So we may assume that $G \neq R_{10}$. By Corollary 6.17, let $\{u_1, u_2\}$ be a pair of twins of G, and $\{v_1, v_2, v_3\}$ the set of neighbors of u_1 and u_2. Then $u_1v_1u_2v_2$ is a cycle of length 4, and hence (i) holds. Suppose that u_1v_1 is a unique chord of a cycle C in G. Then all neighbors of u_1 and v_1 belong to C, and in particular, u_2 belongs to C and has three neighbors in C, a contradiction. Hence (ii) holds.

By Corollary 6.17 we may assume that none of v_1, v_2, v_3 is a cut vertex of G. So there is a connected component C of $G \setminus \{u_1, u_2, v_1, v_2, v_3\}$ such that all of v_1, v_2, v_3 have a neighbor in C. Let C' be a minimal induced subgraph of C that is connected and all of v_1, v_2, v_3 have a neighbor in C'. Since G is cubic, it is easy to see that $V(C') \cup \{u_1, u_2, v_1, v_2, v_3\}$ induces a subdivision of $K_{3,3}$. Therefore, by Kuratowski's Theorem (see for example [14]), G is not planar. □

Chapter 7

Excluding wheels as subgraphs

In this chapter:

- **If *G* and H are graphs, then we say that G is *H-free* if G does not contain H as a subgraph (not necessarily induced).**
- **K_4 is a wheel.**

The work described in this chapter comes from two different papers. Results presented in Section 7.3 (concerned with 3-wheel-free graphs) come from a joint work with Frédéric Havet and Nicolas Trotignon [4] (unpublished), results presented in Section 7.4 is accepted in Journal of Graph Theory [1].

Recall that a k-wheel is a graph formed by a chordless cycle and a vertex, outside the cycle, that has at least k neighbors in the cycle. In this chapter, we study classes of graphs that do not contain k-wheels as subgraphs for several values of k. It appears that arguments from connectivity are more adapted to study these classes than the decomposition method.

Here is the plan of the chapter. In the first section, we state every known result about classes of graphs defined by forbidding k-wheels as subgraphs. In the second chapter we present some tools around connectivity. The third section is devoted to 3-wheel-free graphs and the last section, that contains the main contribution of the author to the area, is concerned with 4-wheel-free graphs.

7.1 State of art

In this section, we give a brief survey of the known results on k-wheel-free graphs. These results are split in two parts. We first present some extremal results that give the maximal number of edges that a k-wheel-free graph with n vertices can contain. Then some results about local structural properties with coloring applications. In the rest of the chapter we are only interested by these last ones.

Recall that a twin is a pair of non adjacent vertices that have the same neighborhood.

The class of 2-wheel-free graphs is very simple, we already precisely described their structure in Chapter 5 and we do not deal with them here.

Extremal results

The following theorem give the maximal number of edges that a 3-wheel-free graph can contain.

Theorem 7.1 (Thomassen [87]) *If G is a 3-wheel-free graphs on n vertices, then it has at most $2n - 3$ edges.*

In the same paper, Thomassen proves that this bound is tight and he precisely describes graphs that reach it. These graphs are called *(K_3, $K_{3,3}$)-cockades* and are defined recursively as follows:

1. K_3 and $K_{3,3}$ are (K_3, $K_{3,3}$)-cockades.

2. If G_1 and G_2 are (K_3, $K_{3,3}$)-cockades and $e_i \in E(G_i)$ for $i = 1, 2$, then the graph obtained by identifying e_1 and e_2 (and their respective extremity) is a (K_3, $K_{3,3}$)-cockade.

The next theorem is an extension of the theorem above to 4-wheel-free graphs.

Theorem 7.2 (Horev [57]) *If G is a 4-wheel-free graph on n vertices, then it has at most $3n - 8$ edges. Moreover, the bound is reached by a unique graph, the graph obtained from $K_{3,n-3}$ by adding an edge to the color class of size 3.*

The following theorem is concerned with k-wheel-free graphs in general but the given bound is not tight.

Theorem 7.3 (Horev and Lomonosov [58]) *If G is a k-wheel-free graph ($k \geq 5$) on n vertices, then it has at most $(2k - 3)(n - k + 1)$ edges.*

The good bound is conjectured to be:

Conjecture 7.4 (Horev and Lomonosov [58]) *If G is a k-wheel-free graph ($k \geq 5$) on n vertices, then it has at most $(r - 1)(n - r + 1) + \lceil \frac{r-1}{2} \rceil$ edges. Moreover, it is conjectured that the only graph reaching this bound is the unique graph obtained from $K_{k-1,n-k+1}$ by adding a maximum matching to the color class of cardinality $k - 1$.*

It is interesting to note that extremal k-wheel-free graphs form a quite rich class when $k = 3$, whereas it is formed by a unique one when $k \geq 4$ (or at least when $k = 4$, for $k \geq 5$ it is only conjectured but there are many reasons to think the conjecture is true). We will also see that, similarly, the class of k-connected k-wheel-free graphs is quite rich when $k = 3$ whereas it is formed by a unique graph (namely $K_{k,k}$) when $k \geq 4$ (see Theorem 7.11).

These differences are certainly explained by the fact that, when $k \geq 3$, the structure around a set of k vertices that do not lie on a common cycle in a $(k+1)$-connected graph is very constraining, whereas it is less restrictive in the case where $k = 2$. See Section 7.2 for more details about that.

Structural and coloring results

Theorem 7.5 (Turner [95]) *For any integer $k \geq 2$, if G is a k-wheel-free graph, then G contains a vertex of degree at most k.*

The proof of the above theorem relies on some elegant arguments on a longest chordless path in a graph that has minimum degree $k+1$. Note that the result stated in [95] is slightly weaker than Theorem 7.5, but the proof given by Turner in [95] proves exactly the version given here.

Theorem 7.5 easily implies the following result.

Corollary 7.6 *For any integer $k \geq 5$, if G is a k-wheel-free graph, then G is (k+1)-colorable.*

Thomassen and Toft showed the following result about 3-wheel-free graphs (an alternative proof of it can be found in [4], we say more about it in Section 7.3):

Theorem 7.7 (Thomassen and Toft [88]) *If G is a 3-wheel-free graph, then either it contains a pair of twins or it contains a vertex of degree at most 2.*

From Theorem 7.7 they easily get the following corollary that settles a conjecture proposed by Toft in [89].

Corollary 7.8 *If G is a 3-wheel-free graph, then G is 3-colorable.*

PROOF — We proceed by induction on the number of vertices of a 3-wheel-free graph G. If $|V(G)| = 1$, then G is 3-colorable. Otherwise, by Theorem 7.7, either G contains a vertex w of degree at most 2, or a pair $\{u, v\}$ of twins. In the first case, we color $G \setminus \{w\}$ by the induction hypothesis, and give to w one of the four colors not used in its neighborhood. In the second case, we color $G \setminus \{u\}$ by the induction hypothesis, and give to u the same color as v. □

This last corollary is easily seen as being tight since odd cycle are 3-wheel-free. Some more results about 3-wheel-free graphs (more precisely about 3-connected 3-wheel-free graphs and about 3-wheel-free planar graphs) are proved in Section 7.3.

The next theorem is an extension of Theorem 7.7 to 4-wheel-free graphs, it is the main contribution of the author to this subject, the proof is given in Section 7.4.

Theorem 7.9 (Aboulker [1]) *If G is a 4-wheel-free graph, then either it contains a pair of twins or it contains a vertex of degree at most 3.*

It implies the following corollary.

Corollary 7.10 *If G is a 4-wheel-free graph, then G is 4-colorable.*

PROOF — The proof is similar to the proof of corollary 7.8 □

The following result is concerned with k-connected k-wheel-free graphs for any $k \geq 4$. A result actually slightly more general is proved in Section 7.4 (see Theorem 7.32).

Theorem 7.11 (Aboulker [1]) *For $k \geq 4$, the only k-connected k-wheel-free graph is $K_{k,k}$.*

Theorems 7.7, 7.9 and 7.11 suggest the following conjecture.

Conjecture 7.12 *If G is a k-wheel-free graph ($k \geq 5$), then either it contains a pair of twins or it contains a vertex of degree at most $k-1$.*

We propose a second conjecture that is easily seen as being weaker than the above one.

Conjecture 7.13 *If G is a k-wheel-free graph ($k \geq 5$), then it is k-colorable.*

Note that concerning the coloring, Corollary 7.10 and Conjecture 7.13 are tight since for any $k \geq 4$, K_k is a k-wheel-free graph.
Let us finish this section by giving a way to construct an infinite number of k-wheel-free graphs of chromatic number k. Let G_1 and G_2 be disjoint graphs, and let $e_1 = u_1v_1$ and $e_2 = u_2v_2$ be edges of G_1 and G_2, respectively. The graph obtained from G_1 and G_2 by identifying u_1 and u_2, deleting e_1 and e_2, and adding a new edge v_1v_2 is called a *Hajós join* of G_1 and G_2.

Property 7.14 *If G_1 and G_2 are k-wheel-free graphs and G is a Hajós join of G_1 and G_2, then G is k-wheel-free.*

PROOF — Let $e_1 = u_1v_1$ and $e_2 = u_2v_2$ be edges of G_1 and G_2, respectively and assume that G has been obtained from G_1 and G_2 by deleting e_1 and e_2, identifying u_1 and u_2, and adding a new edge v_1v_2. We name u the vertex of G obtained from the identification of u_1 and u_2. Suppose by way of contradiction that G contains a k-wheel (x, C) with spokes $xx_1, \dots, xx_k$. Note that deleting v_1v_2 and u in G yield to a disconnected graph with two connected component, H'_1 and H'_2 say. Put $H_i = G[H'_i \cup \{u\}]$ and assume w.l.o.g. that H'_i is a subgraph of G_i for $i = 1, 2$. So H_1 and H_2 are both k-wheel-free. We may assume w.l.o.g. that $x \in V(H_1)$. Since $\{u, v_2\}$ is a cutset of G, $x_1, \dots, x_k$ need to be in H_1. Therefore, by replacing the part of C that goes through H_2 by the edge u_1v_1, we get a k-wheel in G_1, a contradiction. □

A graph G is *k-critical* if $\chi(G) = k$ and, for any proper subgraph H of G, $\chi(H) < k$. It is a well-known (and easy) fact that the Hajós join of two k-critical graphs is k-critical (see [14] for example). Now, since K_k is k-wheel-free and k-critical, by Property 7.14, one can build an infinite number of k-critical k-wheel-free graphs using Hajós join.

7.2 Preliminaries

In this section, we present every tool we need in this chapter.

Around Menger Theorem

Let G be a graph, $k \geq 1$ an integer, $Y \subseteq V(G)$ a set of at least k vertices, and $x \in V(G) \setminus Y$. A family $\mathcal{F}$ of k paths from x to Y whose only common vertex is x and whose internal vertices are not in Y, is called a *k-fan from x to Y*. The set formed by vertices from Y that are endvertices of a path of $\mathcal{F}$ is denoted by $ext(\mathcal{F})$. We take the following convention for the notation of the paths of a k-fan $\mathcal{F}$: if we denote $P_{x-u_1}, \dots, P_{x-u_k}$ the k paths of $\mathcal{F}$, it means that for $i = 1, \dots, k$, x and u_i are the endvertices of P_{x-u_i}. The next result is an easy consequence of Menger Theorem (see [14]).

Lemma 7.15 (Fan Lemma) *If G is a k-connected graph, $x \in V(G)$ and Y is a subset of $V(G) \setminus \{x\}$ of cardinality at least k, then there is a k-fan from x to Y.*

We will also need the following improvement of the fan Lemma.

Lemma 7.16 (Perfect, [76]) *Let G be a k-connected graph, $k \geq 2$, $k_1 \leq k$, $x \in V(G)$, $S \subseteq V(G)$ with $|S| \geq k$ and $\mathcal{F}^1_x$ a k_1-fan from x to S. There exists a k-fan $\mathcal{F}_x$ from x to S such that $ext(\mathcal{F}^1_x) \subseteq ext(\mathcal{F}_x)$.*

Cycles containing prescribed elements

In graph theory, conditions for a set of elements (vertices, edges or paths) to belong to a cycle has been heavily studied (see [52] for a nice survey on this subject). Since a vertex v of a graph G is the center of a k-wheel if and only if there exists a cycle going through at least k vertices of $N(v)$ in $G \setminus v$, these kinds of result are very helpfull for the study of k-wheel-free graphs. The first result in this area is the following.

Theorem 7.17 (Dirac [46]) *In a k-connected graph G,*

1. *given any k vertices, there is a cycle passing through the k vertices;*

2. *given any edge and any $k-1$ vertices there is a cycle passing through all of them.*

In [99], Watkins and Mesner state a sufficient and necessary condition for a set of k vertices in a $(k-1)$-connected graph ($k \geq 3$) to not be contained in a common cycle. As we already mentioned in Section 7.1, there is a deep difference between cases where $k = 3$ and $k \geq 4$.

The statement in the case where $k = 3$ is quite long and complicate to state and, since we do not need it in any proofs presented here, we do not state it formally. A formal statement can be find in [99] or in [4] where an alternative prove of F. Havet, N. Trotignon and the author is given. But, roughly, it says that three vertices x, y, z of a 2-connected graph do not lie on a cycle if they lie on three interiorly distinct paths P_1, P_2, P_3 such that the removal of the endvertices of these three paths pairwise separates x, y and z.

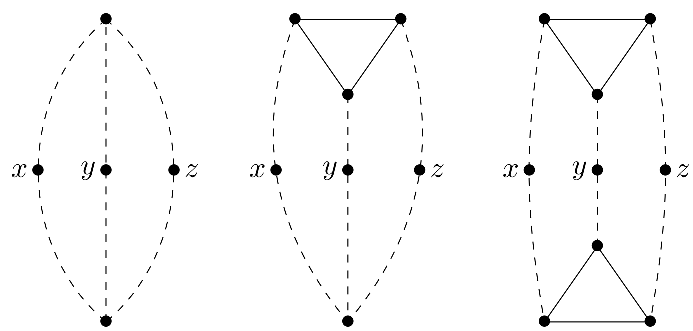


Figure 7.1: In these three graphs, x, y and z do not lie on a common cycle

Note that, interestingly, the three smallest 2-connected graphs that contain three vertices x, y, z not lying in a common cycle of the graph, are the theta, the pyramid and the prism (Figure 7.2). To our knowledge, [99] is the oldest paper mentioning the 3-paths configurations! Note also that Lemma 7.22 gives a sufficient (but not necessary) condition for three vertices in a 2-connected graph to not be contained in a common cycle.

The analogue result in the case $k \geq 4$ says that a set of k vertices is not contained in a cycle of a $(k-1)$-connected graph if and only if there exists a $(k-1)$-cutset that pairwise separates the k vertices (the "if" part is easy to check). This fact is the key tool to prove that the only k-connected k-wheel-free graph is $K_{k,k}$ so we formally state it just before the proof (see Theore 7.29).

Fragments and end blocks

Let G be a graph, $F \subseteq V(G)$ and $x \in V(G)$. We recall that $N(F)$ is the set of vertices from $V(G) \setminus F$ adjacent to at least one vertex of F and $N_F(x) = N(x) \cap F$ is the neighborhood of x in F. We denote by $\overline{F}$ the set $V(G) \setminus (F \cup N(F))$. We say that F is a *fragment* of G if $|N(F)| = \kappa(G)$ and $\overline{F} \neq \emptyset$. Note that if F is a fragment of G, then $\overline{F}$ is a fragment too. Note also that complete graphs are the only graphs that do not contain any fragment. So, if G is a graph of connectivity k, each k-cutset of G defines two fragments.

An *end* of G is a fragment not containing any other fragments as a proper subset (it is a fragment associated with a minimally sided $\kappa(G)$-cutset). It is clear that any fragment F contains an end, and that consequently any graphs that is not a complete graph contain at least two disjoint ends: one in F, another one in $\overline{F}$.

Let F be an end of G. If $|F| = 1$ we say that F is *trivial*. The graph H obtained from $G[F \cup N(F)]$ by adding all (not already existing) edges between vertices of $N(F)$ is called an *end block* of G. Edges with both ends in $N(F)$ are called *marker edges* of H.

Let us finish this section with two easy and essentials properties of the end blocks of a graph.

Property 7.18 *Let G be a graph that is not a clique and let K be a clique in G. Then there exists an end of G disjoint from K.*

PROOF — Let F be a fragment of G. Either $V(K) \subset F \cup N(F)$, or $V(K) \subset \overline{F} \cup N(F)$. The result follows easily from the fact that F and $\overline{F}$ both contain an end. □

The next classical property says that any graph with connectivity k that contains a non trivial end admits a extreme k-cutset i.e. the end block containing the non-trivial end does not admit k-cutset.

Property 7.19 *If G is a graph of connectivity k and F is a non-trivial end of G, then the end block H containing F is (k+1)-connected.*

PROOF — Let G be a graph of connectivity k, F a non-trivial end of G and H the end block of G containing F. Suppose by way of contradiction that $\kappa(H) \leq k$. Since $|V(H)| \geq k+2$, if H is a clique we are done. So we may assume that H is not a clique and thus H admits a fragment F' such that $|N_H(F')| \leq k$. Since $N_H(F)$ induces a clique, by Property 7.18 we may choose F' disjoint from $N_H(F)$. Thus F' is a fragment of G that is a proper subset of F, a contradiction. □

Note that Property 7.19 can be reformulated as follows: every graph of connectivity k that is not a clique admits an extremal k-cutset (where the block of decomposition associated with k cutsets is the end block).

7.3 3-wheel-free graphs

When I started my PHD, the first problem Nicolas Trotignon proposed to me is the study of graphs that do not contain wheels as subgraphs (recall that wheels and 3-wheels are the same). He was interested in graphs that do not contain wheels as induced subgraphs but, this class of graphs being very complicated and hard to tackle (especially at the very beginning of a phd), we decided to start by forbidding them only as subgraphs. This leads to a class easier to handle and, we hoped, might give us some insight on the structure of graphs that do not contain wheels as induced subgraphs.

At the end of the day, we cannot really say that it gave us any useful informations on graphs with no induced wheels, but anyway, it leads us to some very nice results, methods and related open problems.

The main result we obtained is Theorem 7.7. We didn't know at this time it was already proved by Thomassen and Toft. Anyway, our proof (jointly with Havet) is completely different from theirs. We do not include it here because it is very long and technical and the global approach is similar to the approach we use to prove Theorem 7.9 in Section 7.4. On the other hand we include two unknown results about 3-wheel-free graphs: 3-connected 3-wheel-free graphs are minimally 3-connected (see Subsection 7.3.1) and 3-wheel-free planar graphs are 2-colorable (see section 7.3.2). Moreover, the (simple) proofs of these results illustrate some techniques we use in the (way more complicated) proof of Theorem 7.9 (see Section 7.4).

7.3.1 Minimaly 3-connected 3-wheel-free graphs

Recall that a graph G is *minimally 3-connected* if, for any edge e of G, $G \setminus e$ is not 3-connected. An edge e of a 3-connected graph G is said to be *essantial* is $G \setminus \{e\}$ is not 3-connected. So, a graph is minimally 3-connected if and only if it is 3-connected and all its edges are essential.

In this subsection, we give the proof that 3-connected 3-wheel-free graphs are minimally 3-connected. We actually prove a more general statement which is that any end vertex of a non-essential edge is the center of a 3-wheel (see Theorem 7.24).

The fact that 3-connected 3-wheel-free graphs are minimally 3-connected is a step of our proof of Theorem 7.7. It indeed gives some strong information because of the following theorems of Mader.

Theorem 7.20 (Mader [68], see also [13]) *If G is a minimally 3-connected graph, then every cycle of G contains a vertex of degree 3.*

Theorem 7.21 (Mader [68], see also [13]) *If G is a minimally 3-connected graph, then G has at least $\frac{2|V(G)|+2}{5}$ vertices of degree 3.*

The following is the basic tool to characterize the situation when no cycle goes through three given vertices of a 2-connected graph. Note that contrary to Theorem 7.29, it is not an "if and only statement".

Lemma 7.22 *Let G be a 2-connected graph and x, y, z be three vertices of G. Then either*

- *a cycle of G goes through x, y, z; or*
- *x, y, z are distinct and there exist two distinct vertices $t_A, t_B \notin \{x, y, z\}$ and six internally vertex-disjoint paths $P_A = t_A \dots x$, $P_B = t_B \dots x$, $Q_A = t_A \dots y$, $Q_B = t_B \dots y$, $R_A = t_A \dots z$ and $R_B = t_B \dots z$.*

PROOF — Since G is 2-connected, we know that x, y and z are distinct (or a cycle goes through them) and there exists a cycle C that goes through x, z. Cycle C is edge-wise partitioned into two paths S_A and S_B from x to z. Since G is 2-connected, if $y \notin V(C)$, then there exists a 2-fan from

y to C, formed by $Q_A = y \dots t_A$ and $Q_B = y \dots t_B$ say. If $t_A, t_B \in V(S_A)$ then up to symmetry, x, t_A, t_B, y appear in this order along S_A and $xS_At_AQ_AyQ_Bt_BS_AzS_Bx$ is a cycle through x, y, z. Similarly $t_A, t_B \in V(S_B)$ one finds such a cycle. Hence, we may assume $t_A \in V(S_A) \setminus \{x, z\}$ and $t_B \in V(S_B) \setminus \{x, z\}$. We let $P_A = xS_At_A$, $R_A = zS_At_A$, $P_B = xS_Bt_B$ and $R_B = zS_Bt_B$. □

We name $W_3(G)$ the set of vertices that are center of a 3-wheel in G.

Lemma 7.23 *If G is 4-connected, then $W_3(G) = V(G)$.*

PROOF — If a graph G is 4-connected, then any vertex v has at least four neighbors. Since $G \setminus \{v\}$ is 3-connected, by Theorem 7.17, it contains a cycle going through three neighbors of v. Together with v, this cycle forms a wheel. □

Lemma 7.24 *If a 3-connected graph G contains an edge $e = ab$ that is not essential, then $\{a, b\} \subseteq W_3(G)$.*

PROOF — Since $G \setminus ab$ is 3-connected, there exist three vertex-disjoint paths $T_1 = aa_1 \dots b$, $T_2 = aa_2 \dots b$ and $T_3 = aa_3 \dots b$ in $G \setminus ab$.

In $G \setminus a$, which is 2-connected, we may assume that no cycle goes through a_1, a_2 and a_3 (otherwise $a \in W_3(G)$). So, by Lemma 7.22 applied to $G \setminus a$, there exist two vertices u, v and six internally vertex-disjoint paths $P_1 = a_1 \dots u$, $P_2 = a_2 \dots u$, $P_3 = a_3 \dots u$, $Q_1 = a_1 \dots v$, $Q_2 = a_2 \dots v$ and $Q_3 = a_3 \dots v$. We set $X = P_1 \cup P_2 \cup P_3 \cup Q_1 \cup Q_2 \cup Q_3$.

Because of T_1, T_2, T_3, either $b \in X$, in which case we suppose $b \in P_1$, or there exists a 3-fan from b to X in $G \setminus a$. When $b \notin X$, from the pigeon-hole principle, at least two paths from this 3-fan end in $P_1 \cup P_2 \cup P_3$ or in $Q_1 \cup Q_2 \cup Q_3$. So, up to symmetry, if $b \notin X$, then we may assume that there exists a 2-fan from b to $P_1 \cup P_2$. It follows that (wherever b is) there is a cycle in $G \setminus a$ that goes through a_1, a_2 and b. Together with a, this cycle forms a wheel centered at a. This proves $a \in W_3(G)$, and $b \in W_3(G)$ can be proved similarly. □

Corollary 7.25 *If G is a 3-connected 3-wheel-free graph, then G is minimally 3-connected.*

PROOF — Since G is 3-connected, by Lemma 7.23, G has connectivity 3. Let $e = uv$ be an edge of G. Suppose for a contradiction that e is not essential. Then $\deg(u), \deg(v) \geq 4$, and by Lemma 7.24 $u, v \in W(G)$. This contradicts the fact that G is almost wheel-free. Hence, all edges of G are essential and thus G is minimally 3-connected. □

7.3.2 Wheel-free planar graphs

We prove in this section that any wheel-free planar graph on at least two vertices contains at least two vertices of degree at most 2. In fact, the key property that we use is that a planar graph does not contain a subdivision of $K_{3,3}$.

The proof is based on the use of end blocks. Their use is not straightforward because, if G_F is an end block of a 3-wheel-free graph, then it might be that G_F is not 3-wheel-free. Anyway, next lemma shows that, if G_F is not 3-wheel-free, the centers of its wheels are precisely located.

Lemma 7.26 *Let G be a 3-wheel-free graph such that $\kappa(G) = 2$. Let F be an end of G such that $|F| \geq 2$ and G_F the end block containing F. Then $W_3(G_F) \subseteq \{a, b\}$.*

PROOF — Suppose that G_F contains a wheel (w, C) with $w \notin \{a, b\}$. Since G is 3-wheel-free, the edge ab must be an edge of that 3-wheel, and $ab \notin E(G)$. If ab is an edge of C, then a wheel of G is obtained by replacing ab with a path from a to b with internal vertices in $\overline{F}$, a contradiction. □

Lemma 7.27 *If G is a 3-connected graph that contains no subdivision of $K_{3,3}$, then $W_3(G) = V(G)$.*

PROOF — Let v be a vertex of G. It has at least three neighbors x, y, z. If no cycle goes through them, then let $P_A, Q_A, R_A, P_B, Q_B, R_B$ be the six paths of $G \setminus v$ (which is 2-connected) whose existence is proved in Lemma 7.22. Together with v, they form a subdivision of $K_{3,3}$, a contradiction. Hence a cycle C goes through x, y, z, so (v, C) is a 3-wheel centered at v. □

Theorem 7.28 (Aboulker, Havet, Trotignon [4]) *If G is a 3-wheel-free graph on at least two vertices that contains no subdivision of $K_{3,3}$, then G has at least two vertices of degree at most 2.*

PROOF — Our proof is by induction on $|V(G)|$, the result holding trivially when $|V(G)| \leq 4$.

If G is not connected, then by the induction hypothesis, each of its components has at least one vertex of degree at most 2, so G contains at least two vertices of degree at most 2.

If G has a cutvertex a, then let C_1 and C_2 be components of $G \setminus a$. By the induction hypothesis, $G[C_1 \cup \{a\}]$ and $G[C_2 \cup \{a\}]$ have each two vertices of degree at most 2. Thus at least one of them is distinct from a and thus is also a vertex of degree at most 2 in G. Hence, C_1 and C_2 have each at least one vertex of degree at most 2 in G.

By Lemma 7.27, G is not 3-connected. So we can assume that G is of connectivity 2.

We consider two disjoint ends F and F' of G. It is enough to prove that both of them have cardinality 1. So, suppose for a contradiction that F has cardinality at least 2. Let $\{u, v\} = N(F)$, and G_F the end block containing F. So G_F is 3-connected. In addition, it contains no subdivision of $K_{3,3}$. Indeed if a subgraph H of G_F is a subdivision of $K_{3,3}$, then H contains the edge uv. So replacing uv by some path from u to v with internal vertices in $\overline{F}$ yields a subdivision of $K_{3,3}$ in G, a contradiction. Hence, by Lemma 7.27, any vertex of G_F is the center of a 3-wheel. In particular, G_F contains a 3-wheel whose center is not among u, v, a contradiction to Lemma 7.26. □

7.4 4-wheel-free graphs

This section is mostly devoted to the proof of Theorem 7.9. Recall that Theorem 7.9 states that *every 4-wheel-free graph G contains either a pair of twins or a vertex of degree at most 3.*

In the first subsection we show (Theorem 7.32) that the class of k-connected k-wheel-free graph ($k \geq 4$) is reduced to a single graph that is $K_{k,k}$ (we actually show the result on a more general class for technical reasons explain above). In order to prove Theorem 7.9, we only need the result for $k = 4$ but we give here this more general result because it could be a nice start to tackle Conjectures 7.4 and 7.12.

In Subsections 7.4.2, 7.4.3 and 7.4.4 we prove Theorem 7.9 for 4-wheel-free graphs of connectivity 3, 2 and 1 respectively (which together imply Theorem 7.9). In each of these subsections the same method is used: we take an end block of the graph to raise the connectivity and be able to apply results from previous subsections on it.

Note that building end blocks is not class-preserving for the class of 4-wheel-free graphs, the added marker edges may create 4-wheels. This is the reason why the results in subsections 7.4.1-7.4.3 are not dealing directly with 4-wheel-free graphs but with a slightly enlarged classes, namely *almost 4-wheel-free graphs.*

Here is how we define *almost k-wheel-free graphs* ($k \geq 4$) (we define it for any $k \geq 4$ because Theorem 7.32 deals with k-wheel-free graph and not only with 4-wheel-free). We denote by $W_k(G)$ the set of vertices that are centers of at least one k-wheel in G. We say that a graph G is *almost k-wheel-free* if $|W_k(G)| \leq k+1$ and $\alpha(G[W_k(G)]) \leq k-2$.

7.4.1 k-connected k-wheel-free graphs

As we said in Section 7.2, the key tool to prove that the only k-connected k-wheel-free graph when $k \geq 4$ is the following:

Theorem 7.29 (Watkins and Mesner, [99]) *Let G be a graph with $\kappa(G) = k-1 \geq 3$. If $X = \{x_1, \dots, x_k\} \subseteq V(G)$ has the property that no cycle of G goes through all the vertices of X, then G admits a cutset $S = \{s_1, \dots, s_{k-1}\}$ such that $S \cap X = \emptyset$ and $G \setminus S$ has k connected components $C_1, \dots, C_k$ such that $x_i \in C_i$ for $i = 1, \dots, k$.*

In other words, it says that, in a $(k-1)$-connected graph, a set of k vertices is not contained in a cycle if and only if there exists a $(k-1)$-cutset that pairwise separates the k vertices (the "if" part is easy to check). Note that in [99] Watkins and Mesner do not present their result in the same fashion as we do here, but they prove exactly the version given here.

Let us now explain how Theorem 7.29 is to be applied to k-connected graphs with no k-wheels. Let G be a k-connected graph, let $x \in V(G) \setminus W_k(G)$ and let $X = \{x_1, \dots, x_k\} \subseteq N(x)$. Since $x \notin W_k(G)$ there is no cycle going through all vertices of X in $G \setminus \{x\}$. So, by Theorem 7.29 applied to $G \setminus \{x\}$ there exists a set $\{s_1, \dots, s_{k-1}\}$ such that the $x_i's$ for $i = 1, \dots, k$ are in distinct connected components of $G \setminus \{x, s_1, \dots, s_{k-1}\}$. We call the set $S = \{x, s_1, \dots, s_{k-1}\}$ a *Watkins-Mesner-certificate* (*WM-certificate* for short) for $(x, \{x_1, \dots, x_k\})$ and we call, for $i = 1, \dots, k$, $C_S(x_i)$ the connected component containing x_i (see Figure 7.2).

We now prove two easy lemmas before we prove the principle result.

Lemma 7.30 *If G is a (k+1)-connected graph, then $W_k(G) = V(G)$.*

PROOF — Let x be a vertex of G and let $x_1, \dots, x_k$ be k neighbors of x. By Theorem 7.17, there is a cycle C in $G \setminus \{x\}$ going through $\{x_1, \dots, x_k\}$ and thus (x, C) is a k-wheel of G. □

Lemma 7.31 *If G is a k-connected graph, then every vertex contained in a triangle of G is in $W_k(G)$.*

PROOF — Let abc be a triangle of G and let $a_1, \dots, a_{k-2}$ be neighbors of a different from b and c. By Theorem 7.17, there is a cycle C in $G \setminus \{a\}$ passing through $a_1, \dots, a_{k-2}$ and bc and thus $a \in W_k(G)$. Similarly, b and c are in $W_k(G)$. □

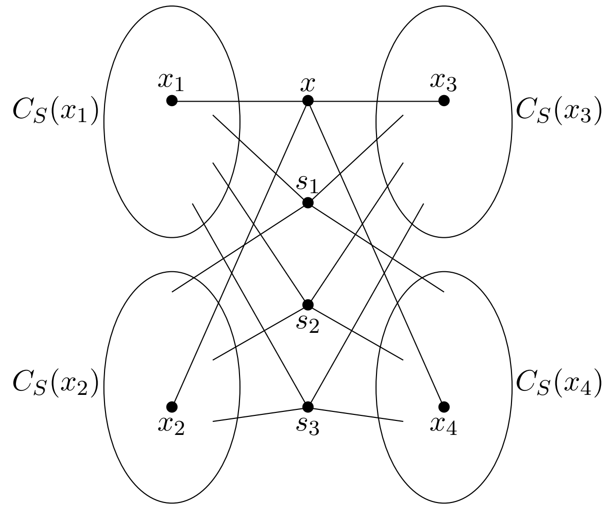


Figure 7.2: $S = \{x, s_1, s_2, s_3\}$ is a WM-certificate for $(x, \{x_1, x_2, x_3, x_4\})$.

Theorem 7.32 *If G is a k-connected almost k-wheel-free graph, then G is isomorphic to either $K_{k,k}$ or K_{k+1}.*

PROOF — Let us argue by way of contradiction and suppose that G is a k-connected k-wheel-free graph that is neither $K_{k,k}$ nor K_{k+1}. Note first that, since K_{k+1} is the unique k-connected graph on at most $k+1$ vertices, $V(G) \setminus W_k(G) \neq \emptyset$. By Lemma 7.30, $\kappa(G) = k$.

(1) G does not contain $K_{k,k}$ as a subgraph.

Suppose by way of contradiction that G contains a subgraph $H = K_{k,k}$ with partition $\{x_1, \dots, x_k\}$ and $\{y_1, \dots, y_k\}$. If H is not induced, say $x_1x_2 \in E(G)$, then, by Lemma 7.31, $\{x_1, x_2, y_1, \dots, y_k\} \subseteq W_k(G)$, a contradiction to the fact that $|W_k(G)| \leq k+1$. So H induces $K_{k,k}$ and, since $\alpha(G[W_k(G)]) \leq k-2$, at most $k-2$ vertices among x_i's (resp. y_i's) are in $W_k(G)$.

If S is a set of vertices of G, we say that a path P is an *S-path* if the endvertices of P are in S and the internal vertices of P are in $G \setminus S$. If there exists an H-path P with both endvertices in X, say with endvertices x_1 and x_2, then $(y_1, x_1Px_2y_2x_3y_3 \dots x_ky_kx_1)$ is a k-wheel and, symmetrically, every vertex in Y are also centers of k-wheels, a contradiction. So there exists no H-path with both endvertices in X and, by symmetry, there exists no H-path with both endvertices in Y.

Since $G \neq H$, we may assume that x_1 say has a neighbor $u \notin V(H)$. There is a 2-fan $\mathcal{F}_u$ from u to $H \setminus \{x_1\}$ in $G \setminus \{x_1\}$. If $\text{ext}(\mathcal{F}_u)$ intersects $X \setminus \{x_1\}$, then there exists an H-path with both endvertices in X, a contradiction. So $\text{ext}(\mathcal{F}_u) \subseteq Y$ and thus there exists an H-path with both endvertices in Y, a contradiction. This proves (1).

(2) Let $x \in V(G) \setminus W_k(G)$ and let $\{x_1, \dots, x_k\} \subseteq N(x)$. If $S = \{x, s_1, \dots, s_{k-1}\}$ is a WM-certificate for $(x, \{x_1, \dots, x_k\})$ and $C_S(x_i) = \{x_i\}$ for some $i \in \{1, \dots, k\}$, then $x_i \in W_k(G)$.

Suppose w.l.o.g. that $C_S(x_1) = \{x_1\}$ and that $x_1 \notin W_k(G)$. So $N(x_1) = \{x, s_1, \dots, s_{k-1}\}$ and there is a WM-certificate $\{x_1, t_1, \dots, t_{k-1}\}$ for $(x_1, \{x, s_1, \dots, s_{k-1}\})$. Now, for $i = 2, \dots, k$, there is a k-fan from x_i to $\{x, s_1, \dots, s_{k-1}\}$ included in $C_S(x_i) \cup \{x, s_1, \dots, s_{k-1}\}$ which implies that $\{x_2, \dots, x_k\} = \{t_1, \dots, t_{k-1}\}$. If $\{x_2, \dots, x_k\} \subseteq N(s_i)$ for $i = 1, \dots, k-1$, then $G[\{x, s_1, \dots, s_{k-1}, x_1, \dots, x_k\}]$ contains a $K_{k,k}$, a contradiction to (1). So we may assume that

s_1 say has a neighbor $u \neq x_2$ in $C_S(x_2)$. There is a 2-fan from u to $\{x, s_2, \dots, s_{k-1}\}$ in $G \setminus \{s_1, x_2\}$ that is included in $C_S(x_2) \cup \{x, s_2, \dots, s_{k-1}\} \setminus \{x_2\}$. So, the two paths of this fan link s_1 with a vertex in $\{x, s_2, \dots, s_{k-1}\}$ and avoid $\{x_1, \dots, x_k\}$, a contradiction to the fact that $\{x_1, \dots, x_k\}$ is a WM-certificate for $(x_1, \{x, s_1, \dots, s_{k-1}\})$. This proves (2).

Observe that, by Lemma 7.31, for any vertex x in $V(G) \setminus W_k(G)$, $N(x)$ is a stable set and thus, by definition of an almost 3-wheel-free graphs, x has at least two neighbors in $V(G) \setminus W_k(G)$.

Let $x \in V(G) \setminus W_k(G)$, let $X = \{x_1, \dots, x_k\} \subseteq N(x)$ and $S = \{x, s_1, \dots, s_{k-1}\}$ a WM-certificate for $(x, \{x_1, \dots, x_k\})$ such that x, X and S are chosen subject to the maximality of $|C_S(x_i)|$ where $x_i \notin W_k(G)$. Assume w.l.o.g. that $C_S(x_1)$ is the one that realizes the maximality.

Since x has at least two neighbors that are not center of a k-wheel, we may assume w.l.o.g. that $x_2 \notin W_k(G)$. By (2), $C_S(x_2) \neq \{x_2\}$. Let $\{y_1, \dots, y_{k-1}\} \subseteq N(x_2) \setminus \{x\}$ and let $T = \{x_2, t_1, \dots, t_{k-1}\}$ be a WM-certificate for $(x_2, \{x, y_1, \dots, y_{k-1}\})$.

Suppose that for some $j \in \{1, \dots, k-1\}$, $C_T(y_j)$ is not included in $C_S(x_2)$ and therefore contains an s_i ($i \in \{1, \dots, k-1\}$). W.l.o.g. $s_1 \in C_T(y_1)$. Since there are $k-1$ internally disjoint paths linking s_1 to x whose interior vertices are included respectively in $C_S(x_1), C_S(x_3), \dots, C_S(x_k)$, $\{t_1, \dots, t_{k-1}\} \subseteq (C_S(x_1) \cup C_S(x_3) \cup \dots \cup C_S(x_k))$. Now, if for some $i \in \{2, \dots, k-1\}$ $C_T(y_i)$ does not contain any vertex in $\{s_2, \dots, s_{k-1}\}$, then x_2 is a cutvertex of G separating $C_T(y_i)$ from the rest of the graph. So we may assume w.l.o.g. that $s_i \in C_T(y_i)$ for $i = 2, \dots, k-1$. Since $|C_S(x_2)| \geq 2$, either $C_T(x)$ or one of the $C_T(y_i)$'s have at least one vertex in $C_S(x_2)$ and thus, either $\{x, x_2\}$ or $\{s_i, x_2\}$ is a cutset of G, a contradiction. So, $C_T(y_i) \subseteq C_S(x_2)$ for $i = 1, \dots, k-1$ and therefore $C_S(x_1) \cup C_S(x_3) \cup \dots \cup C_S(x_k) \subseteq C_T(x)$. This contradicts the maximality of $C_S(x_1)$. □

Note that the above theorem trivially implies Theorem 7.11.

7.4.2 4-wheel-free graphs of connectivity 3

We first give a technical lemma that will be used in the proof of the next theorem.

Lemma 7.33 *Let G be a graph of connectivity 3 and let F be a fragment of G with $N(F) = \{a_1, a_2, a_3\}$. If a_i, for some $i \in \{1, 2, 3\}$ has at least two neighbors in $F \cup N(F)$, then there exists a 2-fan from a_i to $N(F) \setminus \{a_i\}$ in $G[F \cup N(F)]$.*

PROOF — Assume w.l.o.g. that a_1 has two neighbors in $F \cup N(F)$, say x and y. Since $G \setminus \{a_1\}$ is 2-connected, there exist two disjoint paths from $\{x, y\}$ to $\{a_2, a_3\}$ in $G \setminus \{a_1\}$. Since these two paths are clearly included in $F \cup N(F) \setminus \{a_1\}$, there is a 2-fan from a_1 to $\{a_2, a_3\}$ in $G[F \cup N(F)]$. □

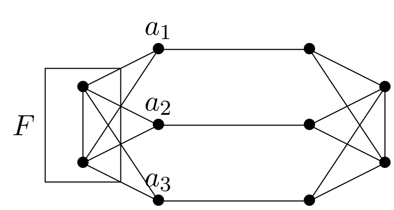


Figure 7.3: A 3-connected 4-wheel-free graph that satisfies the second outcome of Theorem 7.34.

Theorem 7.34 *If G is a graph with $\kappa(G) = 3$ and F is an end of G such that $W(G) \cap F = \emptyset$, then either F is trivial, or $|F| = 2$ and every vertex of $N(F)$ is of degree 3.*

PROOF — Let G be a graph of connectivity 3, let F be an end of G such that $W(G)\cap F = \emptyset$ and let H be the end block containing F. Set $N(F) = \{a_1, a_2, a_3\}$. If F is trivial, then the first outcome of the theorem holds. So we may assume that $|F| \geq 2$ and thus, by Property 7.19, H is 4-connected. Recall that edges a_1a_2, a_2a_3 and a_1a_3 are called the *marker edges* of H.

(1) We may assume that H is not almost 4-wheel-free.

Assume that H is an almost 4-wheel-free graph. By Lemma 7.32, $H = K_5$ (it cannot be $K_{4,4}$ since it contains a triangle). Let x, y be the two vertices of F. Assume that $d_G(a_1) \geq 4$. So a_1 has at least two neighbors in $\overline{F} \cup \{a_2, a_3\}$. Therefore, by Lemma 7.33, there exists a 2-fan $\{P_{a_1-a_2}, P_{a_1-a_3}\}$ from a_1 to $\{a_2, a_3\}$ in $\overline{F} \cup N(F)$. Then $(x, a_1P_{a_1-a_2}a_2ya_3P_{a_1-a_3}a_1)$ is a 4-wheel of G, a contradiction. Thus $d_G(a_1) = 3$ and similarly $d_G(a_2) = d_G(a_3) = 3$. So, every vertex of $N(F)$ is of degree 3 in G and thus the second outcome of the theorem holds. This proves (1).

Observe that (1) implies that there is at least one vertex in F that is the center of some 4-wheel of H. In the rest of the proof, we aim to show that such a vertex is also the center of a 4-wheel in G which is a contradiction.

Let us now give a new definition. Let (x, C) be a 4-wheel of H. We say that (x, C) is an *(i,j,k)-wheel* if $\{i, j, k\} = \{1, 2, 3\}$, $x \in F$, a_ia_j and a_ja_k are in $E(C)$ and xa_j is a spoke of (x, C).

(2) If (x, C) is a 4-wheel of H with $x \in F$, then (x, C) is an (i, j, k)-wheel.

Let (x, C) be a 4-wheel of H with $x \in F$. If C contains no marker edges, then (x, C) is a 4-wheel of G, a contradiction. Assume now that C contains exactly one marker edge, say a_1a_2. Let u be a neighbor of a_1 in $\overline{F}$. Since $G \setminus \{a_1, a_3\}$ is connected, there exists a path P from u to a_2 in $G \setminus \{a_1, a_3\}$. It is clear that P is included in $\overline{F} \cup \{a_2\}$. So, by replacing the marker edge a_1a_2 by a_1uPa_2 in C, we get a 4-wheel of G centered on x, a contradiction. Therefore, we may assume that C contains two marker edges, say a_1a_2 and a_2a_3. If xa_2 is not a spoke of (x, C), then we may replace the path $a_1a_2a_3$ by a_1a_3 in C and we get a 4-wheel of H that contains exactly one marker edge, a contradiction. So xa_2 is a spoke of (x, C) and thus (x, C) is an $(1, 2, 3)$-wheel. This proves (2).

In the rest of the proof, (2) is often used to get a contradiction when a 4-wheel (x, C) of H with $x \in F$ contains only one marker edge. We shall use this without mentioning (2) explicitly.

(3) Let (x, C) be an (i, j, k)-wheel such that $xa_i \notin E(G)$. Set $P = C \setminus \{a_j\}$. Let x_1, x_2, x_3 be three neighbors of x in P such that a_i, x_1, x_2, x_3, a_k appear in this order along P. If there exists a 1-fan $\{Q_{a_j-v}\}$ from $\{a_j\}$ to $\mathring{a}_iP\mathring{a}_k$ in $H \setminus \{x, a_i, a_k\}$, then $x_3 = a_k$ and $v \in \mathring{x}_2P\mathring{a}_k$.

Assume w.l.o.g. that $i = 1$, $j = 2$ and $k = 3$ and let $\{Q_{a_2-v}\}$ be a 1-fan from a_2 to $\mathring{a}_1P\mathring{a}_3$ in $H \setminus \{x, a_1, a_3\}$. If $v \in \mathring{a}_1P\mathring{x}_1$, then $(x, a_2Q_{a_2-v}vPa_3a_2)$ is a 4-wheel of H that contains exactly one marker edge, a contradiction. So $v \notin \mathring{a}_1Px_1$ and by symmetry $v \notin x_3P\mathring{a}_3$. In the case where $x_3 = a_3$, we may assume that $v \in \mathring{x}_1Px_2$ otherwise the outcome of the theorem holds. Observe now that, in the case where $x_3 \neq a_3$, the situation where $v \in \mathring{x}_1Px_2$ is symmetric with the situation where $v \in x_2P\mathring{x}_3$. So we may assume by way of contradiction and w.l.o.g. that $v \in \mathring{x}_1Px_2$. Note that, in what follows, x_3Pa_3 may be reduced to a vertex.

Observe that $\{x_1Pa_1, x_1Pv\}$ is a 2-fan from x_1 to $Q_{a_2-v} \cup vPa_3 \cup \{a_1\}$ in $H \setminus \{x\}$. So, by Lemma 7.16, there is a 3-fan $\mathcal{F}_{x_1} = \{Q_{x_1-a_1}, Q_{x_1-v}, Q_{x_1-w}\}$ in $H \setminus \{x\}$ from x_1 to $Q_{a_2-v} \cup vPa_3 \cup \{a_1\}$

with $\{a_1, v\} \subseteq \mathrm{ext}(\mathcal{F}_{x_1})$. We alter P as follows: $x_1Pa_1 := Q_{x_1-a_1}$ and $x_1Pv := Q_{x_1-v}$. Note that, after this alteration, P is still a path and $P \cup \{a_2\}$ is still the rim of a 4-wheel centered on x in H with spokes xa_2, xx_1, xx_2 and xx_3.

Let us now prove that $w \in \mathring{x}_2P\mathring{x}_3$.

- $w \notin Q_{a_2-v}$ for otherwise $(x, a_2Q_{a_2-v}wQ_{x_1-w}x_1Pa_3a_2)$ is a 4-wheel of H that contains exactly one marker edge, a contradiction.
- $w \notin \mathring{v}Px_2$ for otherwise $(x, a_2Q_{a_2-v}vPx_1Q_{x_1-w}wPa_3a_2)$ is a 4-wheel of H that contains exactly one marker edge, a contradiction.
- $w \notin x_3Pa_3$ for otherwise $(x, a_2Q_{a_2-v}vPwQ_{x_1-w}x_1Pa_1a_2)$ is a 4-wheel of H that contains exactly one marker edge, a contradiction.

So $w \in \mathring{x}_2Px_3$. Observe now that if $v = x_2$, then x is the center of a 4-wheel in H such that the rim contains exactly one marker edge, so $v \neq x_2$.

Now, $\{x_2Pv, x_2Pw\}$ is a 2-fan from x_2 to $Q_{a_2-v} \cup Q_{x_1-w} \cup a_1Pv \cup wPa_3$ in $H \backslash \{x\}$. So, by Lemma 7.16, there is a 3-fan $\mathcal{F}_{x_2} = \{Q_{x_2-v}, Q_{x_2-w}, Q_{x_2-u}\}$ from x_2 to $Q_{a_2-v} \cup Q_{x_2-w} \cup a_1Pv \cup wPa_3$ in $H \backslash \{x\}$. We alter P as follows: $x_2Pv := Q_{x_2-v}$ and $x_2Pw := Q_{x_2-w}$. Note that, after this alteration, P is still a path and $P \cup \{a_2\}$ is still the rim of a 4-wheel centered on x in H with spokes xa_2, xx_1, xx_2 and xx_3. Let us show by considering the possible position of u in $Q_{a_2-v} \cup Q_{x_2-w} \cup a_1Pv \cup wPa_3$ that x is the center of a 4-wheel of H such that the rim contains at most one marker edge.

- If $u \in Q_{a_2-v}$, then $(x, a_2Q_{a_2-v}uQ_{x_2,u}x_2Px_1Q_{x_1,w}wPa_2$ is a 4-wheel of H that contains exactly one marker edge, a contradiction.
- if $u \in Q_{x_2-w}$, then $(x, a_2Q_{a_2-v}vPx_1Q_{x_1-w}uQ_{x_2,u}x_2Pa_3a_2)$ is a 4-wheel of H that contains exactly one marker edge, a contradiction.
- If $u \in a_1Px_1$, then $(x, a_2Q_{a_2-v}vPuQ_{x_2-u}x_2Pa_3a_2)$ is a 4-wheel of H that contains exactly one marker edge, a contradiction.
- If $u \in x_1P\mathring{v}$, then $(x, a_2Q_{a_2-v}vPx_2Q_{x_2-u}uPx_1Q_{x_1-w}wPa_3a_2)$ is a 4-wheel of H that contains exactly one marker edge, a contradiction.
- If $u \in \mathring{w}Px_3$, then $(x, a_2Q_{a_2-v}vPx_1Q_{x_1-w}wPx_2Q_{x_2-u}uPa_3a_2)$ is a 4-wheel of H that contains exactly one marker edge, a contradiction.
- If $u \in x_3Pa_3$, then $(x, a_2Q_{a_2-v}vPx_2Q_{x_2-u}uPwQ_{x_1-w}x_1Pa_1$ is a 4-wheel of H that contains exactly one marker edge, a contradiction.

This proves (3).

(4) If (x, C) is an (i, j, k)-wheel, then x is adjacent to at least two vertices in $\{a_1, a_2, a_3\}$.

Let (x, C) be an (i, j, k)-wheel. Assume w.l.o.g. that $i = 1$, $j = 2$ and $k = 3$. Set $P = C \setminus \{a_2\}$. Let x_1, x_2, x_3 be three neighbors of x in P. Assume a_1, x_1, x_2, x_3, a_3 appear in this order along P. There exists a 1-fan $\{Q_{a_2-v}\}$ from $\{a_2\}$ to $\mathring{a}_1P\mathring{a}_3$ in $H \setminus \{a_1, a_3, x\}$. So, by (3), $a_3 = x_3$. This proves (4).

(5) If (x, C) is an (i, j, k)-wheel, then x is adjacent to exactly two vertices in $\{a_1, a_2, a_3\}$.

Let (x, C) be an (i, j, k)-wheel. Assume w.l.o.g. that $i = 1$, $j = 2$ and $k = 3$. By (4), we may assume that x is adjacent to a_1, a_2 and a_3. Let $u \in N(x) \setminus \{a_1, a_2, a_3\}$. Since H is not almost 4-wheel-free, there exists a vertex $y \in W(H) \setminus \{a_1, a_2, a_3, x\}$.

Assume first that y is adjacent to a_1, a_2 and a_3. If $xy \in E(G)$, then by (1), there exists $z \in W(H) \setminus \{a_1, a_2, a_3, x, y\}$. By (4), z is adjacent to at least two vertices in $\{a_1, a_2, a_3\}$. Assume w.l.o.g. that z is adjacent to a_1 and a_2. Then $(x, a_1za_2ya_3a_1)$ is a 4-wheel of H that contains exactly one marker edge, a contradiction. So $xy \notin E(G)$ and in particular $u \neq y$. There exists a 2-fan (P_1, P_2) from u to $\{a_1, a_2, a_3\}$ in $H \setminus \{x, y\}$. Assume w.l.o.g. that P_1 ends in a_1 and P_2 in a_2. Then $(x, a_1P_1uP_2a_2ya_3a_1)$ is a 4-wheel of H that contains exactly one marker edge, a contradiction.

So we may assume w.l.o.g. that y is adjacent to a_1 and a_2 and not to a_3. Let $v \in N(a_3) \setminus \{a_1, a_2, x\}$. Assume first that $xy \in E(G)$. There exists a 1-fan $\{Q\}$ from v to $\{a_1, a_2\}$ in $H \setminus \{x, y, a_3\}$. Assume w.l.o.g. that Q ends in a_2. Then $(x, a_1ya_2Qva_3a_1)$ is a 4-wheel of H that contains exactly one marker edge, a contradiction. Thus $xy \notin E(G)$ and in particular $u \neq y$. Therefore, there exists a 3-fan $\{R_1, R_2, R_3\}$ from u to $\{a_1, a_2, a_3, y\}$ in $H \setminus \{x\}$. At least two of these paths end in $\{a_1, a_2, a_3\}$. Assume first that one of the paths of the 3-fan, say R_3, ends on a_3. We may assume w.l.o.g. that R_1 ends in a_1. Thus $(x, a_1R_1uR_3a_3a_2ya_1)$ is a 4-wheel of H that contains a unique marker edge, a contradiction. So no path of the fan ends in a_3 and thus, we may assume w.l.o.g. that R_1, R_2, R_3 respectively ends in a_1, a_2, y. There exists a 1-fan $\{T\}$ from v to $R_1 \cup R_2 \cup R_3$ in $H \setminus \{a_3, x, y\}$. It is easy to see that T creates a 4-wheel centered on x that contains exactly one marker edge, a contradiction. This proves (5).

(6) Either there exist two vertices in $W(H) \setminus \{a_1, a_2, a_3\}$ that have at least two common neighbors in $\{a_1, a_2, a_3\}$, or $|W(H) \setminus \{a_1, a_2, a_3\}| \geq 3$.

By (1), there exists a vertex $x \in W(H) \cap F$. Let (x, C) be a 4-wheel of H. By (2), (x, C) is an (i, j, k)-wheel. Assume w.l.o.g. that $i = 1$, $j = 2$ and $k = 3$. By (5), we may assume w.l.o.g. that $N(x) \cap \{a_1, a_2, a_3\} = \{a_2, a_3\}$. Since H is not almost 4-wheel-free, there exists a vertex y in $W(H) \setminus \{a_1, a_2, a_3, x\}$ such that y is the center of an (i, j, k)-wheel. Since y is adjacent to exactly two vertices in $\{a_1, a_2, a_3\}$, if y does not see both a_2 and a_3, then $\alpha(G[\{a_1, a_2, a_3, x, y\}]) = 2$, and therefore there exists a vertex in $W(H) \setminus \{a_1, a_2, a_3, x, y\}$. This proves (6).

So, by (6), we may assume w.l.o.g. that we are in one of the two following cases:

Case 1: There exist two vertices x, y in $W(H) \setminus \{a_1, a_2, a_3\}$ such that x and y are both adjacent to a_2 and a_3.
By (2), there exists $i \in \{2, 3\}$ and a 4-wheel (x, C_i) that is an $(1, i, 5-i)$-wheel. Assume w.l.o.g. that $i = 2$. Set $P_2 = C \setminus \{a_2\}$. Let x_1 and x_2 be two neighbors of x in $P_2 \cap F$ and such that a_1, x_1, x_2 appear in this order along P_2.

If $y \notin V(P_2)$, then $(x, a_1a_2ya_3P_2a_1)$ is a 4-wheel of H containing exactly one marker edge, a contradiction. So $y \notin V(P_2)$. Moreover, since y is adjacent to a_2, by (3), $y \in \mathring{x}_2P\mathring{a}_3$ and we may assume that $yP_2a_3 = ya_3$. Now, define C_3 as the cycle obtained from C_2 by deleting edges a_1a_2 and a_3y and adding edges a_1a_3 and a_2y. Observe that (x, C_3) is an $(1, 3, 2)$-wheel and set $P_3 = C_3 \setminus \{a_3\}$.

Since H is 4-connected, $\{a_1, x, y\}$ does not separate $\{a_2, a_3\}$ from the rest of the graph. Therefore there exists $j \in \{2, 3\}$ such that a_j has a neighbor $z \in F$ distinct from x and y.

Suppose that $z \in V(P_j)$. By (3), $z \notin a_1P_jx_2$. Thus $z \in \mathring{x}_2P_j\mathring{y}$ and thus $(x, a_jzP_ja_1a_{5-j}ya_j)$ is a 4-wheel containing exactly one marker edge, a contradiction. So $z \notin V(P_j)$. There exists a 2-fan

$\{Q_{z-u}, Q_{z-v}\}$ from z to P_j in $H \setminus \{x, a_j\}$. By (3), $\{u, v\} \cap V(a_1 P_j x_2) = \emptyset$. If $u \in \mathring{x}_j P_j \mathring{y}$, then $(x, a_j z Q_{z-u} u P_j a_1 a_{5-j} y a_j)$ is a 4-wheel of H that contains a unique marker edge, a contradiction. So $u \notin \mathring{x}_2 P_j \mathring{y}$ and by symmetry, $v \notin \mathring{x}_2 P_j \mathring{y}$. Therefore $\{u, v\} = \{a_{5-j}, y\}$, say $u = y$ and $v = a_{5-j}$. Then $(x, a_j z Q_{z-v} a_{5-j} a_1 P_j y a_j)$ is a 4-wheel of H that contains a unique marker edge, a contradiction. This completes the proof in Case 1.

Case 2: There exist three vertices x, y and z such that x is adjacent to a_2 and a_3, y is adjacent to a_1 and a_3, and z is adjacent to a_1 and a_2.
By (2), there exists an $i \in \{2, 3\}$ and a 4-wheel (x, C_i) that is an $(1, i, 5-i)$-wheel. Assume w.l.o.g. that $i = 2$. Set $P_2 = C_2 \setminus \{a_2\}$. Let x_1 and x_2 be two neighbors of x in $P_2 \cap F$ and assume that a_1, x_1, x_2 appear in this order along P_2.

If $z \notin V(P_2)$, then $(x, a_2 z a_1 P_2 a_3 a_2)$ is a 4-wheel of H that contains a unique marker edge, a contradiction. Moreover, since z is adjacent to a_2, by (3), $z \notin a_1 P_2 x_2$. So $z \in \mathring{x}_2 P_2 \mathring{a}_3$.

Now set $C' = a_1 a_3 a_2 z P_2 a_1$ and observe that (x, C') is a $(1, 3, 2)$-wheel of H. Let us now discuss the position of y in the graph. If $y \notin V(P_2)$, then $(x, a_1 y a_3 a_2 z P_2 a_1)$ is a 4-wheel of H that contains a unique marker edge, a contradiction. If $y \in x_2 P_2 z$, then $(x, a_1 z a_2 a_3 y P_2 a_1)$ is a 4-wheel of H that contains a unique marker edge, a contradiction. If $y \in z P_2 a_3$, then $(x, a_1 y a_3 a_2 z P_2 a_1)$ is a 4-wheel of H that contains a unique marker edge, a contradiction. So $y \in a_1 P_2 x_2$ which contradicts (3) applied to (x, C'). □

As a trivial corollary of Theorem 7.34 we have the following result on 4-wheel-free graphs of connectivity 3.

Corollary 7.35 *If G is a 4-wheel-free graph of connectivity 3, then G contains at least one vertex of degree 3.*

7.4.3 4-wheel-free graphs of connectivity 2

Theorem 7.36 *If G is a 4-wheel-free graph of connectivity 2 and F is an end of G, then either there is a vertex $v \in F$ of degree at most 3 or the end block containing F is $K_{4,4}$.*

PROOF — Let G be a 4-wheel-free graph of connectivity 2, let F be an end of G with $N(F) = \{a, b\}$ and let H be the end block of G containing F. If F is trivial then the first outcome of the theorem holds. So we may assume that $|F| \geq 2$ and therefore H is 3-connected.

(1) If (x, C) is a 4-wheel of H, then $x \in \{a, b\}$.

Let (x, C) be a 4-wheel of H and assume $x \notin \{a, b\}$. C has to contain the edge ab because it is not a 4-wheel in G, but since we may replace ab by a path linking a to b in $\overline{F} \cup \{a, b\}$, we have a contradiction This proves (1).

Assume first that H is 4-connected. By (1) H is an almost 4-wheel-free graph. By Theorem 7.32 it is either K_5, or $K_{4,4}$. If it is K_5, then vertices of F are center of a 4-wheel in H, a contradiction to (1), so it is $K_{4,4}$ and the second outcome of the theorem holds.

So we may assume that $\kappa(H) = 3$. By Lemma 7.18, there exists an end F_1 of H such that $\{a, b\} \cap F_1 = \emptyset$. By Theorem 7.34, either F_1 is trivial and we are done, or every vertex of $N(F_1)$ is of degree 3 in H. Since $N(F_1)$ contains at least one vertex c distinct from a and b, c is of degree 3 in G. □

Corollary 7.37 *If G is a 4-wheel-free graph of connectivity 2, then either it contains a vertex of degree at most 3, or it contains a pair of twins.*

7.4.4 4-wheel-free graphs of connectivity 1

Theorem 7.38 *If G is a 4-wheel-free graph of connectivity 1, then either G contains a vertex of degree at most 3, or it contains a pair of twins.*

PROOF — Let G be a 4-wheel-free graph of connectivity 1. Let F_1 be an end of G and H_1 the end block containing F_1. Observe that H_1 is actually an induced subgraph of G and is thus 4-wheel-free. If F_1 is trivial, then we are done, so H_1 is 2-connected. If H_1 is a clique, then $|V(H_1)| \leq 4$ and we are done. So, by Lemma 7.18, H_1 admits an end F_2 disjoint from $N(F_1)$. Now, depending on whether H_1 is of connectivity 2, 3 or at least 4, we find the demanded structure in F_2 by Theorem 7.36, 7.34, or 7.32 respectively. □

Chapter 8

Excluding cycles with a fixed number of chords

In this chapter:

- **If *G* and H are graphs, then we say that G is *H-free* if** G **does not contain H as a subgraph.**
- $\mathbf{K_4}$ **is not a wheel.**

Most of this section come from a joint work with Nicolas Bousquet [2] and is concerned with χ-boundedness results.

8.1 Introduction and motivations

We already mentioned χ-bounded classes several times in this document. For example, perfect graphs can be defined as the largest class of graphs that is χ-bounded by the function $f(x) = x$. We also saw that some classes of graphs defined by forbidding induced subgraph, such as triangle-free graphs, are not χ-bounded. So, a natural (and challenging) question is the following:

Question 8.1 *What kind of induced structure is needed to be forbidden in order to get a χ-bounded class?*

Let us now survey some results on χ-boundedness by emphasizing on what different meanings "structure" can take.

If H is a graph, we denote by *Forb(H)* the class of H-free graphs. A first way to tackle the problem is to determine for which graphs H, Forb(H) is χ-bounded. For example, it is proved in [53] that Forb(P_k) is χ-bounded (where P_k denotes the chordless path of length k). In [47], Erdős proved that there exists graphs with arbitrarily large chromatic number and arbitrarily large girth. So, if H contains a cycle, Forb(H) is not χ-bounded. It is actually conjectured in [53] that Forb(H) is χ-bounded if and only if H is a forest. The deeper result concerning this conjecture is certainly a result of Kierstead and Penrice [59] proving that the conjecture holds of every tree of radius at most 2. To get out from this conjecture, we need to forbid a class of graph $\mathcal{H}$ such that $\mathcal{H}$ contains graphs with arbitrarily large girth.

A second way to forbid induced structure is the following: fix a graph H, and forbid every induced subdivision of H. We denote by *Forb*(H)* the class of graphs that does not contain induced subdivisions of H. The class Forb*(H) has been proved to be χ-bounded for a number of examples. The most beautiful one is certainly the proof by Scott [83] that for any forest F, Forb*(F) is χ-bounded. In the same paper he conjectured that, for any graph H, Forb*(H) is χ-bounded. Unfortunately, this conjecture has recently been disproved by Kozik *et al.* in [62]. Based on this work, Chalopin *et al.* [18] gave a precise description of a number of graphs H for which Forb*(H) is not χ-bounded. Since then, no conjecture about which H is needed to ensure Forb*(H) to be χ-bounded has been formulated.

A third way is to forbid a graph H for which some edges can be subdivided but some cannot. More generally, to forbid a class of graphs $\mathcal{H}$ such that, for each $H \in \mathcal{H}$, some edges can be subdivided and some cannot. For example, when we forbid wheels as induced subgraph, this is exactly what we do, the rim can be subdivided but the spokes cannot.

Another class defined like that has already been mentioned in Section 2.3 because it was a proper subclass of wheel-free graphs: graphs that do not contain a cycle with a unique chord as an induced subgraph. In [92], Trotignon and Vušković proved that this class is χ-bounded by the function $max(3, \omega(G))$. Forbidding cycles with a unique chord is equivalent to forbid a diamond (recall that a diamond is a cycle of length 4 with a diagonal) such that every edge but the diagonal can be subdivided.

A *k-cycle* is a chordless cycle with exactly k chords. We call $\mathcal{C}_k$ the class of k-cycle-free graphs i.e. the class of graphs that do not contain cycles with exactly k chords. So, the cited result on the class of graphs that do not contain a cycle with a unique chord may be rephrased as follows : $\mathcal{C}_1$ is χ-bounded.

The two main results of this chapter are that both $\mathcal{C}_2$ (see Theorem 8.5) and $\mathcal{C}_3$ (see Theorem 8.10) are χ-bounded. Note that the statement of Theorem 8.5 concerned a super-class of $\mathcal{C}_2$, see section 8.3 for more details. Class $\mathcal{C}_3$ is particularly interesting to us because, contrary to $\mathcal{C}_2$ that does not admit graphs with cliques larger than the triangle (because K_4 is a 2-cycle), graphs in $\mathcal{C}_3$ might contain any size of cliques. These results suggest the following conjecture :

Conjecture 8.2 *For any integer $k \geq 4$, $\mathcal{C}_k$ is χ-bounded.*

Here is the plan of this chapter: in Section 8.2, we explain the principle tool we use in different proofs of this chapter. Section 8.3 is concerned with the class $\mathcal{C}_2$ and Section 8.4 is concerned with the class $\mathcal{C}_3$.

8.2 Preliminaries

We mentioned that $\mathcal{C}_1$ was already proved to be χ-bounded, we use this result for graphs in $\mathcal{C}_1$ that contain no K_4, which formally give:

Theorem 8.3 (Trotignon and Vušković [92]) *If $G \in \mathcal{C}_1$ and $\omega(G) \leq 3$, then $\chi(G) \leq 3$.*

Let us now explain a classical tool to prove χ-boundedness results for classes of graphs defined by forbidding induced structure and that is extensively use in this chapter.

Let G be a graph. Recall that the *distance* between two vertices x, y of G is the length of a shortest xy-path. Let z be a vertex of G and let i be an integer. The *i-th level* of z is the set of vertices, denoted by $S_i(z, G)$, that are at distance exactly i from z in G. If no confusion is possible, we denote it by $S_i(z)$ in order to avoid too heavy notations. A *father* of a vertex $x \in S_i(z)$ is a vertex in $S_{i-1}(z)$ adjacent to x. For every pair of vertices x, y in S_i, it is easy to see that there exists a chordless xy-path with internal vertices included in $z \cup S_1(z) \cup \cdots \cup S_{i-1}(z)$. Such paths are called *unimodal paths.* Note that interior vertices of unimodal paths are not adjacent to any vertex of $S_i(z)$. This make unimodal paths a key tool to find particular induced structure in a graph. In the rest of the paper, the letter Q will be reserved to denote unimodal paths. If x and y are two vertices in $S_i(z)$, Q_{xy} denotes a unimodal path with endvertices x and y.

The following general remark explains the reason why decomposing a graph into levels (as described above) is a very powerful tool to bound its chromatic number.

Remark 8.4 (Folklore) *Let G be a graph and let z be a vertex of G. There exists an integer k such that $G[S_k(z)]$ has chromatic number at least $\lceil \chi(G)/2 \rceil$.*

PROOF — For any $i \neq j$, there is no edges between a vertex of S_{2i} (resp. of S_{2i+1}) and of S_{2j} (resp. of S_{2j+1}). Indeed adjacent vertices are at distance one, so their level differ by at most one. So,

$$\chi(G) \leq max_{i\,even} \chi(G[S_i]) + max_{j\,odd} \chi(G[S_j])$$

. The result follows.

□

8.3 Graphs that do not contain a cycle with exactly two chords as induced subgraph

Let C be a cycle with exactly two chords $e_1 = a_1 a_2$ and $e_2 = b_1 b_2$. If e_1 and e_2 share an extremity then e_1 and e_2 are *V-chords*. If a_1, a_2, b_1, b_2 are pairwise distinct and appear in the following order along C : a_i, b_j, a_k, b_l with $\{i, k\} = \{j, l\} = \{1, 2\}$, then e_1 and e_2 are *crossing chords*. A 2-cycle with V-chords (resp. crossing chords) is called a *V-cycle* (resp. an *X-cycle*) (see Figure 8.1). A 2-cycle that is not a V-cycle nor an X-cycle is a *parallel cycle*.

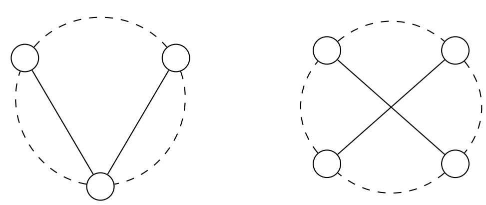


Figure 8.1: A V-cycle and an X-cycle.

The main result of this section is concerned with the class of (X-cycle, V-cycle)-free graphs that of course strictly contains the class of graphs with no 2-cycles.

Theorem 8.5 *Every (X-cycle, V-cycle)-free graph G satisfies $\chi(G) \leq 6$.*

Note that we can bound the chromatic number by a constant here because K_4 is an X-cycle. The proof is built on two steps: Lemmas 8.6 and 8.7.

The first step consists in showing that we can decompose an (X-cycle, V-cycle)-free graph around a complete multipartite graph. More precisely we prove that if a graph is (X-cycle, V-cycle)-free then either it has a clique cutset, or it is a complete tripartite graph, or it is diamond-free. In the second step, we prove that (diamond, V-cycle, X-cycle)-free graphs have chromatic number at most 6 using remark 8.4. This second step is somehow an induction on the number of chords, based on the result of Trotignon and Vušković about $\mathcal{C}_1$ (Lemma 8.3). We finally combine these two lemmas to prove Theorem 8.5.

Note that the first step actually gives us a decomposition theorem for (X-cycle, V-cycle)-free graphs, where the basic classes are complete multipartite graphs and (diamond, X-cycle, V-cycle)-free graphs. Anyway, to get a usable decomposition theorem, one should decompose (diamond, X-cycle, V-cycle)-free graphs that is a too complex class to serve as a basic class.

Lemma 8.6 *If G is an (X-cycle, V-cycle)-free graph, then either G has a clique cutset, or G is isomorphic to a complete tripartite graph, or G is diamond-free.*

PROOF — Assume by way of contradiction that G does not admit a clique cutset, G is not isomorphic to a complete tripartite graph and G contains a diamond. Since G has no clique cutsets, G is 2-connected. Let $H = K_{i,j,k}$ be a maximum (subject to its number of vertices) complete tripartite subgraph of G. Note $A = \{a_1, \dots, a_i\}$ (resp. $B = \{b_1, \dots, b_j\}$, resp. $C = \{c_1, \dots, c_k\}$) the set of the partition of H of cardinality i (resp. j, resp. k). Note that since G contains a diamond, one of the integers i, j, k is strictly greater than 1. Moreover, since G is K_4-free, H is an induced subgraph of G i.e. A, B and C are stable sets.

(1) A vertex $u \notin V(H)$ has at most one neighbor in H.

Assume by way of contradiction that some vertex $u \notin V(H)$ satisfies $d_H(u) \geq 2$. If u has a neighbor in A, B and C, say a_1, b_1 and c_1, then $ua_1b_1c_1$ is a K_4, a contradiction. So we may assume w.l.o.g. that u does not have any neighbor in C. Assume that u has a neighbor in A and a neighbor in B, say a_1 and b_1. By maximality of H, u has at least one non-neighbor in $A \cup B$. Assume w.l.o.g. that a_2 is a non-neighbor of u. Then $ua_1c_1a_2b_1u$ is a V-cycle with chords b_1a_1 and b_1c_1, a contradiction. So we may assume w.l.o.g. that u does not have any neighbor in B and thus have at least two neighbors in A, say a_1 and a_2. Then $ua_1b_1c_1a_2u$ is an X-cycle with chords a_1c_1 and a_2b_2, a contradiction. This proves (1).

Note that $G \neq H$ since otherwise G is a complete tripartite graph. Let K be a connected component of $G \setminus H$. By (1), vertices of K that have a neighbor in H, have a unique neighbor in H. Since G does not contain clique cutsets, $N_H(K)$ must contain two non-adjacent vertices. Therefore, K contains a chordless path $P = p_1 \dots p_k$ such that the neighbors of p_1 and p_k in H are two non-adjacent vertices. Among all such paths, let P be minimal. Assume w.l.o.g. that a_1 and a_2 are the neighbors of respectively p_1 and p_k in H. By minimality of P, no interior vertex of P has a neighbor in A.

If no interior vertex of P is adjacent to a vertex in B or C, then $a_1Pa_2b_1c_1a_1$ is an X-cycle with chords a_1b_1 and a_2c_1, a contradiction. Let i be the smallest integer such that p_i has a neighbor in B or C, say p_i is adjacent to b_1. Then no interior vertices of $p_1 \dots p_i$ is adjacent to any vertices of H and thus $a_1p_1Pp_ib_1a_2c_1a_1$ is a V-cycle with chords b_1a_1 and b_1c_1, a contradiction. □

Lemma 8.7 *If G is a (diamond, X-cycle, V-cycle)-free graph, then for any $z \in V(G)$ and for every integer k, $S_k(z) \in \mathcal{C}_1$.*

PROOF — Let G be a (diamond, X-cycle, V-cycle)-free graph and let $z \in V(G)$. Assume by way of contradiction that there exists an integer k such that $S_k(z)$ contains a 1-cycle C as an induced subgraph. Name a, b the extremities of the unique chord of C. The cycle C is edge-wise partitioned in two ab-path: P^l and P^r (for left and right path, see Figure 8.2).

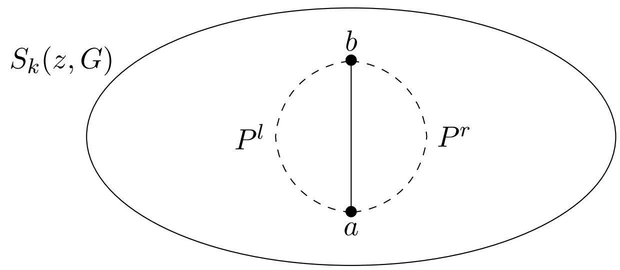


Figure 8.2: In $S_(z, G)$, the cycle C with a unique chord ab and the paths P^r and P^l

Observe that, since G is V-cycle-free, no vertex of G has four neighbors on an induced path. Moreover, no vertex x of G has four neighbors on an induced cycle. Indeed, either two consecutive neighbors are non adjacent, and then x has four neighbors on an induced path, or the cycle is a C_4 and then, G contains a diamond.

(1) For any $x \notin V(C)$, $d_C(x) \leq 3$ and, if x is adjacent to a or b, then $d_C(x) \leq 2$.

Let $x \notin V(C)$.

Suppose first that x is adjacent to a and that $d_C(x) \geq 3$. First assume that xb is an edge. Since $d_C(x) \geq 3$, x has another neighbor x_1 in C. We can assume w.l.o.g. that x_1 is on P^l. If x has no other neighbors on P^l, then $axbP^la$ is an X-cycle with chords xx_1 and ab. So x has another neighbor x_2 on P^l and then it has four neighbors in the chordless cycle aP^lba, a contradiction.

So xb is not an edge. Since $C\backslash\{b\}$ is an induced path, $d_C(x) = 3$. Denote by x_1 and x_2 the two other neighbors of x on C distinct from a. If x_1, x_2 are on P^l, then $aP^lx_1xx_2P^lbP^ra$ (resp. $aP^lx_1xx_2P^lba$) is a V-cycle (resp. an X-cycle) if x_1x_2 is not an edge (resp. is an edge) on a (resp. with chords ax and x_1x_2), a contradiction. Hence, by symmetry, we may assume that $x_1 \in P^l$ and $x_2 \in P^r$. Since G is diamond-free, a is not adjacent to both x_1 and x_2. Assume w.l.o.g. x_1a is not an edge. Thus $x_1xaP^rbP^lx_1$ is an X-cycle with chords xx_2 and ab, a contradiction.

So the second outcome of the claim holds. Now, if x has at least 4 neighbors in C, then x is adjacent neither to a nor to b, and thus it has four neighbors on an chordless path, a contradiction. This proves (1).

(2) Vertices a and b do not have a common father.

Recall that, given two vertices x, y in $S_{k-1}(z)$, Q_{xy} denotes a unimodal path from x to y. And interior vertices of Q_{xy} are not adjacent to any vertex of C.

Assume by way of contradiction that there exists a vertex $x \in S_{k-1}(z)$ that is a common father to a and b. Let c be the neighbor of a on P^r and d be a father of c. By (1), $d \neq x$ and, since G is diamond-free, P^l and P^r have length at least 3 i.e. bc is not an edge.

If d is adjacent to a then $cdQ_{dx}xbac$ is a V-cycle on a. If d is adjacent to b then $cdQ_{dx}bac$ is an X-cycle with chords ax and bd. So d is adjacent neither to a nor to b.

Vertex d has at least one neighbor d_1 on $\mathring{P}^l$, otherwise $cdQ_{dx}bP^lac$ is a V-cycle on a. Moreover, d has a neighbor $d_2 \neq c$ on $\mathring{P}^r$ otherwise $cdQ_{dx}abP^rc$ is an X-cycle with chords ac and xb. By Claim 1, $d_C(x) \leq 3$, so $N_C(d) = \{c, d_1, d_2\}$.

If d_1b is not an edge, then $d_1dcP^rbaP^ld_1$ is an X-cycle with chords ac and dd_2. Otherwise $abxQ_{xd}d_1P^la$ is an X-cycle with chords bd_1 and ax, a contradiction. This proves (2).

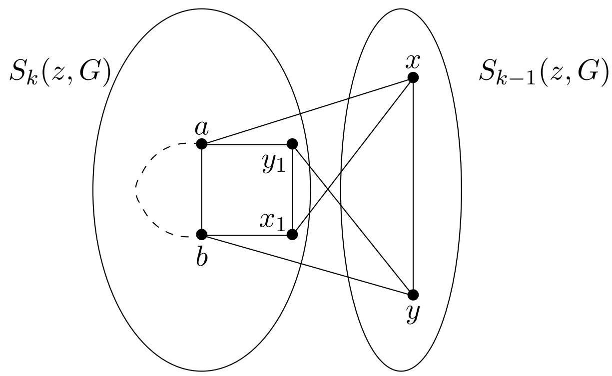


Figure 8.3: The particular graph of Claim 3, note that $P^r = ay_1x_1b$.

(3) Both a and b have a father of degree 1.

Let x and y be respectively a father of a and a father of b. By (2), $x \neq y$. Assume by way of contradiction that say x has a neighbor $x_1 \neq x$ in $\mathring{P}^r$. By (1), $N_C(x) = \{a, x_1\}$. If y has no neighbor in $\mathring{P}^r$, then $axQ_{xy}ybP^ra$ is an X-cycle with chords ab and xx_1, a contradiction. So y has a neighbor, say y_1, in $\mathring{P}^r$ and, by (1), $N_C(y) = \{b, y_1\}$.

Suppose first that $x_1 = y_1$. Since G is diamond-free, bx_1 and ax_1 cannot both be edges. So we may assume w.l.o.g. that bx_1 is not an edge. Then, $aP^rx_1xQ_{xy}yba$ is an X-cycle with chords ax and x_1y, a contradiction. So $x_1 \neq y_1$.

Now, if a, x_1, y_1 appear in this order along P^r, then $axQ_{xy}y_1P^rbP^la$ is a V-cycle on b, a contradiction. So a, y_1, x_1 appear in this order along P^r. If ay_1 is not an edge, then $axQ_{xy}yy_1P^rba$ is an X-cycle with chords xx_1 and by, a contradiction. So ay_1 is an edge and, symmetrically, bx_1 is an edge. If y_1x_1 is not an edge, then $ay_1yQ_{yx}x_1ba$ is an X-cycle with chords ax and by, a contradiction. So x_1y_1 is an edge (i.e. $ax_1y_1x_1b$ is a square). If xy is not an edge, then $abyy_1x_1xa$ is an X-cycle. So xy is an edge (see Figure 8.3).

Let c be the neighbor of a on P_l and let d a father of c. First assume that cb is an edge. Since G is diamond-free, d is adjacent neither to a nor to b. If x_1d is not an edge then $cdQ_{dx}xx_1bac$ is a X-cycle with chords cb and xa, a contradiction. Hence, by symmetry, d is adjacent to both x_1 and y_1 and then $axQ_{xd}dx_1y_1a$ is an X-cycle with chords dy_1 and xx_1.

Hence cb is not an edge. If d is adjacent to both x_1 and y_1, then $axQ_{xd}dx_1y_1a$ is an X-cycle with chords dy_1 and xx_1. If d is adjacent neither to x_1 nor to y_1, $cdQ_{dy}ybx_1y_1ac$ is a X-cycle with chords ab and yy_1. If d is adjacent to x_1 and not to y_1 then $cdQ_{dx}xx_1bac$ is a X-cycle with chords ax and dx_1. If d is adjacent to y_1 and not to x_1 then $cdQ_{dx}xx_1y_1ac$ is a X-cycle with chords yy_1 and xa This proves (3).

By (3), there exist two vertices x and y such that x is a father of a, y is a father of b and $d_C(x) = d_C(y) = 1$. Since G is diamond-free, P^r and P^l cannot be both of length two, so we may assume w.l.o.g. that P^l has length at least 3. Let c be the neighbor of a on P^l and d be a father of

c. Note that $d \neq x$ and $d \neq y$. If $d_C(d) = 1$, then $axQ_{xd}dcP^lbP^ra$ is a V-cycle on a, a contradiction. Hence $d_C(d) \geq 2$.

Assume first d has a neighbor d_1 in P^r. If d_1 is the unique neighbor of d in P^r, then $cdQ_{dy}ybP^rac$ is an X-cycle with chords ab and dd_1 if $a \neq d_1$, or is a V-cycle on a if $d = a$ or $d = b$. So d has a second neighbor d_2 in P^r and thus, by (1), $N_C(d) = \{c, d_1, d_2\}$ and $\{d_1, d_2\} \subseteq \mathring{P}^r$. Assume w.l.o.g. that a, d_1 and d_2 appear in this order along P^r. Now, $axQ_{xd}dd_2P^rbP^la$ is an X-cycle with chords cd and ab, a contradiction. So d has no neighbors in P^r and thus has some neighbors in $\mathring{P}^l$.

If d has exactly one neighbor d_1 in $\mathring{P}^l$, then $axQ_{xd}dcP^lba$ is an X-cycle with chords dd_1 and ac, a contradiction. So d has at least two neighbors in $\mathring{P}^l$ and, by (1), it has exactly two. Put $N_C(d) = \{c, d_1, d_2\}$. Now, $cdQ_{dy}ybP^lc$ is a V-cycle with chords dd_1 and dd_2, a contradiction. □

We are now ready to prove Theorem 8.5, recall that this theorem states that *every (X-cycle, V-cycle)-free graph is 6-colorable.*

PROOF — Assume by way of contradiction that there is some (X-cycle, V-cycle)-free graphs that are not 6-colorable. Let G be minimal with this property. Suppose first that G contains a diamond. Since a complete tripartite graph is 3-colourable, by Lemma 8.6, G admits a clique cutset K. Let C_1 be a connected component of $G \setminus K$, and C_2 the union of all other components of $G \setminus K$. Put $G_1 = G[C_1 \cup K)]$ and $G_2 = G[C_2 \cup K)]$. If G_1 and G_2 are both 6-colourable, then G is 6-colourable, a contradiction. Therefore G_1 or G_2 is not 6-colourable, a contradiction to the minimality of G. So we may assume that G is diamond-free i.e., G is (diamond, X-cycle, V-cycle)-free.

Let z be a vertex of G. Since $\chi(G) = 7$, by Remark 8.4, there is an integer k such that $\chi(S_k(z)) \geq 4$. So, by Theorem 8.3, $S_k(z)$ contains a 1-cycle as an induced subgraph, a contradiction to Lemma 8.7. □

8.4 Graphs that do not contain a cycle with exactly three chords as induced subgraph

The aim of this section is to prove that $\mathcal{C}_3$ is χ-bounded (see Theorem 8.10).

The proof is divided into three parts, according to the clique number. First of all, we prove that every (triangle, 3-cycle)-free graph has chromatic number at most 24. Below, the constant 24 is denoted by c.

For graphs with clique number exactly 3, we prove that the chromatic number is at most $4c$. When the clique number is at least 4, then the chromatic number is close to the clique number. We prove that asymptotically the difference between them is at most one.

Let us state now the exacts statements of these 3 theorems. They are prove in Subsections 8.4.1, 8.4.2 and 8.4.3 respectively.

Theorem 8.8 *A (triangle, 3-cycle)-free graph has chromatic number at most c.*

Theorem 8.9 *A (K_4, 3-cycle)-free graph has chromatic number at most* $4c$.

Theorem 8.10 *A (3-cycle)-free graph has chromatic number at most* $\max(4c, \omega(G) + 1)$.

Note that Theorem 8.10 says that, if a 3-cycle-free graph has a large enough clique (of size at least 96), then $\chi(G) \leq \omega(G) + 1$. Moreover the Hajós join of two cliques shows this bound is tight. Let us recall what the Hajós join of two cliques is and prove it is 3-cycle-free.

Here is how to build the Hajós join of two K_k. Take two disjoint copies H_1 and H_2 of K_{k-1}, add a vertex x complete to H_1 and H_2 and two adjacent vertices a and b, such that a is complete to H_1 and b is complete to H_2. The obtained graph is the Hajós join of K_k and K_k and it is easy to check that it has clique number k and chromatic number $k+1$. Now, let us show it is 3-cycle-free. An ax-path with interior vertices in H_1 is either chordless or has at least two chords. Similarly, a bx-path with interior vertices in H_2 is either chordless or has at least two chords. So a cycle going through both H_1 and H_2 is either chordless, or has exactly two chords, or has more than four chords. Hence the graph is 3-cycle-free.

8.4.1 Clique number 2: proof of Theorem 8.8

Recall that Theorem 8.8 states that *a (triangle, 3-cycle)-free graph has chromatic number at most* $c = 24$.

To prove this result, we need the two following lemmas.

Lemma 8.11 *Let G be a (triangle, 3-cycle)-free graph. For every $z \in V(G)$ and every integer k, $S_k(z)$ is (V-cycle, triangle, 3-cycle)-free.*

Lemma 8.12 *Let G be a (V-cycle, triangle, 3-cycle)-free graph. For every $z \in V(G)$ and every integer k, $S_k(z)$ is (X-cycle, V-cycle, triangle, 3-cycle)-free.*

Before we prove these two lemmas, let us explain how they imply Theorem 8.8. Suppose there exists a (triangle, 3-cycle)-free graph G with $\chi(G) \geq 25$. Let z be a vertex of G. By Remark 8.4, there exists an integer k such that $\chi(G[S_k(z,G)]) \geq 13$. Put $H = G[S_k(z,G)]$. By Lemma 8.11, H is (V-cycle, triangle, 3-cycle)-free.

Let x be a vertex of H. By Remark 8.4, there exists an integer ℓ such $\chi(G[S_\ell(x,H)]) \geq 7$. So, by Theorem 8.5, $G[S_\ell(x,H)]$ contains an X-cycle as an induced subgraph (it cannot contain a V-cycle since it is an induced subgraph of H that is V-cycle-free) which contradicts 8.12.

Proof of Lemma 8.11

Recall that Lemma 8.11 states that, if G is a (triangle, 3-cycle)-free graph and z is a vertex of G, then for every integer k, $S_k(z,G)$ is (V-cycle, triangle, 3-cycle)-free.

PROOF — Let G be a (triangle, 3-cycle)-free graph and z a vertex of G. Assume by way of contradiction that there exists an integer k such that $S_k(z,G)$ contains an induced V-cycle C.

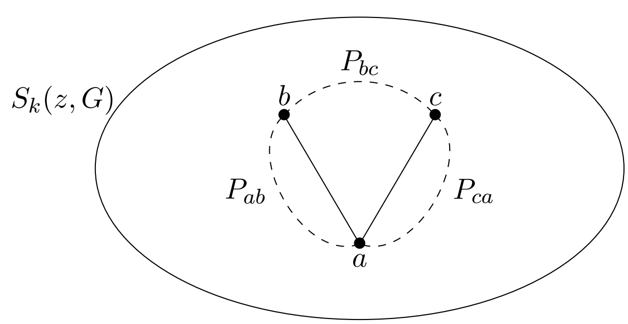


Figure 8.4: The V-cycle C in $S_k(z,G)$.

Let a be the unique vertex of C of degree 4 and let b, c be the two vertices of C of degree 3. Vertices a, b and c are called the *important* vertices of C. We denote by P_{ab} the path from a to b contained in $C\backslash\{ab, ac\}$ that avoids c. Paths P_{bc} and P_{ca} are defined similarly (see Figure 8.4). P_{ab}, P_{bc} and P_{ca} are called the *intervals* of C. By abuse of notation, P_{ab} will sometimes denote $V(P_{ab})$. Also, P_{ab} can be referred to as P_{ba} (and analogously for P_{bc} and P_{ca}). Moreover, if a vertex x is in $V(P_{ab})$, then the path $xP_{ab}b$ (resp. $xP_{ab}a$) can be referred to as P_{xb} or P_{bx} (resp. P_{xa} or P_{ax}). Given two vertices x and y in the same interval, the *external path from x to y* consists in the path from x to y in $C \setminus \{ab, ac\}$ passing through a, b and c. If x_1 and x_2 are two vertices of a path P that have a common father u, we say that x_1 and x_2 are *consecutive neighbors of u along P* if u has no neighbors in $\mathring{x}_1P\mathring{x}_2$.

Note that adjacent vertices of C cannot have a common father otherwise G would contain a triangle. Also note that if a vertex has 5 neighbors on an induced path, there is a 3-cycle.

The proof consists in studying how a vertex not in C can attach on C and then using unimodal paths to get contradictions.

(1) If a vertex $u \notin V(C)$ satisfies $d_C(u) \geq 3$ and all but at most one neighbors of u are contained in an interval of C, then G contains a 3-cycle.

Let u be a vertex not in $V(C)$ such that: $d_C(u) \geq 3$ and all but at most one neighbor of u is contained in an interval of C. So, there exists two vertices u_1 and u_2 in $N_C(u)$ such that u_1 and u_2 are in the same interval of C and u has exactly one neighbor u_3 on the external path P from u_1 to u_2. Then $u_2uu_1Pu_2$ is a 3-cycle with chords ab, ac and uu_3 (note that, since G is triangle-free, u_1u_2 is not an edge). This proves (1).

(2) If $u \notin V(C)$ and $d_C(u) = 3$, then u has exactly one neighbor in each interval i.e. it has one neighbor in each of $\mathring{P}_{ab}$, $\mathring{P}_{bc}$ and $\mathring{P}_{ac}$.

This is immediate by (1). This proves (2).

(3) If $u \notin V(C)$, then u has at most 3 neighbors on $aP_{ab}bP_{bc}c$.

Since G is triangle-free, either a or b is not a neighbor of u. If u has at least 5 neighbors in $aP_{ab}bP_{bc}c$, then u has 5 neighbors on one of the chordless paths $\mathring{a}P_{ab}bP_{bc}c$ or $aP_{ab}bP_{bc}\mathring{c}$ which provides a 3-cycle, a contradiction.

So we may assume that u has exactly 4 neighbors in $aP_{ab}bP_{bc}c$ and w.l.o.g. that u has at least two neighbors in P_{ab}. Let u_1 and u_2 be two consecutive neighbors of u along P_{ab} such that a, u_1, u_2 appear in this order along P_{ab}. Then $u_1uu_2P_{u_2b}bP_{bc}caP_{au_1}u_1$ is a 3-cycle (recall that u_1u_2 is not an edge since G is triangle-free). This proves (3).

Note that by symmetry (3) also holds for $bP_{bc}cP_{ca}a$.

The next claim states the only way a vertex can have at least four neighbors in C.

(4) If $u \notin V(C)$ and $d_C(u) \geq 4$, then $d_C(u) = 4$ and $N_C(u) = \{b, c, y_1, y_2\}$ where y_1 is the neighbor of a in P_{ab} and y_2 is the neighbor of a in P_{ca}.

Let $u \notin V(C)$ and suppose $d_C(u) \geq 4$.

If u has at least three neighbors in P_{bc}, then it has at least four neighbors either in $P_{ab}P_{bc}$ or in $P_{bc}P_{ac}$, which contradicts (4). So u has at most two neighbor in P_{bc}.

Case 1 : u has exactly two neighbors, u_1, u_2 say, on P_{bc}.

Assume w.l.o.g. that b, u_1, u_2, c appear in this order along P_{bc}. By (3), u has at most one neighbor on $aP_{ab}\mathring{b}$ and at most one neighbor on $\mathring{c}P_{ca}a$. Since $d_C(u) \geq 4$, both neighbors exists. Moreover both are distinct from a otherwise there would be 4 neighbors on $P_{ab}P_{bc}$ or $P_{bc}P_{ca}$, contradicting (3). Denote by y_1 (resp. y_2) the neighbor of u in $P_{ab}\mathring{b}$ (resp. in $\mathring{c}P_{ca}$).

If $u_1 \neq b$ then $y_1uu_1P_{u_1c}cP_{ca}P_{ay_1}y_1$ has 3 chords, namely uy_2, uu_2 and ac. So $u_1 = b$ and, by symmetry, $u_2 = c$. If y_2a is not an edge then $y_2uu_1P_{u_1a}acP_{cy_2}y_2$ is a 3-cycle with chords ab, uy_1 and uc. So ay_2, and by symmetry ay_1, are edges, so the outcome holds.

Case 2 : u has exactly one neighbor, u_3 say on P_{bc}.
Since $d_C(u) \geq 4$, u has at least 3 neighbors on $\mathring{b}P_{ba}P_{ac}\mathring{c}$. W.l.o.g. u has at least two neighbors in $P_{ab}\mathring{b}$. By (3), u has exactly two neighbors, u_1, u_2 say, in $P_{ab}\mathring{b}$. Assume that a, u_1, u_2 appear in this order along $P_{ab}\mathring{b}$. Let u_4 be the neighbor of u that is closest from a in $\mathring{P}_{ca}$ (u_4 exists since $d_C(u) \geq 4$). If $u_3 \neq c$ then $u_4uu_3P_{u_3b}bP_{ba}P_{ac}u_4$ is a 3-cycle with chords ab, uu_1 and uu_2. So $u_3 = c$.

Note that a is not a neighbor of u otherwise a, c, u would be a triangle. So $N_C(u) = \{u_1, u_2, c, u_4\}$, otherwise u would have 5 neighbors on the chordless path $V(C) \setminus \{a\}$, i.e. there would be 3-cycle. Hence $u_2ucP_{ca}P_{au_2}u_2$ is a 3-cycle with chords uu_1, ac and uu_4, a contradiction.

Case 3 : u no neighbor on P_{bc}.
Since $d_C(u) \geq 4$, we may assume w.l.o.g. that u has at least two neighbors, u_1, u_2 say, on $P_{ab}\mathring{b}$. By (1), u has at least two other neighbors u_3, u_4 on $\mathring{c}P_{ca}$. Moreover u has no other neighbors in C since otherwise u would have 5 neighbors on the chordless path $V(C) \setminus P_{bc}$. By (1), u_1, u_2, u_3, u_4 are distinct from a. Assume w.l.o.g. that u_2, u_1, a, u_3, u_4 appear in this order along $\mathring{b}P_{ba}P_{ac}\mathring{c}$.

If au_3 is an edge then $u_3uu_2P_{u_2a}acP_{ca}$ is a 3-cycle with chords au_3, uu_4 and uu_1. So we may assume au_3 is not an edge. If u_2b is an edge then $u_2uu_4P_{u_4c}cP_{cb}baP_{au}u_2$ is a 3-ycle with chords uu_1, ac and u_2b. So u_2b is not an edge and hence, $u_2uu_3P_{u_3c}cP_{cb}baP_{au_2}u_2$ is a 3-cycle with chords uu_1, uu_4 and ac, a contradiction.

This proves (4).

(5) Let y be a father of a vertex of C. Then $d_C(y) \leq 3$.

Let y_1 be the neighbor of a in P_{ab} and y_2 be the neighbor of a in P_{ca}. Let y be the father of a vertex in C. Suppose for contradiction that $d_C(y) \geq 4$ By (4), $N_C(y) = \{b, c, y_1, y_2\}$.

Let e be the neighbor of b on P_{ab}. Note that $e \neq y_1$ since otherwise aby_1 is a triangle. Let f be a father of e. By (4), $d_C(f) \leq 3$. If f has no neighbor in $P_{ac} \cup P_{ab} \setminus \{e, b\}$, then $efQ_{fy}ycP_{ca}P_{ab}e$ is a 3-cycle with chords yy_1, yy_2 and ac. So f has at least one neighbor in $P_{ac} \cup P_{ab} \setminus \{e, b\}$.

Assume that f has at least one neighbor f_1 in $P_{ab} \setminus \{e, b\}$. Then it is the only one, otherwise $d_C(f) = 3$ and all neighbors of f are in the same interval contradicting (2). Then $efQ_{fy}baP_{ae}e$ is a 3-cycle with chords eb, yy_1 and ff_1. So f has no neighbors in $P_{ab} \setminus \{e, b\}$.

Hence f has at least one neighbor f_1 in $P_{ca} \setminus \{a\}$ and it is the only one by (2). If f has no neighbor on $\mathring{P}_{bc}$ then $efQ_{fy}yy_2P_{ac}P_{ce}e$ has chords yb, yc and ff_1. So, f has at least one neighbor f_2 in $\mathring{P}_{bc}$, and it is the only one by (2).

If fy is an edge then $f_2feP_{ey_1}y_1yy_2P_{y_2c}P_{cf_2}$ is a 3-cycle with chords fy, ff_1 and yc. Otherwise $eff_2P_{f_2b}byy_2aP_{ae}$ is a 3-cycle with chords, eb, ab and yy_1. This proves (5).

(6) If x is a father of an important vertex of C, then $d_C(x) \leq 2$.

Let x be a father of an important vertex of C. By (5), $d_C(x) \leq 3$. By (2), the father of an important vertex cannot have exactly three neighbors in C. So $d_C(x) \leq 2$. This proves (6).

(7) Let e be the neighbor of a on P_{ab} and let f be a father of e. Then $d_C(f) \leq 2$.

Assume for contradiction that $d_C(f) = 3$. By (5), f has a exactly one neighbor, f_1 say, in $\mathring{P}_{bc}$ and exactly one, f_2 say, in $\mathring{P}_{ca}$. All of them are distinct from important vertices since fathers of important vertices have at most two neighbors on C.

Let x be a father of a. If x has no neighbor on $\mathring{a}P_{ab}P_{bc}$, then $efQ_{fx}acP_{cb}P_{be}e$ is a 3-cycle with chords ab, ae and ff_1. So x has a neighbor, x_1 say in $\mathring{a}P_{ab}P_{bc}$ and, by (6), $N_C(x) = \{a, x_1\}$. If $x_1 \in \mathring{P}_{ab}P_{bf_1}$, then $axQ_{xf}f_1P_{f_1b}P_{ba}a$ is a 3-cycle with chords ab, ef and xx_1. So $x_1 \in f_1P_{f_1c}c$ and then $axQ_{xf}ff_1P_{f_1c}P_{ca}a$ is a 3-cycle with chords ac, ff_2 and xx_1 a contradiction. This proves (7).

(8) Let x be a father of a. Then $d_C(x) = 2$.

By (6), $d_C(x) \leq 2$. So we may assume by way of contradiction that $d_C(x) = 1$. Let e be the neighbor of a on P_{ab} and let f be a father of e. Finally let y be a father of b.

If $d_C(f) = 1$, then $axP_{xf}eP_{eb}P_{bc}P_{ca}a$ is a 3-cycle with chords ae, ab, ac. So $d_C(f) \geq 2$ and thus, by (7), $d_C(f) = 2$. If the second neighbor f_1 of f is on $P_{ab}P_{bc}$, then $axQ_{xf}eP_{eb}P_{bc}ca$ is a 3-cycle with chords ae, ab and ff_1. So $f_1 \in \mathring{P}_{ca}$.

Note that eb is not an edge since otherwise aeb is a triangle. If y has a neighbor in $\mathring{b}P_{bc}P_{ca}e$, then $byQ_{yf}feaP_{ac}P_{cb}b$ is a 3-cycle with chords ab, ac and ff_1. So y has a neighbor, say y_1, in $\mathring{b}P_{bc}P_{ca}e$ and by (6), $N_C(y) = \{b, y_1\}$. If $y_1 \neq e$, then $axQ_{xy}bP_{bc}P_{ca}$ is a 3-cycle with chords ab, ac and yy_1. Hence $y_1 = e$ which contradicts (7). This proves (8).

We now have proved enough claims to finish the proof. Let x be a father of a. By (8), $d_C(x) = 2$. Let x_1 be the neighbor of x distinct from a on C. Let e be the neighbor of a on P_{ab} and let f be a father of e. Finally let y be a father of b. By symmetry we may assume that $x_1 \in P_{bc}P_{ca}\mathring{a}$.

If $d_C(y) = 1$, then $axQ_{xy}bP_{bc}P_{ca}$ would be 3-cycle with chords ab, ac and xx_1. So $d_C(y) \geq 2$ and by (6), $d_C(y) = 2$. Let y_1 be the neighbor of y distinct from b on C.

Assume that both x_1, y_1 are on P_{ca}. If a, y_1, x_1 appears in this order along P_{ac} then $x_1P_{x_1a}P_{ab}yQ_{yx}x_1$ is a 3-cycle with chords ax, ab and yy_1 ($x_1 \neq c$ since otherwise acx is a triangle). So a, x_1, y_1 appear in this order along P_{ac} and $x_1 \neq y_1$. In particular y_1a is not an edge and so $axQ_{xy}yy_1P_{y_1c}P_{cb}P_{ba}a$ is a 3-cycle with chords ab, ac and yb. Moreover, if both x_1, y_1 are on P_{bc} then $axQ_{xy}bP_{bc}a$ is a 3-cycle with chords ab, xx_1 and yy_1. So, either $x_1 \in \mathring{P}_{bc}$ and $y_1 \in \mathring{P}_{ca}$ or $x_1 \in \mathring{P}_{ca}$ and $y_1 \in \mathring{P}_{bc}$.

If $x_1 \in \mathring{P}_{bc}$ and $y_1 \in \mathring{P}_{ca}$ and then $x_1xQ_{xy}y_1P_{y_1a}P_{ab}P_{bx_1}$ is a 3-cycle with chords ax, by and ab. Thus $x_1 \in \mathring{P}_{ca}$ and $y_1 \in \mathring{P}_{bc}$ and then $x_1xQ_{xy}y_1P_{y_1b}P_{ba}P_{bx_1}$ is a 3-cycle with chords ax, by and ab, a contradiction that put an end to the proof. □

Proof of Lemma 8.12

Recall that Lemma 8.12 states that, if G is a (triangle, 3-cycle, V-cycle)-free graph and z is a vertex of G, then for every integer k, $S_k(z, G)$ is X-cycle-free.

PROOF — Let G be a (triangle, 3-cycle, V-cycle)-free graph, z a vertex of G and suppose for contradiction that there exists an integer k such that $S_k(z, G)$ contain an X-cycle C as an induced subgraph.

Let ac and bd be the two chords of C and assume that a, b, c, d, appear in this order along C. Vertices a, b, c, and d are called *important vertices* of C. Two important vertices that do not form a chord of C are said to be *consecutive*. An *interval* is an induced path on $C \setminus \{ac, bd\}$ between two

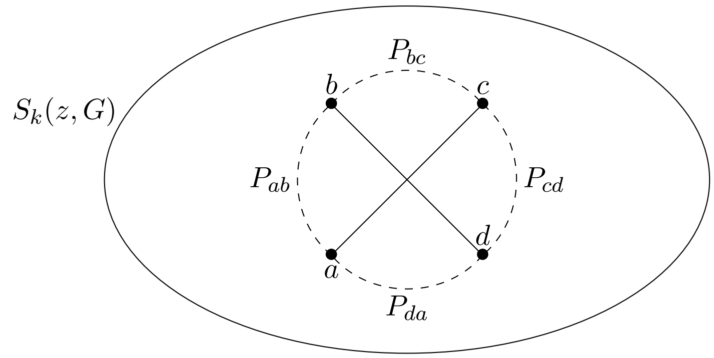


Figure 8.5: The X-cycle C in $S_k(z,G)$.

consecutive important vertices. They are denoted by P_{ab}, P_{bc}, P_{cd} and P_{da} (see Fig. 8.4.1). Note that two intervals share at most one vertex. By abuse of notation, P_{ab} will sometimes denotes $V(P_{ab})$. Also, P_{ab} can ve referred to as P_{ba} (and analogously for P_{cb}, P_{dc} and P_{da}). Given two vertices x and y in the same interval, the *external path from x to y* consists in the path from x to y in $C \setminus \{ac, bd\}$ passing through a, b, c and d.

The proof is divided in two parts. First we prove that no neighbor of the graph has degree larger than 3 on C. We then study more specifically fathers of important vertices and prove that they are neither of degree 3, nor 2 nor 1.

(1) If a vertex $u \notin V(C)$ satisfies $d_C(u) \geq 3$ and all but at most one neighbors of u are contained in an interval of C, then G contains a 3-cycle.

Let u be a vertex not in $V(C)$ such that: $d_C(u) \geq 3$ and all but at most one neighbors of u are contained in an interval of C. So, there exists two vertices u_1 and u_2 in $N_C(u)$ such that u_1 and u_2 are in the same interval of C and u has exactly one neighbor u_3 on the external path P from u_1 to u_2. Then $u_2uu_1Pu_2$ is a 3-cycle with chords ab, ac and uu_3 (note that, since G is triangle-free, u_1u_2 is not an edge). This proves (1).

(2) Every vertex of $u \notin V(C)$, satisfies $d_C(u) \leq 3$. Moreover, if $d_C(u) = 3$, then no interval contains at least two neighbors of u.

Let us first prove a fact. If a vertex v has (at least) 4 neighbors on a path P that has at most one chord, then G contains a V-cycle or a 3-cycle. Indeed let v_1, v_2, v_3, v_4 be consecutive neighbors of v on P. Then $C' = vv_1Pv_4v$ has chords vv_2 and vv_3. If v_1Pv_4 is chordless then C' is a V-cycle otherwise C' is a 3-cycle.

Assume that a vertex $u \notin C$ satisfies $d_C(u) \geq 4$. Since G is triangle free, u is not adjacent to both a and c. By symmetry we can assume that u is not adjacent to a, then u has 4 neighbors on the path $\mathring{a}P_{ab}P_{bc}P_{cd}P_{da}\mathring{a}$ with at most one chord, contradicting the fact proved above.

If $d_C(u) = 3$, then (1) ensures that the neighbors of u are in distinct intervals. This proves (2).

So any vertex $x \notin V(C)$ satisfying $d_C(x) = 3$ is adjacent to at most one important vertex. Indeed two important vertices either are opposite (which would create a triangle), or are in a same interval (which would contradict (2)).

(3) Two adjacent vertices of C do not both have a father of degree one on C.

Let u and v be two adjacent vertices of C. Suppose for contradiction that u (resp. v) admits a father u' (resp. v') such that $d_C(u') = 1$ (resp. $d_C(v') = 1$). We denote by P the external path from v to u. Then $uu'Q_{u'v'}vuP$ is a 3-cycle with chords, uv, ac and bd. This proves (3).

(4) Let x be a father of an important vertex. Then $d_C(x) \leq 2$.

Assume by contradiction that a father x of a satisfies $d_C(x) = 3$. By (2), x has exactly one neighbor, x_1 say, on $\mathring{P}_{bc}$, and exactly one neighbor, x_2 say, on $\mathring{P}_{cd}$. Indeed a is in both P_{ab} and P_{da} and (2) ensures that there is no two neighbors of x in the same interval. Note that it implies that neither bc nor cd are edges.

First assume that ab is an edge. Let y be a father of c. If $d_C(y) = 1$, then $axQ_{xy}cP_{cb}dP_{da}$ is a 3-cycle with chords ab, ac and xx_1. So $d_C(y) \geq 2$. If $d_C(y) \geq 3$, then by (2), $d_C(y) = 3$ and y must have a vertex in $\mathring{P}_{ab}$ which is impossible since we assumed that ab is an edge. So $d_C(y) = 2$. If y_1 is on $P_{ab}P_{bc}$, then $axQ_{xy}cP_{cb}P_{ba}a$ is a 3-cycle with chords ac, xx_1 and yy_1, and if y_1 is on $P_{ad}P_{dc}$, then $axQ_{xy}cP_{cd}P_{da}a$ is a 3-cycle with chords ac, xx_2 and yy_1, a contradiction. So in the following we assume that ab, and by symmetry ad, are not edges.

If bx_1 is an edge then $axx_1P_{x_1c}P_{cd}bP_{ba}a$ is a 3-cycle with chords bx_1, xx_2 and ac. So bx_1, and by symmetry dx_2, are not edges.

Let e be the neighbor of a on P_{ab} and f be a father of e. The cycle $C' = efQ_{fx}x_2P_{x_2c}aP_{ad}bP_{be}e$ has chords xa and xe so it must admit other chords otherwise it is a V-cycle. We already showed that dx_2 nor bc are edges and that the only neighbors of x in C are a, x_1 and x_2. So others chords are due to neighbors of f. Moreover, f must have at least two neighbors that create chords in C', otherwise C' would be a 3-cycle. So, by (2), f has one neighbor on $P_{cd}\mathring{d}$ and one neighbor on $\mathring{d}P_{da}$. Let f_1 be the neighbor of f in $P_{cd}\mathring{d}$. Then $axQ_{xf}eP_{eb}dP_{dc}a$ is a 3-cycle with chords ae, xx_2 and ff_1 (note that ad is not an edge since f has a neighbor on $\mathring{P}_a$). This proves (4).

(5) If ab is an edge, fathers of a have degree exactly two.

Assume by contradiction that a father x of a satisfies $d_C(x) \neq 2$. So, by (4), $d_C(x) = 1$. Let z be a father of c. The cycle $C' = axQ_{xz}cP_{cb}dP_{da}$ has chords ac, ab which, with no additional chords, provides a V-cycle. By (4), $d_C(z) \leq 2$, so C' has at most one other chord due to neighbors of z. Moreover, if cd is an edge, it is a chord of C'. Since C' cannot be a V-cycle nor a 3-cycle, cd is an edge and z has another neighbor z_1 on C'. Since both ab and cd are edges, $z_1 \in P_{bc}$ or in $z_1 \in P_{da}$.

Assume first that $z_1 \in P_{bc}$. If $z_1 \neq b$, then $axQ_{xz}z_1P_{z_1c}dP_{da}$ is a V-cycle with chords zc and ac. So $z_1 = b$. Let e be the neighbor of a in P_{ad} (note that $e \neq d$ otherwise abd is a triangle) and let f be a father of e. Since $d_C(x) = 1$, $d_C(f) \geq 2$ by (3). Such a neighbor, called f_1, is unique, otherwise, since ab and cd are edges, at least two neighbors of f would be in the same interval, contradicting (2). Note that $f_1 \neq a$ otherwise aef_1 is a triangle. Then $efQ_{fz}bP_{bc}dP_{de}$ is a 3-cycle with chords ff_1, zc and bd. So $z_1 \notin P_{bc}$.

Thus, $z_1 \in P_{ad}$. Note that $z_1 \neq a$ since otherwise acz is a triangle. If az_1 is not an edge then $axQ_{xz}z_1P_{z_1d}cP_{cb}a$ is a 3-cycle with chords ac, bd and cz. So az_1 is an edge. Let y be a father of b. We have $d_C(y) \leq 2$ by (4). Then $d_C(y) = 2$ by (3) since $d_C(x) = 1$ and ab is an edge. Moreover, ay is not an edge otherwise aby is a triangle. So y has a neighbor y_1 in $\mathring{b}P_{bc}dP_{dz_1}$. Therefore $byQ_{yz}z_1P_{z_1d}cP_{cb}$ is a 3-cycle with chords zc, bd and yy_1. This proves (5).

(6) If an important vertex has a father of degree one on C, then every father of every important vertex has degree one on C.

Assume w.l.o.g. that a father x of a satisfies $d_C(x) = 1$. We show that it implies that every father of b are of degree one in C which, by symmetry, prove the claim.

Let y be father of b and assume for contradiction that $d_C(y) \neq 1$. Note that by (5), neither ab nor ad are edges. By (4), $d_C(y) = 2$. Let y_1 be the neighbor of y on C distinct from b. Let b_1 be the element of $\{y_1, b\}$ that is nearest from a in P_{ab} and which is distinct from a and let b_2 the other one. Such a vertex exists since b satisfies the conditions. If ab_1 is not an edge, then $axQ_{xy}b_1P_{b_1b}P_{bc}P_{cd}P_{da}$ has 3 chords ac, bd and yb_2. So we may assume that ab_1 is an edge, since ab is not an edge, $b_1 = y_1$.

Let z be a father of d. The cycle $C' = dzQ_{zy}y_1P_{y_1b}P_{bc}P_{cd}$ is, with no additionnal chord, a V-cycle with chords yb, bd. Since $d_C(z) \leq 2$ by (4), there is at most one chord with extremity z, which provides a 3-cycle. This proves (6).

(7) Fathers of important vertices have degree exactly two on C.

Let us prove it by contradiction. By (6), we can assume that all fathers of all important vertices have degree one on C. And by (5), none of ab, bc, cd and da are edges. Let u be a neighbor of a in P_{ab}. Let x be a father of a and y be a father of u. By (3), $d_C(y) \neq 1$ since $d_C(x) = 1$.

By (2), $d_C(y) \leq 3$. If $d_C(y) = 3$ then, by (2) and (6), the neighbors of y are in the interior of distinct intervals. Assume that y has a neighbor in $\mathring{P}_{cd}$ and in $\mathring{P}_{da}$. Let y_1 be the neighbor of y on $\mathring{P}_{cd}$. By (6), a father y' of b satisfies $d_C(y') = 1$. Hence $by'Q_{y'y}y_1P_{y_1d}P_{da}P_{ab}$ has 3 chords: bd and two chords with extremity y. It is easy to see that, since fathers of every important vertex are of degree one in C, we get a contradiction when $d_C(y) = 3$. So $d_C(y) = 2$.

Let u' be the other neighbor of a and let z be a father of u'. Assume first that u' has a father z distinct from y. By symmetry with y, $d_C(z) = 2$. So $u'zQ_{zy}uP_{ub}P_{bc}P_{cd}P_{du'}u'$ has 3 chords: two chords are given by the other neighbors of y and z and the third one is bd. So y is adjacent to u'.

Let w be the neighbor of c in P_{dc} and w' the neighbor of c on P_{cb}. For symmetric reason why y is a father of both u and u', there exists a vertex f that is the father of both w and w'. and that is of degree 2 in C. Then $wfQ_{fy}u'aP_{ab}P_{bc}w$ is a 3-cycle with chords fw', yu and ac (recall that both v and w are distinct from d since none of ad, cd are edges). This proves (7).

We are now armed to finish the proof! By (7), we may assume that fathers of every important vertex have exactly degree 2 on C. Let x and y be some fathers of a and c respectively. Since G is triangle-free, $x \neq y$. Let us denote by x_1 and y_1 the other neighbors of respectively x and y. If x_1 and y_1 are on $P_{ab}P_{bc}$, then $axQ_{xy}cP_{cb}P_{ba}$ is a 3-cycle with chords ac, xx_1 and yy_1, a contradiction. So x_1 and y_1 cannot both be on $P_{ab}P_{bc}$ and, symmetrically, they cannot be on $P_{cd}P_{da}$.

So, we may assume w.l.o.g. that x_1 is on $P_{ab}P_{bc}$ and that y_1 is on $P_{cd}P_{da}$. If $x_1 \in \mathring{P}_{ab}$ then $x_1xQ_{xy}cP_{cd}P_{da}P_{ax_1}$ is a 3-cycle with chords ac, ax and yy_1. Thus x_1 is on P_{bc} and by symmetry y_1 is on P_{da}. More generally, we showed that no father of an important vertex has its second neighbor on the interior of and interval adjacent containing it. If ab and cd are both not edges, then $axP_{xy}cP_{cb}dP_{da}$ is a 3-cycle with chords ac, xx_1, yy_1. So either ab is an edge, or cd is an edge, or both are edges.

Assume w.l.o.g. that ab is an edge. A father w of d has it second neighbor w_1 neither in P_{cd} nor in P_{da} since no father of an important vertex has its second neighbor on the interior of intervals adjacent to it. Since ab is an edge, $w_1 \in P_{bc}$. Now, a father z of b has a unique second neighbor z_1 on $\mathring{P}_{ad}$ (by applying the first part of the proof on b, d instead of a, c). If a, y_1, z_1 appears in this order along P_{ad} then $bzQ_{zy}y_1P_{y_1d}P_{dc}P_{cb}$ is a 3-cycle with chords bd, zz_1 and yc. So a, z_1, y_1 appear in this order along P_{ad} and, in particular, z_1d is not an edge. Symmetrically, b, x_1, w_1 appear in this order along P_{cb} and cx_1 is not an edge. Finnaly $x_1xQ_{xz}z_1P_{z_1a}cP_{cd}bP_{bx_1}$ is a 3-cycle with chords ab, xa and zc, a contradiction. □

8.4.2 Clique number 3 : proof of Theorem 8.9

The proof of Theorem 8.9 is organized as follows. First of all, we prove (see Lemma 8.13) that any (K_4, 3-cycle)-free graph with chromatic number at least $2c$ contains either an butterfly as an induced subgraph or a dragonfly as an induced subgraph (see Figure 8.6). Note that the proof is based on Theorem 8.8.

We then prove that if a graph G is (K_4, 3-cycle)-free and x is a vertex of G, then for any integer k, $S_k(z, G)$ is (dragonfly,butterfly)-free (see Lemmas 8.14 and 8.15).

At the very end, we combine these two result to get the proof of Lemma 8.9.

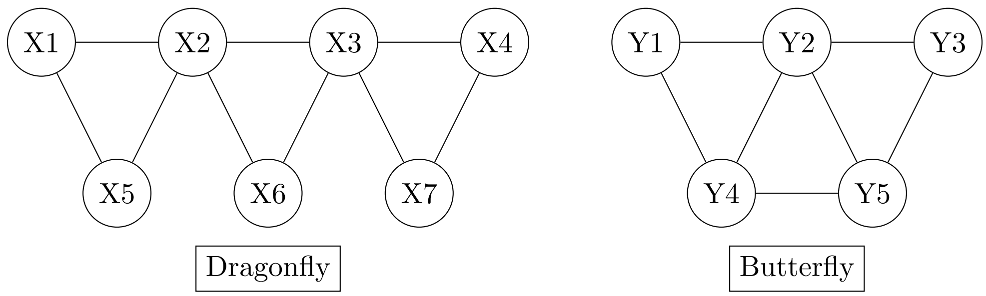


Figure 8.6: The dragonfly and the butterfly

Lemma 8.13 *Let G be a (K_4, 3-cycle)-free graph with $\chi(G) > 2c$. Then G contains a dragonfly or a butterfly as an induced subgraph.*

PROOF — All along the proof, the notations of the vertices of dragonfly and butterfly will fit with notations of Figure 8.6. We first prove that G admits a dragonfly or a butterfly as a subgraph. We then prove that it is induced.

(1) G admits a dragonfly as a subgraph.

Let $T \subseteq V(G)$ be a minimal (by inclusion) subset of vertices such that $G \setminus T$ is triangle-free. By Theorem 8.8, $G \setminus T$ is c-colorable. If $G[T]$ is triangle-free, then $G[T]$ is c-colorable and thus G is $2c$-colorable, a contradiction. Thus, we may assume that $G[T]$ admits a triangle $x_2x_3x_6$. By minimality of T, $(G\setminus T)\cup\{x_2\}$ admits a triangle containing x_2, say $x_1x_2x_5$. Similarly, $(G\setminus T)\cup\{x_3\}$ contains a triangle containing b, say $x_3x_4x_7$.

If $\{x_1, x_5\} = \{x_4, x_7\}$, then $x_1x_2x_3x_5 = K_4$, a contradiction. So $\{x_1, x_5\} \neq \{x_4, x_7\}$.

Assume now that $|\{x_1, x_5\} \cap \{x_4, x_7\}| = 1$ and, w.l.o.g., assume that $x_1 = x_7$ (see Figure 8.4.2). The cycle $C = x_1x_5x_2x_6x_3x_4x_1$ is, if no additional chords exist, a 3-cycle with chords x_1x_2, x_1x_3 and x_2x_3. So C must have at least one more chord. Since G is K_4-free, x_1x_6, x_2x_4 and x_3x_5 are not edges. There remains only 3 possible chords, namely x_4x_5, x_5x_6 and x_4x_6. If say $x_4x_5 \in E(G)$, then $x_1x_2x_5x_4x_3x_1$ is a 3-cycle, a contradiction. So x_4x_5 is not an edge and, by symmetry, x_5x_6 and x_4x_6 are not edges.

So, $\{x_1, x_5\} \cap \{x_4, x_7\} = \emptyset$ and thus $G[\{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}]$ contains a dragonfly as a subgraph. This proves (1).

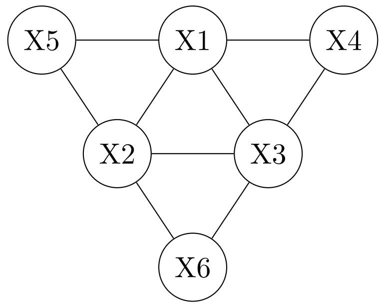


Figure 8.7: Figure in the proof of Claim (1).

(2) G contains either a dragonfly or a butterfly as an induced subgraph.

Observe first that, if G contains a butterfly (see Figure 8.6 for the name of its vertices), then it is induced. Indeed, since G is K_4-free, if it is not induced then $y_1y_3 \in E(G)$ and then $y_1y_4y_5y_3y_2y_1$ is a 3-cycle, a contradiction.

By (1), G admits a dragonfly as a subgraph, name it H and refer to Figure 8.6 for the name of its vertices. We may assume that H is not induced, otherwise we are done. If there exists an edge with one extremity in $\{x_1, x_5\}$ and the other one in $\{x_3, x_6\}$, then $G[\{x_1, x_2, x_3, x_5, x_6\}]$ contains a butterfly as a subgraph and thus as an induced subgraph. So there is no edges with one extremity in $\{x_1, x_5\}$ and the other one in $\{x_3, x_6\}$ and, by symmetry, there is no edges with extremities in $\{x_4, x_7\}$ and the other one in $\{x_2, x_6\}$.

So there exists some edges with one extremity in $\{x_1, x_5\}$ and one extremity in $\{x_4, x_7\}$. By symmetry, we may assume w.l.o.g. that $x_5x_7 \in E(G)$. If it is the only one then, $x_1x_2x_6x_3x_4x_7x_5x_1$ is a 3-cycle, a contradiction. So it is not the only one and thus, some of x_1x_4, x_1x_7 or x_4x_5 are edges of G. If there is exactly one more, then in the three cases $x_1x_2x_3x_4x_7x_5x_1$ is a 3-cycle, a contradiction. So, there is at least two more and there is actually exactly two more, otherwise $x_1x_4x_5x_7 = K_4$. By symmetry between x_1x_7 and x_4x_5, we may assume w.l.o.g. that $x_4x_5 \in E(G)$. So one of the edges x_1x_4 or x_1x_7 exists, but in both cases the cycle $x_2x_6x_3x_4x_7x_5x_2$ is a 3-cycle, a contradiction. This proves (2). □

Lemma 8.14 *Let G be a (K_4, 3-cycle)-free graph and let z be a vertex of H. Then, for every integer i, $G[S_i(z)]$ is dragonfly-free.*

PROOF — Assume by way of contradiction that there exists an integer i such that $G[S_i(z)]$ contains a dragonfly as an induced subgraph. Name it H and refer to Figure 8.6 for the name of its vertices. Let u be a father of x_5 and v be a father of x_7.

The two next claims examine what are the possible neighborhoods of u and v in H.

(1) $N_H(u) \in \{\{x_5\}, \{x_5, x_1\}, \{x_5, x_2\}, \{x_5, x_3\}, \{x_5, x_6\}, \{x_1, x_3, x_5, x_6\}\}$.

First note that u cannot have exactly two neighbors in $\{x_1, x_2, x_3, x_6\}$. Indeed, u cannot see both x_1 and x_2 since otherwise there is a K_4. Thus w.l.o.g. x_6 is a neighbor of u and then $ux_5x_1x_2x_3x_6u$ is a 3-cycle.

Assume now that ux_2 is an edge. Since G is K_4-free, ux_1 is not an edge. Both ux_3, ux_6 are not edges since G is K_4-free. So none of ux_3, ux_6 is an edge since otherwise u has exactly two neighbors in $\{x_1, x_2, x_3, x_6\}$. If ux_7 is an edge then $ux_5x_1x_2x_6x_3x_7u$ is a 3-cycle. So ux_7 and by symmetry ux_4 are not edges. So if ux_2 is an edge, then $N_H(u) = \{x_5, x_2\}$ and one of the outcome holds. So, we may assume from now on that ux_2 is not an edge.

Assume that ux_7 is an edge. By symmetry between x_5, x_2 and x_7, x_3, we can assume that ux_3 is not an edge. Let $S = \{x_1, x_4, x_6\}$. If u has no neighbor on S then $ux_5x_1x_2x_6x_3x_4x_7u$ is a 3-cycle. If u has exactly one neighbor in S, then by symmetry between x_1 and x_6 we may assume that ux_6 is not an edge and thus $ux_5x_1x_2x_6x_3x_7u$ is a 3-cycle. So u has at least two neighbors in S. If u has three neighbors in S, then u has exactly two neighbors on $\{x_1, x_2, x_3, x_6\}$, a contradiction. So u has exactly two neighbors in S. If ux_1 and ux_4 are edges, then $ux_5x_1x_2x_6x_3x_7u$ is a 3-cycle. So, by symmetry between x_1 and x_4, we may assume that the two neighbors of u in S are x_4 and x_6. So then $ux_5x_1x_2x_6x_3x_7u$ is a 3-cycle, a contradiction. So we may assume that ux_7, and by symmetry ux_4 are not an edge..

So, $N_H(u) \subseteq \{x_5, x_1, x_3, x_6\}$ and, since we already proved that u does not have exactly two neighbors in $\{x_1, x_2, x_3, x_6\}$, one of the outcome holds. This proves (1).

(2) $N_H(v) \in \{\{x_7\}, \{x_4, x_7\}, \{x_3, x_7\}, \{x_2, x_7\}, \{x_6, x_7\}, \{x_2, x_4, x_6, x_7\}\}$.

By obvious symmetries in H, the proof is the same as the proof of (1). This proves (2).

Note that by claims (1) and (2), $u \neq v$. In the rest of the proof we show that, whatever the neighborhoods of u and v are, one can find a 3-cycle in $H \cup Q_{uv}$ (recall that Q_{uv} denote a unimodal path linking u and v).

Suppose first that $d_H(u) \leq 2$ and $d_H(v) \leq 2$.
If $d_H(u) = d_H(v) = 1$, then $ux_5x_1x_2x_6x_3x_4x_7vQ_{vu}u$ is a 3-cycle. So we may assume that $d_H(u) = 2$ and thus, by (1), u has exactly one neighbors in $\{x_1, x_3, x_6\}$. Note that by (2), vx_1 is not en edge. If $d_H(v) = 1$, then $ux_5x_1x_2x_6x_3x_7vQ_{uv}u$ is a 3-cycle. Moreover, it is still a 3-cycle if vx_4 is an edge. So $d_H(v) = 2$ and vx_4 is not an edge. Similarly, if ux_1 is an edge, then $vx_7x_4x_3x_6x_2x_5uQ_{uv}v$ is a 3-cycle. So ux_1 is not an edge. Now, by (1) and (2), both u and v has exactly one neighbor in $\{x_2, x_3, x_6\}$ and thus $ux_5x_2x_6x_3x_7vQ_{uv}u$ is a 3-cycle, a contradiction.

So, from now on, we assume that $d_H(u)$ and $d_H(v)$ are not both inferior to 2. Hence we may assume w.l.o.g. that $d_H(u) > 2$, and thus, by (1), $N_H(u) = \{x_1, x_3, x_5, x_6\}$.

Recall that by (2), vx_1 is not an edge. If v has no neighbor in $\{x_2, x_3\}$, then $x_5x_1x_2x_3x_7vQ_{vu}u$ is a 3-cycle. So v has at least one neighbor in $\{x_1, x_2, x_3\}$ and by (2). If $N_H(v) \subseteq \{\{x_2, x_7\}, \{x_3, x_7\}\}$, then $ux_5x_2x_3x_4x_7vQ_{uv}u$ is a 3-cycle. So by (2), $N_H(v) = \{x_2, x_4, x_6, x_7\}$

If uv is not an edge, then $ux_5x_2vx_7x_4x_3u$ is a 3-cycle with chords x_2x_3, x_3x_7 and vx_4, a contradiction. So we may assume that uv is an edge. Let u' and v' be fathers of respectively u and v and note that, since u' and v' are in $S_{i-2}(z)$, they have no neighbors in H. Therefore $u'ux_5x_2x_3x_7vv'Q_{v'u'}u'$ is a 3-cycle, with chords uv, ux_3 and vx_2, a contradiction. □

Lemma 8.15 *Let G be a (K_4, 3-cycle)-free graph and let z be a vertex of H. Then, for every integer i, $G[S_i(z)]$ is butterfly-free.*

PROOF — Assume by way of contradiction that there exists an integer i such that $G[S_i(z)]$ contains a butterfly as an induced subgraph. Name it H and refer to Figure 8.6 for the name of its vertices. Let u be a father of y_4 and v be a father of y_5.

The two next claims examine what are the possible neighborhoods of u and v in H.

(1) $N_H(u) \in \{\{y_4\}, \{y_2, y_4\}, \{y_1, y_3, y_4\}\}$

Assume first that $|N_H(u)| = 2$. If $N_H(u) = \{y_1, y_4\}$, then $uy_1y_2y_3y_5y_4u$ is a 3-cycle, a contradiction. If $N_H(u) = \{y_3, y_4\}$, then $uy_4y_1y_2y_5y_3u$ is a 3-cycle, a contradiction. If $N_H(u) = \{y_4, y_5\}$, then $uy_4y_1y_2y_3y_5u$ is a 3-cycle, a contradiction. So, if $|N_H(u)| = 2$, then $N_H(u) = \{y_2, y_4\}$ and one of the outcome of the theorem holds.

Assume now that $|N_H(u)| = 3$. If $N_H(u) = \{y_2, y_3, y_4\}$, then $uy_4y_2y_5y_3u$ is a 3-cycle, a contradiction. If $N_H(u) = \{y_1, y_3, y_4\}$, then one of the outcome of the theorem holds. So, since G is K_4-free, u has to see y_5. The third neighbor of u is thus y_1 or y_3 and, by symmetry, we may assume that it is y_1. Therefore $uy_4y_1y_2y_5u$ is a 3-cycle, a contradiction.

So we may assume that $|N_H(u)| \geq 4$. Since G is K_4-free, $|N_H(u)| = 4$ and $N_H(u) = \{y_1, y_3, y_4, y_5\}$. Thus $uy_1y_4y_2y_5u$ is a 3-cycle, a contradiction. This proves (1).

(2) $N_H(v) \in \{\{y_5\}, \{y_2, y_5\}, \{y_1, y_3, y_5\}\}$

By obvious symmetries in H, the proof is the same as the proof of (1). This proves (2).

Note that by claims (1) and (2), $u \neq v$. In the rest of the proof we show that, whatever the neighborhoods of u and v are, one can find a 3-cycle in $H \cup Q_{uv}$ (recall that Q_{uv} denote a unimodal path linking u and v).

Case 1 : $N_H(v) = \{y_5\}$.
If $N_H(u) = \{y_4\}$ then $uy_4y_1y_2y_3y_5vQ_{vu}u$ is a 3-cycle, a contradiction. So $N_H(u) \in \{\{y_2, y_4\}, \{y_1, y_3, y_4\}\}$ and thus $uy_4y_2y_3y_5vQ_{vu}u$ is a 3-cycle, a contradiction. This completes the proof in case 1.

So from now on, we may assume that $N_H(v) \neq \{y_5\}$ and, by symmetry, that $N_H(u) \neq \{y_4\}$.

Case 2 : $N_H(v) = \{y_2, y_5\}$.
If $N_H(u) = \{y_2, y_4\}$ then $uy_4y_2y_5vQ_{vu}u$ is a 3-cycle. Otherwise, we may assume that $N_H(u) = \{y_1, y_3, y_4\}$ and then $uy_1y_2y_3y_5vQ_{vu}$ is a 3-cycle, a contradiction.

So from now on, we may assume that $N_H(v) \neq \{y_2, y_5\}$ and by symmetry, $N_H(u) \neq \{y_2, y_4\}$. This leads to the following last case.

Case 3 : $N_H(v) = \{y_1, y_3, y_5\}$ and $N_H(u) = \{y_1, y_3, y_4\}$.
If uv is not an edge then $uy_1y_4y_5vy_3u$ is a 3-cycle, with chords uy_4, vy_1 and y_3y_5, a contradiction. So uv is an edge. Let u' and v' be fathers of respectively u and v. If u' (or v') is adjacent to both u and v we assume that $u' = v'$. Note that since u' and v' are in S_{k-2} they have no neighbors in H. Therefore $u'uy_1y_2y_3vv'Q_{u'v'}u'$ is a 3-cycle, with chords uv, uy_3 and vy_1, a contradiction. This completes the proof in Case 3.

□

We can now give the proof of Theorem 8.9 recall that it states that every (K_4,3-cycle)-free graph has chromatic number at most $4c$.

PROOF — Assume by contradiction that there exists a (K_4, 3-cycle)-free graph G that satisfies $\chi(G) \geq 4c + 1$ and let z be a vertex of G. By Remark 8.4, there exists an integer k such that $S_k(z, G)$ has chromatic number at least $2c + 1$. So, by Lemma 8.13, it must contain a dragonfly or a butterfly, which is a contradiction with Lemma 8.14 or Lemma 8.15. □

8.4.3 Clique number at least 4 : proof of Theorem 8.10

Recall that Theorem 8.9 states that every (3-cycle)-free graph has chromatic number at most $\max(4c, \omega(G)+1)$.

PROOF — Consider by contradiction the smallest (in number of vertices) graph $G \in \mathcal{C}_3$ such that $\chi(G) > \max(\omega(G)+1, 4c)$. By Theorem 8.8 and 8.9, we have $\omega(G) \geq 4$. Put $\omega(G) = \omega$. Let K be a largest clique of G and denote by $x_1, \dots, x_\omega$ the vertices of K.

(1) Every vertex of G is of degree at least $\omega + 1$.

If a vertex v of G is of degree at most ω, then by minimality of G we can color $G \setminus \{v\}$ with $max(\omega(G)+1, 4c)$ colors and extend the coloring to G, a contradiction. This proves (1).

(2) G does not admit clique cutsets.

Assume by contradiction that G has a clique cutset A. Let C_1 be a connected component of $G \setminus A$, and C_2 the union of all others components. By minimality of G, we may color $G[C_i \cup K]$ with $max(\omega(G)+1, 4c)$ colors for $i = 1, 2$. By using the same colors for the vertices of A in the coloring of $G[C_1 \cup K]$ and $G[C_2 \cup K]$, we can extend the coloring to G, a contradiction. This proves (2).

(3) If $u \in N(K)$, then $d_K(u) = 1$ or $\omega - 1$.

Assume by way of contradiction that u has at least two neighbors in K, say x_1 and x_2, and at least two non-neighbors, say x_3 and x_4. Then $ux_1x_3x_4x_2u$ is a 3-cycle, with chords x_1x_2, x_1x_4 and x_2x_3, a contradiction. This proves (3).

Define $S_i = \{u \in N(K) | N_K(u) = \{x_i\}\}$, $T_i = \{u \in N(K) | N_K(u) = V(K) \setminus \{x_i\}\}$ and, for all $i = 1, \dots, \omega$, $U_i = S_i \cup T_i$.

An uv-path P is an *$N(K)$-connection* if no vertex of P is in K and $N(K) \cap P = \{u, v\}$. Note that vertices of $\mathring{P}$ have no neighbors on K and that an $N(K)$-connection can be an edge.

(4) Let P be an $N(K)$-connection with endvertices u and v. Then there exists an integer i such that $\{u, v\} \subseteq U_i$ and $\{u, v\} \not\subseteq T_i$.

Let i, j, k and l be 4 distinct integers in $\{1, \dots, \omega\}$. Such integers exist since $\omega \geq 4$.
If $u \in T_i$ and $v \in T_j$, then $ux_jx_kx_ivPu$ is a 3-cycle, with chords ux_k, vx_k and x_ix_j.
If $u \in S_i$ and $v \in T_j$, then $ux_ix_kx_lvPu$ is a 3-cycle, with chords vx_i, vx_k and x_ix_l.
If $u \in S_i$ and $v \in S_j$, then $ux_ix_kx_lx_jvPu$ is a 3-cycle, with chords x_ix_j, x_ix_l and x_jx_k.
If $u \in T_i$ and $v \in T_i$, then $ux_jx_ix_kvPu$ is a 3-cycle, with chords ux_j, vx_k and x_jx_k. This proves (4).

(5) There is a unique $i \in \{1, \dots, \omega\}$ for which $U_i \neq \emptyset$.

Let us argue by way of contradiction. By (2), $G \setminus K$ is connected, so there exists a path P in $G \setminus K$ from U_i to U_j such that $i \neq j$. Choose P subject to its minimality. It is clear that P is an $N(K)$-connection and thus it contradicts (4). This proves (5).

By (5), we may assume w.l.o.g. that $U_1 \neq \emptyset$ and, for any $i \neq 1$, $U_i = \emptyset$. Moreover, S_1 and T_1 both contain at least two vertices, otherwise x_1 or x_2 have degree at most ω, a contradiction to (1).

We say that a vertex x is *complete* to a set of vertex S is x is adjacent to every vertex in S.

(6) If there exists an $N(K)$-connection from a vertex of T_1 to a vertex $s_1 \in S_1$, then s_1 is complete to T_1.

Let P be a minimal $N(K)$-connection from s_1 to T_1. Denote by $t_1 \in T_1$ the second endvertex of P. Assume by way of contradiction that there exists a vertex $t_2 \in T_1 \setminus \{t_1\}$ that is not adjacent to s_1. Then there is no edge linking t_2 with a vertex of P, otherwise there would be an $N(K)$-connection from t_1 to t_2, contradicting (4). So, $s_1Pt_1x_2t_2x_3x_1s_1$ is a 3-cycle with chords x_1x_2, t_1x_3 and x_2x_3, a contradiction.

So s_1 is complete to $T_1 \setminus \{t_1\}$ and, by minimality of P, s_1 is adjacent to t_1. This proves (6).

(7) $N(T_1) \subseteq S_1 \cup K$.

Assume by contradiction that there exists $t_1 \in T_1$ such that $N(t_1) \nsubseteq S_1 \cup K$. Since t_1 is not a cutvertex by (2), consider a minimal path P' from $N(t_1) \setminus (S_1 \cup K)$ to $N(K)$ in $G \setminus \{t_1\}$. Call t'_1 and x the extremities of P with $t'_1 \in N(t_1) \setminus (S_1 \cup K)$ and put $P = t_1Px$. Observe that if t_1x is not an edge, then t_1Px is an $N(K)$-connection and that in both cases there exists an $N(K)$-connection linking t_1 and x. So, by (4), $x \notin T_1$. Hence, by (6), x is complete to T_1 and in particular xt_1 is an edge. Finally $t_1Ps_1x_1x_2x_3t_1$ is a 3-cycle with chords s_1t_1, x_1x_3 and t_1x_2, a contradiction. This proves (7).

(8) For any vertex $t \in T_1$, $N(t) = S_1 \cup K \setminus \{x_1\}$.

Let $t \in T_1$. By (4), T_1 is a stable set. So if t is not adjacent to any vertex of S_1, $N(t) = K \setminus \{x_1\}$, a contradiction to (1). So t is adjacent to at least one vertex in S_1 and thus, by (6), t is complete to S_1. This proves (8).

Let t_1 and t_2 be two distinct vertices in T_1 (remind that they exist because if $|T_1| = 1$, then $d(x_2) = \omega$, contradicting (1)). By (8), $N(t_1) = N(t_2) = S_1 \cup K \setminus \{x_1\}$. By minimality of G, $G \setminus \{t_2\}$ admits a proper coloring γ with $\max(4c, \omega + 1)$ colors. Since $t_1t_2 \notin E(G)$ and $N(t_1) = N(t_2)$, γ can be extended to G by giving to t_2 the same color as t_1. □

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