# Induced subgraphs of digraphs with large dichromatic number.

Lecture Notes of the Course given at SGT 2022

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## 1 Definitions and notations

Let $G$ be a digraph. We denote by $V(G)$ its set of vertices and by $A(G)$ its set of arcs. For a vertex $x$ of $G$, we denote by $N^+(x)$ (resp. $N^-(x)$, $N^o(x)$) the set of its out-neighbours (resp. in-neighbours, non-neighbours). For a given set of vertices $X \subseteq V(G)$, we denote by $G[X]$ the subgraph of $G$ induced by $X$.

For two disjoint set of vertices $X, Y$, we write $X \Rightarrow Y$ to say that for every $x \in X$ and for every $y \in Y$, $xy \in A(G)$, and we write $X \rightarrow Y$ to say that every arc with one end in $X$ and the other one in $Y$ is oriented from $X$ to $Y$ (but some vertices of $X$ might be non-adjacent to some vertices of $Y$). When $X = \{x\}$ we write $x \Rightarrow Y$ and $x \rightarrow Y$.

A **tournament** is an orientation of a complete graph. A **transitive tournament** is an acyclic tournament and we denote by $TT_n$ the unique acyclic tournament on $n$ vertices. Given a transitive tournament $T$ on $n$ vertices $\{v_1, \dots, v_n\}$, we say that $v_1, \dots, v_n$ is the **topological ordering** of $T$ if, for all $1 \leq i < j \leq n$, we have $v_i v_j \in A(T)$.

Given two tournaments $H_1$ and $H_2$, we denote by $\Delta(1, H_1, H_2)$ the tournament obtained from pairwise disjoint copies of $H_1$ and $H_2$ plus a vertex $x$, and all arcs from $x$ to the copy of $H_1$, all arcs from the copy of $H_1$ to the copy of $H_2$, and all arcs from the copy of $H_2$ to $x$. When $\ell$ and $k$ are integers, we write $\Delta(1, k, H)$ for $\Delta(1, TT_k, H)$ and $\Delta(1, \ell, k)$ for $\Delta(1, TT_\ell, TT_k)$. The tournament $\Delta(1, 1, 1)$ is also denoted by $\vec{C_3}$ and called a **directed triangle**.

Given a class of digraphs $\mathcal{C}$, we say that a digraph $H$ is a **hero in** $\mathcal{C}$ is all $H$-free digraphs in $\mathcal{C}$ have bounded dichromatic number.

Given a set of digraphs $\mathcal{F}$, we denote by $Forb_{ind}(\mathcal{F})$ the class of digraphs which have no member of $\mathcal{F}$ as an induced subgraph. We extend the notation $Forb_{ind}(\mathcal{F})$ by allowing (non-oriented) graphs in $\mathcal{F}$. If $\mathcal{F}$ is such a set, we define $Forb_{ind}(\mathcal{F})$ to be the set of digraphs that does not contain as an induced subdigraph: any digraph of $\mathcal{F}$, and any orientation of a non-oriented graph of $\mathcal{F}$.

Given a class of digraphs $\mathcal{C}$, we define the chromatic number $\chi(\mathbf{C}) = \max\{\chi(\mathbf{G}) \mid \mathbf{G} \in \mathcal{C}\}$ with understanding that $\chi(\mathcal{C}) = \infty$ when it is not bounded.

## 2 Induced subgraphs of digraphs with large dichromatic number

A $k$**-colouring** of a graph $G$ is a mapping $\phi : V(G) \to [1, k]$ such that for all $i \in [1, k]$, $\phi^{-1}(i)$ is a stable set. In other words it is a partition of the vertices of $G$ into at most $k$ stable sets. The **chromatic number** $\chi(G)$ of a graph $G$ is the minimum $k$ such that $G$ admits a $k$-colouring.

It is not an easy task to extend the notion of chromatic number to digraphs in a meaningful way. Anyway, since a few years, more and more results are showing that the (now) so-called notion of **dichromatic number** is the right concept to generalise chromatic number to directed graphs, and more and more efforts are made to extend colouring results from undirected graphs to directed graphs through this notion.

A $k$**-dicolouring** of a digraph $D$ is a mapping $\phi : V(D) \to [1, k]$ such that for all $i \in [1, k]$, $\phi^{-1}(i)$ is acyclic (that is, $\phi^{-1}(i)$ has no directed cycle). In other words, it is a partition of the vertices of $D$ in at most $k$ sets such that each of these sets induce a directed acyclic graph. The **dichromatic number** $\vec{\chi}(D)$ of a digraph $D$ is the minimum $k$ such that $D$ admits a $k$-dicolouring.

The dichromatic number was first introduced by Neumann-Lara [20] in 1982 and was rediscovered by Mohar [18] 20 years later. It is easy to see that for any undirected graph $G$, the *symmetric digraph* $\overleftrightarrow{G}$ obtained from $G$ be replacing each edge by a digon satisfies $\chi(G) = \vec{\chi}(\overleftrightarrow{G})$. This simple fact permits to generalise results on the chromatic number of undirected graphs to digraphs via the dichromatic number.

Let us give an example. The maximum degree of an undirected graph $G$ is denoted by $\Delta(G)$. It is an easy observation that for every graph $G$, $\chi(G) \leq \Delta(G) + 1$. The following classical result of Brooks characterises the (very few) graph for which equality holds.

**Theorem 2.1** (Brooks' Theorem, [8])**.** *A graph $G$ satisfies $\chi(G) = \Delta(G) + 1$ if and only if $G$ is an odd cycle or a complete graph.*

This result was generalised by Mohar[19] (see also [1]) to digraphs via the dichromatic number. First, observe that the maximum degree of a graph does not have a clear analogue for digraphs. We now introduce two ways to measure maximum degree in a digraph that make sense in the context of Brooks' Theorem. Let $v$ be a vertex of a digraph $G$. We define the **maxdegree** of $v$ as $d_{max}(v) = \max(d^+(v), d^-(v))$ and the **mindegree** of $v$ as $d_{min}(v) = \min(d^+(v), d^-(v))$. We can then define the corresponding maximum degrees: $\Delta_{max}(G) = \max_{v \in V(G)}(d_{max}(v))$ and $\Delta_{min}(G) = \max_{v \in V(G)}(d_{min}(v))$.

**Exercise 2.2.** For every digraph $G$, $\vec{\chi}(G) \leq \Delta_{min}(G) + 1 \leq \Delta_{max}(G) + 1$.

We are now ready to state the directed Brooks' Theorem:

**Theorem 2.3** ([19, 1])**.** *Let $G$ be a connected digraph, then $\vec{\chi}(G) \leq \Delta_{max}(G) + 1$ and equality holds if and only if one of the following occurs: $G$ is a directed cycle, or a bidirected cycle of odd length or a bidirected complete graph.*

Observe that we recover the undirected Brooks' Theorem by adding the hypothesis that the digraph is bidirected in the above theorem. It is interesting to not that one cannot replace $\Delta_{max}$ by $\Delta_{min}$ in the above Theorem, it is actually NP-complete to decide if a digraph $D$ is $\Delta_{min}(D)$-dicolourable [1].

More and more theorems of this flavour are being proved this past decade and this is an exciting line of research: take your favourite colouring theorem, and try to generalise it to digraph.

## 3 The Gyárfás-Sumner Conjecture

Given a set of graphs $\mathcal{F}$, we denote by $Forb_{ind}(\mathcal{F})$ the class of graphs which have no member of $\mathcal{F}$ as an induced subgraph. Given a class of graphs $\mathcal{C}$, we define the chromatic number $\chi(\mathbf{C}) = \max\{\chi(\mathbf{G}) \mid \mathbf{G} \in \mathcal{C}\}$ with understanding that $\chi(\mathcal{C}) = \infty$ when it is not bounded.

The following question has been deeply studied: what induced substructures are expected to be found inside a graph if we assume it has very large chromatic number? Or equivalently what are the minimal families $\mathcal{F}$ such that $Forb_{ind}(\mathcal{F})$ has bounded chromatic number? See [21] for a survey on this question. Let us investigate the case where $\mathcal{F}$ is finite. Since complete graphs have unbounded chromatic number and do not contain any induced subgraph other than complete graphs themselves, it is clear that such an $\mathcal{F}$ must contain a complete graph. Moreover, Erdős's celebrated result on the existence of graphs of high girth and high chromatic number [13] implies that if $\mathcal{F}$ is finite, then at least one member of $\mathcal{F}$ must be a forest. These two facts constitute the "only if" part of the following tantalising and still widely open conjecture of Gyárfás and Sumner (see section 3 of [21] for a survey on known results).

**Conjecture 3.1** (Gyárfás-Sumner, [24, 15])**.** Given a set of graphs $\mathcal{F}$, $Forb_{ind}(\mathcal{F})$ has bounded chromatic number if and only if $\mathcal{F}$ contains a complete graph and a forest.

We denote by $\omega(G)$ the size of a maximum clique in $G$. It is clear that for every graph $G$, $\omega(G) \leq \chi(G)$. Given a class of graphs $\mathcal{C}$, we say that $\mathcal{C}$ is $\chi$**-bounded** is for every $G \in \mathcal{C}$, $\chi(G) \leq f(\omega(G))$. This leads to the following equivalent formulation of the Gyárfás-Sumner Conjecture.

**Conjecture 3.2** (Gyárfás-Sumner Conjecture [24, 15])**.** For every forest $T$, $Forb_{ind}(T)$ is $\chi$-bounded.

**Exercise 3.3.** It is enough to prove the Gyárfás-Sumner Conjecture for trees.

# 4 Extending the Gyárfás-Sumner conjecture to digraphs

The whole course is on the following problem:

**Problem 4.1.** What are the finite sets $\mathcal{F}$ of digraphs for which the class $Forb_{ind}(\mathcal{F})$ has bounded dichromatic number?

Harutyunyan and Mohar [16] proved the existence of digraphs with large dichromatic number, and whose underlying graph has large girth. Hence, $Forb_{ind}(\mathcal{F})$ has bounded dichromatic number only if $\mathcal{F}$ contains an oriented forest.

Whereas complete graphs are somehow trivial objects regarding chromatic number, tournaments are already a complex and rich family regarding dichromatic number. Observe for example that the *transitive tournament* on $n$ vertices (i.e. the unique up to isomorphism tournament on $n$ vertices that contains no directed cycle), denoted by $TT_n$, has dichromatic number 1. On the other hand, there exists tournaments with arbitrarily large dichromatic number, as we explain now.

Let us construct a sequence of tournaments $S_k$ such that $\vec{\chi}(S_k) = k$. Let $S_1$ be the tournament on one vertex. Having defined $S_k$, set $S_{k+1} = \Delta(1, S_k, S_k)$. If you try to $k$-dicolour $S_{k+1}$, then the two copies of $S_k$ must use the $k$ colours, and no colour is available to colour the last vertex. Hence, $\vec{\chi}(S_k) = k$ for every $k$.

This shows that a finite set $\mathcal{F}$ such that $Forb_{ind}(\mathcal{F})$ has bounded dichromatic number must contain a tournament.

## 4.1 Heroes in tournaments

In a seminal paper [7], Berger et al. gave a full characterisation of tournaments $H$ such that every $H$-free-tournament has bounded dichromatic number. Such tournaments are said to be **heroes in tournaments**.

**Theorem 4.2** (Berger et al. [7])**.** *A digraph $H$ is a hero in tournaments if and only if:*

- $H = K_1$*, or*
- $H = H_1 \Rightarrow H_2$*, where $H_1$ and $H_2$ are heroes in tournaments, or*
- $H = \Delta(1, k, H_1)$ *or* $H = \Delta(1, H_1, k)$*, where $k \geq 1$ and $H_1$ is a hero in tournaments.*

## 4.2 Strong connected components of heroes

The goal of the subsection is to prove this second bullet of Theorem 4.2:

**Theorem 4.3.** *If $H_1$ and $H_2$ are heroes in tournaments, then so is $H_1 \Rightarrow H_2$.*

Observe that it implies that a digraph is a hero in tournaments if *and only if* each of its strong connected components are. Indeed, the *only if* part of the assertion holds because a subgraph of a hero in tournament (or in any class of digraphs) is a hero in tournament (or in this other class of digraphs).

We actually prove the following stronger result:

**Theorem 4.4.** *Let $H_1$, $H_2$ and $F$ be digraphs such that $H_1 \Rightarrow H_2$ is a hero in $Forb_{ind}(F)$ and $H_1$ and $H_2$ are heroes in $Forb_{ind}(K_1 + F)$. Then $H_1 \Rightarrow H_2$ is a hero in $Forb_{ind}(K_1 + F)$.*

To see that Theorem 4.4 implies Theorem 4.3, take $F = K_1$ and observe that $Forb_{ind}(K_1)$ is empty (so all digraphs is a hero in it), and that $Forb_{ind}(K_1 + K_1)$ is the class of tournaments.

The rest of this subsection is devoted to the proof of Theorem 4.4. The proof comes from [**?**], and is inspired but simpler (we got rid of the intricate notion of *r-mountain*) than the proof of Theorem 4.4 in [7], even though it is more general.

We start with a few definitions and notations. First, in order to simplify statements of the lemmas, we assume $H_1$, $H_2$ and $F$ are fixed all along the subsection and are as in the statement of Theorem 4.4. So there exists a constant $c$ such that:

- $H_1$ and $H_2$ have at most $c$ vertices,
- digraphs in $Forb_{ind}(F, H_1 \Rightarrow H_2)$ have dichromatic number at most $c$,
- for $i = 1, 2$, digraphs in $Forb_{ind}(K_1 + F, H_i)$ have dichromatic number $c$.

If $G$ is a digraph and $uv \in E$, we set $C_{uv} = N^+(v) \cap N^-(u)$, that is the of vertices that form a directed triangle with $u$ and $v$. Finally, for $t \geq 1$, we say that a digraph $K$ is a *t-cluster* if $\vec{\chi}(K) \geq t$ and $|V(K)| \leq f(t)$, where $f(t)$ is the function defined recursively by $f(1) = 1$ and $f(t) = 1 + f(t-1)(1 + f(t-1))$.

The structure of the proof is very simple, we prove that digraphs in $Forb_{ind}(K_1 + F, H_1 \Rightarrow H_2)$ that do not contain a $t$-cluster for some $t$ have bounded dichromatic number (Lemma 4.5), and then that the ones that contains a $t$-cluster for some $t$ also have bounded dichromatic number (Lemma 4.6).

**Lemma 4.5.** *There exists a function $\phi$ such that if $t$ is an integer and $G$ is a digraph in $Forb_{ind}(K_1 + F, H_1 \Rightarrow H_2)$ which contains no $t$-cluster as a subgraph, then $\vec{\chi}(G) \leq \phi(c, t)$*

*Proof.* We prove this by induction on $t$. For $t = 1$ the result is trivial as a 1-cluster is simply a vertex. Assume the existence of $\phi(c, t-1)$, and assume $G$ is a digraph in $Forb_{ind}(K_1 + F, H_1 \Rightarrow H_2)$ which contains no $t$-cluster. Say an arc $uv$ is *heavy* if $C_e$ contains a $(t-1)$-cluster, and *light* otherwise. For a vertex $u$ we define $N_h(u) = \{v \in V(G) \mid uv \text{ or } vu \text{ is a heavy arc}\}$.

**Claim 1.** *For any vertex $u$, $N_h(u)$ contains no $(t-1)$-cluster, and thus $\vec{\chi}(N_h(u)) \leq \phi(c, t-1)$.*

*Proof.* Assume by contradiction that $K$ is a $(t-1)$-cluster in $N_h(u)$. By definition of $N_h(u)$, for every $v \in V(K)$, there exists a $(t-1)$-cluster $K_v$ in $C_{uv}$ or $C_{vu}$ (depending on which of $uv$ or $vu$ is an arc). Let $K' = \{u\} \cup V(K) \cup (\cup_{v \in K} V(K_v))$. We claim that $K'$ is a $t$-cluster. First note that the number of vertices of $K'$ is at most $1 + f(t-1) + f(t-1) \cdot f(t-1) = f(t)$. We need to prove that $K'$ is not $(t-1)$-colourable, so let us consider for contradiction a $(t-1)$-colouring of its vertices, and without loss of generality assume $u$ gets colour 1. Because $K$ is a $(t-1)$-cluster, some vertex $v$ in $K$ must also receive colour 1, and since $K_v$ is also a $(t-1)$-cluster, some vertex $w$ in $K_v$ must also receive colour 1, which produces a monochromatic directed triangle. So $K'$ is indeed a $t$-cluster, a contradiction. ♦

**Claim 2.** *For any vertex $u$, $\min(\vec{\chi}(N^-(u)), \vec{\chi}(N^+(u))) \leq (c+1) \cdot (c + \phi(c, t-1))$.*

*Proof.* Let $u \in V(G)$. By the previous claim and the induction hypothesis, $N_h(u)$ induces a digraph of dichromatic number at most $\phi(c, t-1)$, so it is enough to prove that one of the sets $N_\ell^-(u) := (N^-(u) \setminus N_h(u))$ and $N_\ell^+(u) := (N^+(u) \setminus N_h(u))$ induces a digraph with dichromatic number at most $c \cdot \phi(c, t-1) + c \cdot (c+1)$.

If $N_\ell^+(u)$ induces a $H_2$-free digraph, then it has dichromatic number at most $c < c \cdot \phi(c, t-1) + c \cdot (c+1)$, so we can assume that there exists $V_2 \subseteq N_\ell^+(u)$ such that $G[V_2] = H_2$. We now partition $N_\ell^-(u)$ into three sets $A, B, C$, each of which will have bounded dichromatic number.

Let $A = N_\ell^-(u) \cap (\cup_{v \in V_2} N^+(v)) = N_\ell^-(u) \cap (\cup_{v \in V_2} C_{uv})$. For every $v \in V_2$, $uv \in E$ is light (because $V_2 \subseteq u_\ell^-$), so $G[C_{uv} \cap A]$ does not contain a $(t-1)$-cluster and is thus $\phi(c, t-1)$-colourable by induction. Now, since $H_2$ contains at most $h$ vertices, we get $\vec{\chi}(A) \leq c \cdot \phi(c, t-1)$.

Let $B = N_\ell^-(u) \cap (\cup_{v \in V_2} v^0)$. Since $G$ is $(K_1 + F, H_1 \Rightarrow H_2)$-free, for every $v \in V_2$, $N^o(v)$ is $(F, H_1 \Rightarrow H_2)$-free and thus $\vec{\chi}(G[N^o(v)]) \leq c$. Hence, $\vec{\chi}(B) \leq c^2$.

Finally, consider $C = N_\ell^-(u) \setminus (A \cup B)$. By definition of $A$ and $B$, we get $C \Rightarrow V_2$. Since $G$ is $H_1 \Rightarrow H_2$-free, $G[C]$ is $H_1$-free, and therefore $\vec{\chi}(C) \leq c$.

All together, we get $\vec{\chi}(N_\ell(x)^-) \leq c \cdot \phi(c, t-1) + c \cdot (c+1)$ as desired. ♦

By the previous claim, we can partition the set of vertices into the two sets $V^-$ and $V^+$ defined by:

$$V^- = \{u \in V \mid \vec{\chi}(N^-(u)) \leq (c+1) \cdot (c + \phi(c, t-1))\}$$
$$V^+ = \{u \in V \mid \vec{\chi}(N^+(u)) \leq (c+1) \cdot (c + \phi(c, t-1))\}$$

If $G[V^-]$ is $H_1$-free and $G[V^+]$ is $H_2$-free, then $\vec{\chi}(G) \leq 2c < \phi(c, t)$ and we are done. Assume that there exists $V_1 \subseteq V^-$ such that $G[V_1] = H_1$ (the case where $V^+$ contains an induced copy of $H_2$ is symmetrical).

We now partition $V(G) \setminus V_1$ into three sets of vertices depending on their relation with $V_1$ and prove that each of these set induces a digraph with bounded dichromatic number.

Let $A = \bigcup_{v \in V_1} N^-(v)$. By definition of $N^-(v)$ and since $V_1 \subseteq N^-(v)$, for every $v \in V_1$, $N^-(v)$ has dichromatic number at most $(c+1)(c+\phi(c, t-1))$, and since $H_1$ has at most $c$ vertices we get that $\vec{\chi}(A) \leq c \cdot (c+1) \cdot (c + \phi(c, t-1))$.

Let $B = \bigcup_{v \in V_1} v^0$. Since $G$ is $(K_1 + F, H_1 \Rightarrow H_2)$-free, for every $v \in V_1$, $v^0$ is $(F, H_1 \Rightarrow H_2)$-free and thus $\vec{\chi}(G[v^0]) \leq c$. Hence, $\vec{\chi}(B) \leq c \cdot h$.

Finally, let $C = V(G) \setminus (A \cup B \cup V_1)$. By definition of $A$ and $B$, we have $V_1 \Rightarrow C$, hence $C$ is $H_2$-free and thus $\vec{\chi}(C) \leq c$.

All together, we get that $\vec{\chi}(G) \leq h + h \cdot (h+1) \cdot (c + \phi(c, t-1)) + ch + c := \phi(c, t)$.

□

The proof of the theorem will follow from the second lemma below.

**Lemma 4.6.** *If $G \in \mathcal{C}$ contains a $(3c+1)$-cluster, then $\vec{\chi}(G) \leq c \cdot 2^{f(3c+1)+1}$.*

*Proof.* Let $K$ be a $(3c+1)$-cluster in $G$. Assume there exists a vertex $u \in V(G)$ such that $N^-(u) \cap V(K)$ is $H_1$-free and $N^+(u) \cap V(K)$ is $H_2$-free. Since $u^0 \cap V(K)$ is by assumption $(F, H_1 \Rightarrow H_2)$-free, we get a partition of $V(K)$ into three sets that induce digraphs with dichromatic number at most $c$, a contradiction (this still holds if $u \in K$ as we can add it to any of the sets without increasing the dichromatic number).

So, for every $u \in V(G)$, either $N^-(u) \cap V(K)$ contains a copy of $H_1$, or $N^+(u) \cap V(K)$ contains a copy of $H_2$. Now for every $V_1 \subseteq V(K)$ such that $G[V_1]$ is isomorphic to $H_1$, the set of vertices $u$ such that $V_1 \subset N^-(u)$ is $H_2$-free and therefore has dichromatic number at most $c$. Similarly, for every $V_2 \subset V(K)$ such that $G[V_2]$ is isomorphic to $H_2$, the set of vertices $u$ such that $V_2 \subset N^+(u)$ is $H_1$-free and therefore has dichromatic number at most $c$. By doing this for every possible copy of $H_1$ or $H_2$ inside $V(K)$ we can cover every vertex of $V(G)$. Moreover, the number of subsets of $V(K)$ that induces a copy of $H_1$ (resp. of $H_2$) is at most $2^{f(3c+1)}$. Hence, we get that $\vec{\chi}(G) \leq c \cdot 2^{f(3c+1)+1}$. □

*Proof of Theorem 4.4.* By Lemma 4.5 and Lemma 4.6, we get that every digraph in $Forb_{ind}(K_1 + F, H_1 \Rightarrow H_2)$ has dichromatic number at most $\max(\phi(c, 3c+1), 2^{f(3c+1)+1}c)$, which proves Theorem 4.4. □

## 4.3 Strong heroes

A strongly connected hero is called a **strong hero**.

To prove Theorem 4.2, it remains to prove that $H$ is a strong hero of and only if $H = \Delta(1, k, H_1)$ or $H = \Delta(1, H_1, k)$ for some hero $H_1$.

The proof is quite hard, we are only going to prove the *only if* part.

**Lemma 4.7.** *If $H$ is a strong hero, then $H = \Delta(1, H_1, H_2)$ for some heroes $H_1$ and $H_2$.*

*Proof.* Let $H$ be a hero. Let $S_1 = TT_1$ and, having defined $S_k$, set $S_{k+1} = \Delta(1, S_k, S_k)$. As we already said, $\vec{\chi}(S_k) = k$, so there is a smallest integer $k$ such that $H$ is a subtournament of $S_k$. Name $A$ and $B$ the vertices of the copies of $S_{k-1}$ in $S_k$, and $x$ the last vertex of $S_{k+1}$. Since $H$ is strong, the copy of $H$ in $S_{k+1}$ must contain $x$ and intersect both $A$ and $B$. Together with the fact that a subtournament of a hero is a hero, it implies the result. □

Let $D$ be a tournament with an order on its vertices $v_1, \dots, v_n$. The **backedge graph** $B$ of $D$ is the undirected graph induced by arcs of $D$ that are in the wrong direction. Observe that a stable set of $B$ is an acyclic induced subgraph of $D$. Hence:

**Proposition 4.8.** *Let $D$ be a digraph and $B$ a backedge graph of $D$. Then $\vec{\chi}(D) \leq \chi(B)$.*

Backedge graph is a powerful tool to bound the dichromatic number of a digraph.

Next two lemmas give a construction, based on the backedge graph, of tournaments with arbitrarily large dichromatic number.

**Lemma 4.9.** *Let $B$ be a triangle-free graph with an ordering on its vertices and let $D$ be the tournament with backedge graph $B$. Then $\vec{\chi}(D) \geq \chi(B)/2$.*

*Proof.* It is enough to prove that all transitive tournament of $D$ is the union of two stable sets of $B$. Let $T$ be a transitive subtournament of $D$. Let $<$ be the ordering on $V(D)$ such that $B$ is the backedge graph of $D$ with respect to $<$. If $u, v, w \in V(T)$, $u < v < w$ and $wv, vu \in A(T)$, then either $uw \in A(T)$, a contradiction to the fact that $T$ is transitive, or $wu \in A(T)$, a contradiction to the fact that $B$ is triangle-free. Hence, this cannot happen and thus, setting $X = \{x \in T : \text{no backedge of T ends in } x\}$ and $Y = \{y \in T : \text{no backedge of T starts in } y\}$, we get that $V(T) = X \cup Y$. Moreover, it is clear that both $X$ and $Y$ are stable sets of $B$. □

**Lemma 4.10.** *Every hero is 2-colourable. In particular $\Delta(1, \vec{C}_3, \vec{C}_3)$ is not a hero.*

*Proof.* Let $H$ a hero on at least 4 vertices (tournament with at most 3 vertices are 2-colourable). Let $B$ an undirected graph with large chromatic number, and girth at least $V(H)$. Let $D$ the tournament with backedge graph $B$. By Lemma 4.9 $\vec{\chi}(D) \geq \chi(B)/2$ i.e. $D$ has large dichromatic number. So $D$ contains $H$. Since $B$ has no cycle of length at most $V(H)$, the backedge graph of $H$ is a forest. Hence, by Proposition 4.8, $\vec{\chi}(H) \leq 2$. □

Now, since a tournament with not $\vec{C}_3$ is transitive, Lemmas 4.7 and 4.10, we get that a tournament $H$ is a hero only if $H = \Delta(1, k, H_1)$ or $H = \Delta(1, H_1, k)$ for some hero $H_1$.

## 4.4 Family of digraphs with large dichromatic number

Let us go back to Problem 4.1, that is: *What are the finite sets $\mathcal{F}$ of digraphs for which the class $Forb_{ind}(\mathcal{F})$ has bounded dichromatic number?* We know that such a set $\mathcal{F}$ must contain a hero in tournaments.

Harutyunian and Mohar proved the following:

**Theorem 4.11.** *[16] Given positive integers $g$ and $k$ there exists an oriented graph whose underlying graph has girth at least $g$ and whose dichromatic number is at least $k$.*

This implies that a set of digraphs $\mathcal{F}$ such that $Forb_{ind}(\mathcal{F})$ has bounded dichromatic number must contain an oriented forest.

Now, the potential candidate for problem 4.1 are sets $\{H, F\}$ where $H$ is a hero in tournaments, and $F$ is an oriented forest. We now give a simple construction that rules out a lot of such couple.

We first need to extend our notation $Forb_{ind}(\mathcal{F})$ by allowing (non-oriented) graphs in $\mathcal{F}$. If $\mathcal{F}$ is a set of digraphs and graphs, we define $Forb_{ind}(\mathcal{F})$ to be the set of digraphs that does not contain as an induced subdigraph: any digraph of $\mathcal{F}$, and any orientation of any graph of $\mathcal{F}$. For example $Forb_{ind}(K_3)$ is the class of digraphs with no $\vec{C}_3$ nor $TT_3$.

We denote by $P_k$ the (undirected) path on $k$ vertices.

**Theorem 4.12.** *[4] Digraphs in $Forb_{ind}(\vec{C}_3, P_4)$ have arbitrarily large dichromatic number.*

*Proof.* We construct inductively a sequence $D_1, D_2, \dots$, if digraphs such that $D_i \in Forb_{ind}(\vec{C}_3, P_4)$ and $\vec{\chi}(D_i) = i$. Set $D_1 = K_1$, and having defined $D_i$, define $D_{i+1}$ as follows: take three disjoint copies $V_1,\ V_2,\ V_3$ of $D_i$ plus a vertex $x$, and add all arcs from $V_1$ to $V_2$, from $V_2$ to $V_3$, from $V_3$ to $x$ and from $x$ to $V_1$.

It is easy to check the $D_i$'s has the announced properties. □

Now, since a digraph with no $P_4$ is a forest of oriented stars, and a hero in tournaments with no $\vec{C}_3$ is a transitive tournament, the possible finite sets $\mathcal{F}$ such that $Forb_{ind}(\mathcal{F})$ has bounded dichromatic number are the following:

1. $\{TT_k, F\}$, where $T$ is an oriented forest,

2. $\{H, S\}$ such that $H$ is a hero in tournaments and $S$ is a forest of oriented stars.

We are going to see that suspects of the second bullet are not all guilty, while no suspect of the first bullet have proved their innocence.

As for undirected graph, there is some kind of $\chi$-boundedness point of view for these questions: given a class of digraphs $\mathcal{C}$, what are the tournaments $H$ that are heroes in $\mathcal{C}$?

# 5 $\vec{\chi}$-boundedness

In this Section, we look at the following question: *for which couple $\{TT_k, F\}$ does $Forb_{ind}(\{TT_k, F\})$ have bounded dichromatic number.* Since no counter-example is known, and because if true it would be beautiful, in [4] the authors venture to conjecture that it is true for every such couple. That is, they conjecture the following:
*(*) For every oriented forest F and every integer k, $Forb_{ind}(F, TT_k)$ has bounded dichromatic number.*

Note that this is the same as saying that, for every integer $k$ and every oriented forest $F$, $TT_k$ is a hero in $Forb_{ind}(F)$. As we have seen, heroes in tournaments are way more complicated then undirected complete graphs. But in this case, since the sequence $(TT_k)_{k\in\mathbb{N}}$ is totally ordered, we can define a natural analogue of $\chi$-boundedness as we explain now.

Let $D$ be a digraph. We denote by $\omega(D)$ the clique number of the underlying graph of $D$. We say that a class of digraph $\mathcal{C}$ is **$\vec{\chi}$-bounded** if there exists a function $f$ such that for every $D \in \mathcal{C}$, $\vec{\chi}(D) \leq f(\omega(D))$.

By Ramsey Theorem, $Forb_{ind}(TT_k, T) \subseteq Forb_{ind}(K_{2^k}, T)$, hence, $(*)$ is equivalent to: *For every oriented forest $F$, $Forb_{ind}(F)$ is $\vec{\chi}$-bounded.* And finally, as in the undirected case, it is enough to prove it for oriented trees. Hence, we get the following Conjecture, that can be seen as an oriented analogue of the Gyárfás-Sumner Conjecture 3.2.

**Conjecture 5.1.** [4] For every oriented tree $T$, $Forb_{ind}(T)$ is $\vec{\chi}$-bounded.

Let $\vec{P}_k$ be the directed path on $k$ vertices. The first natural case to look at is $Forb_{ind}(\vec{P}_k)$:

**Conjecture 5.2.** [4] For every integer $k$, $Forb_{ind}(\vec{P}_k)$ is $\vec{\chi}$-bounded.

For $k = 1, 2$ it is trivial. The conjecture is true for $k = 3$, but since $\vec{P}_3$ is an oriented star we will study this case in Section 6. The conjecture was proved very recently for $k = 4$ and is open for $k \geq 5$.

**Theorem 5.3** (Cook, Pilipczuk, Masařik, Reinald, Souza (2022+), [11])**.** *$Forb_{ind}(\vec{P}_4)$ is $\vec{\chi}$-bounded.*

In order to show proof techniques, we will first prove that $Forb_{ind}(K_3, \vec{P}_4)$ has bounded dichromatic number, and we will then give a very rough idea of the proof of the whole result.

## 5.1 The levelling technique

Let $D$ be a digraph and let $x, y$ be two vertices of $D$. The *distance* between $x$ and $y$ is the distance between $x$ and $y$ in the underlying graph of $D$. The *out-distance* from $x$ to $y$ is the length of a shortest directed path from $x$ to $y$. The *in-distance* from $x$ to $y$ is the length of a shortest directed path from $y$ to $x$.

The levelling technique is a method to bound the dichromatic number of a digraph. The idea is the following: Let $D$ be a digraph and $x$ a vertex of $D$. Let $L_i$ be the set of vertices at distance exactly $i$ from $x$. If all layers as $c$-dicolourable, then $D$ is $2c$-dicolourable. Thsi is true because no arc jumping from a level $L_i$ to a level $L_j$ as soon as $|i - j| \geq 2$. Note that the same holds if one replace distance by out-distance or by in-distance. This technique is a straightforward adaptation of a classic technique used in the undirected case.

**Theorem 5.4.** $\vec{\chi}(Forb_{ind}(K_3, P^+(3)) = 2$.

*Proof.* Let $D \in Forb_{ind}(K_3, P^+(3))$. Assume $D$ is strongly connected (otherwise just take the strong connected component with largest dichromatic number). Let $x \in V(D)$. For $i \geq 0$, set $L_i$ to be the set of vertices at out-distance $i$ from $x$. Since $D$ is strongly connected, the collection of $L_i$'s is a partition of $V(D)$. We are going to prove that each layer induces a stable set. Let $k$ be the maximum integer such that each $L_i$ is a stable set for $i = 1, \ldots, k$. Since $D$ is $K_3$-free, $L_1$ is a stable set, so $k \geq 1$. If $L_{k+1}$ is empty, we are done. So assume $L_{k+1}$ is not empty, and by maximality of $k$, $L_{k+1}$ contains an arc $ab$. There exists $a_1 \in L_k$ and $a_2 \in L_{k-1}$ such that $a_2 \to a_1 \to a$. Since $a_2 \to a_1 \to a \to b$ cannot be induced and $a_1$ and $b$ are non-adjacent (because $D$ is triangle-free), $b \to a_2$. There exists $b_1 \in L_k$ and $b_2 \in L_{k-1}$ such that $b_2 \to b_1 \to b$. Since $D$ is $K_3$-free, $b_1 \neq a_1$, $b_2 \neq a_2$ and $b_1$ is not adjacent with $a_2$. Moreover, since $L_{k-1}$ is a stable set, $a_2$ is not adjacent with $b_2$. Hence $b_2b_1ba_2$ is an induced $P^+(3)$, a contradiction.

Now, color every vertex at odd out-distance from $x$ with color 1, every vertex at even out-distance from $x$ with color 2 and $x$ with color 2. It is easy to check that this gives a proper dicoloring. □

## 5.2 Nice sets: a tool to bound the dichromatic number of a class of digraphs

Nice sets was first introduced in [4] and then used in [11] to prove that $Forb_{ind}(\vec{P}_4)$ is $\vec{\chi}$-bounded. It is a tool to prove that a class of digraphs has bounded dichromatic number. This one is not adapted from the undirected case.

**Definition 5.5.** Let $D$ be a digraph. A set of vertices $S$ of $D$ is said to be *nice* if each vertex in $S$ either has no out-neighbour in $V(D) \setminus S$ or has no in-neighbour in $V(D) \setminus S$. The set of vertices in $S$ with no out-neighbour in $V(D) \setminus S$ is called the *in-part of* $S$, and the set of vertices in $S$ with no in-neighbour in $V(D) \setminus S$ is the *out-part of* $S$.

The next lemma gives a sufficient condition for a class of digraph to have bounded dichromatic number.

**Lemma 5.6.** *Let $\mathcal{C}$ be a hereditary class of digraphs. Assume that there exists two integers $c_1$ and $c_2$ such that every digraph in $\mathcal{C}$ contains a nice set $S$ such that the in-part of $S$ has dichromatic number at most $c_1$ and its out-part has dichromatic number at most $c_2$. Then $\vec{\chi}(\mathcal{C}) \leq c_1 + c_2$. In particular, if there exists $c$ such that every digraph in $\mathcal{C}$ admits a nice set with dichromatic number at most $c$, then $\vec{\chi}(\mathcal{C}) \leq 2c$.*

*Proof.* Let $\mathcal{C}$ be a class of digraph as in the statement. Let $D \in \mathcal{C}$ be a minimal counter example, that is: $\vec{\chi}(D) = c_1 + c_2 + 1$ and for every proper subdigraph $H$ of $D$, $\vec{\chi}(H) \leq c_1 + c_2$. By hypothesis, $D$ admits a nice set $S$, with in-part $S_1$ and out-part $S_2$ such that $\vec{\chi}(S_1) = c_1$ and $\vec{\chi}(S_2) = c_2$.

The key observation is that a directed cycle that intersects both $S$ and $V(D) \setminus S$ must intersect both $S_1$ and $S_2$ (note that if $S_1$ (resp. $S_2$) is empty, then no directed cycle can intersect both $V(D) \setminus S$ and $S$). Hence, by minimality of $D$ we can dicolour $V(D) \setminus S$ with $c_1 + c_2$ colours, and extend this dicoloring to $D$ by using colours $1, \ldots, c_1$ for $S_1$ and $c_1 + 1, \ldots, c_1 + c_2$ for $S_2$. □

In [11], the following is proved and implies, together with Lemma 5.6, that $Forb_{ind}(P_4)$ is $\vec{\chi}$-bounded.

**Theorem 5.7.** *For every integer $k$ and for every digraph $D$ in $Forb_{ind}(K_k, \vec{P}_4)$ has a nice set with bounded dichromatic number.*

*Proof.* (**Rough sketch**). The proof goes by induction on $k$. Let $D \in Forb_{ind}(K_k, \vec{P_4})$. Start with a maximum clique clique $K$ of $D$ (so $K$ has size at most $k$). If it is not strongly connected, then it has a strongly connected component $C_1$ such that vertices in $C_1$ have no in-neighbour in $K - C_1$ and a strongly connected component $C_2$ such that vertices in $C_2$ have no out-neighbour in $K - C_2$. Let $P$ be a shortest directed path from $C_2$ to $C_1$ and let $C = D[K \cup P]$. Observe that $D[C]$ is strongly connected.

The authors call such an object $C$ a **closed clique**. Here, an important feature of closed cliques is that they have bounded dichromatic number. Indeed, $K$ has bounded number of vertices, so bounded dichromatic number, and $P$ being a shortest directed path, it has a backedge graph isomorphic to a path, and thus $D[P]$ is 2-dicolourable. Hence, we can use $C$ to find a nice set.

We are going find a nice set using $C$, neighbours of $C$ and some neighbours of neighbours of $C$. Partition $N(C)$ into three sets as follows:

- $X^+ = \{x \in N(C) : x$ has only out-neighbour in $C\}$
- $X^- = \{x \in N(C) : x$ has only in-neighbour in $C\}$
- $Y = \{x \in N(C) : x$ has both in- and out-neighbour in $C\}$

Finally, define:

- $Z = N(Y) - (C \cup X^+ \cup X^-)$.

The authors prove that $S = C \cup X^+ \cup X^- \cup Y \cup Z$ is a nice set with bounded dichromatic number. Showing that it is a nice set is not hard, and rely on the fact that $C$ is strongly connected.

The proof that $S$ has bounded dichromatic number is quite involved technically. By induction, the neighbourhood of a vertex has bounded dichromatic number, so the union of the neighbourhood of a bounded number of vertices have bounded dichromatic number. Unfortunately, the path $P$ (used to make $C$ strongly connected) may have arbitrarily many vertices. The hard part is to prove that neighbours of $P$ have bounded dichromatic number. $\square$

### 5.3 Forbidding all directed cycle except one

We call $t$-chordal the class of digraphs in which all induced directed cycle have length exactly $t$. Quite surprisingly, the following holds:

**Theorem 5.8.** *[9] The class of $t$-chordal digraphs is not $\vec{\chi}$-bounded.*

**Theorem 5.9.** *[9] For every integer $k$, the class of $t$-chordal digraphs with no induced $\vec{P_k}$ is $\vec{\chi}$-bounded.*

## 6 Forbidding a forest of oriented stars

Recall that, given a class of digraphs $\mathcal{C}$, a digraph $H$ is a hero in $\mathcal{C}$ if $H$-free digraphs have bounded dichromatic number. In this section, we study the following problem: *Given a forest of oriented stars $F$, what are the heroes in $Forb_{ind}(S)$?*

In [4], the authors venture to conjecture that for every forest of oriented stars $F$, heroes in $Forb_{ind}(S)$ are the same as hero in tournaments. We are going to see that this is not true.

Chudnovsky, Scott and Seymour proved that for every $k$ and every oriented forest of stars, $TT_k$ is a hero in $Forb_{ind}(F)$. Actually, they proved the following stronger result:

**Theorem 6.1.** *[10] For every $k$ and every oriented forest of stars, the chromatic number of the underlying graphs of digraphs in $Forb_{ind}(TT_k, F)$ have bounded chromatic number.*

Hence, we are only interested in tournaments containing $\vec{C_3}$.

What is the simplest forest of oriented stars that you can think of? Well, it is the digraph with no arc! Let $\overline{K_t}$ denotes the digraph on $n$ vertices and no arc.

**Theorem 6.2** (Harutyunyan, Le, Newman, Thomassé (2019) [17])**.** *For every $t \geq 2$, heroes in $Forb_{ind}(\overline{K}_t)$ are the same as heroes in tournaments.*

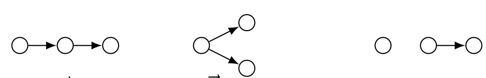
Observe that $Forb_{ind}(\overline{K}_2)$ is the class of tournaments, so this result generalises Theorem 4.2. The proof is by induction on $t$, is very involved and follows the strategy of the proof of Theorem 4.2. The main reason why it works is that, for every $D \in Forb_{ind}(\overline{K}_t)$, the non-neighbourhood of a vertex belonging to $Forb_{ind}(\overline{K}_{t-1})$, it has bounded dichromatic number by induction. Anyway, an interesting difference between the two proofs is that the following beautiful result is proved and used [17]:

**Theorem 6.3** (Harutyunyan, Le, Newman, Thomassé (2019) [17])**.** *For every $D \in Forb_{ind}(\overline{K}_t)$, if for every $x \in V(D)$, $\vec{\chi}(N^+(x))$ is bounded, then $\vec{\chi}(D)$ is bounded.*

Let us formalise the notion. A digraphs $D$ is **$t$-local** if for every $x \in V(D)$, $\vec{\chi}(N^+(x)) \leq t$. A class of digraphs $\mathcal{C}$ has the **local to global property** if there exists a function $f$ such that for every integer $t$ and for every $t$-local digraph $G$ in $\mathcal{C}$, $\vec{\chi}(G) \leq f(t)$. Hence, Theorem 6.3 says that $Forb_{ind}(\overline{K}_t)$ has the local to global property. Now, a natural problem is:

**Problem 6.4.** What classes of digraphs have the local to global property?

The next step is the study of heroes in $Forb_{ind}(F)$ when $F$ is very small. The cases when $F$ has only three vertices is already hard and interesting.



$\vec{P}_3$ $\quad$ $\vec{K}_{1,2}$ $\quad$ $K_1 + TT_2$

In the newt three subsections, we are going to study heroes in $Forb_{ind}(\vec{P}_3)$, $Forb_{ind}(\vec{K}_{1,2}$ and $Forb_{ind}(K_1 + TT_2)$.

## 6.1 Heroes in $Forb_{ind}(\vec{P}_3)$ = heroes in quasi-transitive digraphs

The goal of this subsection is to prove that heroes in $Forb_{ind}(\vec{P}_3)$ are the same as heroes in tournaments.

A digraph $G$ is **quasi-transitive** if for every triple of vertices $x, y, z$, if $xy, yz \in A(G)$, then $xz \in A(G)$ or $zx \in A(G)$. Hence, the class of quasi-transitive digraphs is precisely $Forb_{ind}(\vec{P}_3)$.

We now introduce an interesting operation called **substitution**. Given two digraphs $G_1$ and $H_1$ with disjoint vertex sets, a vertex $u \in G_1$, and a digraph $G$, we say that $G$ is obtained by **substituting $H_1$ for $u$ in $G_1$**, provided that the following hold:

- $V(G) = (V(G_1) \setminus u) \cup V(H_1)$,
- $G[V(G_1) \setminus u] = G_1 \setminus u$,
- $G[V(H_1)] = H_1$
- for all $v \in V(G_1) \setminus u$ if $v$ sees (resp. is seen by, resp. is non-adjacent to) $u$ in $G_1$, then $v$ sees (resp. is seen by, resp. is non-adjacent with) every vertex in $V(G_2)$ in $G$.

Let $\mathcal{T}$ be the class of tournaments and $\mathcal{A}$ the class of acyclic digraphs. Let $(\mathcal{A} \cup \mathcal{T})^*$ be the closure of $\mathcal{A} \cup \mathcal{T}$ under taking substitution, that is to say digraphs in $(\mathcal{A} \cup \mathcal{T})^*$ are the digraphs obtained from a vertex by repeatedly substituting vertices by digraphs in $\mathcal{A} \cup \mathcal{T}$. A classic result of Bang-Jensen and Huang [6] (see also Proposition 8.3.5 in [5]), implies that quasi-transitive digraphs are all in $(\mathcal{A} \cup \mathcal{T})^*$.

**Theorem 6.5.** *Heroes in $(\mathcal{A} \cup \mathcal{T})^*$ are the same as heroes in tournaments. In particular, heroes in $Forb_{ind}(\vec{P}_3)$ are the same as heroes in tournaments.*

*Proof.* Let $H$ be a hero in tournaments and $c$ be the maximum dichromatic number of an $H$-free tournament. We prove by induction on the number of vertices that $H$-free digraphs in $(\mathcal{A} \cup \mathcal{T})^*$ are also $c$-dicolourable. Let $G \in (\mathcal{A} \cup \mathcal{T})^*$ on $n \geq 2$ vertices and assume that all digraphs in $(\mathcal{A} \cup \mathcal{T})^*$ on at most $n-1$ vertices are $c$-dicolourable.

There exist $G_1, \dots, G_s$, $H_1, \dots, H_{s-1}$ and vertices $v_1 \dots, v_{s-1}$ such that the $G_i$'s and the $H_i$'s are digraphs of $\mathcal{A} \cup \mathcal{T}$ with at least two vertices, $G_1 = K_1$, $G_s = G$, $v_i \in V(G_i)$ and for $i = 1, \dots s-1$, $G_{i+1} = G_i(v_i \leftarrow H_i)$.

If all $H_i$ are tournaments, then $G$ is a tournament and is thus $c$-dicolourable. So we may assume that there exists $1 \leq i \leq s-1$ such that $H_i$ is an acyclic digraph. Let $x_1, \dots, x_t$ be the vertices of $H_i$. There exist $t$ digraphs $X_1, \dots, X_t$ in $(\mathcal{A} \cup \mathcal{T})^*$ such that $G$ is obtained from $G_{i+1}$ by substituting $x_1$ by $X_1$, $x_2$ by $X_2$, …, $x_t$ by $X_t$ and some vertices of $V(G_{i+1}) \setminus \{x_1, \dots, x_t\}$ by digraphs in $(\mathcal{A} \cup \mathcal{T})^*$. Note that the order in which these substitutions are performed does not matter.

Let $X = \cup_{1 \leq i \leq t} V(X_i)$. So $V(G) \setminus X$ can be partitioned into 3 sets $S^+$, $S^-$, $S^0$ such that for every $v \in X$, $v$ sees all vertices of $S^+$, is seen by all vertices of $S^-$ and is non-adjacent with all vertices of $S^0$.

For $i = 1, \dots, t$, let $D_i = G[G_i \setminus (X \setminus X_i)]$. By induction, the $D_i$'s are $c$-dicolourable. For $i = 1, \dots, t$, let $\phi_i$ be a $c$-dicolouring of $D_i$. Assume without loss of generality that $|\phi_1(X_1)| \geq |\phi_i(X_i)|$ for $1 \leq i \leq t$. In particular $\vec{\chi}(X_i) \leq |\phi_1(X_1)|$ for $i = 1, \dots, t$. Extend $\phi_1$ to a $c$-dicolouring of $D$ by dicolouring each $X_i$ (independently) with colours from $\phi_1(X_1)$. We claim that this gives a $c$-dicolouring of $G$.

Let $C$ be an induced directed cycle of $G$. If $C$ is included in $X$ or $V(G) \setminus X$, then $C$ is not monochromatic. So we may assume that $C$ intersects both $V(G) \setminus X$ and $X$. Since vertices in $X$ share the same neighbourhood outside $X$ and $C$ is induced, $C$ must intersect $X$ on exactly one vertex, and this vertex can be chosen to be any vertex of $X$. In particular we may assume that it is in $X_1$. Hence $C$ is not monochromatic. □

## 6.2 Heroes in $Forb_{ind}(\vec{K}_{1,2})$ = heroes in local out-tournament

A digraph $D$ is a **local out-tournament** if for every vertex $x \in V(D)$, $N^+(x)$ is a tournament. Hence, local out-tournaments are precisely digraphs in $Forb_{ind}(\vec{K}_{1,2})$.

The following result was proved indepently by Steiner [23] and Aboulker, Aubian and Charbit [2]

**Theorem 6.6** ([2] and [23])**.** *$K_1 \Rightarrow C_3$ (and thus $C_3$ too) is a hero in $Forb_{ind}(\vec{K}_{1,2})$, and digraphs in $Forb_{ind}(K_1 \Rightarrow C_3, \vec{K}_{1,2})$ are 2-dicolourable.*

Both proofs are similar, and a precise description of digraphs in $Forb_{ind}(K_1 \Rightarrow C_3, \vec{K}_{1,2})$.

**Conjecture 6.7.** Heroes in local out-tournaments are the same as heroes in tournaments.

In order to prove this conjecture, a first step should be the following:

**Conjecture 6.8.** If $H_1$ and $H_2$ are heroe sin local out-tournaments, then so is $H_1 \Rightarrow H_2$.

## 6.3 Heroes in $Forb_{ind}(K_1 + TT_2)$ = heroes in oriented complete multipartite digraphs

Result of this section comes from [3].

It is easy to see that $Forb_{ind}(K_1 + TT_2)$ is the same as the class of oriented complete multipartite graphs, that is orientations of complete multipartite graphs.

It was conjectured in [4] that heroes in oriented complete multipartite graphs are the same as heroes in tournaments. This conjectures was proved wrong:

**Theorem 6.9.** *[3] The digraphs $\Delta(1, 2, \vec{C}_3)$, $\Delta(1, \vec{C}_3, 2)$, $\Delta(1, 2, 3)$ and $\Delta(1, 3, 2)$ are not heroes in tournaments.*

On the positive side, it is proved that:

**Theorem 6.10.** *[3] A digraph $H$ is a hero in oriented complete multipartite graphs if:*

- *$H = K_1$,*
- *$H = H_1 \Rightarrow H_2$, where $H_1$ and $H_2$ are heroes in oriented complete multipartite graphs, or*

- $H = \Delta(1, 1, H_1)$ *where $H_1$ is a hero in oriented complete multipartite graphs.*

Observe that the second bullet of the theorem above implies that a digraph is a hero in oriented complete multipartite graphs if *and only if* each of its strong connected components are. Indeed, the *only if* part of the assertion holds because a subgraph of a hero in any class is a hero in this class. Moreover, it is implied by Theorem 4.4.

Since a hero in oriented complete multipartite graphs must be a hero in tournaments, Theorem 4.2, Theorem 6.9 and Theorem 6.10 imply that, to get a full characterisation of heroes in oriented complete multipartite graphs, it suffices to decide whether $\Delta(1, 2, 2)$ is a hero in oriented complete multipartite graphs or not. If it is not, then heroes in oriented complete multipartite graphs are precisely the ones described in Theorem 6.10. If it is, then a digraph $H$ is a hero in oriented complete multipartite graphs if and only if:

- $H = K_1$ or $H = \Delta(1, 2, 2)$,
- $H = H_1 \Rightarrow H_2$, where $H_1$ and $H_2$ are heroes in oriented complete multipartite graphs, or
- $H = \Delta(1, 1, H_1)$ where $H_1$ is a hero in oriented complete multipartite graphs.

**Question 6.11.** Is $\Delta(1, 2, 2)$ a hero in oriented complete multipartite graphs?

The proof that $\Delta(1, 1, H_1)$ where $H_1$ is a hero in oriented complete multipartite graphs follows the strategy of the proof that $\Delta(1, k, H_1)$ where $H_1$ is a hero in tournaments is a hero in tournaments [7]. It is not easy.

Let us prove Theorem 6.9 that is $\Delta(1, 2, \vec{C}_3)$, $\Delta(1, \vec{C}_3, 2)$, $\Delta(1, 2, 3)$ and $\Delta(1, 3, 2)$ are not heroes in oriented complete multipartite graphs. Since reversing all arcs of a $\Delta(1, 2, \vec{C}_3)$-free oriented complete multipartite graph results in a $\Delta(1, \vec{C}_3, 2)$-free oriented complete multipartite graph and does not change the dichromatic number, if $\Delta(1, 2, \vec{C}_3)$ is not a hero in oriented complete multipartite graphs then $\Delta(1, \vec{C}_3, 2)$ is not either. Similarly, if $\Delta(1, 2, 3)$ is not a hero in oriented complete multipartite graphs then $\Delta(1, 3, 2)$ is not either. Hence, it is enough to prove that $\Delta(1, 2, C_3)$ nor $\Delta(1, 2, 3)$ are heroes in oriented complete multipartite graphs. This is implied by the existence of $\{\Delta(1, 2, C_3), \Delta(1, 2, 3)\}$-free oriented complete multipartite graphs with arbitrarily large dichromatic number. The rest of this section is dedicated to the description of such digraphs.

A **feedback arc set** of a given digraph $G$ is a set of arcs $F$ of $G$ such that their deletion from $G$ yields an acylic digraph. The idea of the construction comes from the fact that a feedback arc set of $\Delta(1, 2, \vec{C}_3)$ or of $\Delta(1, 2, 3)$ must induce a digraph with at least one vertex of in- or out-degree at least 2. We then describe an oriented complete multipartite graph with large dichromatic number in which every subtournament has a feedback arc set inducing disjoint directed paths, implying that it does not contain $\Delta(1, 2, \vec{C}_3)$ nor $\Delta(1, 2, 3)$ by the fact above.

Given an undirected graph $H$, a *$k$-colouring* of $H$ is a partition of $V(G)$ into $k$ independent sets. The *chromatic number* of $H$ is the minimum $k$ such that $H$ is $k$-colourable. Let $G$ be a digraph. We denote by $\chi(G)$ the chromatic number of the underlying graph of $G$. The (undirected) *line graph* of $G$ is denoted by $L(G)$ and defined as follows: its vertex set is $A(G)$, and two of its vertices vertices $ab, cd \in A(G)$ are adjacent if and only if $b = c$.

Be aware that the next lemma deals with chromatic number and not dichromatic number. We think it appears for the first time in [14].

**Lemma 6.12.** *[14] For every digraph G, we have $\chi(L(G)) \geq \log(\chi(G))$.*

*Proof.* Let $G$ be a digraph and assume $L(G)$ admits a $k$-colouring. Observe that a colouring of $L(G)$ is the same as a colouring of the arcs of $G$ in such a way that no $\vec{P}_3$ is monochromatic. Consider the following colouring of $G$: for each $v \in V(G)$, colour $v$ with the set of colours received be the arcs entering in $v$. This is a $2^k$-colouring of $G$ because the colouring of $A(G)$ does not have monochromatic $\vec{P}_3$. □

Let $s \geq 3$ be an integer and let us describe the graph $L(L(TT_s))$. Assuming the vertices of $TT_s$ are numbered $v_1, \ldots, v_s$ in the topological ordering (that is, for all $1 \leq i < j \leq s$, we have $v_iv_j \in A(T)$), for any $i < j < k$, $\{v_i, v_j, v_k\}$ induces a $\vec{P}_3$ in $TT_s$. This way, we get a natural name for the vertices of $L(L(TT_s))$, namely $V(L(L(TT_s))) = \{(v_i, v_j, v_k) \mid \text{ for every } i < j < k\}$. Moreover, edges of $L(L(TT_s))$ are of the form $(v_i, v_j, v_k)(v_j, v_k, v_\ell)$ for every $i < j < k < \ell$. For $2 \leq j \leq s - 1$, set $V_j = \{(v_i, v_j, v_k)\} : i < j < k\}$. So $V_j$'s partition the vertices of $L(L(TT_s))$ into stable sets.

We now define the digraph $D_s$ from $L(L(TT_s))$ as follows. The vertices of $D_s$ are the same as the vertices of $L(L(TT_s))$ and $D_s$ is an oriented complete multipartite graph with parts $(V_2, V_3, \dots, V_{s-1})$ and we orient the arcs as follow: given $j < k$, the edges of $L(L(TT_s))$ are oriented from $V_j$ to $V_k$ and all the other arcs are oriented from $V_k$ to $V_j$. This complete the description of $D_s$.

The arcs $v_iv_j$ such that $i < j$ are called the *forward arcs* of $D_s$, and the other arcs the *backward arcs* of $D_s$. Observe that the underlying graph of the graphs induced by the forward arcs of $D_s$ is $L(L(TT_s))$.

The following remark is the crucial feature of $D_s$.

**Remark 6.13.** Given a vertex $(v_i, v_j, v_k)$ of $D_s$, the forwards arcs going out $(v_i, v_j, v_k)$ are included in $V_k$ and the forward arcs going in $(v_i, v_j, v_k)$ are included in $V_i$.

An *out-star* (resp. *in-star*) is a connected digraph made of one vertex of in-degree 0 (resp. of out-degree 0) and vertices of in-degree 1 (resp. out-degree 1). Observe that a digraph that does not contain $\vec{P}_3$ as a subgraph is a disjoint union of in- and out-stars.

**Lemma 6.14.** *For every integer s,* $\vec{\chi}(D_s) \geq \frac{1}{2}\log(\log(s))$.

*Proof.* Let $V_2, \dots, V_{s-1}$ be the partition of $D_s$ as in the definition. Recall that $V(D_s) = \{(v_i, v_j, v_k) : 1 \leq i < j < k \leq s\}$. Denote by $F_s$ the digraph induced by the forward arcs of $D_s$. So the underlying graph of $F_s$ is $L(L(TT_s))$ and by Lemma 6.12, $\chi(F_s) \geq log(log(s))$.

Let $R$ be an acyclic induced subgraph of $D_s$. Observe that a directed path on 3 vertices in $D_s$ using only arcs in $F_s$ must be of the form $(v_{i_1}, v_{i_2}, v_{i_3}) \to (v_{i_2}, v_{i_3}, v_{i_4}) \to (v_{i_3}, v_{i_4}, v_{i_5})$ where $1 \leq i_1 < i_2 < i_3 < i_4 < i_5 \leq s$ and is thus contained in a directed triangle of $D_s$ (because $(v_{i_1}, v_{i_2}, v_{i_3})(v_{i_3}, v_{i_4}, v_{i_5})$ is not an edge of $L(L(TT_s))$, and thus is not an arc of $F_s$, and thus $(v_{i_3}, v_{i_4}, v_{i_5})(v_{i_1}, v_{i_2}, v_{i_3})$ is an arc of $D_s$). Hence, $A(R) \cap A(F_s)$ does not contain $\vec{P}_3$ as a subgraph and is thus a disjoint union of out- and in-stars. So $A(R) \cap A(F_s)$ can be partitioned into two stable sets of $F_s$. Hence, a $t$-dicolouring of $D_s$ implies a $2t$-(undirected) colouring of $F_s$. As we have that $\chi(F_s) \geq log(log(s))$, the result follows. □

**Lemma 6.15.** *If $T$ is a tournament contained in $D_s$, then $T$ has a feedback arc set formed by disjoint union of directed paths.*

*Proof.* Let $T$ be a subgraph of $D_s$ inducing a tournament. Then each vertex of $T$ belongs to a distinct $V_i$ and thus, by Remark 6.13, the forward arcs of $D_s$ that are in $T$ induce a disjoint union of directed paths (i.e. every vertex have in- and out-degree at most 1) and clearly form a feedback arc set of $T$. □

**Lemma 6.16.** *For every $s \geq 1$, $D_s$ does not contain $\Delta(1, 2, C_3)$ nor $\Delta(1, 2, 3)$.*

*Proof.* Observe that the two digraphs $\Delta(1, 2, C_3)$ and $\Delta(1, 2, 3)$ only differ on the orientation of one arc: reversing an arc of the copy of $C_3$ in $\Delta(1, 2, C_3)$ leads to $\Delta(1, 2, 3)$ and reversing an arc of the copy of $TT_3$ in $\Delta(1, 2, 3)$ leads to $\Delta(1, 2, C_3)$. Our argument does not make any use of the orientations between the vertices inside this oriented $K_3$. Let $H$ be one of $\Delta(1, 2, C_3)$ or $\Delta(1, 2, 2)$, and let $x$ be the vertex in the copy of $K_1$, and $y_1$ and $y_2$ the vertices in the copy of $TT_2$. See Figure 1.

Thanks to Lemma 6.15, it is enough to prove that in every feedback arc set of $H$, there exists a vertex with in- or out-degree at least 2. Let $F$ be a feedback arc set of $H$ and assume for contradiction that it induces a disjoint union of directed paths. Then both $xy_1$ and $xy_2$ cannot belong to $F$. So we may assume without loss of generality that $xy_1 \notin F$. But then $F$ must intersect the three disjoint paths of length 2 that go from $y_1$ to $x$, which necessarily implies that $F$ contains either two arcs coming out of $y_1$ or two arcs coming in $x$. □

By Lemma 6.14 and Lemma 6.16, $\Delta(1, 2, C_3)$ and $\Delta(1, 2, 3)$ are not heroes in oriented complete multipartite graphs.

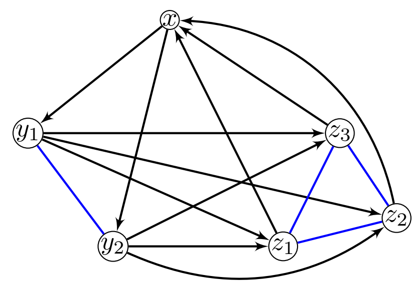



Figure 1: whatever the orientations of blue edges, $D_s$ does not contain this tournament and hence does not contain $\Delta(1, 2, C_3)$ nor $\Delta(1, 2, 3)$.

# 7 Orientations of chordal graphs

We are now going to turn our attention on class of digraphs defined as follows: take a class of (undirected) graphs $\mathcal{C}$, and look at all possible orientations of all graphs of $\mathcal{C}$.

A graph $G$ is **chordal** if it contains no induced cycle of length at least 4. Chordal graphs have been studied for the first time in the pioneer work of Dirac [12] who proved that every chordal graph $G$ is either a complete graph, or contains a clique $S$ such that $G \setminus S$ is disconnected. This easily implies that all chordal graphs can be obtained by gluing complete graphs along cliques. From this point of view, it is natural to try to generalise results on tournaments to orientations of chordal graphs.

**Lemma 7.1** ([12] Dirac, 1961)**.** *Let $G_1$ and $G_2$ be two chordal graphs such that $V(G_1) \cap V(G_2)$ induces a complete graph both in $G_1$ and $G_2$. Then their union is a chordal graph.*

This implies for undirected graph colouring that chordal graphs are perfect graphs, and thus their chromatic numbers and colouring properties are determined solely by the (largest) cliques contained in them. It is then natural to ask whether also for the dichromatic number of oriented chordal graphs important characteristics are determined by the largest dichromatic numbers of their subtournaments. In particular, it is a natural problem to characterise the heroes in oriented chordal graphs and to see whether they are the same as for tournaments.

It is actually not all the case, next theorem gives a full characterisation of heroes in oriented chordal graphs

**Theorem 7.2.** *A digraph $H$ is a hero in oriented chordal graphs if and only if $H$ is a transitive tournament or is isomorphic to $\Delta(1, 1, k)$ for some integer $k \geq 1$.*

Moreover, the constructions showing that some heroes in tournaments are not heroes in oriented chordal graphs exhibit some oriented chordal graphs with arbitrarily large dichromatic number and in which all subtournaments are 2-dicolourable.

Here is how we prove Theorem 7.2. By Theorem 4.2, it is easy to see that a hero in tournaments is either a transitive tournament, or is $\Delta(1, 1, k)$ for some integer $k \geq 1$, or contains one of the heroes $\Delta(1, 2, 2)$, $K_1 \Rightarrow \vec{C_3}$ or $\vec{C_3} \Rightarrow K_1$ as a subtournament. Moreover, since reversing all arcs of a $(\vec{C_3} \Rightarrow K_1)$-free oriented chordal graph results in a $(K_1 \Rightarrow \vec{C_3})$-free oriented chordal graph and does not change the dichromatic number, proving that $\vec{C_3} \Rightarrow K_1$ is not a hero in oriented chordal graphs implies that $K_1 \Rightarrow \vec{C_3}$ is not either. Hence, to prove Theorem 7.2, it will be enough to prove the following:

- Transitive tournaments and $\Delta(1, 1, k)$ for $k \geq 1$ are heroes in oriented chordal graphs. This is done in Section 7.0.1.
- $\Delta(1, 2, 2)$ and $\vec{C_3} \Rightarrow K_1$ are not heroes in oriented chordal graphs. This is respectively done in subsections 7.0.2 and 7.0.3.

### 7.0.1 $\Delta(1,1,k)$ and transitive tournaments are heroes in oriented chordal graphs

We'll need the two following results:

**Theorem 7.3** (Stearns, [22]). *For each integer $n \geq 1$, a tournament with at least $2^{n-1}$ vertices contains a transitive tournament with $n$ vertices.*

A vertex is **simplicial** if its neighbourhood induces a complete graph.

**Lemma 7.4.** *[12] Every chordal graph has a simplicial vertex.*

In the following, we define the **triangle degree** of a vertex $x$ in a digraph $G$ as the maximum size of a collection of directed triangles that pairwise share the common vertex $x$ but no further vertices.

**Lemma 7.5.** *Every vertex of a $\Delta(1,1,k)$-free tournament has triangle degree less than $2^{2k-2}$.*

*Proof.* Let $G$ be a $\Delta(1,1,k)$-free tournament and $x$ a vertex of $G$. Assume for contradiction that $x$ has triangle degree at least $2^{2k-2}$, that is, there exist pairwise distinct vertices $a_1, b_1, \ldots, a_{2^{2k-2}}, b_{2^{2k-2}}$ such that $x \to a_i \to b_i \to x$. By Theorem 7.3 we can find a transitive tournament $T$ in $G[\{a_1, \ldots, a_{2^{2k-2}}\}]$ of size at least $2k-1$. Up to renaming the vertices, we may assume that $T = G[\{a_1, \ldots, a_{2k-1}\}]$ and that $a_1, \ldots, a_{2k-1}$ is the topological ordering of $T$. Then look at $b_{2k-1}$. Set $b_{2k-1}^+ \cap T = T^+$ and $b_{2k-1}^- \cap T = T^-$ and observe that $V(T) = T^+ \cup T^-$ since we are in a tournament. If $|T^+| \geq k$, then $T^+$ together with $b_{2k-1}$ and $a_{2k-1}$ contains a $\Delta(1,1,k)$, a contradiction. So $|T^+| \leq k-1$. If $|T^-| \geq k$, then $T^-$ together with $b_{2k-1}$ and $x$ contains $\Delta(1,1,k)$, a contradiction. So $|T^+| \leq k-1$. Hence, $|V(T)| \leq 2k-2$, a contradiction. □

**Theorem 7.6.** *Transitive tournaments and $\Delta(1,1,k)$ are heroes in oriented chordal graphs. More precisely, $TT_k$-free oriented chordal graphs have dichromatic number at most $2^{k-1}-1$ and $\Delta(1,1,k)$-free oriented chordal graphs have dichromatic number at most $2^{2k-2}$.*

*Proof.* A $TT_k$-free oriented chordal graph has no clique of size at least $2^{k-1}-1$ by Theorem 7.3, and since chordal graphs are perfect graphs, its underlying graph has chromatic number at most $2^{k-1}-1$ and thus dichromatic number at most $2^{k-1}-1$.

We now prove that $\Delta(1,1,k)$-free oriented chordal graphs have dichromatic number at most $2^{2k-2}$. We proceed by induction on the number of vertices. Let $G$ be a $\Delta(1,1,k)$-free oriented chordal graph. Let $x$ be a simplicial vertex of the underlying graph of $G$. Note that the triangle degree of $x$ in $G$ is equal to the triangle degree of $x$ in the subtournament $G[\{x\} \cup x^+ \cup x^-]$, which by Lemma 7.5 is less than $2^{2k-2}$.

We can then find an acyclic colouring of $G \backslash x$ with $2^{2k-2}$ colours by induction, and since the triangle degree of $x$ in $G$ is less than $2^{2k-2}$, there is a colour $i \in \{1, \ldots, 2^{2k-2}\}$ such that assigning $i$ to $x$ does not produce a monochromatic directed triangle. The resulting colouring is thus an acyclic colouring of $G$: For if there existed a monochromatic directed cycle in this colouring of $G$, there would also have to exist an *induced* monochromatic directed cycle, and since all induced cycles in $G$ have length 3, this cycle would have to be a monochromatic directed triangle. However, such a triangle does not exist, neither through $x$ nor in $G \setminus x$ (by inductive assumption). □

### 7.0.2 $\Delta(1,2,2)$ is not a hero in orientations of chordal graphs

We inductively construct a sequence $(G_k)_{k \in \mathbb{N}}$ of digraphs such that for each $k \geq 1$, the digraph $G_k$ is an orientation of a chordal graph with no copy of $\Delta(1,2,2)$ and satisfying $\vec{\chi}(G_k) = k$.

Let $G_1$ be the digraph on one vertex, and having defined $G_k$, define $G_{k+1}$ as follows. Start with a copy $T$ of $TT_{k+1}$, and for each arc $e = uv$ of $T$, create a distinct copy $G_k^e$ of $G_k$ (vertex-disjoint for different choices of the arc $e \in A(T)$, and all vertex-disjoint from $T$). Next, for each $e = uv \in A(T)$, we add all the arcs $vy$ and $yu$ for every $y \in V(G_k^e)$.

One can prove that for every $k \geq 1$, $G_k$ is a $\Delta(1,2,2)$-free oriented chordal graph with dichromatic number $k$. Hence, $\Delta(1,2,2)$ is not a hero in oriented chordal graphs.

The fact that $\vec{\chi}(G_k) \geq k$ is implied by the facts that in any $(k-1)$-dicolouring of $G_k$, we should have

- The copy of $TT_k$ must have a monochromatic arc, and

- Each copy of $G_{k-1}$ must use all the $k-1$ colours.

which easily implies that no $(k-1)$-dicolouring can exist.

### 7.0.3 $\vec{C}_3 \Rightarrow K_1$ is not a hero in orientations of chordal graphs

All along this subsection, we denote by $\mathcal{C}$ the class of $(\vec{C}_3 \Rightarrow K_1)$-free oriented chordal graphs. The goal of this subsection is to construct digraphs in $\mathcal{C}$ with arbitrarily large dichromatic number.

In the following, given a $k$-colouring $c : V(F) \to \{1, \ldots, k\}$ of a digraph $F$, we say that a subdigraph of $F$ is **rainbow** (with respect to $c$), if its vertices are assigned pairwise distinct colours.

**Lemma 7.7.** *Let $G \in \mathcal{C}$ such that $\vec{\chi}(G) = k$. There exists a digraph $F = F(G) \in \mathcal{C}$ with $\vec{\chi}(F) = k$ satisfying the following property: For every $k$-dicolouring of $F$, there exists a rainbow transitive tournament of size $k$ contained in $F$.*

*Proof sketch.* We prove the lemma by showing the following statement using induction on $i$ (the lemma then follows by setting $F := F^{(k)}$).

$(\star)$ For every $i \in \{1, \ldots, k\}$, there exists a digraph $F^{(i)} \in \mathcal{C}$ such that $\vec{\chi}(F^{(i)}) = k$, and for every $k$-dicolouring of $F^{(i)}$, there exists a copy of $TT_i$ contained in $F^{(i)}$ which is rainbow.

The statement of $(\star)$ is trivially true for $i = 1$, since we may put $F^{(1)} := G$, and in every $k$-dicolouring of $F^{(1)}$ any single vertex forms a rainbow $TT_1$.

For the inductive step, let $i \in \{1, \ldots, k-1\}$ and suppose we have established the existence of a digraph $F^{(i)} \in \mathcal{C}$ of dichromatic number $k$ such that every $k$-dicolouring of $F^{(i)}$ contains a rainbow copy of $TT_i$.

We now construct a digraph $F^{(i+1)}$ from $F^{(i)}$ as follows: Let $\mathcal{X}$ denote the set of all $X \subseteq F^{(i)}$ such that $X$ induces a $TT_i$ in $F^{(i)}$. Now, for every $X \in \mathcal{X}$ create a distinct copy $G_X$ of the digraph $G$ (pairwise vertex-disjoint for different choices of $X$, and all vertex-disjoint from $F^{(i)}$). Finally, for every $X \in \mathcal{X}$, add all the arcs $xy$ with $x \in X$ and $y \in V(G_X)$. This complete the description of $F^{i+1}$. One can prove that $F^{i+1}$ has the desired properties. □

**Theorem 7.8.** *The digraph $\vec{C}_3 \Rightarrow K_1$ is not a hero in oriented chordal graphs.*

*Proof.* We construct a sequence of digraphs $(G_k)_{k \in \mathbb{N}}$ such that $\vec{\chi}(G_k) = k$ and $G_k \in \mathcal{C}$. Let $G_1$ be the one-vertex-digraph and, having defined $G_k$, define $G_{k+1}$ as follows. Let $F_k := F(G_k) \in \mathcal{C}$ be the digraph given by Lemma 7.7, so $\vec{\chi}(F_k) = k$ and every $k$-dicolouring of $F_k$ contains a rainbow copy of $TT_k$.

Let $\mathcal{T}$ denote the set of subdigraphs of $F_k$ that are transitive tournaments. Now, for each transitive subtournament $T \in \mathcal{T}$, add a copy $F_k^T$ of $F_k$ (vertex-disjoint for different choices of $T$, and all vertex-disjoint from $F_k$). Next, for every $T \in \mathcal{T}$, add all the arcs $xy$ with $x \in V(T)$ and $y \in V(F_k^T)$. Finally, for every choice of $T \in \mathcal{T}$ and every transitive subtournament $T'$ of $F_k^T$, add a vertex $x_{T,T'}$ that is seen by every vertex of $T'$ and that sees every vertex of $T$. One can proof that the $G_k$'s have the desired properties. □

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