Extending the Gyárfás-Sumner conjecture to digraphs

Pierre Aboulker — ENS Paris

Mai 2021

Non-oriented world

- $\chi(G)$: chromatic number of G.

[1] Size of a smallest cycle

Non-oriented world

- $\chi(G)$: chromatic number of G.

- Let $\mathcal{F}$ be a set of graphs. $G \in \mathit{Forb}(\mathcal{F})$ if G does not contains any member of $\mathcal{F}$ as an induced subgraph.

[1]Size of a smallest cycle

Non-oriented world

- $\chi(G)$: chromatic number of G.
- Let $\mathcal{F}$ be a set of graphs. $G \in Forb(\mathcal{F})$ if G does not contains any member of $\mathcal{F}$ as an induced subgraph.

Question: for which **finite** set of graphs $\mathcal{F}$, $Forb(\mathcal{F})$ has bounded chromatic number?

[1] Size of a smallest cycle

Non-oriented world

• $\chi(G)$: chromatic number of G.

• Let $\mathcal{F}$ be a set of graphs. $G \in \mathit{Forb}(\mathcal{F})$ if G does not contains any member of $\mathcal{F}$ as an induced subgraph.

Question: for which **finite** set of graphs $\mathcal{F}$, $\mathit{Forb}(\mathcal{F})$ has bounded chromatic number?

► $\mathcal{F}$ must contain a complete graph.

[1] Size of a smallest cycle

Non-oriented world

• $\chi(G)$: chromatic number of G.

• Let $\mathcal{F}$ be a set of graphs. $G \in Forb(\mathcal{F})$ if G does not contains any member of $\mathcal{F}$ as an induced subgraph.

Question: for which **finite** set of graphs $\mathcal{F}$, $Forb(\mathcal{F})$ has bounded chromatic number?

- $\mathcal{F}$ must contain a complete graph.
- $\mathcal{F}$ must contain a forest.

Because there is graphs with arbitrarily large girth[1] and chromatic number [Erdős, 60's]

[1] Size of a smallest cycle

Non-oriented world

- $\chi(G)$: chromatic number of G.
- Let $\mathcal{F}$ be a set of graphs. $G \in Forb(\mathcal{F})$ if G does not contains any member of $\mathcal{F}$ as an induced subgraph.

Question: for which **finite** set of graphs $\mathcal{F}$, $Forb(\mathcal{F})$ has bounded chromatic number?

- ► $\mathcal{F}$ must contain a complete graph.
- ► $\mathcal{F}$ must contain a forest.

Because there is graphs with arbitrarily large girth[1] and chromatic number [Erdős, 60's]

Gyárfás-Sumner conjecture (1987)
For every integer k and every forest F, $Forb(K_k, F)$ has bounded chromatic number.

[1] Size of a smallest cycle

χ-boundedness

- $\omega(G)$: size of a maximum clique of G.

χ-boundedness

- $\omega(G)$: size of a maximum clique of G.

$$\omega(G) \leq \chi(G) \quad \text{for every graph } G$$

χ-boundedness

- $\omega(G)$: size of a maximum clique of G.

$$\omega(G) \leq \chi(G) \quad \text{for every graph } G$$

A hereditary class of graphs is χ-bounded if $\chi(G) \leq f(\omega(G))$ for every G in the class.

Gyárfás-Sumner conjecture (1987)
$Forb(F)$ is χ-bounded if and only if F is a forest.

χ-boundedness

- $\omega(G)$: size of a maximum clique of G.

$$\omega(G) \leq \chi(G) \quad \text{for every graph } G$$

A hereditary class of graphs is χ-bounded if $\chi(G) \leq f(\omega(G))$ for every G in the class.

Gyárfás-Sumner conjecture (1987)
Forb(F) is χ-bounded if and only if F is a forest.

Result: It is enough to prove it for trees.

Directed world, dichromatic number

- *Digraphs*: no loop, no multiple arc.
- *Oriented graphs*: no digon.
- *Symmetric digraphs*: only digons.

Directed world, dichromatic number

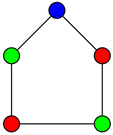
- *Digraphs*: no loop, no multiple arc.
- *Oriented graphs*: no digon.
- *Symmetric digraphs*: only digons.

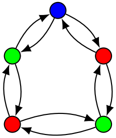
- Coloring a digraph D: no monochromatic directed cycle.
- $\vec{\chi}(D)$: the *dichromatic number* of D.

In other words: **partition** D **in acyclic induced subdigraphs** instead of stable sets.

Dichromatic number generalises chromatic number

Property: For every graph G, $\chi(G) = \vec{\chi}(\overleftrightarrow{G})$.





Heroic sets

Let $\mathcal{F}$ be a set of oriented graphs.

Forb $(\mathcal{F})$ is the class of oriented graphs containing no member of $\mathcal{F}$ as an induced subdigraph.

Problem: What are the finite sets $\mathcal{F}$ for which *Forb* $(\mathcal{F})$ has bounded dichromatic number?

Such sets are heroic.

Heroic sets

Let $\mathcal{F}$ be a set of oriented graphs.

Forb $(\mathcal{F})$ is the class of oriented graphs containing no member of $\mathcal{F}$ as an induced subdigraph.

Problem: What are the finite sets $\mathcal{F}$ for which *Forb* $(\mathcal{F})$ has bounded dichromatic number?

Such sets are heroic.

- Tournament = orientation of a complete graph.
- $\vec{C}_3$ is the directed triangle.
- Transitive tournament: tournaments with no $\vec{C}_3$

Oriented graphs that must be contained in all heroic sets

Problem: What are the finite sets $\mathcal{F}$ for which $Forb(\mathcal{F})$ has bounded dichromatic number?

- $\mathcal{F}$ must contain a tournament T.

Oriented graphs that must be contained in all heroic sets

Problem: What are the finite sets $\mathcal{F}$ for which $Forb(\mathcal{F})$ has bounded dichromatic number?

- $\mathcal{F}$ must contain a tournament T.
- $\mathcal{F}$ must contain an oriented forest F.

Oriented graphs that must be contained in all heroic sets

Problem: What are the finite sets $\mathcal{F}$ for which $Forb(\mathcal{F})$ has bounded dichromatic number?

- $\mathcal{F}$ must contain a tournament T.

- $\mathcal{F}$ must contain an oriented forest F.

Harutyunyan and Mohar (2012): there is oriented graph with large dichromatic number and such that its underlying graph has large girth.

Tournaments and Heroes

► A tournament H is a hero if and only if the class of H-free tournaments have bounded dichromatic number.

For example, TT_k and $\overrightarrow{C}_3$ are heroes.

Tournaments and Heroes

► A tournament H is a hero if and only if the class of H-free tournaments have bounded dichromatic number.

For example, TT_k and $\overrightarrow{C}_3$ are heroes.

Theorem: [Berger, Choromanski, Chudnovsky, Fox, Loebl, Scott, Seymour and Thomassé, 2015]

- A strong tournament is a hero if and only if it is equal to $\Delta(H, TT_k, TT_1)$ or $\Delta(H, TT_1, TT_k)$, where H is a hero.
- A tournament H is a hero if and only if all its strong connected components are heroes.

Problem: for which hero H and oriented forest F, $Forb(H, F)$ has bounded dichromatic number?

Problem: for which hero H and oriented forest F, $Forb(H, F)$ has bounded dichromatic number?

Theorem: $Forb(\overrightarrow{C}_3, P_4)$ has arbitrarily large dichromatic number.

Problem: for which hero H and oriented forest F, $Forb(H, F)$ has bounded dichromatic number?

Theorem: $Forb(\overrightarrow{C}_3, P_4)$ has arbitrarily large dichromatic number.

Conjecture [Aboulker, Charbit, Naserasr, 2020]: The set $Forb(H, F)$ has bounded dichromatic number if and only if:

- H is a hero and F is the disjoint union of stars or

Problem: for which hero H and oriented forest F, $Forb(H, F)$ has bounded dichromatic number?

Theorem: $Forb(\overrightarrow{C}_3, P_4)$ has arbitrarily large dichromatic number.

Conjecture [Aboulker, Charbit, Naserasr, 2020]: The set $Forb(H, F)$ has bounded dichromatic number if and only if:

- H is a hero and F is the disjoint union of stars or
- H is a transitive tournament and F is any oriented forest.

Problem: for which hero H and oriented forest F, $Forb(H, F)$ has bounded dichromatic number?

Theorem: $Forb(\overrightarrow{C}_3, P_4)$ has arbitrarily large dichromatic number.

Conjecture [Aboulker, Charbit, Naserasr, 2020]: The set $Forb(H, F)$ has bounded dichromatic number if and only if:

- H is a hero and F is the disjoint union of stars FALSE or
- H is a transitive tournament and F is any oriented forest.

Some partial answers

Theorem [Chudnovsky, Scott, Seymour, 2019]
For every integer k and disjoint unions of stars F, $Forb(TT_k, F)$ has bounded chromatic number.

Some partial answers

Theorem [Chudnovsky, Scott, Seymour, 2019]
For every integer k and disjoint unions of stars F, $Forb\,(TT_k, F)$ has bounded chromatic number.

Theorem [Harutyunyan, Le, Newman, Thomassé, 2019]
For every integer t and every hero H, $Forb\,(H, \overline{K}_t)$ has bounded dichromatic number.

$Forb\,(\overline{K}_2)$ is the class of tournaments.

First part of the conjecture

Conjecture: for every hero H and every disjoint union of stars F, $Forb(H, F)$ has bounded dichromatic number.

What about forest on three vertices.

Complete multipartite oriented graphs

$Forb\,(\overrightarrow{K}_2 + K_1)$ is the class of complete multipartite oriented graphs.

Conjecture: for every hero H, H-free complete multipartite graph has bounded dichromatic number.

Theorem: $\vec{\chi}(Forb\,(\overrightarrow{C}_3, \overrightarrow{K}_2 + K_1)) = 2$ (Aboulker, Charbit, Naserasr, 2021)

Quasi-transitive graphs

$Forb\,(\overrightarrow{P}_3)$ is the class of **quasi-transitive oriented graphs**.

Quasi-transitive graphs

$Forb(\overrightarrow{P}_3)$ is the class of **quasi-transitive oriented graphs**.

Transitive oriented graphs are transitive orientation of co-graphs.

Theorem [Bang-Jensen and Huang, 1995]
The class of quasi-transitive oriented graph is equal to the closure of $\mathcal{C} = \{$tournaments $\cup$ transitive oriented graphs$\}$ under taking substitution.

Quasi-transitive graphs

$Forb(\overrightarrow{P}_3)$ is the class of **quasi-transitive oriented graphs**.

Transitive oriented graphs are transitive orientation of co-graphs.

Theorem [Bang-Jensen and Huang, 1995]
The class of quasi-transitive oriented graph is equal to the closure of $\mathcal{C} = \{$tournaments $\cup$ transitive oriented graphs$\}$ under taking substitution.

Corolary: for every hero H, H-free quasi-transitive graphs have bounded dichromatic number.

Local out-tournament

G is a **local out-tournament** if for every vertex x, $N^+(x)$ is a tournament.

It corresponds to $Forb(S_2^+)$.

Theorem: $\vec{\chi}(Forb(\overrightarrow{C}_3, S_2^+)) = 2$ [Steiner / Aboulker, Aubian, Charbit, 2021]

Conjecture: for every hero H, H-free local out-tournament have bounded dichromatic number.

Conjecture: For every k and every oriented forest F, $Forb(TT_k, F)$ has bounded dichromatic number.

Conjecture: For every k and every oriented forest F, $Forb(TT_k, F)$ has bounded dichromatic number.

It is equivalent to:

Conjecture: For every k and every oriented tree T, $Forb(TT_k, T)$ has bounded dichromatic number.

Conjecture: For every k and every oriented forest F, $Forb(TT_k, F)$ has bounded dichromatic number.

It is equivalent to:

Conjecture: For every k and every oriented tree T, $Forb(TT_k, T)$ has bounded dichromatic number.

It is equivalent to:

Conjecture: For every k and every oriented tree T, $Forb(K_k, T)$ has bounded dichromatic number.

This is because: $Forb(TT_k, T) \subseteq Forb(K_{2^k}, T)$

Conjecture: For every k and every oriented forest F, $Forb(TT_k, F)$ has bounded dichromatic number.

It is equivalent to:

Conjecture: For every k and every oriented tree T, $Forb(TT_k, T)$ has bounded dichromatic number.

It is equivalent to:

Conjecture: For every k and every oriented tree T, $Forb(K_k, T)$ has bounded dichromatic number.

This is because: $Forb(TT_k, T) \subseteq Forb(K_{2^k}, T)$

So we get a notion of $\vec{\chi}$-boundedness!

Conjecture: for every oriented tree T, $Forb(T)$ is $\vec{\chi}$-bounded

i.e. there is a function f such that for all $G \in Forb(T)$, $\vec{\chi}(G) \leq f(\omega(G))$.

Forbidding a path

Theorem [Gyárfás, 80's]: $\mathit{Forb}\,(P_k)$ is χ-bounded.

Proof that in a triangle-free (connected) graph with sufficiently large chromatic number, every vertex is the starting point of a long induced path.

Directed path

Conjecture: $Forb(\overrightarrow{P}_k)$ is $\vec{\chi}$-bounded.

- In a triangle-free (strongly connected) oriented graph with large $\vec{\chi}$, it is not true that every vertex is the starting point of a long induced path.

- Even if an oriented graph is strongly connected, there does need to be induced directed path between any pair of vertices.

Forbidding $\overrightarrow{P}_4$

First open case:

Conjecture: $Forb(\overrightarrow{P_4})$ is $\vec{\chi}$-bounded.

- $Forb(K_3, \overrightarrow{P_4})$ has dichromatic number at most 2.
- $Forb(K_4, \overrightarrow{P_4})$ has dichromatic number at most 414

The levelling technic

Let x be a vertex.

Let L_i the set of vertices at distance i from x.

If $\vec{\chi}(L_i) \leq k$ for every i, then $\vec{\chi}(G) \leq 2k$.

The levelling technic

Let x be a vertex.

Let L_i the set of vertices at distance i from x.

If $\vec{\chi}(L_i) \leq k$ for every i, then $\vec{\chi}(G) \leq 2k$.

Theorem: If $G \in Forb\,(K_3, \overrightarrow{P_4})$, then $\vec{\chi}(G) \leq 2$ because every L_i is a stable set.

Nice sets

Theorem: If $G \in Forb\,(K_4, \overrightarrow{P}_4)$, then $\vec{\chi}(G) \leq 414$.

Proof: G has a **nice set** with bounded dichromatic number.

Definition: A nonempty set of vertices S is nice if each vertex in S either has no out-neighbor in $V(D) \setminus S$ or has no in-neighbor in $V(D) \setminus S$.

Recap

Conjecture: For every hero H and every disjoint union of stars F, digraphs in $Forb(H, F)$.

Conjecture: or every integer k and every oriented tree T, $Forb(K_k, T)$ has bounded dichromatic number.

- $Forb(K_k, \overrightarrow{P_4})$ has bounded dichromatic number ($k \geq 5$)?
- $Forb(K_3, \overrightarrow{P_t})$ has bounded dichromatic number ($t \geq 5$)?

THANK YOU FOR YOUR ATTENTION