Clique number of tournaments

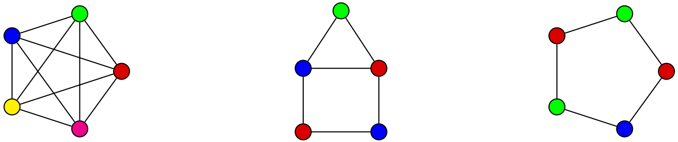
Pierre Aboulker — ENS Paris
joint work with Guillaume Aubian, Pierre Charbit, Samuel Coulomb, Stéphan Thomassé, Raul Wayne

The chromatic number

Colouring: adjacent vertices receive distinct colours.

$\Leftrightarrow$

Partition the vertices into independent sets.



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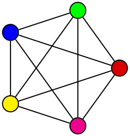

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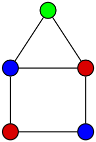

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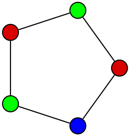

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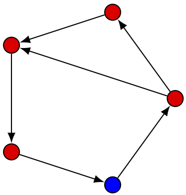
Chromatic number of $G = \chi(G)$: minimise the number of colours.

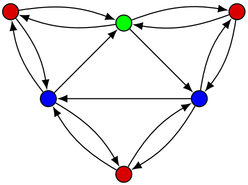
Question: How could we define directed graph colouring?

The dichromatic number

- Coloring a digraph D: no monochromatic (induced) directed cycle.
- $\vec{\chi}(D)$: the *dichromatic number* of D.

In other words: **partition D in acyclic induced subdigraphs** instead of stable sets.

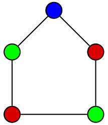


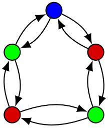


- Being acyclic is the same as having a topological ordering.

Dichromatic number generalises chromatic number

Property: For every graph G, $\chi(G) = \vec{\chi}(\overleftrightarrow{G})$.





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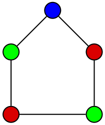
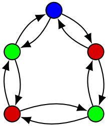


G



$\overleftrightarrow{G}$

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- Brooks' Theorem, Gallaï Theorem, Wilf Theorem (algebraic graph theory)...
- Extremal graph theory,
- List dichromatic number,
- Substructure forced by large dichromatic number,
- Dicolouring digraphs on surfaces.

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Perfect graphs: χ-bounded by the function $f(x) = x$.

Gyárfás-Sumner Conjecture:
Let H be a graph. The class of H-free graphs is χ-bounded if and only if H is a forest.

Theorem [Folklor]: If $\mathcal{C}$ is χ-bounded, then so is $\mathcal{C}^{subst}$

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WHAT IS THE CLIQUE NUMBER OF A DIGRAPH?

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Conjecture [PA, Charbit, Naserasr, 2020]: Let H be an oriented graph. H-free oriented graphs are $\vec{\chi}$-bounded if and only if H is an oriented forest.

Backedge graph

Given a digraph D, and a total ordering $\prec$ on $V(D)$, let $D^{\prec}$ be the (undirected) graph with vertex set $V(D)$ and edge uv if $u \prec v$ and $vu \in A(D)$.

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For every $\prec$:

$$\vec{\chi}(D) \leq \chi(D^{\prec})$$

Moreover, there exists $\prec$ such that $\chi(D^{\prec}) \leq \vec{\chi}(D)$.

Hence:

$$\vec{\chi}(D) = \min \left\{ \chi(D^{\prec}) : \prec \text{ is a total ordering of } V(D) \right\}$$

Clique number of digraphs

So we have a new definition of the dichromatic number:

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This leads a natural definition of the clique number of a digraph:

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We clearly have:

- $\omega(G) = \vec{\omega}(\overleftrightarrow{G})$ (because for every $\prec$, $\overleftrightarrow{G}^{\prec} = G$), and
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Goal of the talk: to investigate the clique number of tournaments

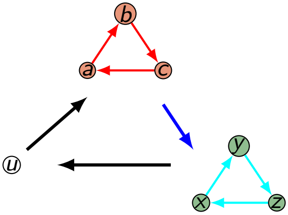
Tournaments

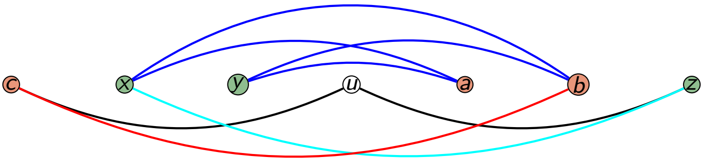
- Tournament = orientation of a complete graph.
- $\vec{C}_3$ is the directed triangle.
- Transitive tournament (TT_k) = acyclic tournament = tournaments with no $\vec{C}_3$
- Dicolour a tournament $\Leftrightarrow$ no monochromatic $\vec{C}_3$ $\Leftrightarrow$ partition into transitive tournaments.

- Tournaments can have large dichromatic number:

Define the S_k recursively as follows:

Let $S_1 = TT_1$, $S_k = \Delta(TT_1, S_{k-1}, S_{k-1})$. We have $\vec{\chi}(S_k) = k$





A triangle-free ordering of S_3. So $\vec{\omega}(S_3) = 2$.

Tournaments with clique number 1 or 2

$$\overrightarrow{\omega}(T) = \min \left\{\omega(T^{\prec}) : \prec \text{ is a total ordering of } V(T)\right\}$$

Properties:

- $\overrightarrow{\omega}(TT_n) = 1$.
- $\overrightarrow{\omega}(\vec{C}_3) = 2$.

Let T be a tournament.

- $\overrightarrow{\omega}(T) = 1$ if and only if T is a transitive tournament.
- $\overrightarrow{\omega}(T) \geq 2$ if and only if T contains a $\vec{C}_3$.

Question: what is the complexity of deciding if $\overrightarrow{\omega}(T) \geq 3$?

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Property: The clique number of a digraph is equal to the maximum clique number of its strong components.

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Fundamental inequality [Nguyen, Scott, Seymour, 2023]:
For every tournament T and every ordering $\prec$ of $V(T)$.

$$\frac{\chi(T^{\prec})}{\omega(T^{\prec})} \quad \leq \quad \overrightarrow{\chi}(T) \quad \leq \quad \chi(T^{\prec})$$

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Application: construction of interesting tournaments from undirected graphs.

$\overrightarrow{\omega}$-ordering and $\overrightarrow{\chi}$-ordering

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So $\overrightarrow{\omega}$-orderings give a good approximation of $\overrightarrow{\chi}$.

Proof: For every $\overrightarrow{\omega}$-ordering: $\quad \chi(T^{\prec}) \quad \leq \quad \overrightarrow{\chi}(T) \cdot \overrightarrow{\omega}(T) \quad \leq \quad \overrightarrow{\chi}(T)^2$

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Property: there is $\overrightarrow{\chi}$-orderings that does not give a good approximation of $\overrightarrow{\omega}$.

Question: Is there always an ordering $\prec$ that is both an $\overrightarrow{\omega}$-ordering and a $\overrightarrow{\chi}$-ordering?

Tournaments with arbitrarily large clique number

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Lemma: For any integer n, $\overrightarrow{\omega}(B_n) \geq n$.

Proof: By induction on n. Let $\prec$ be an $\overrightarrow{\omega}$-ordering. Look at the in-neighbourhood of the first vertex in $\prec$.

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Question: what is the smallest $f(n)$ such that every n-vertex tournament T has $\overrightarrow{\omega}(T) \leq f(n)$?

Relation with the domination number

Domination number: size of the smallest $X \subseteq V(T)$ such that $N^+[X] = V(T)$.

Property: For every tournament T,

$$\mathrm{dom}(T) \quad \leq \quad \overrightarrow{\omega}(T) \quad \leq \quad \overrightarrow{\chi}(T)$$

$\vec{\chi}$-bounded class of tournaments

A class of tournaments $\mathcal{T}$ is $\vec{\chi}$-bounded if there exists a function f such that, for every $T \in \mathcal{T}$,

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Conjecture: Let $\mathcal{D}$ be a class of digraphs. If $\mathcal{D}$ is $\overrightarrow{\chi}$-bounded, then so is $\mathcal{D}^{subst}$.

Relation between $\overrightarrow{\chi}$-boundedness and χ-boundedness

Given a class of tournaments $\mathcal{T}$, let us denote by $\mathcal{T}^{\prec}$ the class of all backedge graphs of tournaments in $\mathcal{T}$:

$$\mathcal{T}^{\prec} = \{T^{\prec} \mid T \in \mathcal{T}, \prec \text{ an ordering of } T\}$$

For example, if $\mathcal{T} = \{\text{transitive tournaments}\}$, then $\mathcal{T}^{\prec} = \{\text{permutation graphs}\}$.

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Theorem: Let $\mathcal{T}$ be a class of tournaments. $\mathcal{T}$ is $\overrightarrow{\chi}$-bounded if and only if $\mathcal{T}^{\prec}$ is χ-bounded.

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- $\mathcal{T}$ is $\overrightarrow{\chi}$-bounded $\Rightarrow$ $\mathcal{T}^{\prec}$ is χ-bounded.

Proof: let f be a function such that for every $T \in \mathcal{T}$, we have $\overrightarrow{\chi}(T) \leq f(\overrightarrow{\omega}(T))$.
Now, for every $T^{\prec} \in \mathcal{T}^{\prec}$:

$$\begin{aligned}
\chi(T^{\prec}) &\leq \omega(T^{\prec}) \cdot \overrightarrow{\chi}(T) && \text{by the fundamental inequality} \\
&\leq \omega(T^{\prec}) \cdot f(\overrightarrow{\omega}(T)) \\
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- $\mathcal{T}^{\prec}$ is χ-bounded $\Rightarrow \mathcal{T}$ is $\overrightarrow{\chi}$-bounded.

Proof: Let g be a function such that for every $T^{\prec} \in \mathcal{T}^{\prec}$, $\chi(T^{\prec}) \leq g(\omega(T^{\prec}))$.
Now, for any $T \in \mathcal{T}$ and every ordering $\prec$ of T.

$$\overrightarrow{\chi}(T) \leq \chi(T^{\prec}) \leq g(\omega(T^{\prec})) \leq g(\overrightarrow{\omega}(T))$$

Classes of tournaments defined by forbidding a single tournament

Given a tournament H, Forb(H) is the **class of tournaments** T such that T does not contain H as a subtournament.

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Given a tournament H, Forb(H) is the **class of tournaments** T such that T does not contain H as a subtournament.

Question: for which tournament H is $Forb(H)$ $\overrightarrow{\chi}$-bounded?

i.e. there is a function f such that, for every $T \in Forb(H)$, $\overrightarrow{\chi}(T) \leq f(\overrightarrow{\omega}(T))$

We say that such an H is $\overrightarrow{\chi}$-binding.

Heroes

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The most simple case of χ-bounding function is a constant function.

Question: for which tournament H there is a number c_H such that, for every $T \in \mathit{Forb}(H)$, $\overrightarrow{\chi}(T) \leq c_H$?

Heroes

Question: for which tournament H is $Forb(H)$ $\overrightarrow{\chi}$-bounded?

The most simple case of χ-bounding function is a constant function.

Question: for which tournament H there is a number c_H such that, for every $T \in Forb(H)$, $\overrightarrow{\chi}(T) \leq c_H$?

Answer: such tournaments are called heroes and have been characterised by Berger, Choromanski, Chudnovsky, Fox, Loebl, Scott, Seymour and Thomassé in 2013.

Tournaments and Heroes

- A tournament H is a hero if there exists a number c_H such that every H-free tournaments T has $\overrightarrow{\chi}(T) \leq c_H$.

For example, $\vec{C}_3$ and TT_k are heroes .

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Theorem: *[Berger, Choromanski, Chudnovsky, Fox, Loebl, Scott, Seymour and Thomassé, 2013]*

A digraph H is a hero if and only if:

- $H = K_1$.
- $H = (H_1 \Rightarrow H_2)$
- $H = \Delta(1, k, H)$ or $H = \Delta(1, H, k)$, where $k \geq 1$ and H is a hero.

Gentlemen

► A tournament H is a gentlemen if there exists a number c_H such that every H-free tournaments T has $\overrightarrow{\omega}(T) \leq c_H$.

Question: Who are the gentlemen?

Of course, all heroes are gentlemen.

Gentlement and heroes are the same

Theorem [PA, Aubian, Charbit, Lopes, 2023]: Heroes and gentlemen are the same.

Proof:

- We want to prove that all gentlemen are heroes.
- Take a minimal counter-example H (in particular H is a gentlemen but not a hero).
- All subtournaments of H are gentlemen, and thus heroes by induction.
- Consider the sequence of tournaments $S_1, S_2, S_3, \dots$.
- We proved that they have arbitrarily large $\overrightarrow{\omega}$.
- So H is of the form $\Delta(1, A, B)$.
- Nguyen, Scott and Seymour proved that $S_3 = \Delta(1, \vec{C}_3, \vec{C}_3)$ is not a gentlemen.
- So one of A or B is a transitive tournament, so H is a hero.

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- Let H be a tournament such that no backedge graph of H is a forest.

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  - So T is H-free.
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- Moreover, by a celebrated theorem of Erdős, graph in $\mathcal{C}$ can have arbitrarily large chromatic number.
- Hence, by the fundamental inequality, tournaments in $\mathcal{T}[\mathcal{C}]$ can have arbitrarily large dichromatic number.
- So $\mathcal{T}[\mathcal{C}]$ is not $\overrightarrow{\chi}$-bounded, and thus the class of H-free tournaments is not $\overrightarrow{\chi}$-bounded.

Gyárfás-Sumner Conjecture for tournaments

Recall that:

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Conjecture: Let H be a tournament. *Forb*(H) is $\overrightarrow{\chi}$-bounded if and only if H has an ordering $\prec$ for which $H^{\prec}$ is a forest.

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How to find counter-example

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A counter-example on 5 vertices was found by Guillaume Aubian. More counter-examples were found by Samuel Coulomb.

Way to prove that a given tournament H is not $\overrightarrow{\chi}$-binding:

- Start with Blanche-Descarte construction $G_1, \dots, G_k, \dots$ (or any other triangle-free constructions with large χ).
- Order (smartly) the vertices of each G_i and transform each G_i into a tournament T_i.
- These tournaments have clique number 2 and arbitrarily large dichromatic number by the fundamental inequality.
- Prove that the T_i are H-free.

$\overrightarrow{\chi}$-binding tournaments

Question: For which tournament H is $Forb(H)$ $\overrightarrow{\chi}$-binding?

What is known:

- if H_1 and H_2 are $\overrightarrow{\chi}$-binding, then so is $H_1 \Rightarrow H_2$,

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- If there exists $\prec$ such that $H^{\prec}$ is a matching, then H is $\overrightarrow{\chi}$-binding (corollary of a result announced by Briański, Davies and Walczak).
- If H is $\overrightarrow{\chi}$-binding, then so is the tournament obtained from H by reversing every arc of T,

Theorem [Le, Harutyunyan, Thomassé and Wu, 2017]
There exists a function λ such that, if for every vertex v, $\overrightarrow{\chi}(v^+) \leq t$, then $\overrightarrow{\chi}(T) \leq \lambda(t)$.

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Proof by induction on $\overrightarrow{\omega}$. Assume $\overrightarrow{\omega}(T) = k$. Two steps:

Step 1: If $N(xy) = N(y^+) \cap N(x^-)$ has large $\overrightarrow{\chi}$, then xy is a backedge in every $\overrightarrow{\omega}$-ordering.

Step 2: for every vertex x, $\overrightarrow{\chi}(x^+)$ or $\overrightarrow{\chi}(x^-)$ is small.

Theorem: if $Forb(H)$ is $\overrightarrow{\chi}$-bounded, then so is $Forb(rev(H))$, where $rev(H)$ is obtained by reversing every arc if H.

Proof:

- Recall that: $Forb(H)$ is $\overrightarrow{\chi}$-bounded $\Leftrightarrow$ $Forb(H)^{\prec}$ is χ-bounded.
- Observe that $T^{\prec} = rev(T)^{rev(\prec)}$.
- So $Forb(H)^{\prec} = Forb(rev(H))^{\prec}$.

Relations with χ-boundedness of classes of ordered graphs

From a tournament H, we can define the following set of ordered undirected graphs:

$$\{H\}_o^{\prec} = \{(H^{\prec}, \prec) : \prec \text{ is an ordering of } H\}$$

Theorem:
Forb(H) is $\overrightarrow{\chi}$-bounded **if and only if** $Forb_o(\{H\}_o^{\prec})$

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Let $(M, \prec)$ be an ordered graph with maximum degree 1. Then $Forb_o(M, \prec)$ is χ-bounded.

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Adventurous Conjecture: A tournament H is $\overrightarrow{\chi}$-binding **if and only if** there is an ordering $\prec$ of H such that $(H^{\prec}, \prec)$ is χ-binding.

Relation with the Erdős-Hajnal Conjecture

Erdős-Hajnal Conjecture (1981): Let H be a graph. there exists a number c_H such that every H-free graph G has a clique or a stable set of size $|V(G)|^{c_H}$.

Alon, Pach, Solymosi (2001) proved that it is equivalent with:

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Theorem: Let H be a tournament. If $Forb(H)$ is polynomially $\overrightarrow{\chi}$-bounded, then H has the Erdős-Hajnal property.

The BIG $\Rightarrow$ BIG Conjecture

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A class of tournament $\mathcal{T}$ has the *BIG $\Rightarrow$ BIG* property if for every $T \in \mathcal{T}$, if $\overrightarrow{\chi}(T) \geq f(t)$, then T contains A and B such that $\overrightarrow{\chi}(A), \overrightarrow{\chi}(B) \geq t$ and $A \Rightarrow B$.

BIG $\Rightarrow$ BIG Conjecture [Nguyen, Scott, Seymour, 2023]: The class of all tournaments has the BIG $\Rightarrow$ BIG property.

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Theorem: If a class of tournaments $\mathcal{T}$ is $\overrightarrow{\chi}$-bounded, then $\mathcal{T}$ has the *BIG* $\Rightarrow$ *BIG* property.

Clusters

Given a tournament parameter γ, a γ-cluster of a tournament T is a subtournament X of T of bounded size with large γ.

Question: for which parameters γ_1 and γ_2 we have that, for all tournaments T with sufficiently large γ_1, T has a γ_2-cluster.

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The following states that: Large domination number implies a $\overrightarrow{\chi}$-cluster.

Theorem [Thomassé, Le, Harutyunyan and Wu, 2019]
There is two functions f and ℓ such that, for every integer k, every tournament T with $\text{dom}(T) \geq f(k)$ has a subtournament X with $|X| \leq \ell(k)$ and $\overrightarrow{\chi}(X) \geq k$

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Corollary [Local to global for $\overrightarrow{\chi}$]
Let T be a tournament. If $\overrightarrow{\chi}(x^+) \leq t$ for every vertex x, then $\overrightarrow{\chi}(T) \leq f(t)$.

Proof: If T has small domination number, vertices are covered by a small number of out-neighbours. Otherwise apply the Theorem with $k := t + 1$ and find a vertex with a large out-neighbourhood.

Clusters

Recall that for every tournament T, $dom(T) \leq \overrightarrow{\omega}(T) \leq \overrightarrow{\chi}(T)$.

Theorem [Thomassé, Le, Harutyunyan and Wu, 2019]
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Theorem: Large dichromatic number does **not** imply a $\overrightarrow{\chi}$-cluster.

Conjecture [Thomassé, Le, Harutyunyan and Wu, 2019]:
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Here is a weaker version of the above conjecture:

Conjecture [A, Aubian, Charbit, Wayne, 2024]
Large domination number implies a $\overrightarrow{\omega}$-cluster.

Conjecture [Local to global for $\overrightarrow{\omega}$]
Let T be a tournament. If $\overrightarrow{\omega}(x^+) \leq k$ for every vertex x, then $\overrightarrow{\omega}(T) \leq f(k)$.

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A tournament H is a rebel if there is a number c_H such that every H-free tournament has domination number at most c_H.

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A poset tournament if a tournament that has backedge graph that is a comparability graph.

Conjecture [Chudnovsky, Kim, Liu, Seymour and Thomassé, 18]
A tournament is a rebel **if and only** if it is a poset tournament.

The above conjecture implies that *large domination number implies an $\overrightarrow{\omega}$-cluster*.

Indeed, the S_k are poset tournament. Moreover, they have arbitrarily large $\overrightarrow{\omega}$. Hence, if the S_k are rebel, then tournaments with sufficiently large domination number contains an S_k as a subtournament, which forms a $\overrightarrow{\omega}$-cluster.

Complexity

Observation: $\overrightarrow{\omega}(T) = 1$ if and only T is a transitive tournament.

Theorem [Aubian, 2024]: For every $k \geq 3$, deciding if $\overrightarrow{\omega}(T) \leq k$ is NP-complete.

Open Question: what is the complexity of deciding if a tournament T has $\overrightarrow{\omega}(T) \leq 2$?

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Theorem [PA, Aubian, Charbit, Thomassé]: We can decide in poly-time if, given a tournament T, $\overrightarrow{\omega}(T) > 2$ or $\overrightarrow{\omega}(T) \leq 10^{10}$

Question: Is there a function f such that for every integer k, there is a poly-time algorithm that, given a tournament T decide if $\overrightarrow{\omega}(T) \geq k$, or $\overrightarrow{\omega}(T) \leq f(k)$

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Given a class of (undirected) graph $\mathcal{C}$, we say that a FAS is a $\mathcal{C}$-FAS if $F \in \mathcal{C}$.

$\mathcal{C}\text{-FAS}$ *Problem* the associated decision problem, that is deciding if a tournament has a $\mathcal{C}$-FAS.

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$\mathcal{C}$-FAS *Problem* the associated decision problem, that is deciding if a tournament has a $\mathcal{C}$-FAS.

It is NP-hard when:

- $\mathcal{C}_k = \{k\text{-colourable graphs}\}$, when $k \geq 2$ (Bokal, Fijavz, Juvan, Kayll and Mohar, 2004)
- $k \geq 4$, $\mathcal{C} = \{K_k\text{-free graphs}\}$ (Aubian, 2024)
- $\mathcal{C} = \{\text{forests}\}$ (PA, Aubian, Lopes, 2024)
- $\mathcal{D}_k = \{\text{graphs with max degree } k\}$ when $k \geq 2$ (Davot, Isenmann, Roy, and Thiebaut, 2023) (and polynomial when $k \leq 1$)

Question: what is the complexity when:

- ▶ $\mathcal{C}$ is the set of all paths?
- ▶ $\mathcal{C}$ is the set of triangle-free graphs?

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THANK YOU FOR YOUR ATTENTION