# Tutorial on FPT algorithms

## 1. Feedback Vertex Set in bipartite tournaments

A Feedback Vertex Set of a directed graph G is a set of vertices S such that G - S has no directed cycle. A bipartite tournament is an orientation of a complete bipartite graph.

- (a) Show that if a bipartite tournament has a directed cycle, then it has a directed cycle of length 4.
- (b) Detail an algorithm with running time  $4^k n^{O(1)}$  that decides whether a bipartite tournament has a feedback vertex of size at most k. Justify the running time and the correctness of your algorithm.

# 2. Polynomial kernel for Feedback Arc set in tournaments

Let G be a directed graph and F be a subset of arcs of T. We denote by  $G \otimes F$  the graph obtained from G after reversing each arc of F. F is a **Feedback Arc Set** (FAS for short) if G - F is a directed acylic digrah (DAG for short).

Given a set of arcs F, you can define  $rev(F) = \{yx : xy \in F\}$ .

- (a) Prove that if  $G \otimes F$  is a DAG, then F is a FAS.
- (b) Give an example of a digraph D with a FAS F such that  $D \otimes F$  is not a DAG.
- (c) Let G be a directed graph and F a subset of arcs of G. Show that F is a minimal FAS if and only if F is a minimal set of arcs such that  $G \otimes F$  is a DAG.

A **tournament** is a directed graph T such that for each pair of vertices (u, v), exactly one of the arcs uv and vu is in T. In other words it is an orientation of a complete graph. We are intereste in the k-FEEDBACK ARC SET IN TOURNAMENTS problem that takes a tournament T and an integer k as input and return YES if and only if Thas a FAS of size at most k.

- (d) Prove that a tournament is acyclic if and only it does not contain a directed triangle (that is a directed cycle on 3 vertices).
- (e) Find two simple reduction rules that will permit to get a kernel of size k(k+2) for the k-FEEDBACK ARC SET IN TOURNAMENTS problem. Hint: The rules look a lot as the two rules saw in class for k-VERTEX COVER in graphs.

#### 3. Polynomial kernel for the k-CONNECTED VERTEX COVER PROBLEM

In the k-CONNECTED VERTEX COVER PROBLEM, we are given a graph G and an integer k, and the objective is to decide if G contains a vertex cover S of size at most k and such that G[S] is connected (where G[S] is the subgraph of G induced by S).

(a) Give a simple graph for which the kernelization procedure seen in class for k-VERTEX-COVER fails for k-CONNECTED VERTEX COVER.

(b) Show that k-CONNECTED VERTEX COVER admits a kernel with at most  $2^k + O(k^2)$  vertices.

# 4. Triangle Packing

Given a graph G and an integer k, the goal is to decide if G contains k vertex-disjoint triangles. Using colour coding, show that the problem can be solved in times  $2^{O(k)}n^{O(1)}$ .

# 5. Closest String Problem

The closest string is an NP-Hard problem which tries to find the geometrical center of a set of input strings.

Given n strings  $s_1, \ldots, s_n$  each of length m and an integer d, find a string x of length m such that  $d_H(x, s_i) \leq d$  for  $i = 1, \ldots, m$ , where  $d_H$  denotes the Hamming distance. Find an FPT algorithm for Closest String Problem running in time  $O(nm+nd(d+1)^d)$ .

# 6. Randomized algorithm for Feedback Vertex set

A feedback vertex set is a set of vertices S such that  $G \setminus S$  is a forest.

- (a) Let G be a multigraph with minimum degree at least 3 and let S be a feedback vertex set of G. Prove that more than half of the edges are incident to at least one vertex of S.
- (b) Prove that there exists a polynomial-time randomized Monte-Carlo algorithm with one-sided error that, given a FEEDBACK VERTEX SET instance (G, k), either reports a FALSE or finds a feedback vertex set in G of size at most k. If the algorithm is given a YES-instance, it returns a solution with probability at least  $4^{-k}$ .
- (c) Prove that there exists a randomized algorithm that, given a FEEDBACK VERTEX SET instance (G, k), in time  $4^k \cdot n^{O(1)}$ , either reports a failure or finds a feedback vertex set in G of size at most k. Moreover, if the algorithm is given a yes-instance, it returns a solution with constant probability.

# 7. Iterative compression for FEEDBACK VERTEX SET in tournaments

FEEDBACK VERTEX SET in tournament is the following problem:

Input: A tournament T and an integer k

Question: Is there  $S \subseteq V(T)$  such that  $T \setminus S$  is acyclic?

The goal of the exercise is to use iterative compression to get a  $2^k \cdot n^{O(1)}$ -time algorithm for FVS in tournaments.

- (a) Design a  $3^k \cdot n^{O(1)}$ -time algorithm using a simple branching.
- (b) Define the problems FVS COMPRESSION and DISJOINT FVS in tournaments.
- (c) Prove that if we can solve DISJOINT FVS in tournaments in time  $n^{O(1)}$ , then we can solve FVS in tournaments in time  $2^k \cdot n^{O(1)}$ .
- (d) Design a polynomial-time algorithm to solve DISJOINT FVS in tournament.