Clique number of tournaments

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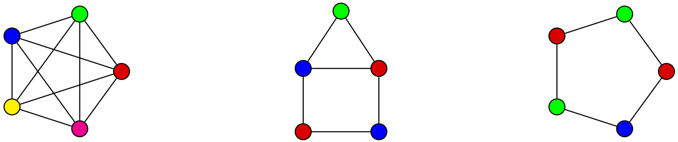
The chromatic number

Colouring: adjacent vertices receive distinct colours.

$\Leftrightarrow$

Partition the vertices into independent sets.



$\chi = 5$ $\chi = 3$ $\chi = 3$

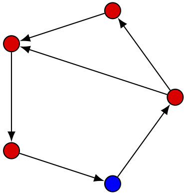
Chromatic number of $G = \chi(G)$: minimise the number of colours.

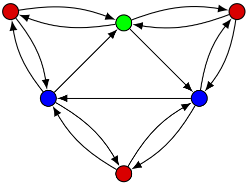
Question: How could we define directed graph colouring?

The dichromatic number

- Coloring a digraph D: no monochromatic (induced) directed cycle.
- $\vec{\chi}(D)$: the *dichromatic number* of D.

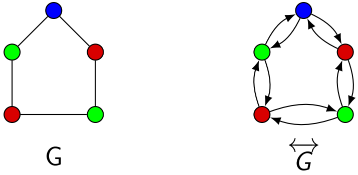
In other words: **partition D in acyclic induced subdigraphs** instead of stable sets.





Dichromatic number generalises chromatic number

Property: For every graph G, $\chi(G) = \vec{\chi}(\overleftrightarrow{G})$.



There is more and more results on the dichromatic number of digraphs for which, in the special case of symmetric digraphs, we recover an existing result on undirected graph.

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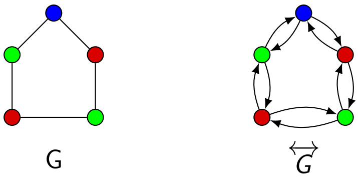

There is more and more results on the dichromatic number of digraphs for which, in the special case of symmetric digraphs, we recover an existing result on undirected graph.

- Brooks' Theorem, Gallaï Theorem, Wilf Theorem (algebraic graph theory)...
- Extremal graph theory,
- List dichromatic number,
- Substructure forced by large dichromatic number,
- Dicolouring digraphs on surfaces.

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Perfect graphs: χ-bounded by the function $f(x) = x$.

Gyárfás-Sumner Conjecture:
Let H be a graph. The class of H-free graphs is χ-bounded if and only if H is a forest.

Theorem[Folklor]: If $\mathcal{C}$ is χ-bounded, then so is $\mathcal{C}^{subst}$

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WHAT IS THE NOTION OF CLIQUE NUMBER OF A DIGRAPH?

What is the clique number of a digraph?

We would like that, for every graph G and every digraph D:

$$\omega(G) = \overrightarrow{\omega}(\overleftrightarrow{G}) \quad \text{and} \quad \overrightarrow{\omega}(D) \leq \overrightarrow{\chi}(D)$$

First attempt:

$\overrightarrow{\omega}(D) =$ size of a maximum symmetric clique in D.
But for every oriented graphs G, $\overrightarrow{\omega}(G) = 1$, not very satisfying.

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Conjecture [PA, Charbit, Naserasr, 2020]: Let H be an oriented graph. H-free oriented graphs are $\vec{\chi}$-bounded if and only H is an oriented forest.

Backedge graph

Given a digraph D, and a total order $\prec$ on $V(D)$, let $D^{\prec}$ be the (undirected) graph with vertex set $V(D)$ and edge uv if $u \prec v$ and $vu \in A(D)$.

$D^{\prec}$: *backedge graph* of D with respect to $\prec$

For every $\prec$:

$$\overrightarrow{\chi}(D) \leq \chi(D^{\prec})$$

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For every $\prec$:

$$\vec{\chi}(D) \leq \chi(D^{\prec})$$

Moreover, there exists $\prec$ such that $\chi(D^{\prec}) \leq \vec{\chi}(D)$.

Hence:

$$\vec{\chi}(D) = \min\left\{\chi(D^{\prec}) : \prec \text{ is a total order of } V(D)\right\}$$

Clique number of digraphs

So we have a new definition of the dichromatic number:

$$\overrightarrow{\chi}(D) = \min \left\{\chi(D^{\prec}) : \prec \text{ is a total order of } V(D)\right\}$$

This leads a natural definition of the clique number of a digraph:

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We clearly have:

- $\vec{\omega}(\overleftrightarrow{G}) = \omega(G)$ (because for every $\prec$, $\overleftrightarrow{G}^{\prec} = G$), and
- $\vec{\omega}(D) \leq \vec{\chi}(D)$.

Tournaments

- Tournament = orientation of a complete graph.
- $\vec{C}_3$ is the directed triangle.
- Transitive tournament (TT_k): acyclic tournament $\Leftrightarrow$ tournaments with no $\vec{C}_3$

- Tournaments can have large dichromatic number.

Let $S_1 = TT_1$, $S_k = \Delta(TT_1, S_{k-1}, S_{k-1})$. We have $\overrightarrow{\chi}(S_k) = k$

Tournaments with clique number 1 or 2

$$\overrightarrow{\omega}(T) = \min \left\{\omega(T^{\prec}) : \prec \text{ is a total order on } V(T)\right\}$$

Properties:

- $\overrightarrow{\omega}(TT_n) = 1$.
- $\overrightarrow{\omega}(\vec{C}_3) = 2$.

Let T be a tournament.

- $\overrightarrow{\omega}(T) = 1$ if and only if T is a transitive tournament.
- $\overrightarrow{\omega}(T) \geq 2$ is and only if T contains a $\vec{C}_3$.

Question: what is the complexitiy of deciding if $\overrightarrow{\omega}(T) \geq 3$?

First properties of $\overrightarrow{\omega}$

Property: The clique number of a digraph is equal to the maximum clique number of its strong components.

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Fundamental inequality [Nguyen, Scott, Seymour, 2023]:
For every tournament T and every ordering $\prec$ of $V(T)$.

$$\frac{\chi(T^{\prec})}{\omega(T^{\prec})} \leq \overrightarrow{\chi}(T) \leq \chi(T^{\prec})$$

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Application: construction of interesting tournaments from undirected graphs.

$\overrightarrow{\omega}$-ordering and $\overrightarrow{\chi}$-ordering

Let T a tournament and $\prec$ be an ordering of $V(T)$. It is a:

$\overrightarrow{\omega}$-ordering if $\omega(T^{\prec}) = \overrightarrow{\omega}(T)$ $\qquad$ $\overrightarrow{\chi}$-ordering if $\chi(T^{\prec}) = \overrightarrow{\chi}(T)$

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Property: For every tournament T and every $\overrightarrow{\omega}$-ordering $\prec$ we have:

$$\overrightarrow{\chi}(T) \quad \leq \quad \chi(T^{\prec}) \quad \leq \quad \overrightarrow{\chi}(T) \cdot \overrightarrow{\omega}(T) \quad \leq \quad \overrightarrow{\chi}(T)^2$$

Question: Is there always an ordering $\prec$ that is both an $\overrightarrow{\omega}$-ordering and a $\overrightarrow{\chi}$-ordering?

Tournaments with arbitrarily large clique number

Question: Can you find tournaments with arbitrarily large clique number?

Let $\tilde{S}_1 = TT_1$ and inductively, for $n \geq 1$, let $\tilde{S}_n = \Delta(\tilde{S}_{n-1}, \tilde{S}_{n-1}, \tilde{S}_{n-1})$.

Lemma: For any integer n, $\overrightarrow{\omega}(\tilde{S}_n) \geq n$.

Proof: By induction on n. Let $\prec$ be an $\overrightarrow{\omega}$-ordering. Look at the in-neighbour of the first vertex in $\prec$.

Relations with the dominating set number

Dominating number: size of the smallest $X \subseteq V(T)$ such that $N^+[X] = V(T)$.

Property: For every tournament T,

$$\text{dom}(T) \quad \leq \quad \overrightarrow{\omega}(T) \quad \leq \quad \overrightarrow{\chi}(T)$$

$\overrightarrow{\chi}$-bounded class of tournaments

A class of tournaments $\mathcal{T}$ is $\overrightarrow{\chi}$-bounded if there exists a function f such that, for every $T \in \mathcal{T}$,

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Conjecture: Let $\mathcal{D}$ be a class of digraphs. If $\mathcal{D}$ is $\overrightarrow{\chi}$-bounded, then so is $\mathcal{D}^{subst}$.

Relation between $\overrightarrow{\chi}$-boundedness and χ-boundedness

Given a class of tournaments $\mathcal{T}$, let us denote by $\mathcal{T}^{\prec}$ the class of all backedge graphs of tournaments in $\mathcal{T}$:

$$\mathcal{T}^{\prec} = \{T^{\prec} \mid T \in \mathcal{T}, \prec \text{ an ordering of } T\}$$

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Theorem: Let $\mathcal{T}$ be a class of tournaments. The following properties are equivalent:

(i) $\mathcal{T}$ is $\overrightarrow{\chi}$-bounded.

(ii) $\mathcal{T}^{\prec}$ is χ-bounded.

(iii) $\mathcal{T}^{\prec_{\overrightarrow{\omega}}}$ is χ-bounded.

$$\mathcal{T}^{\prec} = \{T^{\prec} \mid T \in \mathcal{T}, \prec \text{ an ordering of } T\}$$

- $\mathcal{T}$ is $\overrightarrow{\chi}$-bounded $\Rightarrow \mathcal{T}^{\prec}$ is χ-bounded.

Proof: let f be a function such that for every $T \in \mathcal{T}$, we have $\overrightarrow{\chi}(T) \leq f(\overrightarrow{\omega}(T))$. Now, for every $T^{\prec} \in \mathcal{T}^{\prec}$:

$$\begin{aligned} \chi(T^{\prec}) &\leq \omega(T^{\prec}) \cdot \overrightarrow{\chi}(T) && \text{by the fundamental inequality} \\ &\leq \omega(T^{\prec}) \cdot f(\overrightarrow{\omega}(T)) \\ &\leq \omega(T^{\prec}) \cdot f(\omega(T^{\prec})) \end{aligned}$$

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- $\mathcal{T}^{\prec_{\overrightarrow{\omega}}}$ is χ-bounded $\Rightarrow$ $\mathcal{T}$ is $\overrightarrow{\chi}$-bounded.

Proof: Let g be a function such that for every $T^{\prec} \in \mathcal{T}^{\prec_{\overrightarrow{\omega}}}$, $\chi(T^{\prec}) \leq g(\omega^{\prec}(T))$. Now, for any $T \in \mathcal{T}$ and every $\overrightarrow{\omega}$-ordering $\prec$ of T.

$$\overrightarrow{\chi}(T) \leq \chi(T^{\prec}) \leq g(\omega(T^{\prec})) = g(\overrightarrow{\omega}(T))$$

Classes of tournaments defined by forbidding a single tournament

Given a tournament H, Forb(H) is the class of tournaments T such that T does not contain H as a subtournament.

Question: for which tournament H is $Forb(H)$ $\overrightarrow{\chi}$-bounded?

i.e. there is a function f such that, for every $T \in Forb(H)$,

$$\overrightarrow{\chi}(T) \leq f(\overrightarrow{\omega}(T))$$

We say that such that H are $\overrightarrow{\chi}$-binding.

Heroes

Question: for which tournament H is $Forb(H)$ $\overrightarrow{\chi}$-bounded?

The most trivial case of χ-bounding function is a constant function.

Question: for which tournament H there is a number c_H such that, for every $T \in Forb(H)$, $\overrightarrow{\chi}(T) \leq c_H$?

Answer: such tournaments are called heroes and have been characterised by Berger, Choromanski, Chudnovsky, Fox, Loebl, Scott, Seymour and Thomassé in 2013.

Gentlemen

- A tournament H is a gentlemen if there exists a number c_H such that every H-free tournaments T has $\overrightarrow{\omega}(T) \leq c_H$.

Question: Who are the gentlemen?

Of course, all heroes are gentlemen.

Tournaments and Heroes

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Theorem: *[Berger, Choromanski, Chudnovsky, Fox, Loebl, Scott, Seymour and Thomassé, 2013]*

A digraph H is a hero if and only if:

- $H = K_1$.
- $H = (H_1 \Rightarrow H_2)$
- $H = \Delta(1, k, H)$ or $H = \Delta(1, H, k)$, where $k \geq 1$ and H is a hero.

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Gentlement and heroes are the same

Theorem [PA, Aubian, Charbit, Lopes, 2023]: Heroes and gentlement are the same.

Proof:

- Take a minimal counter-example H.
- All subtournaments of H are gentlemen, and thus heroes by induction.
- Consider the sequence of tournaments $S_1, S_2, S_3, \ldots$.
- We proved that they have arbitrarily large $\overrightarrow{\omega}$.
- So H is of the form $\Delta(1, A, B)$.
- Nguyen, Scott and Seymour proved $S_3 = \Delta(1, \vec{C_3}, \vec{C_3})$ is not a gentlemen.
- So one of A or B is a transitive tournament, so H is a hero.

Gyárfás-Sumner Conjecture for tournaments

Conjecture: Let H be a tournament. $Forb(H)$ is $\overrightarrow{\chi}$-bounded if and only if H has an ordering $\prec$ for which $H^{\prec}$ is a forest.

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Recall that:

Gyárfás-Sumner Conjecture, 1981:
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We proved:

- the only if part,
- it is enough to prove it for trees instead of forests,
- if it holds for a tournament T then it holds for the tournaments obtained by reversing every arc of T,
- If H_1 and H_2 are $\overrightarrow{\chi}$-binding, then so is $H_1 \Rightarrow H_2$,
- It holds for $H = T[\vec{P}_k]$.

Theorem: *Forb*(H) is $\overrightarrow{\chi}$-bounded $\Rightarrow$ H has an ordering $\prec$ such that $H^{\prec}$ is a forest.

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  - Hence, for every $X \subseteq T$ such that $|X| = |V(H)|$, $T^{\prec}[X]$ is a forest, and thus distinct from H.
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- Observe that every $T \in \mathcal{T}[\mathcal{C}]$ has $\overrightarrow{\omega}(T) \leq 2$.

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- Observe that every $T \in \mathcal{T}[\mathcal{C}]$ has $\overrightarrow{\omega}(T) \leq 2$.
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Theorem: *Forb*(H) is $\overrightarrow{\chi}$-bounded $\Rightarrow$ H has an ordering $\prec$ such that $H^{\prec}$ is a forest.

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- Moreover, by a celebrated theorem of Erdős, graph in $\mathcal{C}$ can have arbitrarily large chromatic number.
- Hence, by the fundamental inequality, tournaments in $\mathcal{T}[\mathcal{C}]$ can have arbitrarily large dichromatic number.
- So $\mathcal{T}[\mathcal{C}]$ is not $\overrightarrow{\chi}$-bounded, and thus H-free tournaments is not $\overrightarrow{\chi}$-bounded.

Theorem: If H_1 and H_2 are $\overrightarrow{\chi}$-binding, then so it $H_1 \Rightarrow H_2$.

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There exists a function λ such that, if for every vertex v), $\overrightarrow{\chi}(T[N^+(v)]) \leq t$, then $\overrightarrow{\chi}(T) \leq \lambda(t)$.

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Theorem [Briański, Davies and Walczak, 2023+] Let $(M, \prec)$ be an order graph with maximum degree 1. Then *Forb*$_o(M, \prec)$ is χ-bounded.

Theorem: *Forb*$(H_{\vec{P}_k})$ is $\overrightarrow{\chi}$-bounded.

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Arc local to global Theorem [Klingelhoefer and Newman, 2023] If G is an oriented graph such that $\alpha(G) \leq \alpha$, and $N_{\vec{C}_3}(xy) \leq k$, then $\overrightarrow{\chi}(G) \leq f(\alpha, k)$.

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Parameterised complexity, approximation algorithms etc, nothing is known.

Relation with the Erdős-Hajnal Conjecture

Erdős-Hajnal Conjecture (1981): Let H be a graph. there exists a number c_H such that every H-free graph G has a clique or a stable set of size $|V(G)|^{c_H}$.

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Theorem: If *Forb*(H) is polynomially $\overrightarrow{\chi}$-bounded, then H has the Erdő-Hajnal property.

The BIG $\Rightarrow$ BIG Conjecture

$\mathcal{T}$ has the *BIG $\Rightarrow$ BIG* property if for every $T \in \mathcal{T}$, if $\overrightarrow{\chi}(T) \geq f(t)$, then T contains two disjoint subtournaments A and B such that $\overrightarrow{\chi}(A), \overrightarrow{\chi}(B) \geq t$ and $A \Rightarrow B$.

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Some more open questions

Conjecture: The class of tournaments with twinwidth at most k is $\overrightarrow{\chi}$-bounded.

Conjecture: For every integer $k \geq 3$, there is an infinite number of k-$\overrightarrow{\omega}$-critical tournaments. True for $k \leq 4$, maybe entirely solved by Aubian.

Conjecture (Large $\overrightarrow{\omega}$ implies a $\overrightarrow{\omega}$-cluster)
There exists two functions f and ℓ such that, for every integer k, every tournament T with $\overrightarrow{\omega}(T) \geq f(k)$ contains a subtournament X with $|X| \leq \ell(k)$ and $\overrightarrow{\omega}(X) \geq k$.

Conjecture: There exists a function g such that, if $\overrightarrow{\omega}(N^+(v)) \leq t$ for every vertex v, then $\overrightarrow{\omega}(T) \leq g(t)$.

Conjecture: for every n-vertex tournament T, $\overrightarrow{\omega}(T) = O(log(n))$

THANK YOU FOR YOUR ATTENTION