Extending the Gyárfás-Sumner conjecture to digraphs

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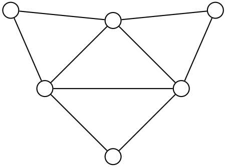
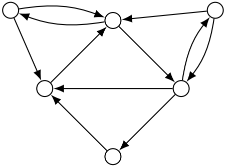
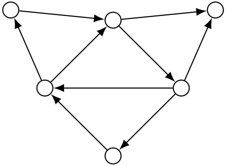
Graph and directed graph theory



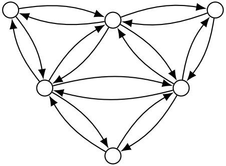
A graph



A digraph



An oriented graph



A symmetric digraph

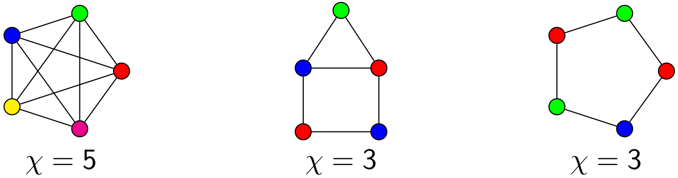
The chromatic number

The chromatic number

Colouring: adjacent vertices receive distinct colours.

$\Leftrightarrow$

Partition the vertices into independent sets.



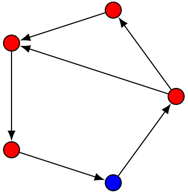
Chromatic number of $G = \chi(G)$: minimise the number of colours.

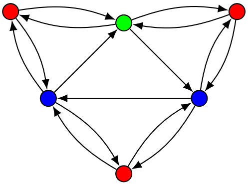
Question: How could we define directed graph colouring?

The dichromatic number

- Coloring a digraph D: no monochromatic (induced) directed cycle.
- $\vec{\chi}(D)$: the *dichromatic number* of D.

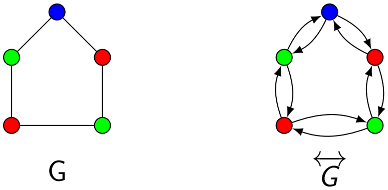
In other words: **partition D in acyclic induced subdigraphs** instead of stable sets.





Dichromatic number generalises chromatic number

Property: For every graph G, $\chi(G) = \vec{\chi}(\overleftrightarrow{G})$.



There is more and more results on the dichromatic number of digraphs for which, in the special case of symmetric digraphs, we recover an existing result on undirected graph.

Brooks' Theorem

$\Delta(G)$: maximum degree of G.

Property: $\chi(G) \leq \Delta(G) + 1$

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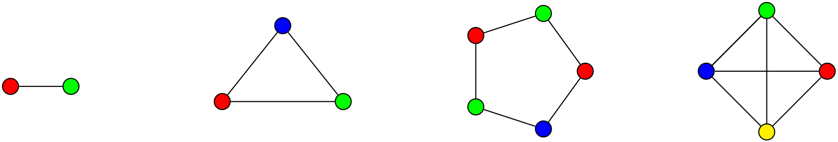
Brooks' Theorem (1932):
$\chi(G) = \Delta(G) + 1$ except if G is a complete graph or an odd cycle.



Directed Brook's Theorem

$d_{max}(v) = max(d^+(v), d^-(v))$

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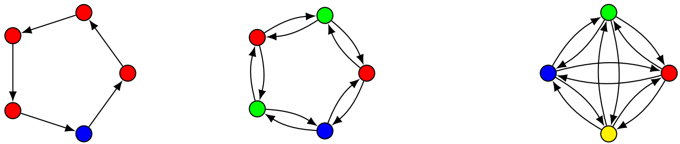
Directed Brooks' Theorem:
$\vec{\chi}(D) = \Delta_{max}(D) + 1$ except if D is a directed cycle, a symmetric odd cycle, or a symmetric complete graph.



$\vec{\chi} = 2$
$\Delta_{max} = 1$

$\vec{\chi} = 3$
$\Delta_{max} = 2$

$\vec{\chi} = 4$
$\Delta_{max} = 3$

Line of research: take your favourite theorem on chromatic number, and generalise it to digraphs via the dichromatic number.

From now on, digraphs will be supposed to be digon-free.

Non-oriented world

• Let $\mathcal{F}$ be a set of graphs. $G \in Forb(\mathcal{F})$ if G does not contains any member of $\mathcal{F}$ as an induced subgraph.

[1] Size of a smallest cycle

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- $\mathcal{F}$ must contain a forest.

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Gyárfás-Sumner conjecture (1987)
For every integer k and every forest F, $\mathit{Forb}(K_k, F)$ has bounded chromatic number.

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χ-boundedness

- $\omega(G)$: size of a maximum clique of G.

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A hereditary class of graphs is χ-bounded if $\chi(G) \leq f(\omega(G))$ for every G in the class.

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Result: It is enough to prove it for trees.

Directed world, dichromatic number

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Heroic sets

Let $\mathcal{F}$ be a set of oriented graphs.

Forb $(\mathcal{F})$ is the class of oriented graphs containing no member of $\mathcal{F}$ as an induced subdigraph.

Problem: What are the finite sets $\mathcal{F}$ for which *Forb* $(\mathcal{F})$ has bounded dichromatic number?

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- Tournament = orientation of a complete graph.
- $\vec{C}_3$ is the directed triangle.
- Transitive tournament: tournaments with no $\vec{C}_3$

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Harutyunyan and Mohar (2012): there is oriented graph with large dichromatic number and such that its underlying graph has large girth.

Tournaments and Heroes

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Theorem: *[Berger, Choromanski, Chudnovsky, Fox, Loebl, Scott, Seymour and Thomassé, 2015]*
All heros can be constructed as follows:

- T_1 is a hero.
- If H_1 and H_2 are heroes, then $H_1 \Rightarrow H_2$ is a hero.
- If H is a hero, then $\Delta(H, TT_k, TT_1)$ and $\Delta(H, TT_1, TT_k)$ are heros.

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Theorem *[Chudnovsky, Scott, Seymour, 2019]*
For every integer k and disjoint unions of stars F, $Forb(TT_k, F)$ has bounded chromatic number.

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Conjecture: for every disjoint union of stars S, heroes in $Forb(S)$ are the same as heroes in tournaments FALSE

Conjecture: for every oriented tree T and every k, TT_k is a hero in $Forb(T)$.

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Theorem *[Harutyunyan, Le, Newman, Thomassé, 2019]*
Heroes in $\mathit{Forb}\,(\overline{K}_t)$ has bounded dichromatic number.

$\mathit{Forb}\,(\overline{K}_2)$ is the class of tournaments.

Small forests

What about forest on three vertices.

Quasi-transitive graphs

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Corolary: Heroes in $Forb(\overrightarrow{P}_3)$ are the same as heroes in tournaments.

Complete multipartite oriented graphs

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Theorem[A., Aubian, Charbit 2021+]:
There exists heros H such that H-free oriented complete multipartite graphs have arbitrarily large dichromatic number.

- Define the line graph $L(G)$ of an oriented graph.
- Prove that $\chi(L(G)) \geq log(\chi(G))$.
- Build a oriented complete multipartite graphs from $L(L(TT_n))$.
- Prove it does not contain some heros.

Complete multipartite oriented graphs

We say that a tournament H is a hero in $\mathcal{C}$ if all H-free digraphs in $\mathcal{C}$ have bounded dichromatic number.

Theorem*[A., Aubian, Charbit 2022+, B. Walczak, 2022+]*:

A digraph H is a hero in $Forb(\overrightarrow{K}_2 + K_1)$ if and only if:

- $H = K_1$,
- $H = H_1 \Rightarrow H_2$, where H_1 and H_2 are heroes in $Forb(\overrightarrow{K}_2 + K_1)$, or
- $H = \Delta(1, 1, H_1)$ where H_1 is a hero in $Forb(\overrightarrow{K}_2 + K_1)$.

Local out-tournament

G is a **local out-tournament** if for every vertex x, $N^+(x)$ is a tournament.

It corresponds to $Forb(S_2^+)$.

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Theorem: $\vec{\chi}\,(Forb\,(\overrightarrow{C}_3,\, S_2^+)) = 2$ [Steiner / Aboulker, Aubian, Charbit, 2021]

Conjecture: heroes in $Forb\,(S_2^+)$ are the same as heroes in tournaments.

Recall the big conjecture:

Conjecture *[Aboulker, Charbit, Naserasr, 2020]*: The set $Forb(H, F)$ has bounded dichromatic number if and only if:

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So we get a notion of $\vec{\chi}$-boundedness!

Conjecture: for every oriented tree T, $Forb(T)$ is $\vec{\chi}$-bounded

i.e. there is a function f such that for all $G \in Forb(T)$, $\vec{\chi}(G) \leq f(\omega(G))$.

Forbidding a path

Theorem [Gyárfás, 80's]: *Forb* (P_k) is χ-bounded.

Proof that in a triangle-free (connected) graph with sufficiently large chromatic number, every vertex is the starting point of a long induced path.

Directed path

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Theorem [Cook, Masarík, Pilipczuk, Reinald, Souza, 2022+]:
$Forb\,(\overrightarrow{P_4})$ is $\vec{\chi}$-bounded.

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Theorem [A. Aubian, Charbit, Thomassé, 2022+]
$Forb(K_3, \vec{P}_6)$ have bounded dichromatic number.

The levelling technic

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If $\vec{\chi}(L_i) \leq k$ for every i, then $\vec{\chi}(G) \leq 2k$.

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If $\vec{\chi}(L_i) \leq k$ for every i, then $\vec{\chi}(G) \leq 2k$.

Theorem: If $G \in Forb\,(K_3, \overrightarrow{P_4})$, then $\vec{\chi}(G) \leq 2$ because every L_i is a stable set.

Nice sets

Definition: A nonempty set of vertices S is nice if each vertex in S either has no out-neighbor in $V(D) \setminus S$ or has no in-neighbor in $V(D) \setminus S$.

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Lemma: Given a class of digraphs $\mathcal{C}$, if every digraph in $\mathcal{C}$ has a nice set with dichromatic number at most c, then digraphs in $\mathcal{C}$ have dichromatic number at most $2c$.

Partial recap of open cases

Conjecture: Heroes in $Forb(S_2^+)$ are the same as heroes in tournaments.

Conjecture: $Forb(\vec{C}_3, S)$ has bounded dichromatic number for every oriented star S.

Conjecture: For every k, $Forb(\vec{P}_k)$ is $\vec{\chi}$-bounded.

First open cases:

- $Forb(K_4, \vec{P}_5)$ is $\vec{\chi}$-bounded.
- $Forb(K_3, \vec{P}_7)$ has bounded dichromatic number.

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THANK YOU FOR YOUR ATTENTION