Induced subgraphs in oriented graphs with large dichromatic number

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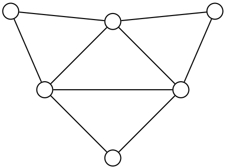
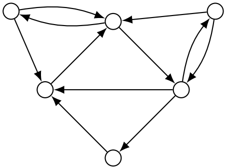
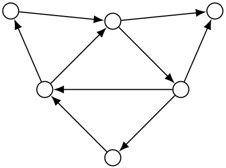
Graph and directed graph theory



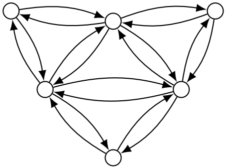
A graph



A digraph



An oriented graph



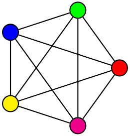
A symmetric digraph

The chromatic number

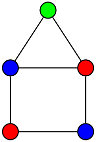
Colouring: adjacent vertices receive distinct colours.

$\Longleftrightarrow$

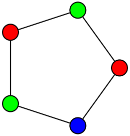
Partition the vertices into independent sets.



$\chi = 5$



$\chi = 3$



$\chi = 3$

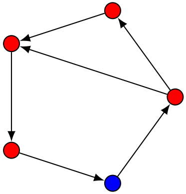
Chromatic number of $G = \chi(G)$: minimise the number of colours.

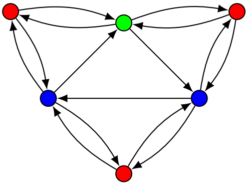
Question: How could we define directed graph colouring?

The dichromatic number

- Coloring a digraph D: no monochromatic (induced) directed cycle.
- $\vec{\chi}(D)$: the *dichromatic number* of D.

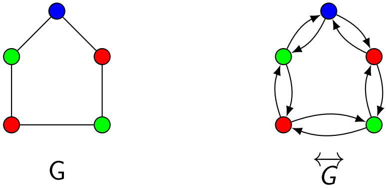
In other words: **partition D in acyclic induced subdigraphs** instead of stable sets.





Dichromatic number generalises chromatic number

Property: For every graph G, $\chi(G) = \vec{\chi}(\overleftrightarrow{G})$.



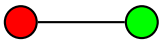
There is more and more results on the dichromatic number of digraphs for which, in the special case of symmetric digraphs, we recover an existing result on undirected graph.

Brooks' Theorem

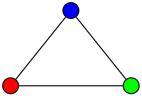
$\Delta(G)$: maximum degree of G.

Property: $\chi(G) \leq \Delta(G) + 1$

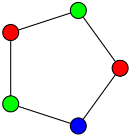
Brooks' Theorem (1932):
$\chi(G) \leq \Delta(G)$ except if G is a complete graph or an odd cycle.



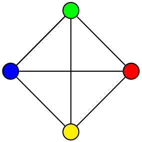
$\chi = 2$
$\Delta = 1$



$\chi = 3$
$\Delta = 2$



$\chi = 3$
$\Delta = 2$



$\chi = 4$
$\Delta = 3$

Directed Brook's Theorem

$d_{max}(v) = max(d^+(v), d^-(v))$

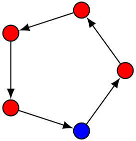
$\Delta_{max}(D) = max(d_{max}(v) : v \in D)$

$d_{min}(v) = min(d^+(v), d^-(v))$

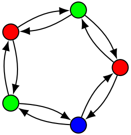
$\Delta_{min}(D) = max(d_{min}(v) : v \in D)$

Property: $\vec{\chi}(G) \leq \Delta_{min}(D) \leq \Delta_{max}(D) + 1$

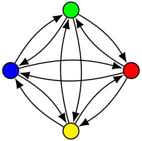
Directed Brooks' Theorem:
$\vec{\chi}(D) \leq \Delta_{max}(D)$ except if is a directed cycle, symmetric odd cycle, symmetric complete graph.



$\vec{\chi} = 2$
$\Delta_{max} = 1$



$\vec{\chi} = 3$
$\Delta_{max} = 2$



$\vec{\chi} = 4$
$\Delta_{max} = 3$

Line of research: take your favourite theorem on chromatic number, and generalise it to digraphs via the dichromatic number.

From now on, digraphs will be supposed to be digon-free.

Extending the Gyárfás-Sumner Conjecture I

Question: what can we say about the induced subgraphs of graphs with very large chromatic number?

Induced subgraphs of graphs with large chromatic number

• Let $\mathcal{F}$ be a set of graphs. $G \in Forb_{ind}(\mathcal{F})$ if G does not contains any member of $\mathcal{F}$ as an induced subgraph.

Question: for which **finite** set of graphs $\mathcal{F}$, $Forb_{ind}(\mathcal{F})$ has bounded chromatic number?

- $\mathcal{F}$ must contain a complete graph.
- $\mathcal{F}$ must contain a forest.

Because there is graphs with arbitrarily large girth[1] and chromatic number [Erdős, 60's]

Gyárfás-Sumner conjecture (1987)
For every integer k and every forest F, $Forb_{ind}(K_k, F)$ has bounded chromatic number.

[1] Size of a smallest cycle

χ-boundedness

- $\omega(G)$: size of a maximum clique of G.

$$\omega(G) \leq \chi(G) \quad \text{for every graph } G$$

A hereditary class of graphs is χ-bounded if there exists a function f such that $\chi(G) \leq f(\omega(G))$ for every G in the class.

Remark: $Forb_{ind}(F)$ is χ-bounded $\Leftrightarrow$ $Forb_{ind}(K_k, F)$ has bounded chromatic number for every k.

Gyárfás-Sumner conjecture (1987)
$Forb_{ind}(F)$ is χ-bounded if and only if F is a forest.

Result: It is enough to prove it for trees.

Induced subgraph of digraphs with large dichromatic number

Let $\mathcal{F}$ be a finite set of digraphs.

$Forb_{ind}(\mathcal{F})$ is the class of digraphs containing no member of $\mathcal{F}$ as an induced subdigraph.

Problem: What are the finite sets $\mathcal{F}$ for which $Forb_{ind}(\mathcal{F})$ has bounded dichromatic number?

Definitions

- Tournament = orientation of a complete graph.
- $\overrightarrow{C}_3$ is the directed triangle.
- Transitive tournament: tournament with no $\overrightarrow{C}_3$ and thus no directed cycle.
- TT_n: the unique transitive tournament on n vertices.

- Given two disjoint set of vertices X, Y, $X \Rightarrow Y$ means all arcs from X to Y.
- $\Delta(1, H_1, H_2)$ denotes the following digraph: $TT_1 \Rightarrow H_1 \Rightarrow H_2 \Rightarrow TT_1$.
- $\vec{C}_4(1, H_1, H_2, H_3)$ denotes the following digraph: $TT_1 \Rightarrow H_1 \Rightarrow H_2 \Rightarrow H_3 \Rightarrow TT_1$

Oriented graphs that must be contained in all heroic sets

Problem: What are the finite sets $\mathcal{F} = \{F_1, F_2, \ldots, F_t\}$ for which $Forb_{ind}(\mathcal{F})$ has bounded dichromatic number?

- $\mathcal{F}$ must contain a tournament T.

$D_1 = TT_1$, $D_k = \Delta(TT_1, D_{k-1}, D_{k-1})$.

- $\mathcal{F}$ must contain an oriented forest F.

Harutyunyan and Mohar (2012): there is oriented graphs with large dichromatic number and such that their underlying graphs have large girth.

Remark: A large tournament does not need to have a large dichromatic number. Hence, unlike the undirected case, we don't necessarily needs to bound the size of a maximum tournament to bound the dichromatic number.

Tournaments and Heroes

A tournament H is a hero if and only if the class of H-free tournaments have bounded dichromatic number.

Problem: What are the finite sets $\mathcal{F}$ for which $Forb_{ind}(\mathcal{F})$ has bounded dichromatic number?

- $\mathcal{F}$ must contain a hero H.
- $\mathcal{F}$ must contain an oriented forest F.

Hence, to have a clean picture, we need to understand heroes.

Who are the heroes?

A tournament H is a hero if and only if the class of H-free tournaments have bounded dichromatic number.

Exercise:

- Prove that $\vec{C}_3$ is a hero.
- Prove that TT_k is a hero.

Is there more heroes?

Heroes

Results of this section come from *Tournaments and colouring* of Berger, Choromanski, Chudnovsky, Fox, Loebl, Scott, Seymour and Thomassé, 2013.

A basic and useful Lemma

Lemma: A subtournament of a hero is a hero.

Proof: Let H_2 be a hero, and H_1 a subtournament of H_2.
Then H_1-free tournaments are also H_2-free.
So H_1-free tournaments have bounded dichromatic number and thus H_1 is a hero.

Strategy:

- Look at non strongly connected heroes
- Look at strongly connected heroes.

Strong components of heroes

Theorem [Berger at al.]: A tournament is a hero if and only is all its strong components are.

The *if* part follows from the fact that a subtournament of a hero is a hero.

To prove the *only if* part, it is enough to prove that:

Theorem: If H_1 and H_2 are heroes, then $H_1 \Rightarrow H_2$ is a hero.

The proof is not easy, and I unfortunately don't have enough time to present it.

Strong heroes

A stongly connected hero is called a strong hero.

How to know what a hero looks like?

Strategy: let H be a hero. Find a class of tournaments $\mathcal{T}$ with arbitrarily large dichromatic number. All tournaments in $\mathcal{T}$ with sufficiently large dichromatic must contain H.

Lemma [Berger et al.]: If H is a strong hero, then $H = \Delta(1, H_1, H_2)$ where H_1, H_2 are heroes.

Proof:

- Set $D_1 = TT_1$, $D_{k+1} = \Delta(1, D_k, D_k)$.
- H must appear in some D_k. $\square$

Strong heroes

Next Goal: prove that $\Delta(1, \vec{C}_3, \vec{C}_3)$ is not a hero (which implies that all heroes are $\Delta(1, \vec{C}_3, \vec{C}_3)$-free.

Observe that $\Delta(1, \vec{C}_3, \vec{C}_3)$ is not 2-colourable.

Backedge graph: a tool to prove that a digraph has large dichromatic number

Let D be a digraph with an order on its vertices $v_1, \ldots, v_n$.

The backedge graph B of D is the **undirected graph** induced by arcs of D that are in the **wrong direction**.

Key remark: A stable set of B is an acyclic induced subgraph of D.

So $\vec{\chi}(D) \leq \chi(B)$

Morality: a way to bound the dichromatic number of a digraph is to bound the chromatic number of one of its backedge graph.

Backedge graph: a tool to construct tournaments with large dichromatic number

Lemma: Let B be a **triangle-free (undirected) graph** with an ordering on its vertices and let D be the tournament with backedge graph B.
Then $\vec{\chi}(D) \geq \chi(B)/2$.

Proof:

- Let T be a transitive subtournament of D.
- It is enough to prove that T is the union of two stable sets of B.
- Since B is triangle-free, if $u < v < w$ and $wv, vu \in A(D)$, then $uw \notin A(D)$ and thus $uw \in A(D)$ and (u, v, w) is a $\vec{C}_3$.
- Set $X = \{x \in T$: no backedge of T ends in $x\}$
- Set $Y = \{y \in T$: no backedge of T starts in $y\}$
- It is clear that X and Y are stable sets in B.
- Since B is triangle-free, $V(T) = X \cup Y$. □

Strong heroes

Lemma [Berger et al.]: Every hero is 2-colourable. In particular $\Delta(1, \vec{C_3}, \vec{C_3})$ is not a hero.

- Let H a hero on at least 4 vertices.
- Let B an undirected graph with large chromatic number, and girth at least $V(H)$.
- Let D the tournament with backedge graph B.
- By the previous Lemma: $\vec{\chi}(D) \geq \chi(B)/2$ i.e. D has large dichromatic number.
- So D contains H.
- Since B has no cycle of length at most $V(H)$, the backedge graph of H is a forest.
- So $\vec{\chi}(H) \leq 2$.

Strong heroes

Lemma [Berger et al.]: If H is a strong hero, then $H = \Delta(1, H_1, H_2)$ where H_1, H_2 are heroes.

Lemma [Berger et al.]: $\Delta(1, \vec{C}_3, \vec{C}_3)$ is not a hero.

Corollary: If H is a strong hero, then $H = \Delta(1, H_1, TT_k)$ or $H = \Delta(1, TT_k, H_1)$ where H_1 is a hero.

From now on, We write $\Delta(1, k, H)$ for $\Delta(1, TT_k, H)$

Theorem [Berger at al.]: H is a strong hero if and only is $H = \Delta(1, k, H_1)$ or $H = \Delta(1, H_1, k)$ for some hero H_1.

Proof: hard (but important to understand).

Full characterisation of heroes

Theorem: *[Berger, Choromanski, Chudnovsky, Fox, Loebl, Scott, Seymour and Thomassé, 2015]*

A tournament H is a hero if and only if:

- $H = TT_1$, or
- $H = H_1 \Rightarrow H_2$ where H_1 and H_2 are heroes.
- $H = \Delta(1, H_1, k)$ or $H = \Delta(1, k, H_1)$ where H_1 is a hero.

Extending Gyarfas-Sumner Conjecture

Problem: What are the finite sets $\mathcal{F}$ for which $Forb_{ind}(\mathcal{F})$ has bounded dichromatic number?

- $\mathcal{F}$ must contain a hero H.
- $\mathcal{F}$ must contain an oriented forest F.

Problem: for which hero H and oriented forest F, $Forb_{ind}(H, F)$ has bounded dichromatic number?

Extending Gyarfas-Sumner Conjecture

Problem: for which hero H and oriented forest F, $Forb_{ind}(H, F)$ has bounded dichromatic number?

Theorem: $Forb_{ind}(\overrightarrow{C}_3, P_4)$ has arbitrarily large dichromatic number.

Proof: Set $D_1 = TT_1$ and $D_k = \vec{C}_4(TT_1, D_{k-1}, D_{k-1}, D_{k-1})$.
$\vec{\chi}(D_k) = k$ and $D_k \in Forb_{ind}(\overrightarrow{C}_3, P_4)$. $\square$

Remark: a digraph with no P_4 is a forest of oriented stars.

Conjecture *[Aboulker, Charbit, Naserasr, 2020]*: The set $Forb_{ind}(H, F)$ has bounded dichromatic number if and only if:

- H is a transitive tournament and F is any oriented forest, or
- H is a hero and F is a forest of oriented stars FALSE.

A first result

Conjecture *[Aboulker, Charbit, Naserasr, 2020]*: The set $Forb_{ind}(H, F)$ has bounded dichromatic number if and only if:

- H is a transitive tournament and F is any oriented forest.
- H is a hero and F is the disjoint union of stars FALSE or

Theorem *[Chudnovsky, Scott, Seymour, 2019]*
For every integer k and forest of oriented stars F, $Forb_{ind}(TT_k, F)$ has bounded chromatic number.

$\vec{\chi}$-BOUNDEDNESS

In this section, we study the following conjecture:

Conjecture: For every k and every oriented tree T, $Forb_{ind}(K_k, T)$ has bounded dichromatic number.

$\vec{\chi}$-boundedness

Given a digraph D, we denote by $\omega(D)$ the size of a largest clique in the underlying graph of D.

We say that a hereditary class of digraphs $\mathcal{C}$ is $\vec{\chi}$-bounded if there exists a function f such that for every $D \in \mathcal{C}$, $\vec{\chi}(D) \leq f(\omega(D))$.

Conjecture: For every k and every oriented forest F, $Forb_{ind}(TT_k, F)$ has bounded dichromatic number.

It is equivalent to:

Conjecture: For every k and every oriented tree T, $Forb_{ind}(TT_k, T)$ has bounded dichromatic number.

It is equivalent to:

Conjecture: For every k and every oriented tree T, $Forb_{ind}(K_k, T)$ has bounded dichromatic number.

This is because: $Forb_{ind}(TT_k, T) \subseteq Forb_{ind}(K_{2^k}, T)$

So we get a notion of $\vec{\chi}$-boundedness!

Conjecture: for every oriented tree T, $Forb_{ind}(T)$ is $\vec{\chi}$-bounded

i.e. there is a function f such that for all $G \in Forb_{ind}(T)$, $\vec{\chi}(G) \leq f(\omega(G))$.

Forbidding a directed path

Conjecture: for every oriented tree T, $Forb_{ind}(T)$ is $\vec{\chi}$-bounded.

Equivalently: for every oriented tree T and every integer k, $Forb_{ind}(K_k, T)$ has bounded dichromatic number.

Theorem [Chudnovsky, Scott, Seymour, 2019]: for every oriented star S, $Forb_{ind}(S)$ is $\vec{\chi}$-bounded.

Let $\vec{P}_k$ be the directed path on k vertices.

Conjecture: for every k, $Forb_{ind}(\vec{P}_k)$ is $\vec{\chi}$-bounded

Forbidding a path in the undirected world

This slide is on **undirected graphs**.

Theorem [Gyárfás, 80's]: $Forb_{ind}(P_t)$ is χ-bounded.

Equivalently: for every integers k, t, $Forb_{ind}(K_k, P_t)$ has bounded chromatic number.

Sketch of Proof that in a triangle-free (connected) graph with sufficiently large chromatic number, every vertex is the starting point of a long induced path.

- Let $x_0 \in V(G)$
- Since $N(x_0)$ is triangle-free, $G - N[x_0]$ has large chromatic number.
- So there exists a connected component C_1 of $G - N[x_0]$ with large chromatic number.
- Choose a vertex $x_1 \in N(x_0)$ such that x_1 has neighbours in C_1.
- Repeat the same operation in $C \cup \{x_1\}$, starting with x_1.

Forbidding a directed path

Conjecture: For every integer k, $Forb_{ind}(\overrightarrow{P}_k)$ is $\vec{\chi}$-bounded.

For $k = 1, 2$: trivial
For $k = 3$: Chudnovsky, Scott, Seymour (2019).
For $k = 4$: Cook, Pilipczuk, Masařik, Reinald, Souza (2022+)

But the Gyárfás path technique does not work in the directed case. Indeed:

- Even if an oriented graph is strongly connected, there does need to be induced directed path between any pair of vertices.

- In a triangle-free (strongly connected) oriented graph with large $\vec{\chi}$, it is not true that every vertex is the starting point of a long induced path.

Observation [Steiner]: If P is a shortest directed path, then the digraph induced by P is 2-dicolourable.

Proof: since it is a shortest directed path, it has a backedge graph isomorphic to a path.

The levelling technic

We want to prove the following:
For every integer k, $Forb_{ind}(K_k, \vec{P_4})$ has bounded dichromatic number.

Let's do it first for $k = 3$.

Let x be a vertex.

Let L_i the set of vertices at out-distance i from x.

If $\vec{\chi}(L_i) \leq k$ for every i, then $\vec{\chi}(G) \leq 2k$.

Theorem: If $G \in Forb_{ind}(K_3, \overrightarrow{P_4})$, then $\vec{\chi}(G) \leq 2$ because every L_i is a stable set.

Proof: on board

Nice sets: A tool to prove that a class of digraphs has bounded dichromatic number

Definition: A nonempty set of vertices S is nice if each vertex in S either has no out-neighbor in $V(D) \setminus S$ or has no in-neighbor in $V(D) \setminus S$.

Proposition: Let $\mathcal{C}$ be a class of digraphs such that for every $D \in \mathcal{C}$, D has a nice set S such that $\vec{\chi}(S) \leq c$. Then $\vec{\chi}(\mathcal{C}) \leq 2c$.

Proof:

- Let $G \in \mathcal{C}$ and let S be a nice set of G with dichromatic number at most c.
- Let S_{out} the set of vertices of S with no in-neighbour outside S
- Let S_{enter} the set of vertices of S with no out-neighbour outside S
- Colour $G - S$ with $2c$ colours by induction.
- colour vertices of S with colours $\{1, \dots, c\}$
- colour vertices of S with colours $\{c + 1, \dots, 2c\}$

Theorem [Cook, Pilipczuk, Masařik, Reinald, Souza (2022+)]:
If $G \in Forb_{ind}(K_k, \overrightarrow{P}_4)$, then $\vec{\chi}(G)$ is bounded.

Strategy of the proof: Every digraph in $Forb_{ind}(K_k, \overrightarrow{P}_4)$ has a nice set with bounded dichromatic number.

- Assume by induction that $Forb_{ind}(K_{k-1}, \overrightarrow{P}_4)$
- Let $G \in Forb_{ind}(K_k, \overrightarrow{P}_4)$.
- Start with a maximum tournament K of G.
- If K is not strongly connected, it has a source and a sink strongly connected component.
- Let P be a shortest directed path from the sink to the source.
- Set $C = K \cup P$, this is closed a **closed clique**.
- Let X be the set of vertices that have both an in- and an out-neighbour in C.
- Let $N = C \cup N(C) \cup N(X)$.
- They prove that N is a nice set with bounded dichromatic number.

Theorem [Cook, Pilipczuk, Masařik, Reinald, Souza (2022+)]:
for every orientation H of P_4, $Forb_{ind}(H)$ is $\vec{\chi}$-bounded.

Next step: prove that $Forb_{ind}(\vec{P}_k)$ is $\vec{\chi}$-bounded for $k \geq 5$.

Very first open case: does $Forb_{ind}(K_3, \vec{P}_5)$ has bounded dichromatic number?

Only one cycle length

We call *t*-chordal the class of digraphs in which all induced directed cycle have length exactly t. Quite surprisingly, the following holds:

Theorem [A. Carbonero, P. Hompe, B. Moore and S. Spirkl, 2022]:
The class of t-chordal digraphs is not $\vec{\chi}$-bounded.

The same authors also proved the following, using the Gyárfás path technique:

Theorem [A. Carbonero, P. Hompe, B. Moore and S. Spirkl, 2022]:
The class of t-chordal digraph with no induced $\vec{P}_t$ is $\vec{\chi}$-bounded.

Forbidding a forest of stars

In this section, we study the following problem:

Problem: For which hero H and forest of oriented stars F does $Forb_{ind}(H, F)$ have bounded dichromatic number.

Generalisation of heroes

Problem: For which hero (in tournaments) H and forest of stars F does $Forb_{ind}(H, S)$ have bounded dichromatic number.

Recall that, for very forest of oriented stars S, $Forb_{ind}(TT_k, S)$ has bounded dichromatic number (Chudnovsky, Scott and Seymour, 2019).

So we are only interested in heroes containing a $\vec{C}_3$.

But this kind of heroes are not linearly ordered as transitive tournaments, so we lose the notion of $\vec{\chi}$-boundedness.

This is the reason why we introduce the following definition:

Definition: Let $\mathcal{C}$ a class of digraphs. A digraph H is a hero in $\mathcal{C}$ if every H-free digraph in $\mathcal{C}$ has bounded dichromatic number.

Problem:
Who are the heroes in $Forb(S)$, when S is a forest of oriented stars?

Problem:
Who are the heroes in $Forb_{ind}(S)$, when S is a forest of oriented stars?

Remark: Since $Forb_{ind}(S)$ contains all tournaments, heroes in $Forb_{ind}(S)$ must be, in particular, heroes in tournaments.

Theorem: *[Berger et al]*
A digraph H is a hero in tournaments if and only if:

- $H = TT_1$, or
- $H = H_1 \Rightarrow H_2$ where H_1 and H_2 are heroes in tournaments.
- $H = \Delta(1, H_1, k)$ or $H = \Delta(1, k, H_1)$ where H_1 is a hero in tournaments.

It is tempting to conjecture that, for every forest of oriented stars S, heroes in $Forb_{ind}(S)$ are the same as heroes in tournaments, but it is unfortunately not true.

Let's try to solve this problem for the simplest types of forest of oriented stars.

Bounding the independence number

$\overline{K}_t$: t vertices, no arc. The simplest forest of stars.

$Forb_{ind}(\overline{K}_2)$ is the class of tournaments.

Question: what are the heroes in $Forb(\overline{K}_t)$?

Theorem *[Harutyunyan, Le, Newman, Thomassé, 2019]*
Heroes in $Forb_{ind}(\overline{K}_t)$ are the same as heroes in tournaments.

Equivalently, it says that a digraph in $Forb_{ind}(\overline{K}_t)$ in which all subtournaments have bounded dichromatic number also have bounded dichromatic number.

Their proof looks a lot like the proof of Berger et al. with a major difference, they use and prove the following beautiful result:

Theorem *[Harutyunyan, Le, Newman, Thomassé, 2019]*
For every $D \in Forb_{ind}(\overline{K}_t)$, if for every $x \in V(D)$, $\vec{\chi}(N^+(x))$ is bounded, then $\vec{\chi}(D)$ is bounded.

The local to global property

A digraphs is *t-local* if for every $x \in V(D)$, $\vec{\chi}(N^+(x)) \leq t$.

A class of digraphs $\mathcal{C}$ has the local to global property if there exists a function f such that for every integer t and for every t-local digraph G in $\mathcal{C}$, $\vec{\chi}(G) \leq f(t)$.

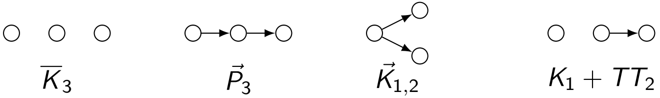
We know that tournaments, and more generally $Forb_{ind}(\overline{K}_t)$ has the local to global property.

Problem: What other classes of digraphs have the local to global property?

Small forests

Problem:
What are the heroes in $Forb(S)$, when S is a forest of oriented stars?

What about forest of oriented stars on three vertices?



We are now going to study heroes in $Forb_{ind}(\vec{P}_3)$, $Forb_{ind}(\vec{K}_{1,2})$, $Forb_{ind}(K_1 + TT_2)$.

Quasi-transitive digraphs

A digraph is **quasi-transitive** if whenever *ab* and *bc* are arcs, one of *ac* or *ca* is too.
Quasi-transitive digraphs are the same as $Forb_{ind}(\overrightarrow{P}_3)$.

A nice operation: substitution

Given two digraphs G_1 and H_1 with disjoint vertex sets, a vertex $u \in G_1$, we say that the digraph $G = G_1(u \leftarrow H_1)$ is obtained by substituting H_1 for u in G_1, provided that the following hold:

- $V(G) = (V(G_1) \setminus u) \cup V(H_1)$,
- $G[V(G_1) \setminus u] = G_1 \setminus u$,
- $G[V(H_1)] = H_1$
- for all $v \in V(G_1) \setminus u$ if v sees (resp. is seen by, resp. is non-adjacent to) u in G_1, then v sees (resp. is seen by, resp. is non-adjacent with) every vertex in $V(G_2)$ in G.

Given a class of digraphs $\mathcal{C}$, the closure of $\mathcal{C}$ under substitution denoted $\mathcal{C}^*$ is the class of digraphs that can be obtained from a vertex by repeatedly substitute some vertices by digraphs in $\mathcal{C}$.

Quasi-transitive graphs

Let $\mathcal{T}$ be the class of tournaments and $\mathcal{A}$ the class of acyclic digraphs.

Theorem [Bang-Jensen and Huang, 1995]
The class of quasi-transitive oriented graph is contained in $(\mathcal{A} \cup \mathcal{T})^*$.

Theorem [Aboulker, Aubian, Charbit, 2022]:
Heroes in $(\mathcal{A} \cup \mathcal{T})^*$ digraphs are the same as heroes in tournaments.

A equivalent way to say it is:
if $D \in (\mathcal{A} \cup \mathcal{T})^*$ is a quasi-transitive digraph in which every subtournament have bounded dichromatic number, then D has bounded dichromatic number.

Theorem: if H-free tournaments are c-dicolourable, then H-free digraphs in $(\mathcal{A} \cup \mathcal{T})^*$ are also c-dicolourable.

LOCAL OUT-TOURNAMENTS

Local out-tournament

G is a **local out-tournament** if for every vertex x, $N^+(x)$ is a tournament.

The class of local out-tournaments is the same as $Forb_{ind}(\vec{K}_{1,2})$.

Theorem [Steiner / Aboulker, Aubian, Charbit, 2021]: $K_1 \Rightarrow \vec{C}_3$ is a hero in local out-tournaments.

We even have: $\vec{\chi}(Forb_{ind}(K_1 \Rightarrow \overrightarrow{C}_3, \vec{K}_{1,2})) = 2$

Heroes in local out-tournaments

Conjecture: for every hero H, H-free local out-tournaments have bounded dichromatic number.

- One possible strategy:
  - Prove that if H_1 and H_2 are heroes in local out-tournaments, the so is $H_1 \Rightarrow H_2$.
  - Prove that if H is a hero in local out-tournaments, than so is $\Delta(1, k, H)$ and $\Delta(1, H, k)$.
- Another strategy:
  - Prove that if D is a local out-tournament in which every subtournament has bounded dichromatic number, then D has bounded dichromatic number.
- A last strategy:
  - Construct a counter example.

Oriented complete multipartite graphs

Heroes in complete multipartite oriented graphs

$Forb_{ind}\,(\overrightarrow{K}_2 + K_1)$ is the class of oriented complete multipartite graphs, OCMG for shorts.

Theorem *[A., Aubian, Charbit 2021+]*:
If H_1 and H_2 are heroes in OCMG, then so is $H_1 \Rightarrow H_2$.

Theorem *[A., Aubian, Charbit 2021+]*:
If H is a hero in OCMG, then so is $\Delta(1, 1, H)$.

Heroes in complete multipartite oriented graphs

Theorem*[A., Aubian, Charbit, 2021+]*: A digraph H is a hero in OCMG if:

- $H = K_1$,
- $H = H_1 \Rightarrow H_2$, where H_1 and H_2 are heroes in OCMG, or
- $H = \Delta(1, 1, H_1)$ where H_1 is a hero in OCMG.

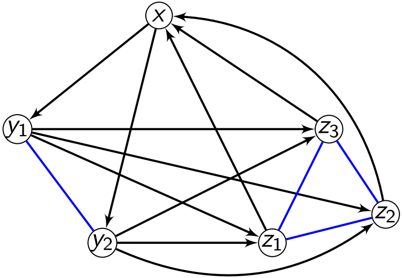
Question: What about $\Delta(1, 2, H)$?

So the first open cases are:

- $\Delta(1, 2, 2)$
- $\Delta(1, 2, \vec{C_3})$, $\Delta(1, 2, 3)$

Heroes in complete multipartite oriented graphs

Theorem *[A., Aubian, Charbit 2021+]*:
$\Delta(1,2,3)$ and $\Delta(1,2,\vec{C}_3)$ are not heroes in OCMG.



In particular: there exists OCMG with arbitrarily large dichromatic number in which all subtournaments have bounded dichromatic number.

Heroes in complete multipartite oriented graphs

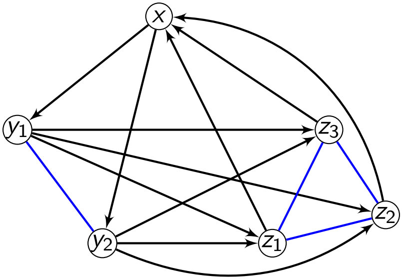
Strategy:

- Define the line graph $L(G)$ of an oriented graph.
- Prove that $\chi(L(G)) \geq log(\chi(G))$.
- Build a oriented complete multipartite graphs from $L(L(TT_n))$.
- Prove it has large dichromatic number.
- Prove it does not contain $\Delta(1, 2, 3)$ nor $\Delta(1, 2, \vec{C}_3)$.

Feedback arc set of $\Delta(1, 2, 3)$

Feedback arc set: set of arcs F such that their deletion leads to an acyclic digraph.

Observation: all feedback arc sets of $\Delta(1, 2, 3)$ and $\Delta(1, 2, \vec{C}_3)$, contain a vertex of out- or in-degree at least 2.



Line graph of digraphs

The line graph $L(D)$ of a digraph D is the following digraph:

- vertex set is $A(D)$.
- ef is an arc of $L(D)$ if $e = uv$ and $f = vw$.

Be aware that the following Lemma is on the chromatic number of the underlying graphs.

Lemma: for every digraph D, $\chi(L(D)) \geq \log(\chi(D))$.

Proof:

- Assume $L(D)$ admits a k-colouring.
- Observe that a colouring of $L(D)$ is a colouring of the arcs of D is such a way that no $\vec{P}_3$ is monochromatic.
- For each $v \in V(D)$, colour v with the set of colours used by the arcs entering in v.
- Prove that is it a 2^k-colouring of D.

Let's have a look at $L(L(TT_s))$.

Set $V(TT_s) = (v_1, v_2, \ldots, v_s)$.

So the vertices of $L(L(TT_s))$ are: $\{(v_i, v_j, v_k) : 1 \leq i < j < k \leq s\}$.

And its set of arcs is: $\{(v_i, v_j, v_k)(v_j, v_k, v_\ell) : 1 \leq i < j < k < \ell \leq s\}$.

Set $V_j = \{(v_i, v_j, v_k) : i < j < k\}$.

Define the oriented complete multipartite graphs D_s with parts $V_1, \ldots, V_s$ like that:

- Edges of $L(L(TT_s))$ are oriented from left to right: forward arcs
- All the other edges from right to left: backward arcs.

Observation: given a vertex (v_i, v_j, v_k) of D_s:

- the forwards arcs going out (v_i, v_j, v_k) are included in V_k
- the forward arcs going in (v_i, v_j, v_k) are included in V_i.

Hence, subtournaments of D_s cannot be equal to $\Delta(1, 2, 3)$ nor to $\Delta(1, 2, \vec{C}_3)$

Lemma: $\vec{\chi}(D_s) \geq \frac{1}{2} \log(\log(s))$

Proof:

- An acyclic subgraph of D_s is made of disjoint out- or in-stars of $L(L(TT_s))$.
- Hence, an acyclic subgraph of D_s can be partitioned into two stable sets of $L(L(TT_s))$
- So $\log(\log(s)) \leq \chi(L(L(TT_s))) \leq 2\vec{\chi}(D_s)$

Chararterization of heroes in OCMG

Theorem*[A., Aubian, Charbit 2021+]*: A digraph H is a hero in OCMG if:

- $H = K_1$,
- $H = H_1 \Rightarrow H_2$, where H_1 and H_2 are heroes in OCMG, or
- $H = \Delta(1, 1, H_1)$ where H_1 is a hero in OCMG.

$\Delta(1, 2, 3)$ and $\Delta(1, 2, \vec{C_3})$ are not heroes in OCMG

Open Question: is $\Delta(1, 2, 2)$ a hero in OCMG?

If it is not, then the above theorem is a characterisation of heroes in OCMG, otherwise the following is:
A digraph H is a hero in OCMG if and only if:

- $H = K_1$ or $H = \Delta(1, 2, 2)$,
- $H = H_1 \Rightarrow H_2$, where H_1 and H_2 are heroes in OCMG, or
- $H = \Delta(1, 1, H_1)$ where H_1 is a hero in OCMG.

Heroes in oriented chordal graphs

Results of this section is a joint work with Raphael Steiner and Guillaume Aubian.

Heroes in oriented chordal graphs

A chordal graph is a graph with no induced directed cycle of length at least 4.

Lemma [Dirac, 60's]:
A graph G is chordal if and only:

- G is a clique, or
- G admits a clique cutset.

In other words: Chordal graph can be obtained by gluing complete graphs along cliques.

Oriented chordal graphs: all orientations of chordal graphs.

Question: Who are the heroes in oriented chordal graph?

Question: is it true that an oriented chordal graph in which every subtournament have bounded dichromatic number, itself has bounded dichromatic number? NO!

$\Delta(1,2,2)$ is not a hero in oriented chordal graphs

We construct a sequence of oriented chordal graph $D_1, D_2, \ldots$ such that

- $\vec{\chi}(D_k) = k$,
- D_k does not contain $\Delta(1,2,2)$,
- and actually all tournaments in D_k have dichromatic number at most 2.

$\Delta(1, 2, 2)$ is not a hero in oriented chordal graphs

$G_1 = K_1$. Assuming D_k is known, construct D_{k+1} as follows:

- Start with a $TT_{k+1} = T$.
- For each arc $ab \in A(T)$, add a copy R_{ab} of D_{k+1} and the following arcs:
  - $b \Rightarrow R_{ab}$ and
  - $R_{ab} \Rightarrow a$.
- that is, each vertex of R_{ab} form a triangle with a and b.

We need to prove that:

- D_k is an oriented chordal graph,
- $\vec{\chi}(D_k) = k$,
- D_k is $\Delta(1, 2, 2)$-free.

$\vec{C}_3 \Rightarrow K_1$ is not a hero in oriented chordal graphs

We construct a sequence of oriented chordal graph $G_1, G_2, \dots$ such that

- $\vec{\chi}(G_k) = k$,
- G_k does not contain $\vec{C}_3 \Rightarrow K_1$.

Given a dicolouring of a digraph G, a subset of $S \subset V(G)$ is rainbow if each vertex of S receives distinct colours.

Lemma: Assume G_k is known. There exists a digraph $F = F(G_k)$ such that:

- $\vec{\chi}(F) = k$ and
- For every k-dicolouring of F, F contains a rainbow TT_k.

Heroes in oriented chordal graphs

Question: Who are the heroes in oriented chordal graphs?

We know that:

- It must be a hero in tournaments
- It does not contain $\Delta(1,2,2)$, nor $K_1 \Rightarrow \vec{C}_3$, nor $\vec{C}_3 \Rightarrow K_1$.

Hence, the only candidate are TT_k and $\Delta(1,1,k)$.

Exercise: prove that TT_k is a hero in oriented chordal graphs.

Theorem: Heroes in oriented chordal graphs are precisely transitive tournaments and $\Delta(1,1,k)$.

Proof: Look at a simplicial vertex.

Interval graphs

Theorem: Heroes in oriented unit interval graph are the same as heroes in tournaments.

Open Question: Is it true that heroes in oriented interval graphs are the same as heroes in tournaments?

Digraphs with all directed cycle of the same length are not $\vec{\chi}$-bounded