Clique number of tournaments

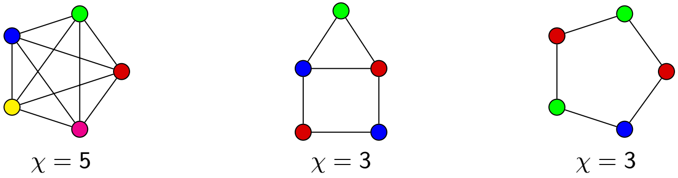
Pierre Aboulker — ENS Paris

The chromatic number

Colouring: adjacent vertices receive distinct colours.

$\Leftrightarrow$

Partition the vertices into independent sets.



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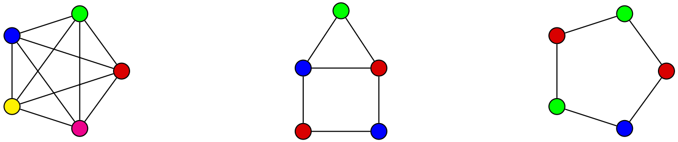


$\chi = 5$ $\chi = 3$ $\chi = 3$

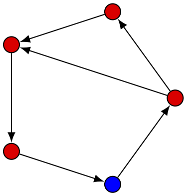
Chromatic number of $G = \chi(G)$: minimise the number of colours.

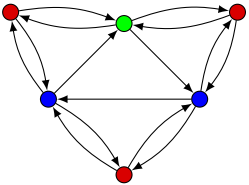
Question: How could we define directed graph colouring?

The dichromatic number

- Coloring a digraph D: no monochromatic (induced) directed cycle.
- $\vec{\chi}(D)$: the *dichromatic number* of D.

In other words: **partition D in acyclic induced subdigraphs** instead of stable sets.

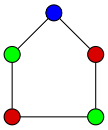


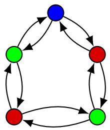


- Being acyclic is the same as having a topological ordering.

Dichromatic number generalises chromatic number

Property: For every graph G, $\chi(G) = \vec{\chi}(\overleftrightarrow{G})$.





There is more and more results on the dichromatic number of digraphs for which, in the special case of symmetric digraphs, we recover an existing result on undirected graph.

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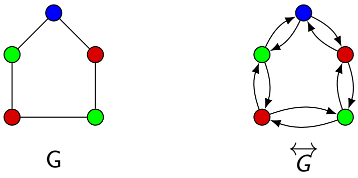

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- Brooks' Theorem, Gallaï Theorem, Wilf Theorem (algebraic graph theory)...
- Extremal graph theory,
- List dichromatic number,
- Substructure forced by large dichromatic number,
- Dicolouring digraphs on surfaces.

Clique number versus chromatic number

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Perfect graphs: χ-bounded by the function $f(x) = x$.

Gyárfás-Sumner Conjecture:
Let H be a graph. The class of H-free graphs is χ-bounded if and only if H is a forest.

Theorem[Folklor]: If $\mathcal{C}$ is χ-bounded, then so is $\mathcal{C}^{subst}$

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WHAT IS THE NOTION OF CLIQUE NUMBER OF A DIGRAPH?

What is the clique number of a digraph?

We would like that, for every graph G and every digraph D:

$$\omega(G) = \vec{\omega}(\overleftrightarrow{G}) \quad \text{and} \quad \vec{\omega}(D) \leq \vec{\chi}(D)$$

First attempt:

$\vec{\omega}(D) =$ size of a maximum symmetric clique in D.

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Interesting, but does not satisfy $\vec{\omega}(D) \leq \vec{\chi}(D)$.

Conjecture [PA, Charbit, Naserasr, 2020]: Let H be an oriented graph. H-free oriented graphs are $\vec{\chi}$-bounded if and only if H is an oriented forest.

Backedge graph

Given a digraph D, and a total ordering $\prec$ on $V(D)$, let $D^{\prec}$ be the (undirected) graph with vertex set $V(D)$ and edge uv if $u \prec v$ and $vu \in A(D)$.

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For every $\prec$:

$$\vec{\chi}(D) \leq \chi(D^{\prec})$$

Moreover, there exists $\prec$ such that $\chi(D^{\prec}) \leq \vec{\chi}(D)$.

Hence:

$$\vec{\chi}(D) = \min\left\{\chi(D^{\prec}) : \prec \text{ is a total ordering of } V(D)\right\}$$

Clique number of digraphs

So we have a new definition of the dichromatic number:

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This leads a natural definition of the clique number of a digraph:

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We clearly have:

- $\vec{\omega}(\overleftrightarrow{G}) = \omega(G)$ (because for every $\prec$, $\overleftrightarrow{G}^{\prec} = G$), and
- $\vec{\omega}(D) \leq \vec{\chi}(D)$ (because for every graph G, $\omega(G) \leq \chi(G)$).

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Goal of the talk: to investigate the clique number of tournaments

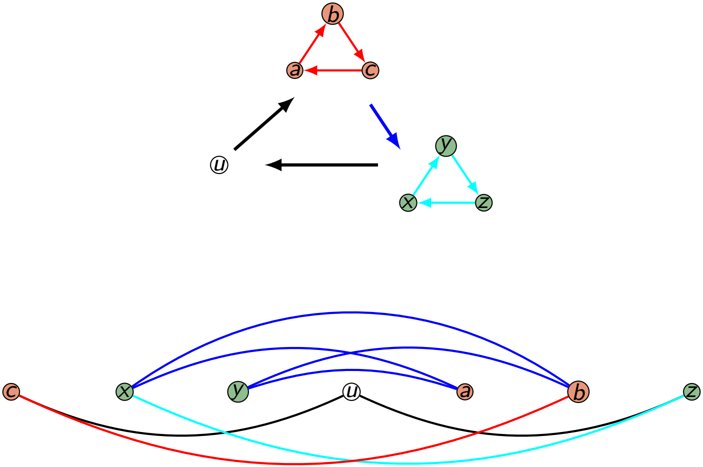
Tournaments

- Tournament = orientation of a complete graph.
- $\vec{C}_3$ is the directed triangle.
- Transitive tournament (TT_k) = acyclic tournament = tournaments with no $\vec{C}_3$
- Dicolour a tournament $\Leftrightarrow$ no monochromatic $\vec{C}_3$.

- Tournaments can have large dichromatic number:

Define the S_k recursively as follows:

Let $S_1 = TT_1$, $S_k = \Delta(TT_1, S_{k-1}, S_{k-1})$. We have $\overrightarrow{\chi}(S_k) = k$



A triangle-free ordering of S_3. So $\overrightarrow{\omega}(S_3) = 2$.

Tournaments with clique number 1 or 2

$$\overrightarrow{\omega}(T) = \min \left\{\omega(T^{\prec}) : \prec \text{ is a total ordering of } V(T)\right\}$$

Properties:

- $\overrightarrow{\omega}(TT_n) = 1$.
- $\overrightarrow{\omega}(\vec{C}_3) = 2$.

Let T be a tournament.

- $\overrightarrow{\omega}(T) = 1$ if and only if T is a transitive tournament.
- $\overrightarrow{\omega}(T) \geq 2$ if and only if T contains a $\vec{C}_3$.

Question: what is the complexity of deciding if $\overrightarrow{\omega}(T) \geq 3$?

First properties of $\overrightarrow{\omega}$

Property: The clique number of a digraph is equal to the maximum clique number of its strong components.

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Fundamental inequality [Nguyen, Scott, Seymour, 2023]:
For every tournament T and every ordering $\prec$ of $V(T)$.

$$\frac{\chi(T^{\prec})}{\omega(T^{\prec})} \quad \leq \quad \overrightarrow{\chi}(T) \quad \leq \quad \chi(T^{\prec})$$

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Application: construction of interesting tournaments from undirected graphs.

$\overrightarrow{\omega}$-ordering and $\overrightarrow{\chi}$-ordering

Let T a tournament and $\prec$ be an ordering of $V(T)$. It is a:

$\overrightarrow{\omega}$-ordering if $\omega(T^{\prec}) = \overrightarrow{\omega}(T)$ $\overrightarrow{\chi}$-ordering if $\chi(T^{\prec}) = \overrightarrow{\chi}(T)$

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Property: For every tournament T and every $\overrightarrow{\omega}$-ordering $\prec$ we have: $\chi(T^{\prec}) \leq \overrightarrow{\chi}(T)^2$.
So $\overrightarrow{\omega}$-orderings give a good approximation of $\overrightarrow{\chi}$.

Question: Is it true that every $\overrightarrow{\chi}$-ordering is not too bad for $\overrightarrow{\omega}$?

Question: Is there always an ordering $\prec$ that is both an $\overrightarrow{\omega}$-ordering and a $\overrightarrow{\chi}$-ordering?

Tournaments with arbitrarily large clique number

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Let $B_1 = TT_1$ and inductively, for $n \geq 1$, let $B_n = \Delta(B_{n-1}, B_{n-1}, B_{n-1})$.

Lemma: For any integer n, $\overrightarrow{\omega}(B_n) \geq n$.

Proof: By induction on n. Let $\prec$ be an $\overrightarrow{\omega}$-ordering. Look at the in-neighbourhood of the first vertex in $\prec$.

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We know that $n \leq \overrightarrow{\omega}(B_n) \leq \left(\frac{3}{2}\right)^n = \overrightarrow{\chi}(B_n)$.

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Question: what is the smallest $f(n)$ such that every n-vertex tournament T has $\overrightarrow{\omega}(T) \leq f(n)$?

Relations with the dominating set number

Dominating number: size of the smallest $X \subseteq V(T)$ such that $N^+[X] = V(T)$.

Property: For every tournament T,

$$\mathrm{dom}(T) \quad \leq \quad \overrightarrow{\omega}(T) \quad \leq \quad \overrightarrow{\chi}(T)$$

$\overrightarrow{\chi}$-bounded class of tournaments

A class of tournaments $\mathcal{T}$ is $\overrightarrow{\chi}$-bounded if there exists a function f such that, for every $T \in \mathcal{T}$,

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Conjecture: Let $\mathcal{D}$ be a class of digraphs. If $\mathcal{D}$ is $\overrightarrow{\chi}$-bounded, then so is $\mathcal{D}^{subst}$.

Relation between $\overrightarrow{\chi}$-boundedness and χ-boundedness

Given a class of tournaments $\mathcal{T}$, let us denote by $\mathcal{T}^{\prec}$ the class of all backedge graphs of tournaments in $\mathcal{T}$:

$$\mathcal{T}^{\prec} = \{T^{\prec} \mid T \in \mathcal{T}, \prec \text{ an ordering of } T\}$$

For example, if $\mathcal{T} = \{\text{transitive tournaments}\}$, then $\mathcal{T}^{\prec} = \{\text{permutation graphs}\}$.

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Theorem: Let $\mathcal{T}$ be a class of tournaments. The following properties are equivalent:

(i) $\mathcal{T}$ is $\overrightarrow{\chi}$-bounded.

(ii) $\mathcal{T}^{\prec}$ is χ-bounded.

(iii) $\mathcal{T}^{\prec_{\varpi}}$ is χ-bounded.

$$\mathcal{T}^{\prec} = \{T^{\prec} \mid T \in \mathcal{T}, \prec \text{ an ordering of } T\}$$

- $\mathcal{T}$ is $\overrightarrow{\chi}$-bounded $\Rightarrow$ $\mathcal{T}^{\prec}$ is χ-bounded.

Proof: let f be a function such that for every $T \in \mathcal{T}$, we have $\overrightarrow{\chi}(T) \leq f(\overrightarrow{\omega}(T))$.
Now, for every $T^{\prec} \in \mathcal{T}^{\prec}$:

$$\begin{aligned} \chi(T^{\prec}) &\leq \omega(T^{\prec}) \cdot \overrightarrow{\chi}(T) && \text{by the fundamental inequality} \\ &\leq \omega(T^{\prec}) \cdot f(\overrightarrow{\omega}(T)) \\ &\leq \omega(T^{\prec}) \cdot f(\omega(T^{\prec})) \end{aligned}$$

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- $\mathcal{T}^{\prec\overrightarrow{\omega}}$ is χ-bounded $\Rightarrow$ $\mathcal{T}$ is $\overrightarrow{\chi}$-bounded.

Proof: Let g be a function such that for every $T^{\prec} \in \mathcal{T}^{\prec\overrightarrow{\omega}}$, $\chi(T^{\prec}) \leq g(\omega^{\prec}(T))$.
Now, for any $T \in \mathcal{T}$ and every $\overrightarrow{\omega}$-ordering $\prec$ of T.

$$\overrightarrow{\chi}(T) \leq \chi(T^{\prec}) \leq g(\omega(T^{\prec})) = g(\overrightarrow{\omega}(T))$$

Classes of tournaments defined by forbidding a single tournament

Given a tournament H, Forb(H) is the class of tournaments T such that T does not contain H as a subtournament.

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Question: for which tournament H is $Forb(H)$ $\overrightarrow{\chi}$-bounded?

i.e. there is a function f such that, for every $T \in Forb(H)$, $\overrightarrow{\chi}(T) \leq f(\overrightarrow{\omega}(T))$

We say that such an H is $\overrightarrow{\chi}$-binding.

Heroes

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Answer: such tournaments are called heroes and have been characterised by Berger, Choromanski, Chudnovsky, Fox, Loebl, Scott, Seymour and Thomassé in 2013.

Tournaments and Heroes

- A tournament H is a hero if there exists a number c_H such that every H-free tournaments T has $\overrightarrow{\chi}(T) \leq c_H$.

For example, $\vec{C_3}$ and TT_k are heroes .

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Theorem: *[Berger, Choromanski, Chudnovsky, Fox, Loebl, Scott, Seymour and Thomassé, 2013]*

A digraph H is a hero if and only if:

- $H = K_1$.
- $H = (H_1 \Rightarrow H_2)$
- $H = \Delta(1, k, H)$ or $H = \Delta(1, H, k)$, where $k \geq 1$ and H is a hero.

Gentlemen

- A tournament H is a gentlemen if there exists a number c_H such that every H-free tournaments T has $\overrightarrow{\omega}(T) \leq c_H$.

Question: Who are the gentlemen?

Of course, all heroes are gentlemen.

Gentlement and heroes are the same

Theorem [PA, Aubian, Charbit, Lopes, 2023]: Heroes and gentlemen are the same.

Proof:

- We want to prove that all gentlemen are heroes.
- Take a minimal counter-example H (in particular H is a gentlemen but not a hero).
- All subtournaments of H are gentlemen, and thus heroes by induction.
- Consider the sequence of tournaments $S_1, S_2, S_3, \dots$.
- We proved that they have arbitrarily large $\overrightarrow{\omega}$.
- So H is of the form $\Delta(1, A, B)$.
- Nguyen, Scott and Seymour proved that $S_3 = \Delta(1, \vec{C_3}, \vec{C_3})$ is not a gentlemen.
- So one of A or B is a transitive tournament, so H is a hero.

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Theorem: $Forb(H)$ is $\overrightarrow{\chi}$-bounded $\Rightarrow$ H has an ordering $\prec$ such that $H^{\prec}$ is a forest.

- Let H be a tournament such that no backedge graph of H is a forest.

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- Observe that every $T \in \mathcal{T}[\mathcal{C}]$ has $\overrightarrow{\omega}(T) \leq 2$.
- Moreover, by a celebrated theorem of Erdős, graph in $\mathcal{C}$ can have arbitrarily large chromatic number.
- Hence, by the fundamental inequality, tournaments in $\mathcal{T}[\mathcal{C}]$ can have arbitrarily large dichromatic number.
- So $\mathcal{T}[\mathcal{C}]$ is not $\overrightarrow{\chi}$-bounded, and thus the class of H-free tournaments is not $\overrightarrow{\chi}$-bounded.

Recall that:

Gyárfás-Sumner Conjecture, 1981:
Let H be a graph. Forb(H) is χ-bounded if and only if H is a forest.

Gyárfás-Sumner Conjecture for tournaments

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How to find counter-example

Conjecture: Let H be a tournament. *Forb*(H) is $\overrightarrow{\chi}$-bounded if and only if H has an ordering $\prec$ for which $H^{\prec}$ is a forest. HEAVILY FALSE

A counter-example on 5 vertices was found by Guillaume Aubian. More counter-examples were found by Samuel Coulomb.

Way to prove that a given tournament H is not $\overrightarrow{\chi}$-binding:

- Start with Blanche-Descarte construction $G_1, \dots, G_k, \dots$ (or any other triangle-free constructions with large χ).
- Order (smartly) the vertices of each G_i and transform each G_i into a tournament T_i.
- These tournaments have clique number 2 and arbitrarily large dichromatic number by the fundamental inequality.
- Prove that the T_i are H-free.

$\overrightarrow{\chi}$-binding tournaments

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What is known:

- if H_1 and H_2 are $\overrightarrow{\chi}$-binding, then so is $H_1 \Rightarrow H_2$,

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- if H_1 and H_2 are $\overrightarrow{\chi}$-binding, then so is $H_1 \Rightarrow H_2$,
- if H is $\overrightarrow{\chi}$-binding, then so is $\Delta(1, 1, H)$ (corollary of a result of Alantha and Felix).

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- If there exists $\prec$ such that $H^{\prec}$ is a matching, then H is $\overrightarrow{\chi}$-binding [Briański, Davies and Walczak, announced in 2023].

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- If there exists $\prec$ such that $H^{\prec}$ is a matching, then H is $\overrightarrow{\chi}$-binding [Briański, Davies and Walczak, announced in 2023].
- if H is $\overrightarrow{\chi}$-binding, then so is the tournament obtained from H by reversing every arc of T,

Theorem [Le, Harutyunyan, Thomassé and Wu, 2017]
There exists a function λ such that, if for every vertex v, $\overrightarrow{\chi}(v^+) \leq t$, then $\overrightarrow{\chi}(T) \leq \lambda(t)$.

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Proof by induction on $\vec{\omega}$. Assume $\vec{\omega}(T) = k$. Two steps:

Step 1: If $N(uv) = N(v^+) \cap N(u^-)$ is big, then uv is a backedge in every $\vec{\omega}$-ordering.

Step 2: for every vertex x, $\vec{\chi}(x^+)$ or $\vec{\chi}(x^-)$ is small.

Theorem: if *Forb*(H) is $\overrightarrow{\chi}$-bounded, then so is *Forb*$(rev(H))$, where $rev(H)$ is obtained by reversing every arc if H.

Proof:

- Recall that: *Forb*(H) is $\overrightarrow{\chi}$-bounded $\Leftrightarrow$ *Forb*$(H)^{\prec}$ is χ-bounded.
- Observe that $T^{\prec} = rev(T)^{rev(\prec)}$.
- So $Forb(H)^{\prec} = Forb(rev(H))^{\prec}$.

Relations with χ-boundedness of classes of ordered graphs

From a tournament H, we can define the following class of ordered (undirected) graphs:

$$\{H\}_o^{\prec} = \{(T^{\prec}, \prec) : \prec \text{ is an ordering of } T\}$$

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Theorem [Briański, Davies and Walczak, 2024+]
Let $(M, \prec)$ be an ordered graph with maximum degree 1. Then $Forb_o(M, \prec)$ is χ-bounded.

Relation with the Erdős-Hajnal Conjecture

Erdős-Hajnal Conjecture (1981): Let H be a graph. there exists a number c_H such that every H-free graph G has a clique or a stable set of size $|V(G)|^{c_H}$.

Alon, Pach, Solymosi (2001) proved that it is equivalent with:

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Theorem: If $\mathit{Forb}(H)$ is polynomially $\overrightarrow{\chi}$-bounded, then H has the Erdős-Hajnal property.

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Observation: $\overrightarrow{\omega}(T) = 1$ if and only T is a transitive tournament.

Theorem [Aubian, 2024]: For every $k \geq 3$, deciding if $\overrightarrow{\omega}(T) \leq k$ is NP-complete.

Open Question: what is the complexity of deciding if a tournament T has $\overrightarrow{\omega}(T) \leq 2$?

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Theorem [PA, Aubian, Charbit, Thomassé]: We can decide in poly-time if, given a tournament T, $\overrightarrow{\omega}(T) > 2$ or $\overrightarrow{\omega}(T) \leq 10^{10}$

Question: Is there a function f such that for every integer k, there is a poly-time algorithm that, given a tournament T decide if $\overrightarrow{\omega}(T) \geq k$, or $\overrightarrow{\omega}(T) \leq f(k)$

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It is NP-hard when:

- $\mathcal{C}_k = \{k\text{-colourable graphs}\}$, when $k \geq 2$ (Bokal, Fijavz, Juvan, Kayll and Mohar, 2004)
- $k \geq 4$, $\mathcal{C} = \{K_k\text{-free graphs}\}$ (Aubian, 2024)
- $\mathcal{C} = \{\text{forests}\}$ (PA, Aubian, Lopes, 2024)
- $\mathcal{D}_k = \{\text{graphs with max degree } k\}$ when $k \geq 2$ (Davot, Isenmann, Roy, and Thiebaut, 2023) (and polynomial when $k \leq 1$)

Question: what is the complexity when:

- $\mathcal{C}$ is the set of all paths?
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There exists two functions f and ℓ such that, for every integer k, every tournament T with $\overrightarrow{\omega}(T) \geq f(k)$ contains a subtournament X with $|X| \leq \ell(k)$ and $\overrightarrow{\omega}(X) \geq k$.

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Thank You For Your Attention