Generalizations of the geometric the de Bruijn - Erdős Theorem

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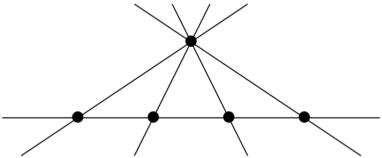
Outline

Sylvester-Gallai and de Bruijn-Erdős

Sylvester-Gallai Theorem (1944) : n non-collinear points in the plan induce at least one line with exactly two points in it.

De Bruijn-Erdős Theorem : n non-collinear points in the plan induce at least n distinct lines.



The true DBE Theorem

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A **finite geometry** is a hypergraph $H = (V, E)$ such that every **pair of vertices** belong to **exactly one hyperedge**.

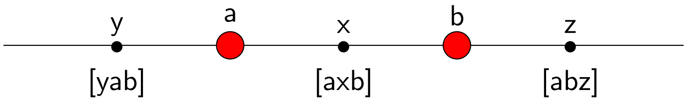
So the hyperedges can be seen as **lines**.

Theorem [**de Bruijn, Erdős, 1948**] : if $H = (V, E)$ is a finite geometry, then $|E| \geq |V|$ and equality occurs iff H is a near pencil or a finite projective plane.

In a metric space, a point **b** is between two points **a** and **c** iff

$$dist(a,b) + dist(b,c) = dist(a,c)$$

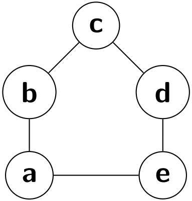
(we write [abc] and we say that $\{a,b,c\}$ is collinear).



$$\overline{ab} = \{a,b\} \cup \{x : [xba] \text{ or } [axb] \text{ or } [abx]\}$$
$$= \{a,b\} \cup \{x : \{a,b,x\} \text{ is collinear}\}$$

Weirdness of lines in metric spaces

- Two lines may intersect on more then 2 points.
- A lines might be the proper subset of another.
- Sylvester Gallai does not hold in metric spaces.



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$$\overline{ac} = \{a, b, c\} \subsetneq \overline{ab}$$

No line with exactly two points

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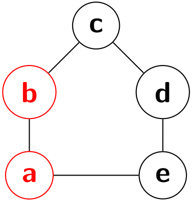

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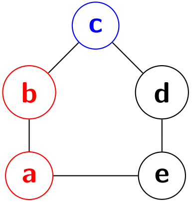

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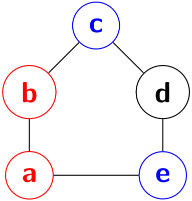

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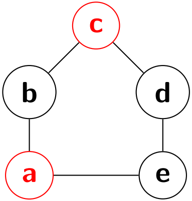

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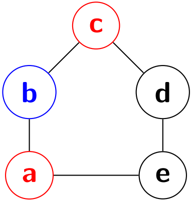

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No line with exactly two points

Chen-Chvátal Conjecture

Chen-Chvátal Conjecture, 2004 :

Any finite metric space on n points either induce at least n distinct lines, or induce a universal line, that is a line containing all the points.

Theorem (P.A., Chen, Huzhang, Kapadia, Supko, 2015) :

A metric space either has a universal line or induces at least $(\frac{1}{\sqrt{2}} - o(1)) \cdot \sqrt{n}$ distinct lines.

What is the primitive notion behind the concept of lines ?

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Three notions to define lines :

- Distances
- Betweenness : $[abc] \leftrightarrow dist(a,b) + dist(b,c) = dist(a,c)$
- Collinearity : $\{a,b,c\}$ is a collinear triple iff $[abc]$ or $[acb]$, or $[bac]$

Outline

Lines in 3-graphs

A 3-graph $H = (V, E)$ is a hypergraph with hyperedges of size 3.

Interpret the fact that $\{a, b, c\} \in E$ as $\{a, b, c\}$ is **collinear**.

We now can define lines in 3-graphs :

$$\overline{ab} = \{a, b\} \cup \{c : \{a, b, c\} \in E\}$$

Question : Do 3-graphs have the DBE property ?

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Observation : A 3-graph with no universal line define at least $\log n$ distinct lines.

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Question : what is the minimum number of lines induced by an n-point 3-graph with no universal line ?

The original de Bruijn-Erdős Theorem

A **finite geometry** is a hypergraph $(V, \mathcal{L})$ such that every pair of vertices belong to exactly one line.

DBE Theorem : If $(V, \mathcal{L})$ is a finite geometry, then $|\mathcal{L}| \geq |V|$ or $\mathcal{L}$ contains a unique line made of all the points. Moreover equality holds iff $(V, \mathcal{L})$ is a near-pencil or a finite projective plan.

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Let $H = (V, E)$ be a 3-graph and $\mathcal{L}(H)$ the set of lines induced by H.

Then $(V, \mathcal{L}(H))$ is a finite geometry iff any four points of H belong to 0, 1 or 4 hyperedges of H.

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Theorem (Beaudou, Bondy, Chen, Chiniforooshan, Chudnovsky, Chvátal, Fraiman, Zwols, 2013) :
If $H = (V, E)$ is a 3-graph such that any four points of V belong to 0, $1, 3$ or 4 hyperedges, then $|\mathcal{L}(H)| \geq |V|$ and equality holds iff $\mathcal{L}(H)$ is a near pencil, a finite projective plan or if H is the complement of a Steiner triple system.

Forbidding induced sub-3-graph

Let $I \subseteq \{0, 1, 2, 3, 4\}$.
We call $\mathcal{H}_I$ the class of 3-graph such that any 4 vertices of $\mathcal{H}$ carry i edges for an $i \in I$.

DBE Theorem : $H_{\{0,1,4\}}$ have the DBE property.

Chen an Chvátal proved that $H_{\{0,1,2,4\}}$ fails to have the DBE property.

Theorem (Beadou et al.) : $H_{\{0,1,3,4\}}$, $H_{\{0,1,2,3\}}$, $H_{\{0,2,4\}}$ have the DBE property.

Forbidding induced sub-3-graph : open questions

Questions : Do $H_{\{1,2,4\}}$, $H_{\{1,2,3,4\}}$, $H_{\{2,3,4\}}$ and $H_{\{0,2,3,4\}}$ have the DBE property ?

Question : which class of 3-graphs $\mathcal{F}$ must be forbidden in order to ensure that $\mathcal{F}$-free 3-graphs have the DBE property ?

Theorem (P.A., Lagarde, Malec, Methuku, Tompkins, 2014)

Let G be a graph and let H be a 3-graph such that $\{a, b, c\} \in E(H)$ iff abc is a triangle of G.
Then H has the DBE property.

Outline

Metric and pseudo metric betweenness

Metric spaces induce naturally a betweenness relation :
$[abc] \Leftrightarrow dist(a,b) + dist(b,c) = dist(a,c)$

Every metric betweenness satisfies the following axioms :

B1 : if $[abc]$, then $[cba]$ (*symmetry*).

B2 : if $[abc]$, then $[acb]$ does not hold.

T1 : if $[abc]$ and $[bxc]$, then $[abxc]$ (*inner transitivity*).

Definition : A Betweenness relation is a pseudo metric betweenness if it satisfies B1, B2 and T1.

Conjecture : Pseudo metric betweenness have the de Bruijn-Erdős property.

$n^{2/5}$ in Pseudo Metric betweenness

Theorem (P.A., Chen, Huzhang, Kapadia, Supko, 2015) : A pseudo metric betweenness with no universal line has a $\Omega(n^{2/5})$ distinct lines.

Recall that : $\overline{ab} = \{a, b\} \cup \{x : [xab] \text{ or } [axb] \text{ or } [abx]\}$

$[x_1 x_2 \dots x_k]$ means that $[x_r x_s x_t]$ holds for any $1 \leq r < s < t \leq k$.

The sequence $(x_1, x_2, \dots, x_k)$ is said to be a *geodesic*.

Lemma : if $\mathcal{B}$ is a pseudo metric betweenness with no universal line and $[x_1 x_2 \dots x_k]$ holds, then $\mathcal{B}$ has at least k lines.

Proof : For $i = 1, \dots, k-1$, let $p_i \notin \overline{x_i x_{i+1}}$.

We prove that lines in :

$$\{\overline{x_1 x_2}, \overline{p_1 x_1}, \overline{p_2 x_2}, \dots, \overline{p_{k-1} x_{k-1}}\}$$

are pairwise distinct.

Proof of $n^{2/5}$ in Pseudo Metric betweenness

Let $x \in V$, define the following poset on $V \setminus x$: $a \prec b$ iff $[xab]$

• If there is a chain of size $n^{2/5}$, then there is a geodesic of size $n^{2/5}$ and we win by the Lemma.
So there is an antichain A of size $n^{3/5}$.

• Partition A in $A_1, \dots, A_t$, where two points $u, v \in A_i$ iff $\overline{xu} = \overline{xv}$

• If $t \geq n^{2/5}$, it is over, so there exists A_i of size at least $n^{1/5}$.

• Three points in A_i cannot be colinear, so there is at least $\binom{|A_i|}{2} = \binom{n^{1/5}}{2} = \Omega(n^{2/5})$ lines.

Lines in betweenness relations

Four important axioms :

- T1 if $[abc]$ and $[bxc]$, then $[abxc]$ (*inner transitivity*).
- T2 if $[abc]$ and $[bcx]$, then $[abcx]$ (*outer transitivity*).
- $\square$: $[abc]$ and $[abd]$ then $[abcd]$ or $[abdc]$.
- γ : $[abc]$ and $[adc]$ and $[abdc]$ or $[adbc]$.

Question : Which properties must be satisfied by a betweenness relation $\mathcal{B}$ to ensure it has the DBE property ?

Lines in metric spaces

Theorem (A., Chen, Huzhang, Kapadia, Supko, 2015) :
A metric space with no universal line has $(\frac{1}{\sqrt{2}} - o(1)) \cdot \sqrt{n}$ distinct lines.

Proof : Let (V, d) be a metric space on n points.

Let $a, b \in V$ such that $d(a, b) = D$ and D is maximal.
Define :

$$X_a = \{x \in V : d(a, x) > D/2\},$$
$$X_b = \{x \in V : d(b, x) > D/2\},$$
$$Y = \{x \in V : d(a, x) = d(b, x) = D/2\}.$$

$V = X_a \cup X_b \in Y$.

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Metric spaces with particular properties

Observation : It is enough to prove the conjecture for metric spaces with integral distances.

Metric spaces with bounded number of distances

Theorem (A., Chen, Huzhang, Kapadia, Supko, 2015) :
A metric space with D distinct distances and with no universal line has $n/5D$ distinct lines.

Proof : Let $(V, dist)$ be a metric space on n points of diameter D.

Prove that there is $\Omega(n^2)$ pairs of vertices at distance D.

So there is a line ℓ generated by $\Omega(n)$ pairs of vertices such that vertices in a pair are at distance exactly $D/2$:

$$\ell = \overline{u_1 v_1} = \overline{u_2 v_2} = \cdots = \overline{u_t u_t} \text{ and } d(u_i, v_i) = D/2$$

Particular metric spaces

Particular types of metric spaces that verify the Chen-Chvátal conjecture :

- (Chvátal, 2013) Metric spaces where all distances are in $\{0, 1, 2\}$.
- (Kantor and Patkós, 2013) Metric spaces consisting of points in general position in the plane with the L_1 metric. (at least $n/47$ lines when the points are not in general position).

When two pairs of points define the same line in pseudometric betweenness

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Let $\mathcal{B}$ be a pseudometric betweenness on a set V.

We define :

$I(a,b) = \{x : [axb]\}, \qquad E(a,b) = \{x : [xab] \text{ or } [abx]\}$

If $\overline{ab} = \overline{cd}$, then one of the following is true :

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If $\overline{ab} = \overline{cd}$, then one of the following is true :

(α) $\{a,b,c,d\}$ (a set of size 3 or 4) is geodesic ;

(β) $[abc]$, $[bcd]$, $[cda]$, $[dab]$ and $I(a,b) = I(c,d) = \emptyset$.

(γ) $[acb]$, $[cbd]$, $[bda]$, $[dac]$ and $E(a,b) = E(c,d) = \emptyset$.

Graph metrics

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Theorem (A., Chen, Huzhang, Kapadia, Supko, 2015) :

An n-point graph metric with no universal line and diameter D has $(n/D)^{4/3}$ distinct lines.

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An n-point graph metric with no universal line and diameter D has $(n/D)^{4/3}$ distinct lines.

Theorem (A., Chen, Huzhang, Kapadia, Supko, 2015) :

An n-point graph metric with no universal line has a $\Omega(n^{4/7})$ distinct lines.

Graph classes

Special classes of graphs that satisfy the Chen-Chvátal Conjecture :

- Chrodal graph ahs the DBE property ((Beaudou, Bondy, Chen, Chiniforooshan, Chudnovsky, Chvátal, Fraiman, Zwols, 2012).
- Distance hereditary graphs have the DBE property (P.A, Kapadia, 2014)
- A super class of distance hereditary graph and chordal graph have the DBE property (P.A, Matamala, Rocher, Zamora, 2016).
- Graphs where no line is a proper subset of another line (Chen, Huzhang, Miao, Yang, 2014).

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Outline

The average intersection

n stands for the number of points and m for the number of lines.

Let V be n points in the plane inducing lines $\ell_1, \dots, \ell_m$.

We note $i_{av}(V)$ the average intersection, that is :

$$i_{av}(V) = \frac{\sum_{1 \leq i < j \leq m} |\ell_i \cap \ell_j|}{\binom{m}{2}} \leq 1$$

where ℓ_i are the lines defines by V.

DBE : $m \geq n$

Conjecture : for every set V of n points in the plane :

$$i_{av}(V) \cdot m \geq n$$

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or equivalently

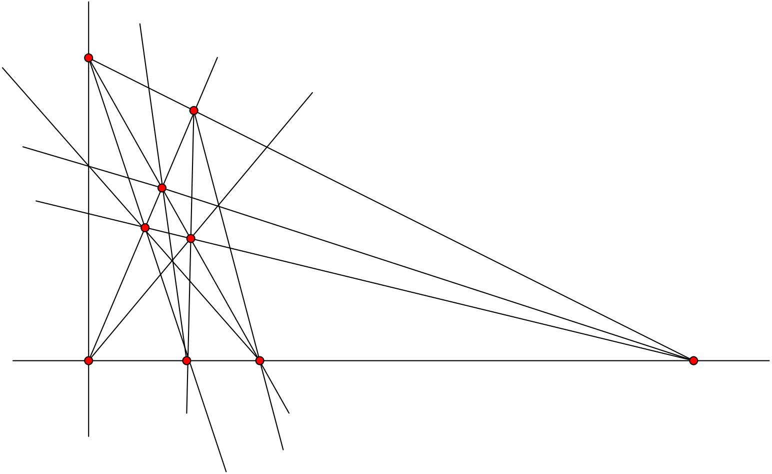
$$\sum_{v \in V} d(v) \cdot (d(v) - 1) \geq n \cdot (m - 1)$$

Where $d(v)$ is the number of lines that goes through v.

Observe that if the degree is constant it is equivalent to :

$$d(d-1) \geq m - 1$$

A tight example



13 lines, 9 points, all of degree 4.

Thank you for your attention