Graph Minor Theory and its algorithmic consequences
MPRI Parametrized Complexity

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6 Hours' Programm

- Wagner conjectures, Minors and Topological Minors
  - Classes of graphs defined by fordding some graphs as minors or topological minors.
  - The k-disjoint path problem and its links with minors and topological minors.
  - Wagner Conjecture and its links with minor clases classes.
  - Wagner Conjecture, Well Quasi Orders and Kruskal Theorem (mostly out of program).
- Treewidth
  - Definition and basic properties.
  - Duality of treewidth: brumble and the game of cops and robber.
  - Grid minor Theorem and treewidth of classes of rgaphs defined by forbidding a minor.
- The Graph Minor Theorem.
- FPT algorithm using the Graph Minor Theorem.

1 - Characterization of graph classes by forbidden configurations

Graph theory

Graphs: a mathematical object and an efficient modeling tool.

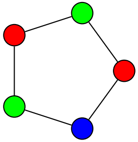
Important questions:

- What classes of graphs have good algorithmic properties? (colouring, clique max...)
- What classes of graphs have good structural properties? (decomposition theorem, elimination ordering...)

Forbidding a substructure:

- Minors: Robertson and Seymour, 1983-2012
- Topological minors
- Induced subgraphs

Chromatic number



$\chi(G)$= minimum number of colors needed to color the vertices in such a way that adjacent vertices receive distinct colors. In other words its a partitioning of the vertex sets into stable sets, minimizing the number of stable sets.

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What is the chromatic of $K_{a,b}$? K_n? C_n?

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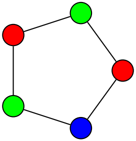

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Solution: easy.

Let k an integer. A class of graphs $\mathcal{C}$ is **k-degenerate** if for all $G \in \mathcal{C}$, G has a vertex of degree at most k.

Exercice 2

Let $\mathcal{C}$ be a k-degenerate class of graphs closed under taking induced subgraph. Prove that all graphs in G has chromatic number at most $k+1$.

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Solution: We do it by induction on the number of vertices. Graphs on 1 vertex are easy to handle. Assume every graph in $\mathcal{C}$ on at most $n-1$ vertices is $(k+1)$-colourable.

Let $G \in \mathcal{C}$ on n vertices. Since $\mathcal{C}$ is k-degenerate, it has a vertex x of degree at most k. Since $\mathcal{C}$ is closed under taking induced subgraph, $G \setminus x \in \mathcal{C}$ and is thus $(k+1)$-colourable by induction. Now, since $d(x) \leq k$, neighbours of x use at most k colours. So we can extend the $(k+1)$-colouring of $G \setminus x$ to a $(k+1)$-colouring of G.

Containment relations

We define four operations on a graph G:

1. **Remove a vertex** v (and all its incident edges), denoted $G \setminus v$.
2. **Remove an edge** e (but not its end vertices), denoted $G \setminus e$.
3. **Contract an edge** $e = xy$, denoted G/e:
   (i.e. remove x and y, add a new vertex z with neighbourhood $N(z) = (N(x) \cup N(u)) \setminus \{z\}$ (no loops))
4. **Topological contraction** is a contraction of edge e that has an endvertex of degree 2. Its inverse is the **subdivision operation** which consists in removing an edge xy, adding a new vertex z, and adding the edges xz and zy.

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Definition

Let G and H be two graphs.

- H **induced subgraph** of G if H obtained from G by the repeated use of 1.
- H **subgraph** of G if H obtained from G by the repeated use of 1 and 2.
- H **topological minor** of G if H is a minor of G and every contraction used was topological.
- H **minor** of G if H obtained from G by the repeated use of rule 1,2 and 3.

Partial orders

Each of these containments relations define a partial order on graphs:

- H induced subgraph of G: $H \subseteq_i G$
- H subgraph of G: $H \subseteq G$
- H topological minor of G: $H \preceq_t G$
- H minor of G: $H \preceq_m G$

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Let $\preceq$ be any of these orders. We say that a **class of graphs** $\mathcal{C}$ is **$\preceq$-closed** (subgraph-closed, minor-closed...) if **for all** $G \in \mathcal{C}$**:** $H \preceq G \Rightarrow H \in \mathcal{C}$.

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- The class of planar graphs is minor closed (and thus topological-minor-closed, subgraph-closed and induced-subgraph-closed).
- The class of bipartite graphs is subgraph-closed, but not topological-minor-closed.
- The class of all graphs whose connected components are cliques is induced-subgraph-closed, but not subgraph-closed.

Minors

Here is an equivalent definition for minors that is often useful:

Lemma

Let G and H be two graphs, and denote $V(H) = \{v_1, \dots, v_p\}$. Then H is a minor of G if and only if there exists p **connected** *and disjoint subgraphs $G_1, \dots, G_p$ of G such that for every edge $v_i v_j$ of H, there exists an edge between G_i and G_j. The graphs induced by $G_1, \dots, G_p$ is called a* ***H-model*** *of G.*

Exercice 3

Show that the (3×3)-grid has a K_4-minor by showing it has a K_4-model.

Topological Minors

A topological minor is also called **subdivision**.
Here is an equivalent definition of topological minor.

Definition

A graph H is a **topological minor** of a graph G if there exists a injective mapping f from $V(H)$ to $V(G)$ such that for each edge uv of H, there exists in G a path P_{uv} connecting $f(u)$ and $f(v)$ in G with the property that all these path are internally disjoint.

Exercice 4

Describe the graphs that do not contain the following graphs as topological minors: K_3, $K_{1,3}$, $K_{1,4}$. (For fun, do the same exercise for other containments relations).

Classes of graph defined by forbidden configurations

For a set $\mathcal{F}$ of graphs, let $Forb_{\preccurlyeq}(\mathcal{F}) = \{G : \forall F \in \mathcal{F}, F \not\preccurlyeq G\}$ i.e. the class of graphs not containing any graphs of $\mathcal{F}$ under $\preccurlyeq$-relation. We say such graph are $\mathcal{F}$-$\preccurlyeq$-free.

- $Forb_{\preccurlyeq_t}(K_5, K_{3,3}) =$ planar graphs $= (K_5, K_{3,3})$-topological minor free graphs.
- $Forb_{\subseteq}(C_3, C_5, C_7, ...) =$??
- $Forb_{\subseteq_i}(K_{1,2}) =$??

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A graph F is a **$\preccurlyeq$-obstruction** for a class $\mathcal{C}$ if $F \notin \mathcal{C}$ but for every $H \preccurlyeq F$, $H \in \mathcal{C}$.

Let $Obst_{\preccurlyeq}(\mathcal{C})$ be the set of all $\preccurlyeq$-obstruction of $\mathcal{G}$.

- K_5 is a topological-minor-obstruction for planar graphs since K_5 is not planar, but every proper topological-minor of K_5 is.
- K_6 is not a topological-minor-obstruction for planar graphs since $K_5 \preccurlyeq_t K_6$ and K_5 is not planar.

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Exercises

Exercice 5

Let $\mathcal{C}$ be a class of graphs and $\preccurlyeq$ a containment relation on graphs.
Prove that $\mathcal{C}$ is $\preccurlyeq$-closed **if and only if** there exists a (possibly infinite) set of graphs $\mathcal{F}$ such that $\mathcal{C} = Forb_{\preccurlyeq}(\mathcal{F})$.

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Exercice 8

1. Prove that every graph with average degree at least 2^{r-2} contains K_r as a minor.
2. For r fixed, does there exist K_r minor-free graphs with arbitrarily large chromatic number?

Solution

Solution Exercise 8:

1- Recall that the average degree of a graph is $\frac{\sum_{v \in V(G)} d(v)}{|V(G)|} = \frac{2|E(G)|}{|V(G)|}$.
We proceed by induction on r. Easy when $r = 1$ or 2. Let G be a graph of average degree at least 2^{r-2}. Therefore $\frac{|E(G)|}{|V(G)|} \geq 2^{r-3}$. Let H be minimal amongst all minors of G such that $\frac{|E(H)|}{|V(H)|} \geq 2^{r-3}$. It implies that when one contracts an edge in H, one must loose at least 2^{r-3} edges (otherwise the inequality would still be satisfied, and H would not be minor minimal). Hence, for any xy edge of H, x and y have at least 2^{r-3} common neighbours. In other words, if x is a vertex in H, then the minimum degree in its neighbourhood is at least 2^{r-3}, so by induction it contains a K_{r-1} minor, which yields with x the desired K_r minor.

2- Hence K_r-minor-free graphs has a vertex of degree at most 2^{r-2} and is thus $2^{r-2} + 1$ colourable.

2 - Three Algorithmic Problems

A Classical Connectivity Problem

Consider the following problem of connectivity.

Problem (k disjoint paths problem)

Input : A graph G, an integer k and two subsets of vertices A and B of size k
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From a structural point of view, the maximum number of paths linking A and B corresponds to a minimum cut-vertex separating A and B and is a classical result of **Menger**.

Theorem (Menger,1927)

Let x and y be distinct vertices of a graph G. Then the minimum number of vertices whose deletion separates x from y is equal to the maximum number of internally disjoint paths linking x and y.

Exercise on connectivity

Let G be a graph, $x \in V(G)$ and $Y \subseteq V(G) \setminus \{x\}$. A family of k internally disjoint (x, Y)-paths whose terminal vertices are distinct is referred to as a ***k*-fan** from x to Y.

Exercice 9

Let G be a k-connected graph.

1. Let x be a vertex of G, and let $Y \subseteq V \setminus \{x\}$ be a set of at least k vertices of G. Then there exists a k-fan in G from x to Y. (This property is known as the **Fan Lemma**).
2. Let S be a set of k vertices in a k-connected graph G, where $k \geq 2$. Then there is a cycle in G which includes all the vertices of S.

For a very good presentation of Menger Theorem and its consequences, see the book *Graph Theory* of J. A. Bondy and U. S. R. Murty, chapters 9.1 and 9.2.

A similar problem.

Problem (k-disjoint rooted paths problem)

Input : A graph G, an integer k, and two subsets of vertices $S = \{s_1, s_2, \dots, s_k\}$ and $T = \{t_1, t_2, \dots, t_k\}$
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- Crucial role in VLSI design, related to commodity flow problem, many applications.
- With $k \geq 2$ part of the input, this problem is NP-complete, even restricted to the class of planar graphs.
- Nevertheless, in the Graph Minor series of papers, Robertson and Seymour proved a polynomial algorithm for fixed k.

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  - Complexity: $O(f(k)n^k)$, where $k = |V(H)|$, and $n = |V(G)|$. Therefore **polynomial for every fixed** k. So the problem is in (**XP**).

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  - Complexity: $O\big(f(k)n^k\big)$, where $k = |V(H)|$, and $n = |V(G)|$. Therefore **polynomial for every fixed k**. So the problem is in (**XP**).
  - In 2010, Grohe, Kawabarayashi, Marx, and Wollan proved much better: $O\big(f(k)n^3\big)$. So the problem is actually **FPT**.

Topological Minor Detection II

Theorem

Let H be a fixed graph with k edges. One can decide whether H is a topological minor of a given graph G in time $O\big(f(k)n^k\big)$.

Sketch proof:
Let $f : V(H) \to V(G)$ be an injection.
Observe that there is $\binom{n}{|V(H)|}$ such injections.
Do the following for each injection.
We want to decide if there exists disjoint paths in G between the $f(v)$ corresponding to edges of H.
To do that, we replace (in G) each vertex $f(v)$ by $d_H(v)$ copies of $f(v)$ (having the same neighbours).
Now, for $k = |E(H)|$, solving the k-Rooted Disjoint Paths Problem for well chosen sets solve the problem.

Consequences

In particular, the previous theorem implies that any family of graphs that is defined with **forbidding a FINITE family of graphs as topological minors is polynomially testable**.

In other words if $\mathcal{C} = Forb_{\preccurlyeq_t}(\mathcal{F})$ where $\mathcal{F}$ is a finite set of graphs, than we can decide in polynomial time if a graph G belongs to $\mathcal{C}$.

Example of such class?

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Example of such class?

Theorem (Kuratowski, 1930)

A graph G is planar if and only if it does not contain K_5 nor $K_{3,3}$ as a topological minor.

Note that one does not need to solve k rooted paths problem to get polytime algorithms for recognizing planar graphs (there exist even linear algorithms to do that).

3 - Wagner Conjecture and minor closed classes

Wagner Conjecture (now *Graph Minor Theorem*)

Our goal in this last chapter is a single theorem, one which dwarfs any other result in graph theory and may doubtless be counted among the deepest theorems that mathematics has to offer: *in every infinite set of graphs there are two such that one is a minor of the other.* This *graph minor theorem* (or *minor theorem* for short), inconspicuous though it may look at first glance, has made a fundamental impact both outside graph theory and within. Its proof, due to Neil Robertson and Paul Seymour, takes well over 500 pages.

Reinhart Diestel

Minors Vs Topological Minors

- By definition: H topological minor of $G \Rightarrow H$ minor of G
- **Exercise:** converse not true: find a pair of graphs G and H such that H is a minor of G but H is not a topological minor of G.

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- By definition: H topological minor of $G \Rightarrow H$ minor of G
- **Exercise:** converse not true: find a pair of graphs G and H such that H is a minor of G but H is not a topological minor of G.
  Solution: Set H to be two disjoint $K_{1,2}$ and link their vertices of degree 2 by an edge. Then H is a minor of $K_{1,4}$, but not a topological minor.

When H is subcubic (maximum degree at most 3), this is nevertheless true.

Theorem

Let H be a graph with maximum degree at most 3. Then a graph G has an H-minor if and only if it contains an H-subdivision.

Proof on the next slide.

Theorem

Let H be a graph with maximum degree at most 3. Then a graph G has an H-minor if and only if it contains an H-subdivision.

Sketch proof:

- Assume H is a minor of G
- Let G' be a **minimal** topological minor of G such that H is a minor of G' ($[V(G)| + |E(G)|$ is minimized).
- Note that G' is a topological minor of G means that G' is btained from G by deleting vertices, edges, a contracting edges with at least one extremity of degree at most 2.
- Look at an H-model $(G_1, \ldots, G_p)$ (where $p = |V(H)|$) of G.
- By minimality of G, each G_i is a tree with at most 3 leaves and no vertex of degree 2 (at most three leaves because if it has 4, then one is not used to connect G_i to another G_j, and if there is a vertex of degree 2 (resp. a cycle) we can contract an edge (resp. delete an edge) and have a smaller topological minor of G that still contains an H-model).
- Each such tree must be a star, so we get the topological minor.

Minors Vs Topological Minors

A similar argument proves this more general result.

Theorem

For every graph F, there exists a **finite** *family of graphs $\mathcal{F}$ such that:*
G contains F as a minor if and only if it contains some graph in $\mathcal{F}$ as a topological minor.
In other words: $Forb_{\preccurlyeq_m}(F) = Forb_{\preccurlyeq_t}(\mathcal{F})$.

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In other words: $Forb_{\preccurlyeq_m}(F) = Forb_{\preccurlyeq_t}(\mathcal{F})$.

Proof: We start the proof exactly as in the previous result, and by again choosing minimal G_i, we now get for each G_i a tree with at most $|H|$ leaves and no vertex of degree 2. There is finitely many such trees (why?). So by replacing the vertices of H by these trees in all possible ways, we obtain a finite collection of graphs $\mathcal{H}$ with the desired properties.

Minor detection

This result combined with the theorem on topological minor detection now clearly implies the following theorem.

Theorem (Robertson and Seymour, 1995)

Let H be a fixed graph. There exists a polynomial time algorithm to decide whether H is a minor of a given graph G.

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Theorem (Robertson and Seymour, 1995)

Let H be a fixed graph. There exists a polynomial time algorithm to decide whether H is a minor of a given graph G.

Corollary

If $\mathcal{C}$ is a class of graphs defined by forbidding finitely many minors, then there exists a polynomial algorithm to decide wether an input graph belongs to $\mathcal{C}$

Wagner Conjecture

Question

What are the families defined by finitely many forbidden minors?

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Examples :

Graph Class	Minor minimal graphs
Forests	triangle
Union of Paths	triangle, claw
Planar	K_5 , $K_{3,3}$
Toric	≥ 17523 (but finite)

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Toric	$\geq$ 17523 (but finite)

Exercice 10

For each of the following classes, decide if it is minor closed or not. If it is, find the set of obstructions, if it is not not, try to describe the smallest minor closed class containing it: cliques, paths, cycles, graphs of max degree k?

Minor Closed Classes

Question

What are the families defined by finitely many forbidden minors?

- A trivial fact is that such families are **closed under taking minors** (every minor of a graph in the family is in the family).

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Theorem (Graph Minor Theorem, Robertson and Seymour, XX)

Any minor closed class of graphs is defined by a finite list of forbidden minors

With an important consequence (among many others):

Corollary

If $\mathcal{C}$ is a minor closed class, then there exists a polynomial time algorithm to decide if a given graph belongs to $\mathcal{C}$.

Recap

- The k-routed disjoint path problem is solvable in FPT time parametrized by k ($f(k)n^3$).

This implies that:

- Given a finite set of graphs $\mathcal{F}$, deciding if a given graph G belongs to $Forb_{\prec_t}(\mathcal{F})$ can be done in FPT time parametrized by the size of $\mathcal{F}$ (we have seen that the k-routed disjoint path problem implies XP-time, but Grohe et al proved FPT-time: $f(\mathcal{F})n^3$).

- Given a finite set of graphs $\mathcal{F}$, deciding if a given graph G belongs to $Forb_{\prec_t}(\mathcal{F})$ can be done in FPT time parametrized by the size of $\mathcal{F}$.

Then, by the graph minor theorem, we have the final super strong result:

- Given a minor-closed class $\mathcal{C}$, one can decide in polytime if a given graph G belongs to $\mathcal{C}$.

More detailed: Since $\mathcal{C}$ is minor-closed, there is a finite set of graphs $\mathcal{F}$ such that $\mathcal{C} = Forb_{\prec_m}(\mathcal{F})$, and thus there is a $(\mathcal{F})n^3$-time algorithme to decide if a rgaph G belongs to $Forb_{\prec_m}(\mathcal{F})$

Exercises

Exercice 11

Prove that the following problems are solvable in time $O(f(k)n^3)$.

- k-**Vertex Cover**
  Input : A graph G.
  Output : TRUE if there exists a set S of at most k vertices such that $G \setminus S$ has no edge.
- k-**Feedback vertex set**
  Input : A graph G.
  Output : TRUE if there exists a set S of at most k vertices such that $G \setminus S$ has no cycle.
- k-**leaf Spanning Tree**
  Input : A graph G.
  Output : TRUE if there exists in G a spanning tree T with at least k leaves.

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  Output : TRUE if there exists in G a spanning tree T with at least k leaves.

HINT: Observe that for each of these problems, the set of TRUE instances is closed under taking minor.

The idea is that any property closed under taking minor is testable in time $O(f(k))\, n^3$

4 - Well Quasi Orders and Wagner Conjecture

Introduction

WE SKIP THIS SECTION BECAUSE WE HAVE NO TIME. FEEL FREE TO READ IT IF YOU LIKE THE TOPIC!

In this section we will try to understand some of the ideas behind the proof of Wagner's conjecture by proving similar but (much) easier results.

We first introduce the notion of well quasi order that gives an equivalent way to state Wagner Conjecture.

Then we will prove a theorem due to Kruskal saying that trees are well quasi ordered for the minor relation.

In the next section, we will explain through the notion of treewidth why Kruskal Theorem and its proof is central in Robertson and Seymour's proof.

Wagner's Conjecture

Definition (Obstructions)

For a given minor closed class $\mathcal{C}$, a graph H is said to be an **obstruction** of $\mathcal{C}$ if H is not in $\mathcal{C}$ but every strict minor of H is.

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Proposition

Let $\mathcal{C}$ be a minor closed class, and $\mathcal{O}$ be its (possibly infinite) set of obstructions. Then $G \in \mathcal{C}$ if and only if G does not contain any graph of $\mathcal{O}$ as a minor
In particular $\mathcal{C} = Forb_{\preccurlyeq_m}(\mathcal{O})$. Moreover, $\mathcal{O}$ is the smallest set of graphs with this property.

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- Observe that by definition the set of obstructions forms an **antichain** of the minor partial order: no obstruction is a minor of another obstruction.

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- So Wagner's conjecture is to prove that a set of obstructions is always finite.
- Observe that by definition the set of obstructions forms an **antichain** of the minor partial order: no obstruction is a minor of another obstruction.
- Is it true that **every antichain is finite?**

Wagner's Conjecture - continued

Proposition

The following are equivalent :

- *Every minor closed class has a finite set of obstructions.*
- *There is no infinite antichain for the minor relation.*

Wagner's Conjecture - continued

Proposition

The following are equivalent :

- *Every minor closed class has a finite set of obstructions.*
- *There is no infinite antichain for the minor relation.*

Definition

A partial order $\preccurlyeq$ defined on a set X is a **well quasi order** (WQO) if there is no infinite strictly decreasing sequence and no infinite antichain.

Wagner's conjecture is equivalent to say that the class of all graphs with the minor relation is a WQO.

Exercises on well quasi ordering

Exercice 12

For each of these, say if it is a wqo.

- $(\mathbb{N}, \leqslant)$.
- $(\mathbb{R}, \leqslant)$.
- $(\mathbb{N}^2, \preccurlyeq)$ where $(x, y) \preccurlyeq (x'y')$ iff $(x \preccurlyeq x'$ and $y \preccurlyeq y')$,
- $(\mathcal{G}, \subseteq_i)$ where $\mathcal{G}$ is the class of all graphs (recall that $H \subseteq_i G$ means H is an induce subgraph of G).
- Finite trees ordered by subgraph relation.
- $(\mathcal{G}, \preccurlyeq)$ where $G \preccurlyeq H$ if G topological minor of H

Some solution one the next slide.

Finite trees ordered by subgraph relation. No: take double broom: paths with end vertices of degree 3.

$(\mathcal{G}, \preccurlyeq)$ **where** $G \preccurlyeq H$ **if** G **topological minor of** H: NO, take the family of thick cycle, where a thick cycle is a cycle where each edge is doubles (parallel edges).

Dealing with WQO: a first tool

Proposition: Let $(X, \preccurlyeq)$ be a partially ordered set and $(x_i)_{i\in\mathbb{N}}$ be any sequence. Then $(x_i)_{i\in\mathbb{N}}$ has an infinite subsequence that is either strictly increasing, or strictly decreasing or an antichain.

Dealing with WQO: a first tool

Proposition: Let $(X, \preccurlyeq)$ be a partially ordered set and $(x_i)_{i \in \mathbb{N}}$ be any sequence. Then $(x_i)_{i \in \mathbb{N}}$ has an infinite subsequence that is either strictly increasing, or strictly decreasing or an antichain.

Proof: By Ramsey, or: Let (x_i) be any sequence. Starts with x_1, and consider

- $A_1 = \{j,\ j > 1 \text{ and } x_1 \preccurlyeq x_j\}$
- $B_1 = \{j,\ j > 1 \text{ and } x_1 \preccurlyeq x_i\}$
- $C_1 = \{j,\ j > 1 \text{ and } x_1 \text{ and } x_j \text{ are incomparable}\}$

If A_1 is infinite we say that x_1 is of type A and delete all elements that are not in A_1. If not, but B_1 is infinite, say that x_1 is of type B and delete all elements that are not in B_1. Finally in the last case, say that x_1 is of type C and delete all vertices not in C_1.
Up to extracting a subsequence and renaming, we can assume no element were deleted, so that the x_i with $i \geq 2$ were all in A_1, or all in B_1, or all in C_1. We do this sequentially for x_2, then x_3, ... I.e., at each step, we define A_i, B_i, C_i as

- $A_i = \{j,\ j > i \text{ and } x_i \preccurlyeq x_j\}$
- $B_i = \{j,\ j > i \text{ and } x_j \preccurlyeq x_i\}$
- $C_i = \{j,\ j > i \text{ and } x_i \text{ and } x_j \text{ are incomparable}\}$

and at each step we define the type of x_i to be one of A, B, C depending on which is infinite. Then we extract by keeping only the elements in the infinite set.
Eventually we have a type for each element of the sequence (which is in fact a subsequence of the original sequence). Now there must be a type with infinitely number of elements and to each type clearly corresponds one of the three possible type of infinite subsequence.

Dealing with WQO: a first tool

Corollary

Let $(X, \preccurlyeq)$ be a partially ordered set. The three assertions are equivalent

1. *$(X, \preccurlyeq)$ is a wqo.*
2. *from every sequence $(x_i)_{i \in \mathbb{N}}$ one can extract an infinite increasing subsequence.*
3. *from every sequence $(x_i)_{i \in \mathbb{N}}$ one can extract $i < j$ such that $x_i \preccurlyeq x_j$.*

This will be useful: in order to prove that a given partial order is a WQO, we will only prove the third statement, but when we use the fact that an order is a WQO (for example in a proof by induction), we can use the second statement which is (in appearance) much stronger.

(x_i, x_j) is a **good pair** if $i < j$ and $x_i \preccurlyeq x_j$.

Hence, $(X, \preccurlyeq)$ is a WQO if and only if every sequence $(x_i)_{i \in \mathbb{N}}$ has a good pair.

Second tool: extending a partial order

Let $(X, \leq)$ be a partial order. For finite subsets $A, B \subset X$, write $A \preccurlyeq B$ if there is an injective mapping $f : A \to B$ such that $a \leq f(a)$ for all $a \in A$.

This naturally extends $\leq$ to a partial order on $[X]^\omega$, the set of all finite subsets of X.

Second tool: extending a partial order

Let $(X, \leq)$ be a partial order. For finite subsets $A, B \subset X$, write $A \preccurlyeq B$ if there is an injective mapping $f : A \to B$ such that $a \leq f(a)$ for all $a \in A$.

This naturally extends $\leq$ to a partial order on $[X]^\omega$, the set of all finite subsets of X.

Lemma

If X is a WQO, then so is $[X]^\omega$.

Proof [see Diestel, Lemma 12.1.3]: Main idea: start with a "minimum" infinite antichain.

Proof sketch

- Assume for contradiction that $[X]^w$ has a bad sequence, i.e. an infinite sequence with no good pair.
- We construct a "minimal" bad sequence $(A_n)_{n\in\mathbb{N}}$ as follows:
- Assume inductively that A_i has been defined for every $i < n$, and that there exists a bad sequence in $[X]^w$ starting with $A_0, \dots, A_{n-1}$.
- Choose A_n such that some bad sequence starts with $(A_0, \dots, A_{n-1}, A_n)$ and $|A_n|$ is minimum with this property.
- For each n, pick en element $a_n \in A_n$, and set $B_n = A_n \setminus \{a_n\}$.
- Since X is WQO, $(a_n)_{n\in\mathbb{N}}$ has an infinite increasing subsequence $(a_{n_i})_{i\in\mathbb{N}}$
- Now look at sequence $(A_0, \dots, A_{n_0-1}, B_{n_0}, B_{n_1}, \dots)$.
- By the the minimal choice of A_n, it is not a bad sequence, i.e. there exists a good pair (X, Y), i.e. $X \prec Y$.
- If $X = A_i$ and $Y = A_j$ where $i < j < n_0$, contradiction since $(A_0, A_1, \dots)$ has no good pair.
- If $X = A_i$ and $Y = B_{n_j}$, then $A_i \prec B_{n_j} \prec A_{n_j}$ so $(A_i, A_{n_j}$ is a good pair of $(A_0, A_1, \dots)$, contradiction.
- If $X = B_{n_i}$ and $Y = B_{n_j}$ with $i < j$, then, since $a_i \prec a_j$, we again have $A_{n_i} \prec A_{n_j}$, again the same contradiction.

The graph minor theorem for trees

Theorem (Kruskal 1960)

The finite trees are WQO by the topological minor relation, i.e. for every infinite sequence of trees $T_0, T_1, \ldots,$ there exists $i < j$ such that $T_i \preccurlyeq_t T_j$.

Proof: See Theorem 12.2.1 In Diestel's book.

Proof

Let T_1 and T_2 be two rooted trees. We say that $T_1 \leq T_2$ if there is a subdivision of T_1 that can be embedded into T_2 so that the root of T_1 is mapped onto the root of T_2.

We are going to prove (on board) that the set of tree is WQO by $\leq$ (which is slightly stronger than the announced result).

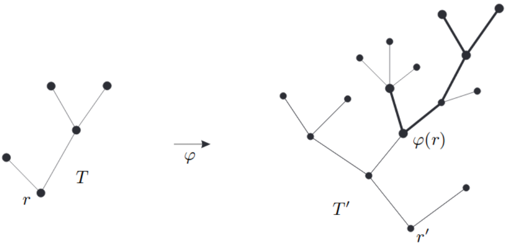

Fig. 12.2.1. An embedding of T in T' showing that $T \leqslant T'$

Reinhart Diestel

5 - TreeWidth

Treewidth

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- This is achieved by the notion of **Treewidth** which is a notion of "treelikeness". In other words it measures how much a graph look like a tree.

  You can understand it like this: if a graph has treewidth 5, then it is at distance 5 from being a tree. Or it is a tree of width 5.

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  You can understand it like this: if a graph has treewidth 5, then it is at distance 5 from being a tree. Or it is a tree of width 5.
- The goal of this section is to introduce treewidth, tree decomposition, and to extend Kruskal Theorem to graphs with bounded treewidth (no proof), look at graphs of treewidth at most 3.

Definition of a tree decomposition and of treewidth

Let G be a graph. A **tree decomposition** of G is a pair (T, W), where T is a tree and $W = (W_t)_{t \in V(T)}$ a collection of subsets of $V(G)$ indexed on $V(T)$ satisfying :

(T_1) For every $v \in V(G)$, there exists $t \in V(T)$ such that $v \in W_t$
- every vertex is in some bag -

(T_2) For every edge $uv \in E(G)$, there exists $t \in V(T)$ such that $u, v \in W_t$
- every edge is in a bag -

(T_3) For every $u \in V(G)$, $T_u = \{t \in V(T) \ , \ u \in W_t\}$ induces a connected subgraph of T.

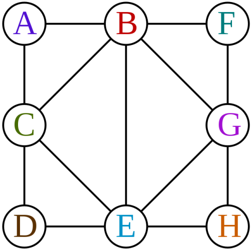
The **width** of a tree decomposition is $\max_{t \in V(T)}(|W_t| - 1)$

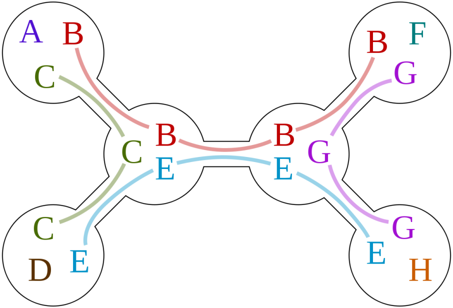
The **tree width** of a graph G, denoted $\mathbf{tw(G)}$, is the **minimum width of a tree decomposition of G**.

Equivalent definitions of tree decomposition

Equivalent definition of tree decomposition: a tree decomposition of G is a tree T along with a collection of subtrees T_v if T, one for each vertex of G, with the condition that T_u and T_v intersect **if** uv is an edge of G.

Example of a tree decomposition





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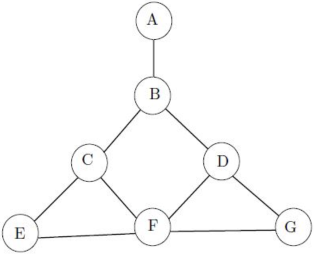
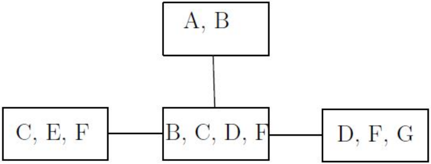
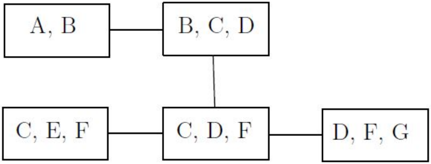


The original graph G



A tree-decompostion of width 3



A tree-decompostion of width 2

Helly Property

Here is a key lemma regarding subtrees intersection; by analogy with Helly's Theorem on convex subsets of $\mathbb{R}^d$, this property is often called **Helly property for subtrees of a tree**.

Lemma (Helly property for subtrees of a tree)

Let $\mathcal{T}$ be a collection of pairwise intersecting subtrees of a given tree T. Then $\cap_{T \in \mathcal{F}} T \neq \emptyset$.

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Proof:

- Assume not. So for each vertex x of T, there is a subtree T_x in $\mathcal{T}$ that does not contain x.
- Therefore T_x is contained in one of the components of $T \setminus x$.
- One edge incident to x corresponds to this component, orient this edge out from x.
- This way, we get an orientation of some edges of T such that each vertex has one outgoing edge.
- Since there are less edges than vertices in a tree, there must be an edge oriented both ways, which results in two non intersecting subtrees in $\mathcal{T}$. Contradiction.

Helly Property

Lemma (Helly property for subtrees of a tree)

Let $\mathcal{T}$ be a collection of pairwise intersecting subtrees of a given tree T. Then $\cap_{T \in \mathcal{F}} T \neq \emptyset$.

Corollary

Let G be a graph and K be a complete subgraph of G. In any tree decomposition (T, W) of G, there exists a vertex t of T such that $K \subseteq W_t$.
in particular, $\mathrm{tw}(G) \geq \omega(G) - 1$

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in particular, $\mathrm{tw}(G) \geq \omega(G) - 1$

Proof: Set $V(K) = \{v_1, \ldots, v_k\}$. For every $i \neq j$, T_{v_i} intersects T_{v_j}, so the T_{v_i} pairwise intersect and thus, by the Helly property for subtrees of a tree, the T_{v_i} have a common intersection, i.e. there is a vertex of T containing all the v_i.

A first lower bound on tree-width

Proposition

For every graph G, there exists a tree decomposition of width $\mathrm{tw}(G)$ such that for every edge $st \in E(T)$, $W_s \not\subset W_t$ and $W_t \not\subset W_s$. Such a tree decomposition is called **irreducible**.

Proof idea: If $st \in E(T)$ and $W_s \subseteq W_t$, contract st.

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Corollary

In every graph G, there exists a vertex of degree at most $\mathrm{tw}(G)$, i.e. $\delta(G) \leq \mathrm{tw}(G)$.

Proof idea: look at a bag corresponding to a leaf.

Corollary

The class of graph with treewidth at most k is k-degenerated.
Hence, for all graphs G, $\chi(G) \leq \mathrm{tw}(G) + 1$.

Separation property of tree decompositions

The following is an easy but fundamental result. It says that a tree decomposition transfers the separation properties of the tree to the decomposed graph.

Proposition (Separation Property)

Let (T, W) be a tree decomposition of G and t_1t_2 be an edge of T and let $S = W_{t_1} \cap W_{t_2}$. For $i = 1, 2$, denote by T_i the connected component of $T \setminus t_1t_2$ containing t_i, and G_i the subgraph of G induced by $\cup_{t \in T_i}(W_t \setminus S)$. Then S is a cutset of G separating G_1 from G_2.

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The following is an easy but fundamental result. It says that a tree decomposition transfers the separation properties of the tree to the decomposed graph.

Proposition (Separation Property)

Let (T, W) be a tree decomposition of G and $t_1 t_2$ be an edge of T and let $S = W_{t_1} \cap W_{t_2}$. For $i = 1, 2$, denote by T_i the connected component of $T \setminus t_1 t_2$ containing t_i, and G_i the subgraph of G induced by $\cup_{t \in T_i}(W_t \setminus S)$. Then S is a cutset of G separating G_1 from G_2.

Sketch of the proof: To prove that S separates G_1 from G_2, it is enough to prove that $V(G_1) \cap V(G_2) = \emptyset$, and there is no edge between $V(G_1)$ and $V(G_2)$.

Assume there exists $u \in V(G_1) \cap V(G_2)$. Then there exists $x_i \in T_i$ such that $u \in W_{x_1} \setminus S$, and $u \in W_{x_2} \setminus S$. Let P be the unique path linking x_1 and x_2 in T. P contains the edge $t_1 t_2$. Moreover, since T_u is a connected subtree of T, u is in every bag W_y such that $y \in V(P)$. In particular, $u \in W_{t_1} \cap W_{t_2} = S$, a contradiction. This proves that $V(G_1) \cap V(G_2) = \emptyset$.

The proof that there is no edge between $V(G_1)$ and $V(G_2)$ is similar.

Closure property

Proposition

Let G be a graph, v a vertex of G and e an edge of G.

- $\mathrm{tw}(G \setminus e) \leq \mathrm{tw}(G)$
- $\mathrm{tw}(G \setminus v) \leq \mathrm{tw}(G)$
- $\mathrm{tw}(G/e) \leq \mathrm{tw}(G)$

Proof:

- for $G \setminus e$, do nothing
- for $G \setminus v$, just remove v from every bag containing it.
- for G/e, where $e = uv$: let w be the new vertex. Add w in every bag containing u or v, and delete every occurrence of u and v.

Treewidth and Minors

Here are two corollaries of the proposition of the previous slide.

Corollary

If H is a minor of G, then $\mathrm{tw}(H) \leq \mathrm{tw}(G)$

Corollary

The class of graphs of treewidth at most k is closed under taking minors.

Wagner's conjecture for graphs with bounded tree-width

Graphs with bounded treewidth are sufficiently similar to trees that it becomes possible to adapt the proof of Kruskal Theorem to them.

Very roughly, one has to iterate the "minimal bad sequence" used in Kruskal proof $\mathrm{tw}(G)$ times.

This takes us a step further towards a proof of the Graph Minor Theorem:

Theorem (Robertson and Seymour, IV)

Given an infinite sequence of graphs $G_1, \ldots, G_n, \ldots$, all of treewidth at most k, there exists i, j such that $G_i \prec_m G_j$.

Corollary

The class of graphs of treewidth at most k has a finite number of obstructions. I.e., there exists a finite set of graphs $\mathcal{F}$, such that $Forb_{\preccurlyeq_m}(\mathcal{F})$ is exactly the class of graphs with treewidth at most k.

Obstructions for graphs with treewidth at most 2

So, for every fixed k, the class $\{G : \mathrm{tw}(G) \preccurlyeq k\}$ has a finite number of obstructions.

Let us try to describe the obstructions for small values of k.

Theorem

- $\mathrm{tw}(G) \leq 1 \Leftrightarrow$ *G is a forest* $\Leftrightarrow$ *G does not contain K_3 as a minor* $\Leftrightarrow G \in \mathit{Forb}_{\preccurlyeq_m}(K_3)$
- $\mathrm{tw}(G) \leq 2 \Leftrightarrow$ *G does not contain K_4 as a minor* $\Leftrightarrow G \in \mathit{Forb}_{\preccurlyeq_m}(K_4)$

The first item is easy, let us prove the second.

Proof sketch:

- If G contains K_4 as a minor, then $\mathrm{tw}(G) \geq \mathrm{tw}(K_4) = 3$. So $\mathrm{tw}(G) \leq 2 \Rightarrow G \in Forb_{\preccurlyeq_m}(K_4)$.

Proof sketch:

- If G contains K_4 as a minor, then $\mathrm{tw}(G) \geq \mathrm{tw}(K_4) = 3$. So $\mathrm{tw}(G) \leq 2 \Rightarrow G \in Forb_{\preccurlyeq_m}(K_4)$.
- Let $G \in Forb_{\preccurlyeq_m}(K_4)$ and let us prove that $\mathrm{tw}(G) \leq 2$. We proceed by induction on $V(G)$.

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- If G contains K_4 as a minor, then $\mathrm{tw}(G) \geq \mathrm{tw}(K_4) = 3$. So $\mathrm{tw}(G) \leq 2 \Rightarrow G \in Forb_{\preccurlyeq_m}(K_4)$.
- Let $G \in Forb_{\preccurlyeq_m}(K_4)$ and let us prove that $\mathrm{tw}(G) \leq 2$. We proceed by induction on $V(G)$.
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- Prove first that every 3-connected graph contains K_4 as a minor (Use Menger Theorem).

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- Take the disjoint union of T_1 and T_2 and add an edge between t_1 and t_2, and don't change the bags.
- Check that this gives a tree decomposition of G (i.e. check that the three axioms of the definition of tree decomposition are still satisfied).

- Assume now that $S = \{a, b\}$.
- If $ab \notin E(G)$, then add ab to G and prove that this does not create a K_4-minor.
- To do it, assume that $G + ab$ has a K_4 model, then prove that you can choose it such that it is included in $G[C \cup S]$ for some connected component of $G \setminus S$. Then observe that you can replace the edge ab by a path linking a and b that has interior vertices in a connected component $C' \neq C$. Conclude that this gives a K_4-model in G, contradiction.
- So now S is a clique (we call that a **clique cutset**).
- Let C_1 be a connected component of $G \setminus S$ and $C_2 = G \setminus (S \cup C_1)$.
- For $i = 1, 2$, set $G_i = G[C_i \cup S]$ (The G_i are often called **block decomposition**). By minimality of G, $\mathrm{tw}(G_i) \leq 2$.
- Take a tree decomposition of G_1 and G_2 of width at most 2 and link a bag of G_1 containing ab to a bag of G_2 containing ab.
- Prove that this is a tree decomposition of G of width 2.

Bounds for graphs with treewidth at most 2

Theorem

- $\mathrm{tw}(G) \leq 1 \Leftrightarrow G \in Forb_{\preccurlyeq_m}(K_3)$
- $\mathrm{tw}(G) \leq 2 \Leftrightarrow G \in Forb_{\preccurlyeq_m}(K_4)$

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$$\mathrm{tw}(G) \leq k \text{ iff } G \in Forb_{\preccurlyeq_m}(K_{k+2})$$ **FALSE**

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- One could hope for a general result of the type:

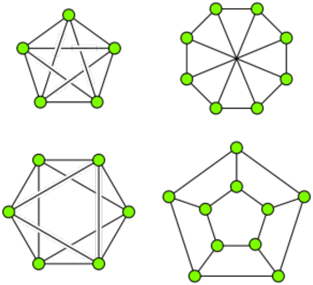
$$\mathrm{tw}(G) \leq k \text{ iff } G \in Forb_{\preccurlyeq_m}(K_{k+2}) \quad \textbf{FALSE}$$

- It is clear that if $tw(G) \leq k$, then $G \in Forb_{\prec_m}(K_{k+2})$.
  But there exists graph with no K_5 minor and with arbitrarily large treewidth.
  (As we will soon see, even planar graphs can have arbitrarily large treewidth).

Obstructions for graphs with treewidth at most 3

Theorem

$\mathrm{tw}(G) \leq 3 \Leftrightarrow G$ *does not contain one of the four following graphs as as a minor :* K_5,W_8, O *and* $C_5 \times K_2$.



Digression : Hadwiger Conjecture

We know that for every graph G:

$$\begin{aligned} \omega(G) \leq \quad & \chi(G) & \leq \mathrm{tw}(G) + 1 \\ \omega(G) \leq \quad & \omega_m(G) & \leq \mathrm{tw}(G) + 1 \end{aligned}$$

where $\omega_m(G)$ is the largest integer k such that G has a K_k minor.

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- For $k = 3$: $\omega_m(G) \leq 3 \Leftrightarrow \mathrm{tw}(G) \leq 2 \Rightarrow \chi(G) \leq 3$ by the above inequalities.
- For $k = 4$: $\omega_m(G) \leq 4 \Rightarrow \chi(G) \leq 4$ contains the Four Colour Theorem since planar graphs are K_5-minor free. In fact it is equivalent (and hence true), thanks to a structural characterisation of graphs with no K_5 minor due to Wagner.

Theorem (Wagner, 1956)

G is K_5-minor free if and only G is a subgraph of some graph built recursively by clique sums operation, starting from planar graphs and W_8.

We will see later in the course that this theorem together with the 4-color theorem implies Hadwiger conjecture for $k = 5$, that is

$$\omega_m(G) \leq 4 \Rightarrow \chi(G) \leq 4$$

Exercises on treewidth

Exercice 13

Prove that if H is a subdivision of G, then $\mathrm{tw}(H) = \mathrm{tw}(G)$

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Prove that if H is a subdivision of G, then $\mathrm{tw}(H) = \mathrm{tw}(G)$

Solution: H is a subdivision of G means that H can be obtained from G be replacing some edges by path.

If G is a tree then H is also a tree and we have $\mathrm{tw}(H) = \mathrm{tw}(G) = 1$. Otherwise $tw(G) \geq 2$. Then for each bag W containing both a and b, add a a new bag $\{a, x, b\}$ adjacent to it.

The following exercise says that classes of graphs with bounded treewidth are sparse.

Exercice 14

Show that graphs G of treewidth at most k with $k \geq 1$ have strictly less than $k|V(G)|$ edges.

Next exercise is very important to design algorithm based on the tree decomposition.

Exercice 15

Show that every graph G admits a tree decomposition of width $\mathrm{tw}(G)$ with at most $|V(G)|$ bags.

Hint: prove the stronger statement that a irreducible tree decomposition has at most n bags.

Exercises on treewidth

Exercice 16

Determine the treewidth of a path, a tree, a complete graph, a complete bipartite graph, the cube.

Exercice 17

Prove that if G contains (as a subgraph) a complete bipartite graph with parts A and B, then in every tree decomposition there exists a bag that contains A or a bag that contains B.

Hint: Delete all vertices but the vertices of the complete bipartite graph. We have a tree decomposition of the complete bipartite. A bag that is not a leaf must be a cutset, and thus contains A or B.

You should be able to do this exercise, moreover the fact it proves is quite important.

Exercice 18

Prove that if x and y are two vertices that are joined by $k+1$ internally vertex disjoints paths, then in every tree decomposition of G of width at most k, there exists a bag containing both x and y.

Hint: Use the separation property of tree decomposition.

Solution on the next slide.

Solution Let (T, W) an irreducible (no bag is included in another one) tree decomposition of width $k = tw(G)$. Assume for contradiction that no bag contains both x and y. Let t and t' two nodes of T such that $x \in W_t$, $y \in W_{t'}$. Let uv be an edge on the unique path linking t and t' in T. Then, by the separation property, $W_u \cap W_v$ is a cutset of G, of size at most k (because the tree decomposition is irreducible) that separates x and y. By Menger Theorem, it contradicts the fact that x and y are linked by $k+1$ internally vertex disjoint paths.

As you have already seen, treewidth also plays a crucial role in algorithmic. We'll come back to it.

6 - Brambles - Duality - Cops and Robbers

In the previous section, we have seen that Wagner Conjecture holds for class of graphs with bounded treewidth.

To make a proof of the general case, we should be able to say stuff about the graphs it does not cover, i.e. to deduce informations about a graph from the assumption it has large treewidth.

The main theorem of this section achieves that: it identifies a canonical obstruction to small treewidth, a structural phenomenon that occurs in a graph if and only if it has large treewidth.

This phenomenon is called **Bramble**.

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The main theorem of this section achieves that: it identifies a canonical obstruction to small treewidth, a structural phenomenon that occurs in a graph if and only if it has large treewidth.

This phenomenon is called **Bramble**.
(In reality, it is mainly used to get certificate on the value of the treewidth of a graph, the notion of tangle is used as an obstruction for large treewidth, but we won't see it during this class).

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Definition (Bramble)

- We say that two connected subgraphs of G **touch** if they have non empty intersection or if they are joined by an edge.

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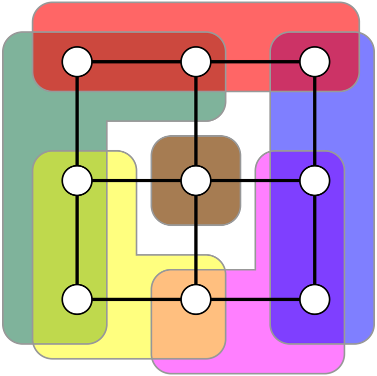
- The **bramble number** of G, denoted $\mathrm{bn}(G)$, is the maximum order of a bramble of G.

Note that if G contains K_p as a minor, then the connected subgraphs of a K_p-model of G form a bramble (no intersection, just touching) of order p.

Note that we have already seen the notion of transversal, for example a vertex cover is a transversal of the edges.

A Bramble

A bramble of order 4 of $G_{3,3}$:



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- But the last vertex of a maximal directed path has no outgoing edge, contradiction.

Duality Theorem II

Proposition

If (T, W) is a tree decomposition of G and $\mathcal{B}$ is a bramble in G, then there exists $t \in T$ such that W_t is a transversal of $\mathcal{B}$.

Therefore

$$\mathrm{bn}(G) \leq tw(G) + 1$$

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The converse inequality is true but harder to prove.
It gives the following sort of minmax theorem (in fact maxmin=minmax).

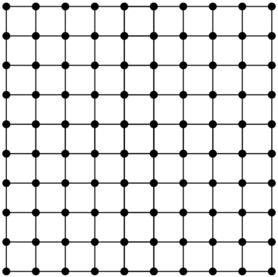
Theorem (Seymour and Thomas, 1993)

For every graph G, $\mathrm{bn}(G) = \mathrm{tw}(G) + 1$

Grids

Now, we know that if a graph has large treewidth, then it also has a large brumble. But is it so usefull?

We are going to see later that it also has a large grid (as a minor), which is often way more useful. For the moment, let us just prove that grids have large treewidth.



What is the treewidth of the grid?

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The treewidth of the grid $G_{n,n}$ is equal to n.

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- It is easy to check that the following is a bramble of order $n+1$:
  - $A = \{x_{i,1},\, 1 \leq i \leq n\}$, the last row,
  - $B = \{x_{1,j},\, 1 \leq j < n\}$ the last column minus its last element,
  - $C_{ij} = \{x_{kj},\ 1 \leq k < n\} \cup \{x_{ik},\ 1 \leq k < n,\}$ (crosses minus the last element of row and column).

A Game of Cops and Robber

- 2 player game on a graph: one controls the Robber, the other control Cops
- Goal of the cops is to capture the robber
- Many variants exist

In our variant :

- cops and robbers are standing on vertices of the graph
- at each turn a fraction of the cops can move by helicopter and land on any vertex of the graph.
- The robber sees an helicopter approaching and can instantly move at infinite speed to any other vertex along a path of a graph. The only constraint is that he is not permitted to run through a vertex occupied by some cop.

The cops win if at some point they occupy all vertices adjacent to the position of the robber, and an extra cop lands by helicopter on the robber.

Definition

The **cop number** of a graph G, denoted $\mathrm{cn}(G)$, is the smallest number of cops needed to ensure the capture of the robber from any starting position.

TreeDec = strategy for the cops

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- Cops apply this strategy until it reaches some leaf of the tree and the robber cannot escape.

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- So $V(B_1) \cup V(B_2)$ is not occupy by any cut during the flight, and B_2 is not occupy by any cops after the flight.
- The robber can thus freely move from B_1 to B_2, and stay safe in B_2! This strategy can be applied for ever.