Colouring digraphs and arc-connectivity

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Join work with Guillaume Aubian and Pierre Charbit

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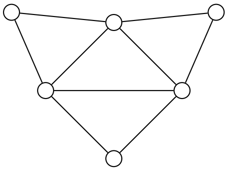
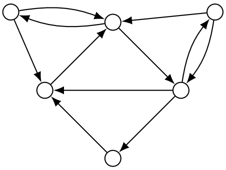
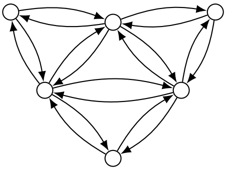
Graph and directed graph theory



A graph



A digraph



A symmetric digraph

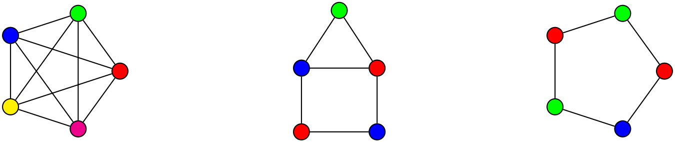
The chromatic number

The chromatic number

Colouring: adjacent vertices receive distinct colours.

$\Leftrightarrow$

Partition the vertices into independent sets.



$\chi = 5$ $\chi = 3$ $\chi = 3$

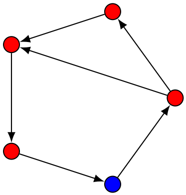
Chromatic number of $G = \chi(G)$: minimise the number of colours.

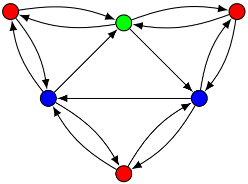
Question: How could we define directed graph colouring?

The dichromatic number

- Coloring a digraph D: no monochromatic (induced) directed cycle.
- $\vec{\chi}(D)$: the *dichromatic number* of D.

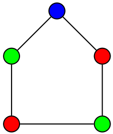
In other words: **partition D in acyclic induced subdigraphs** instead of stable sets.



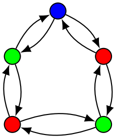


Dichromatic number generalises chromatic number

Property: For every graph G, $\chi(G) = \vec{\chi}(\overleftrightarrow{G})$.



G



$\overleftrightarrow{G}$

There is more and more results on the dichromatic number of digraphs for which, restricted to of symmetric digraphs, we recover an existing result on undirected graph.

Chromatic number vs dichromatic number

$\chi(G)$ is the maximum chromatic number of a **connected component** of G.

$\vec{\chi}(D)$ is the maximum dichromatic number of a **strong component** of D.

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Brooks' Theorem

$\Delta(G)$: maximum degree of G.

Property: $\chi(G) \leq \Delta(G) + 1$

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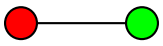
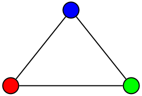
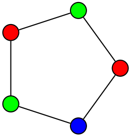
Brooks' Theorem (1932):
$\chi(G) = \Delta(G) + 1$ except if G is a complete graph or an odd cycle.



$\chi = 2$
$\Delta = 1$



$\chi = 3$
$\Delta = 2$



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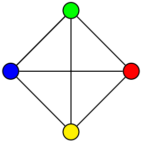

$\chi = 4$
$\Delta = 3$

Directed Brook's Theorem

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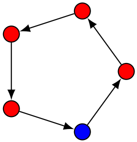
Directed Brooks' Theorem [Mohar, 2010]:
$\vec{\chi}(D) = \Delta_{max}(D) + 1$ except if D is a directed cycle, a symmetric odd cycle, or a symmetric complete graph.



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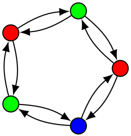

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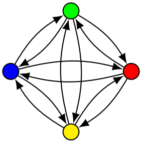

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Line of research: take your favourite theorem on chromatic number, and generalise it to digraphs via the dichromatic number.

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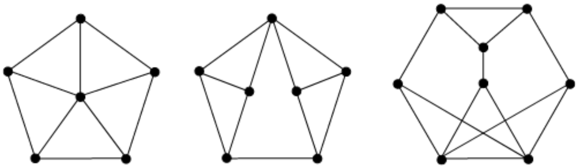
Theorem: [A., Bretell, Havet, Trotignon (k=3) 2015, Stiebitz and Toft, 2016]
A graph G if k-extremal if and only if:

- It is an odd cycle ($k = 2$), or
- An odd wheel ($k = 3$), or
- K_k, or
- It is the Hajós join of two graphs.

Some 3-extremal graphs:



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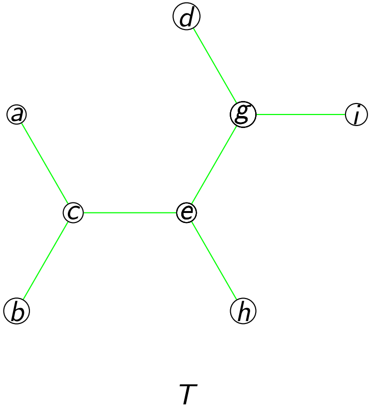
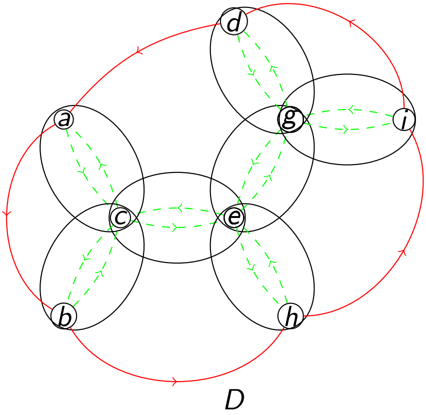
Definition: a digraph D is *k*-extremal if and only if it is:

- $\vec{\chi}(D) = \lambda(D) + 1 = k + 1$,
- biconnected, and
- strong.

How can we generalise Hajós join?

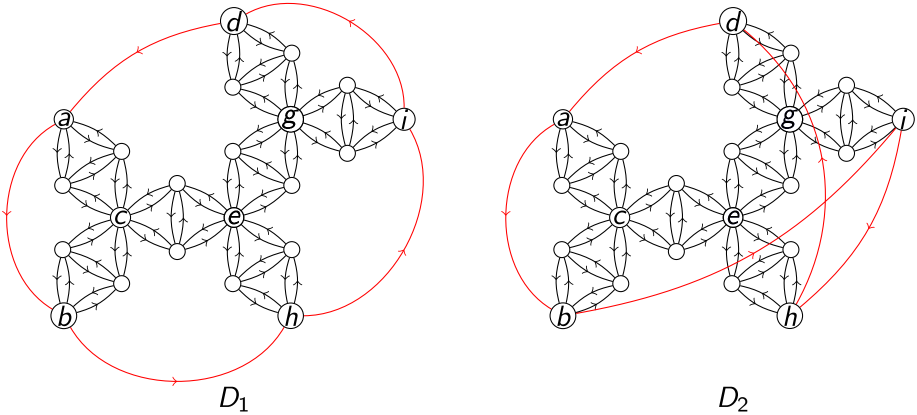
Hajós tree join





D_1 is 3-extremal: $\vec{\chi}(D_1) = \lambda(D_1) + 1 = 4$.

But D_2 is not: $\vec{\chi}(D_2) = 4$, but $\lambda(D_2) = \lambda(g, e) = 4$



Theorem: [A., Aubian, Charbit, 2023+]
$k \geq 3$. A graph G if k-extremal if and only if G is :

- a symmetric odd wheel ($k = 3$), or
- a symmetric K_k, or
- a directed Hajós join of two digraphs, or
- a the Hajós tree join of some digraphs.

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Open question: Characterize 2-digraphs.

k-**extremal**: $\chi(G) = \lambda(G) + 1 = k + 1$, strong and biconnected.

Lemma: A k-extremal digraph is

- $(k+1)$-dicritical (all proper subgraph are k-dicolourable),
- for every x, y, $\lambda(x, y) = \lambda(y, x) = k$, and
- Eulerian ($d^+(x) = d^-(x)$ for every vertex x)

Extended Hajós join

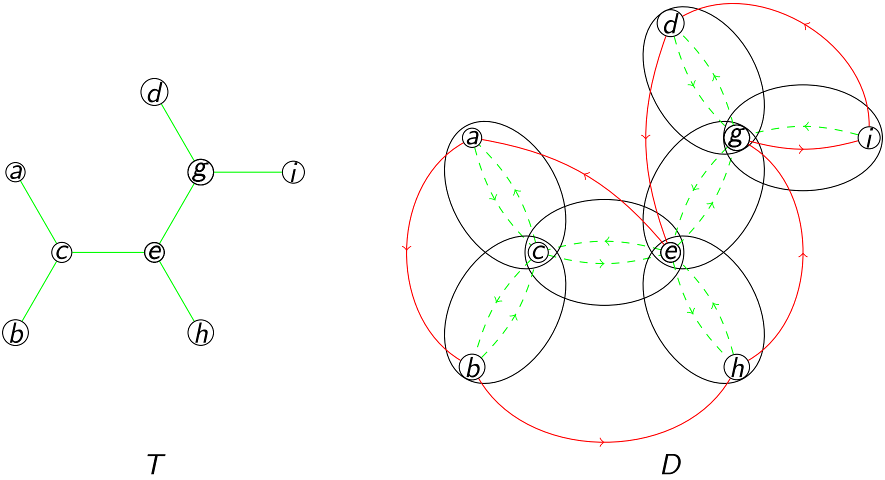

Figure: A cartoonish drawing of an extended Hajós tree join D. Its peripheral cycle is in red. Removed digons are in dashed green. T is the corresponding tree.

Conjecture for 2-extremal digraphs

Given

- a tree T embedded in the plane with at least two edges,
- A partition (A, B) of the edges of T, with $A = \{u_1v_1, \dots, u_av_a\}$ and $B = \{x_1y_1, \dots, x_by_b\}$ such that every leaf to leaf path in T contains an even number of edges of B,
- a circular ordering $C = (x_1, \dots, x_\ell)$ of the leaves of T, taken following the natural ordering given by the embedding of T, and
- for $i = 1, \dots, a$, a digraph D_i such that
  - $V(D_i) \cap V(T) = \{u_i, v_i\}$,
  - $[u_i, v_i] \subseteq A(D_i)$, and
  - for $1 \leq i \neq j \leq a$, $V(D_i) \setminus \{u_i, v_i\} \cap V(D_j) \setminus \{u_j, v_j\} = \emptyset$,

Let $\mathcal{H}_2$ be the smallest class of digraphs containing symmetric odd cycle and closed under taking directed Hajós join and 2-Hajós tree join.

Conjecture: A digraph is 2-extremal if and only if it is in $\mathcal{H}_2$.

Hypergraph case

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Theorem: [Schweser, Stiebitz and Toft , 2018]
A hypergraph H if k-extremal if and only if:

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- An odd wheel ($k = 3$), or
- K_k, or
- It is the Hajós join of two hypergraphs.

A 3-extremal diraph: $\vec{\chi}(D) = 4$ and $\lambda(D) = 3$.
Directed cycles *abc* and *acd* intersect on 2 vertices.