Heroes in orientations of chordal graphs

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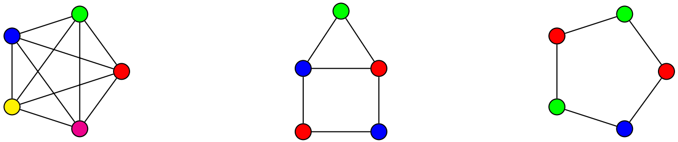
The chromatic number

Colouring: adjacent vertices receive distinct colours.

$\Leftrightarrow$

Partition the vertices into independent sets.



$\chi = 5$ $\chi = 3$ $\chi = 3$

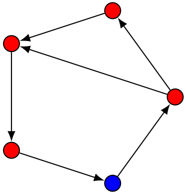
Chromatic number of $G = \chi(G)$: minimise the number of colours.

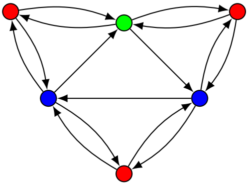
Question: How could we define directed graph colouring?

The dichromatic number

- Coloring a digraph D: no monochromatic (induced) directed cycle.
- $\vec{\chi}(D)$: the *dichromatic number* of D.

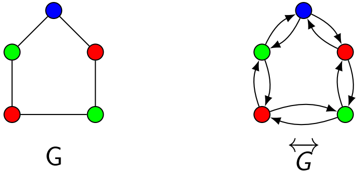
In other words: **partition D in acyclic induced subdigraphs** instead of stable sets.





Dichromatic number generalises chromatic number

Property: For every graph G, $\chi(G) = \vec{\chi}(\overleftrightarrow{G})$.



There is more and more results on the dichromatic number of digraphs for which, in the special case of symmetric digraphs, we recover an existing result on undirected graph.

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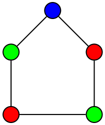
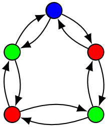


G



$\overleftrightarrow{G}$

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- Brooks' Theorem, Gallaï Theorem, Wilf Theorem (algebraic graph theory)...
- Extremal graph theory,
- List dichromatic number,
- Substructure forced by large dichromatic number,
- Dicolouring digraphs on surfaces.

Tournaments

- Tournament = orientation of a complete graph.
- $\overrightarrow{C}_3$ is the directed triangle.
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Question: why does a tournament have large dichromatic number?

Tournaments and Heroes

- A tournament H is a hero if and only if the class of H-free tournaments have bounded dichromatic number.

For example, $\overrightarrow{C}_3$ and TT_k are heroes (TT_k-free tournaments have at most 2^k vertices by Ramsey).

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Theorem: *[Berger, Choromanski, Chudnovsky, Fox, Loebl, Scott, Seymour and Thomassé, 2013]*

A digraph H is a hero if and only if:

- $H = K_1$.
- $H = (H_1 \Rightarrow H_2)$
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Definition: Let $\mathcal{C}$ be a class of digraphs.
H is a hero in $\mathcal{C}$ if H-free digraphs of $\mathcal{C}$ have bounded dichromatic number.

Chordal graphs

- A graph is chordal if it has no induced cycle of length at least 4.
- Chordal graphs are perfect.

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Answer: NO

Question: who are the heroes in oriented chordal graphs.

Theorem [A., Aubian, Steiner]: H is a hero in oriented chordal graphs if and only if H is a transitive tournament or $\Delta(1, 1, k)$.

Proof:

- TT_k-free oriented chordal graphs have bounded dichromatic number.
- $\Delta(1, 1, k)$-free oriented chordal graphs have bounded dichromatic number.

- $\Delta(1, 2, 2)$-free oriented chordal graphs can have large dichromatic number.
- $(\vec{C}_3 \Rightarrow K_1)$-free oriented chordal graphs can have large dichromatic number.

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Theorem [A., Aubian, Steiner]: heroes in oriented unit interval graphs are the same as heroes in tournaments.

More precisely:
Given an oriented unit interval graph G,
if all its subtournaments have dichromatic number at most c,
then G has dichromatic number at most $2c$.