# Chordal directed graphs are not directed $\chi$-bounded

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**Abstract**

We show that digraphs with no transitive tournament on 3 vertices and in which every induced directed cycle has length 3 can have arbitrarily large dichromatic number. This answers to the negative a question of Carbonero, Hompe, Moore, and Spirkl (and strengthens one of their results).

## 1 Introduction

Throughout this paper, we only consider simple graphs (resp. directed graph) $G$, that is, for every two distinct vertices $u$ and $v$, the graph $G$ does not have multiple edges (resp. both arcs $uv$ and $vu$).

Relations between the chromatic number $\chi(G)$ and the clique number $\omega(G)$ of a graph $G$ have been studied for decades in structural graph theory. In particular, it is well known that there exist triangle-free graphs $G$ with arbitrarily large chromatic number (see e.g. [3, 9]). A hereditary class of graphs is *$\chi$-bounded* if there exists a function $f$ such that for every $G \in \mathcal{G}$, $\chi(G) \leq f(\omega(G))$ (see e.g. a recent survey [7] on the topic). The following question received considerable attention in the last few years: Consider a hereditary class of graphs $\mathcal{G}$ in which every triangle-free graph has bounded chromatic number. Is it true that $\mathcal{G}$ is $\chi$-bounded? Carbonero, Hompe, Moore and Spirkl [2] answered to it by the negative in a recent breakthrough paper.

Their initial motivation was actually to prove a result on digraphs. Let $D$ be a digraph. A *$k$-dicolouring* of $D$ is a $k$-partition $(V_1, \dots, V_k)$ of $V(D)$ such that $D[V_i]$ is acyclic for every $1 \leq i \leq k$. Such a partition is also called an *acyclic colouring* of $D$. The *dichromatic number* of $D$, denoted by $\vec{\chi}(D)$ and introduced by Neumann-Lara in [6], is the smallest integer $k$ such that $D$ admits a $k$-dicolouring. We denote by $\omega(D)$ the size of a largest clique in the underlying graph of $D$. We call *directed triangle* the directed cycle of length 3. As for unoriented graphs, we say that a hereditary class of digraphs $\mathcal{G}$ is *$\vec{\chi}$-bounded* if for every $G \in \mathcal{G}$, $\vec{\chi}(G) \leq f(\omega(G)$.

Carbonero, Hompe, Moore and Spirkl [2] proved that the class of digraphs with no induced directed cycle of odd length at least 5 is not $\vec{\chi}$-bounded by giving a collection of digraphs with no induced directed cycle of odd length at least 5, no $K_4$ and with arbitrarily large dichromatic number. They ask (Question 3.2) if the class of digraphs in which every induced directed cycle has length 3 is $\vec{\chi}$-bounded. These digraphs can be seen as directed analogues of chordal graphs, where a *chordal directed graph* is a directed graph with no induced directed cycle of length at least 4.

We answer negatively to this question (and thus strengthen the construction of [2]). Let us denote by $TT_3$ the *transitive tournament* on 3 vertices (i.e. the triangle which is oriented acyclically). Let $\mathcal{C}_3$ be the class of digraphs with no $TT_3$ nor induced directed cycle of length at least 4. We prove the following.

**Theorem 1.** *For every $k$, there exists $G \in \mathcal{C}_3$ such that $\vec{\chi}(G) \geq k$.*

Since any orientation of a $K_4$ contains a $TT_3$, it answers Question 3.2 of [2].

## 2 Proof of Theorem 1

Our proof technique can be seen as a generalization of the construction of triangle-free graphs with arbitrarily large chromatic number due to Zykov [9]. Assume that we are given a triangle-free graph $G_k$ with chromatic number at least $k$, and let us define $G_{k+1}$ as follows (note that we can set $G_1$ as a single vertex graph). Let $G$ be the graph made of $k$ disjoint copies of $G_k$. Set $\mathcal{I}$ to be the set of all $k$-subsets of vertices of $G$ containing exactly one vertex in each copy of $G_k$. Now, build the graph $G_{k+1}$ from $G$ as follows: for every set $I \in \mathcal{I}$, create a new vertex $x_I$ adjacent to every vertex in $I$. The key observation is that, for any colouring of $G_{k+1}$, for each $I \in \mathcal{I}$, the vertex $x_I$ forces $I$ to miss at least one colour, namely the one received by $x_I$. This easily implies that $G_{k+1}$ is not $k$-colourable. Indeed, if one tries to $k$-colour $G_{k+1}$, since $G_k$ has chromatic number $k$, there must be a vertex $x_i$ coloured $i$ in the $i^{th}$ copy of $G_k$ for every $i \leq k$. A contradiction with the key observation above. Moreover, since each set of $\mathcal{I}$ is an independent set, $G_{k+1}$ is triangle-free.

For digraphs, such a naive construction fails since adjacent vertices are allowed to receive the same colour. A way to force a given independent set $I$ of a digraph $D$ to avoid a colour (without creating induced directed cycle of length at least 4 nor $TT_3$) is to connect each vertex of $I$ to an arc $uv$ (instead of a single vertex as in the directed case) in such a way that each vertex of $I$ forms a directed triangle with $uv$ and then hope that the two vertices $u$ and $v$ receive the same colour. Unfortunately we cannot force an arc to have both endpoints of the same colour. But we have for a slightly weaker property, namely:

**Remark 1.** *Let $G \in \mathcal{C}_3$ be a directed graph with at least one arc. Any $\vec{\chi}(G)$-dicolouring of $G$ contains at least one monochromatic arc.*

*Proof.* The result trivially holds if $\vec{\chi}(G) = 1$, so we may assume that $\vec{\chi}(G) \geq 2$. Let $V_1, \ldots, V_{\vec{\chi}(G)}$ be a $\vec{\chi}(G)$-dicolouring of $G$. The set $V_1 \cup V_2$ must contain an induced directed cycle $C$ since otherwise $G$ would be $(\vec{\chi}(G) - 1)$-colourable. (Indeed, a colouring of the vertices of a digraph is acyclic if and only if none of its induced directed cycle is monochromatic). Hence, by definition of $\mathcal{C}_3$, $V_1 \cup V_2$ contains a directed triangle, and an arc of this directed triangle must have both endpoints in $V_1$ or both endpoints in $V_2$. □

Let $G$ be a $k$-chromatic digraph and $I$ be an independent set of $G$. Using Remark 1, we prove that we can create a graph $G'$ containing many copies of $G$ such that, for every $k$-coloring of $G'$, there is one copy of $G$ in $G'$ where the vertices of $I$ (in that copy) miss at least one color (Lemma 2). We then extend this result for arbitrarily many independent sets (Lemma 3). We then prove Theorem 1 using Lemma 3 as in Zykov's construction.

**Lemma 2.** *Let $k$ be an integer. Let $G \in \mathcal{C}_3$ with $n$ vertices and $m$ arcs, and such that $\vec{\chi}(G) = k$. Let $I$ be an independent set of $G$. Then there exists a digraph $H \in \mathcal{C}_3$ such that $H$ contains $m$ pairwise disjoint copies $G_1, \dots, G_m$ of $G$ and satisfy the following:*

- *For every $1 \leq i \neq j \leq m$, there is no arc between $G_i$ and $G_j$;*
- *For every $k$-dicolouring of $H$, there exists an index $i \leq m$ and a colour $\alpha$ such that no vertex of the copy of $I$ in $G_i$ is coloured with $\alpha$.*

*Moreover $H$ has $n \cdot (m+1) \leq n^4$ vertices and at most $m(m+1) + mn^2 \leq n^4$ arcs.*

*Proof.* Let us first describe the construction of $H$. We first create $m+1$ pairwise disjoint copies of $G$ denoted by $G_1, \dots, G_m, G_{m+1}$. For every $i \leq m$, let $I_i$ be the copy of $I$ in $G_i$. Let us denote $u_1^{m+1}v_1^{m+1}, \dots, u_m^{m+1}v_m^{m+1}$ the arcs of $G_{m+1}$. We add in $H$ some arcs between the $G_i$ ($i \leq m$) and $G_{m+1}$ as follows. For every $i \leq m$ and for every vertex $x \in I_i$, add the arcs $v_i^{m+1}x$ and $xu_i^{m+1}$ in $H$.

Observe that $H$ has $n \cdot (m+1)$ vertices and $m \cdot (m+1) + 2m \cdot |I| \leq n^4$ arcs as announced.

By construction, for every $1 \leq i \neq j \leq m$, there is no arc between $G_i$ and $G_j$, so the first bullet holds.

Let $c$ be a $k$-dicolouring of $H$. By Remark 1, $G_{m+1}$ has a monochromatic arc, say $u_i^{m+1}v_i^{m+1}$. Let $\alpha$ be the colour of $u_i^{m+1}$ and $v_i^{m+1}$ in $c$. Then, for every vertex $x \in I_i$, $x$ is not coloured with $\alpha$ since $H[\{u_i, v_i, x\}]$ is a directed triangle. This proves the second bullet.

To conclude, we simply have to prove that $H \in \mathcal{C}_3$. First assume for contradiction that $H$ contains a copy $X$ of a $TT_3$ as a subgraph. Since there is no arc between $G_i$ and $G_j$ for $1 \leq i \neq j \leq m$, $X$ intersects at most one of the graphs $G_i$ for $i \leq m$. Moreover, since $G$ is in $\mathcal{C}_3$, $X$ is not included in $G_i$ for $i \leq m+1$. So $X$ must intersect $G_{m+1}$ and some $G_i$ for some $i \leq m$. Assume first that $X$ contains two vertices of $G_{m+1}$. By construction, the only vertices of $G_{m+1}$ connected to $G_i$ are $u_i^{m+1}$ and $v_i^{m+1}$. So both vertices are in $X$. Moreover, the only vertices of $G_i$ connected to $G_{m+1}$ are the vertices of $I_i$ so the third vertex must be a vertex $x$ of $I_i$. But by construction, $G[\{x, u_i^{m+1}, v_i^{m+1}\}]$ is a directed triangle, a contradiction. So we can assume that $X$ contains two vertices of $G_i$. Since $X$ is a $TT_3$, they must be adjacent and both be adjacent to a vertex of $G_{m+1}$. But, by construction, the only vertices of $G_i$ connected to $G_{m+1}$ are the vertices of $I_i$ which is an independent set, a contradiction. So $H$ contains no $TT_3$.

Finally, assume for contradiction that $H$ contains a directed cycle $C$ of length at least $4$ as an induced subgraph. Since $G \in \mathcal{C}_3$, $C$ is not contained in $G_i$ for $i = 1, \dots, m+1$. Since there is no arc between $G_i$ and $G_j$ for $1 \leq i \neq j \leq m$, the cycle $C$ intersects $G_{m+1}$ and we may assume without loss of generality that $C$ also intersects $G_1$. So $C$ contains $u_1^{m+1}$ or $v_1^{m+1}$. Since, by construction, $u_1^{m+1}$ has no out-neighbour in $G_1$ and $v_1^{m+1}$ has no in-neighbour in $G_1$, $C$ must contain both $u_1^{m+1}$ and $v_1^{m+1}$ (since the deletion of $u_1^{m+1}$ and $v_1^{m+1}$ disconnects $G_1$ from the rest of the graph). But now all the vertices of $G_1$ incident to $u_1$ or $v_1$ are the vertices $x$ of $I$. And by construction, for every $x \in I$, $H[\{u_1^{m+1}, v_1^{m+1}, x\}]$ is a directed triangle, a contradiction. □

**Lemma 3.** *Let $k, r$ be two integers. Let $G \in \mathcal{C}_3$ such that $\vec{\chi}(G) = k$ and let $I_1, \dots, I_r$ be $r$ independent sets of $G$. There exist an integer $\ell_r$ and a digraph $H \in \mathcal{C}_3$ such that $H$ contains $\ell_r$ pairwise disjoint copies $G_1, \dots, G_{\ell_r}$ of $G$ such that:*

- *For every $1 \leq i \neq j \leq m$, there is no arc between $G_i$ and $G_j$;*

- *For every $k$-dicolouring of $H$, there exists an index $j \leq \ell_r$ such that, for every $s \leq r$, there exists a colour $\alpha_s$ such that no vertex of the copy of $I_s$ in $G_j$ is coloured with $\alpha_s$.*

*Moreover $H$ contains at most $n^{4r}$ vertices and arcs.*

*Proof.* We now have all the ingredients to prove Lemma 3 by induction on $r$. By Lemma 2, the case $r = 1$ holds.

Assume that the conclusion holds for $r \geq 1$ and let us prove the result for $r + 1$. Let $G \in \mathcal{C}_3$ with $\vec{\chi}(G) = k$ and let $I_1, \dots, I_{r+1}$ be $r + 1$ independent sets of $G$. By induction applied on $G$ and independent sets $I_1, \dots, I_r$, there exists an integer $\ell_r$ and a digraph $H_r \in \mathcal{C}_3$ such that $H_r$ contains $\ell_r$ pairwise disjoint copies $G_1, \dots, G_{\ell_r}$ of $G$ such that:

- For every $1 \leq i \neq j \leq \ell_r$, there is no arc between $G_i$ and $G_j$;
- For every $k$-dicolouring of $H_r$, there exists an index $j \leq \ell_r$ such that, for $s = 1, \dots, r$, there exists a colour $\alpha_s$ such that no vertex of the copy of $I_s$ in $G_j$ is coloured with $\alpha_s$.

Note that by induction, $H_r$ has at most $n^{4r}$ vertices and edges. Let us denote by $J$ the union of the vertices of the copies of $I_{r+1}$ in the subgraphs $G_1, \dots, G_{\ell_r}$ and observe that $J$ is an independent set. By Lemma 2 applied on $H_r$ and $J$, there exists a digraph $H_{r+1} \in \mathcal{C}_3$ that contains $m = |E(H_r)|$ pairwise disjoint copies $H_r^1, \dots, H_r^m$ of $H_r$ such that:

- For $1 \leq i \neq j \leq m$, there is no arc between $H_r^i$ and $H_r^j$;
- For every $k$-dicolouring of $H_{r+1}$, there exists an index $j \leq m$ and a colour $\alpha_{r+1}$ such that no vertex of the copy of $J$ in $H_r^j$ is coloured with $\alpha_{r+1}$.

Moreover, $H$ has at most $|V(H_r)|^4 = n^{4(r+1)}$ vertices and arcs.

Let us prove that $H_{r+1}$ satisfies the conclusion of Lemma 3. For every $i \leq m$, $H_r^i$ being a copy of $H_r$, it contains $\ell_r$ copies of $G$, denoted by $G_1^i, \dots, G_{\ell_r}^i$. Thus, by construction of $H_{r+1}$, the graph $H_{r+1}$ contains $\ell_{r+1} := m \cdot \ell_r$ induced copies of $G$ and by construction there is no arc linking any of these copies.

Fix a $k$-dicolouring of $H_{r+1}$. There exists an index $j \leq m$ and a colour $\alpha_{r+1}$ such that no vertex of the copy of $J$ in $H_r^j$ is coloured $\alpha_{r+1}$. Since $H_r^j$ is a copy of $H_r$ there exists an index $k \leq \ell_r$ such that, for $s = 1, \dots, r$, there exists a colour $\alpha_s$ such that no vertex of the copy of $I_s$ in $G_k^j$ is coloured with $\alpha_s$. Hence, the second bullet holds, which completes the proof. □

*Proof of Theorem 1.* Let us construct a sequence $(G_k)_{k \in \mathbb{N}}$ such that for every $k$, $G_k \in \mathcal{C}_3$ and $\vec{\chi}(G_k) \geq k$. Let $G_1$ be the graph reduced to a single vertex and let $G_2$ be the directed triangle. Let $k \geq 2$ and assume that we have obtained a $k$-dichromatic digraph $G_k$ which is in $\mathcal{C}_3$, let us define $G_{k+1}$ as follows. Let $G$ be the digraph consisting of $k$ disjoint copies of $G_k$, denoted by $G_k^1, \dots, G_k^k$. Let $\mathcal{I}$ be the set of independent sets that intersect each $G_k^i$ on a single vertex. Since $\vec{\chi}(G_k) \geq k$, in any $k$-dicolouring of $G$, there exists a vertex $x_i$ coloured $i$ in $G_k^i$ for every $i = 1, \dots, k$. By definition of $\mathcal{I}$, $\{x_1, \dots, x_k\} \in \mathcal{I}$. Hence, for every $k$-dicolouring of $G$, a set of $\mathcal{I}$ receives all the colours.

By Lemma 3 applied on $G$ and $\mathcal{I}$, there exists a digraph $G_{k+1} \in \mathcal{C}_3$ such that, for every $k$-dicolouring of $G_{k+1}$ (if such a colouring exists), there exists a copy of $G$ in $G_{k+1}$ such that each set $\mathcal{I}$ in that copy of $G$ avoids a colour, a contradiction. So $\vec{\chi}(G_{k+1}) \geq k + 1$. □

## 3 Further works

Our $(k+1)$-dichromatic graph has size $n^{2^{poly(n)}}$, which is larger than the graphs obtained using Zykov's construction which have size of order $2^{poly(|G_k|)}$. It would be interesting to know if the size of our example can be reduced.

One can wonder if directed triangles play a particular role in Theorem 1. More formally, one can wonder (as also asked in [2], Question 3.3) for which integer $k$, the class of digraphs which only contain induced directed cycles of length exactly $k$ are $\vec{\chi}$-bounded. Our main result is that it is not the case for $k = 3$. We left the problem open for $k \geq 4$.

On the same flavour, we recall here the following conjecture of Aboulker, Charbit and Naserasr which can be seen as a directed analogue of the well-known Gyárfás-Sumner conjecture [4, 8]. An *oriented tree* is an orientation of a tree.

**Conjecture 1.** *[1] For every oriented tree $T$, the class of digraphs with no induced $T$ is $\vec{\chi}$-bounded.*

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## References

[1] P. Aboullker, P. Charbit, and R. Naserasr. Extension of gyárfás-sumner conjecture to digraphs. *The Electronic Journal of Combinatorics*, 18(2), 2021.

[2] Alvaro Carbonero, Patrick Hompe, Benjamin Moore, and Sophie Spirkl. A counterexample to a conjecture about triangle-free induced subgraphs of graphs with large chromatic number. *arXiv preprint:2201.08204*, 2022.

[3] Blanche Descartes. A three colour problem. *Eureka*, 9(21):24–25, 1947.

[4] A. Gyárfás. On ramsey covering-numbe. In *Infinite and finite sets (Colloq., Keszthely, 1973; dedicated to P. Erdős on his 60th birthday*, pages 801–816. Colloq. Math. Soc. Janos Bolyai 10, North-Holland, Amsterdam, 1975.

[5] A. Gyárfás. "on ramsey covering-numbe. *, Infinite and finite sets (Colloq., Keszthely, 1973; dedicated to P. Erdőos on his 60th birthday*, II, 1975.

[6] V. Neumann-Lara. The dichromatic number of a digraph. *Journal of Combinatorial Theory, Series B*, 33(3):265 – 270, 1982.

[7] A. Scott and P. Seymour. A survey of $\chi$-boundedness. *Journal of Graph Theory*, 95(3), 2020.

[8] D. P. Sumner. Subtrees of a graph and chromatic number. In *The Theory and Applications of Graphs, (G. Chartrand, ed.)*, pages 557–576, New York, 1981. John Wiley & Sons.

[9] A. Zykov. On some properties of linear complexes (in russian). *Mat. Sbornik N.S.*, 24(66), 1949.