Graph Minor Theory and its algorithmic consequences
MPRI Parametrized Complexity

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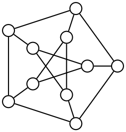
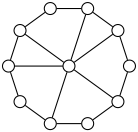
6 Hours' Programm

- Wagner conjectures Minors and topological minors
  - Classes of graphs defined by fordding some graphs as minors or topological minors.
  - The k-disjoint path problem and its links with minors and topological minors.
  - Wagner Conjecture and its links with minor clases classes.
  - Wagner Conjecture, Well Quasi Orders and Kruskal Theorem.
- Treewidth
  - Definition and basic properties.
  - Duality of treewidth: brumble and the game of cops and robber.
  - Grid minor Theorem and treewidth of classes of rgaphs defined by forbidding a minor.
- The Graph Minor Theorem.
- FPT algorithm using the Graph Minor Theorem.

1 - Characterization of graph classes by forbidden configurations

Graph theory





Graphs: a mathematical object and an efficient modeling tool.

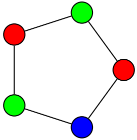
Important questions:

- What classes of graphs have good algorithmic properties? (colouring, clique max...)
- What classes of graphs have good structural properties? (decomposition theorem, elimination ordering...)

Forbidding a substructure:

- Minors: Robertson and Seymour, 1983-2012
- Topological minors
- Induced subgraphs

Chromatic number



$\chi(G)$= minimum number of colors needed to color the vertices in such a way that adjacent vertices receive distinct colors. In other words its a partitioning of the vertex sets into stable sets, minimizing the number of stable sets.

Exercice 1

What is the chromatic of $K_{a,b}$? K_n? C_n?

Let k an integer. A class of graphs $\mathcal{C}$ is **k-degenerate** if for all $G \in \mathcal{C}$, G has a vertex of degree at most k.

Exercice 2

Let $\mathcal{C}$ be a k-degenerate class of graphs closed under taking induced subgraph. Prove that all graphs in G has chromatic number at most $k+1$.

Containment relations

We define four operations on a graph G:

1. **Remove a vertex** v (and all its incident edges) , denoted $G \setminus v$.
2. **Remove an edge** e (but not its end vertices) , denoted $G \setminus e$.
3. **Contract an edge** $e = xy$, denoted G/e :
   (i.e. remove x and y, add a new vertex z with neighbourhood $N(z) = (N(x) \cup N(u)) \setminus \{z\}$ (no loops))
4. **Topological contraction** is a contraction of edge e that has an endvertex of degree 2. Its inverse is the **subdivision operation** which consists in removing an edge xy, adding a new vertex z, and adding the edges xz and zy.

Definition

Let G and H be two graphs.

- H **induced subgraph** of G if H obtained from G by the repeated use of 1.
- H **subgraph** of G if H obtained from G by the repeated use of 1 and 2.
- H **topological minor** of G if H is a minor of G and every contraction used was topological.
- H **minor** of G if H obtained from G by the repeated use of rule 1,2 and 3.

Partial orders

Each of these containment relation defines a partial order on graphs:

- H induced subgraph of G: $H \subseteq_i G$
- H subgraph of G: $H \subseteq G$
- H topological minor of G: $H \preceq_t G$
- H minor of G: $H \preceq_m G$

Let $\preceq$ be any of these orders. We say that a **class of graphs** $\mathcal{C}$ is **$\preceq$-closed** (subgraph-closed, minor-closed...) if **for all** $G \in \mathcal{C}$: $H \preceq G \Rightarrow H \in \mathcal{C}$. .

- The class of planar graphs if minor closed (and thus topological-minor-closed, subgraph-closed and induced-subgraph-closed).
- The class of bipartite graphs is subgraph-closed, but not topological-minor-closed.
- The class of all graphs whose connected components are cliques is induced-subgraph-closed, but not subgraph-closed.

Minors

Here is an equivalent definition for minors that is often useful:

Lemma

Let G and H be two graphs, and denote $V(H) = \{v_1, \ldots, v_p\}$. Then H is a minor of G if and only if there exists p connected and disjoint subgraphs $G_1, \ldots, G_p$ of G such that for every edge $v_i v_j$ of H, there exists an edge between G_i and G_j.
The graphs induced by $G_1, \ldots, G_p$ is called a ***H-model*** *of G.*

Exercice 3

Show that the (3×3)-grid has a K_4-minor by showing it has a K_4-model.

Topological Minors

A topological minor is also called **subdivision**.
Here is an equivalent definition of topological minor.

Definition

A graph H is **topological minor** of a graph G if there exists a injective mapping f from $V(H)$ to $V(G)$ such that for each edge uv of H, there exists in G a path P_{uv} connecting $f(u)$ and $f(v)$ in G with the property that all these path are internally disjoint.

Exercice 4

Describe the graphs that do not contain the following graphs as topological minors : K_3, $K_{1,3}$, $K_{1,4}$.

Classes of graph defined by forbidden configurations

For a set $\mathcal{F}$ of graphs, let $Forb_{\preccurlyeq}(\mathcal{F}) = \{G : \forall F \in \mathcal{F}, F \not\preccurlyeq G\}$ i.e. the class of graphs not containing any graphs of $\mathcal{F}$ under $\preccurlyeq$-relation. We say such graph are $\mathcal{F}$-$\preccurlyeq$-free.

- $Forb_{\preccurlyeq_t}(K_5, K_{3,3})$ = planar graphs = $(K_5, K_{3,3})$-topological minor free graphs.
- $Forb_{\subseteq}(C_3, C_5, C_7, ...)$ = ?? bipartite graphs.
- $Forb_{\subseteq_i}(K_{1,2})$ = ?? graphs whose connected components are cliques.

A graph F is a **$\preccurlyeq$-obstruction** for a class $\mathcal{C}$ if $F \notin \mathcal{C}$ but for every $H \preccurlyeq G$, $H \in \mathcal{C}$.
Let $Obst_{\preccurlyeq}(\mathcal{C})$ be the set of all $\preccurlyeq$-obstruction of $\mathcal{G}$.

- K_5 is a topological-minor-obstruction for planar graphs since K_5 is not planar, but every proper topological-minor of K_5 is.
- K_6 is not a topological-minor-obstruction for planar graphs since $K_5 \preccurlyeq_t K_6$ and K_5 is not planar.

Exercises

Exercice 5

Let $\mathcal{C}$ be a class of graphs and $\preccurlyeq$ a containment relation on graphs. Prove that $\mathcal{C}$ is $\preccurlyeq$-closed if and only if there exists a (possibly infinite) set of graphs $\mathcal{F}$ such that $\mathcal{C} = Forb_{\preccurlyeq}(\mathcal{F})$.

Exercice 6

Prove that a graph G is a forest if and only if it does not contain C_3 as a minor.

Exercice 7

1. Prove that every graph with average degree at least 2^{r-2} contains K_r as a minor.
2. For r fixed, does there exist K_r minor-free graphs with arbitrarily large chromatic number?

Solution

Sketch of proof for Exercise 7: Recall that the average degree of a graph is $\frac{\sum_{v \in V(G)} d(v)}{|V(G)|} = \frac{2|E(G)|}{|V(G)|}$.
We proceed by induction on r. Easy when $r = 1$ or 2. Let G be a graph of average degree at least 2^{r-2}. Therefore $\frac{|E(G)|}{|V(G)|} \geq 2^{r-3}$. Let H be minimal amongst all minors of G such that $\frac{|E(H)|}{|V(H)|} \geq 2^{r-3}$. It implies that when one contracts an edge in H, one must loose at least 2^{r-3} edges (otherwise the inequality would still be satisfied, and H would not be minor minimal). Hence, for any xy edge of H, x and y have at least 2^{r-3} common neighbours. In otherwords, if x is a vertex in H, then the minimum degree in its neighbourhood is at least 2^{r-3}, so by induction it contains a K_{r-1} minor, which yields with x the desired K_r minor.

Hence K_r-minor-free graphs has a vertex of degree at most 2^{r-2} and is thus $2^{r-2} + 1$ colourable.

2 - Three Algorithmic Problems

A Classical Connectivity Problem

Consider the following problem of connectivity.

Problem (k disjoint paths problem)

Input : A graph G, an integer k and two subsets of vertices A and B of size k
Output : TRUE if there exists k vertex disjoint paths from A to B

CLASSIC : Can be solved in time $O((k|E(G)|)$ using **Ford-Fulkerson Algorithm**.

From a structural point of view, the maximum number of paths linking A and B corresponds to a minimum cut-vertex separating A and B and is a classical result of **Menger**.

Theorem (Menger,1927)

Let x and y be distinct vertices of a graph G. Then the minimum number of vertices whose deletion separates x from y is equal to the maximum number of internally disjoint paths linking x and y.

Exercise on connectivity

Let G be a graph, $x \in V(G)$ and $Y \subseteq V(G) \setminus \{x\}$. A family of k internally disjoint (x, Y)-paths whose terminal vertices are distinct is referred to as a ***k*-fan** from x to Y.

Exercice 8

Let G be a k-connected graph.

1. Let x be a vertex of G, and let $Y \subseteq V \setminus \{x\}$ be a set of at least k vertices of G. Then there exists a k-fan in G from x to Y. (This property is known as the **Fan Lemma**).
2. Let S be a set of k vertices in a k-connected graph G, where $k \geq 2$. Then there is a cycle in G which includes all the vertices of S.

For a very good presentation of Menger Theorem and its consequences, see the book *Graph Theory* of J. A. Bondy and U. S. R. Murty, chapters 9.1 and 9.2.

A similar problem.

Problem (k-disjoint rooted paths problem)

Input : A graph G, an integer k, and two subsets of vertices $S = \{s_1, s_2, \dots, s_k\}$ and $T = \{t_1, t_2, \dots, t_k\}$

Output : TRUE iff there exists disjoint paths $P_1, P_2, \dots, P_k$, such that P_i is a path from s_i to t_i.

- Crucial role in VLSI design, related to commodity flow problem, many applications.
- With $k \geq 2$ part of the input, this problem is NP-complete, even restricted to the class of planar graphs.
- Nevertheless, in the Graph Minor series of papers, Robertson and Seymour proved a polynomial algorithm for fixed k.

Theorem (Robertson-Seymour, 1995 (XIII))

The k-disjoint rooted path problem can be solved in time $O\big((f(k).n^3\big)$

(improved to quadratic time by Kawarabayashi, Kobashi and Reed, 2012)

Topological Minor Detection I

Problem (Topological H-minor detection)

Input : A graph G and a graph H.
Output : TRUE if H is a topological minor of G, FALSE otherwise.

- With H part of the input : NP-complete
- With H fixed, polynomial thanks to the k-disjoint path problem algorithm:
  - Complexity : $O\big(f(k)n^k\big)$, where $k = |V(H)|$, and $n = |V(G)|$. Therefore **polynomial for every fixed** k. So the problem is in (**XP**).
  - In 2010, Grohe, Kawabarayashi, Marx, and Wollan proved much better: $O\big(f(k)n^3\big)$. So the problem is actually **FPT**.

Topological Minor Detection II

Theorem

Let H be a fixed graph with k edges. One can decide whether H is a topological minor of a given graph G in time $O\big(f(k)n^k\big)$.

Sketch proof:

Let $f : V(H) \to V(G)$ be an injection.
Observe that there is $\binom{n}{|V(H)|}$ such objects
We want to decide if there exists disjoint paths in G between the $f(v)$ corresponding to edges of H.
To do that, we replace (in G) each vertex $f(v)$ by $d_H(v)$ copies of $f(v)$ (having the same neighbours).
Now, for $k = |E(H)|$, solving the k-Rooted Disjoint Paths Problem for well chosen sets solve the problem.

Consequences

In particular, the previous theorem implies that any family of graphs that is defined with **forbidding a FINITE family of graphs as topological minors is polynomially testable**.

In other words if $\mathcal{C} = Forb_{\preccurlyeq_t}(\mathcal{F})$ where $\mathcal{F}$ is a finite set of graphs, then we can decide in polynomial time if a graph G belongs to $\mathcal{C}$.

Example of such class?

Theorem (Kuratowski, 1930)

A graph G is planar if and only if it does not contain K_5 nor $K_{3,3}$ as a topological minor.

Note that one does not need to solve k rooted paths problem to get polytime algorithms for recognizing planar graphs (there exist even linear algorithms to do that).

3 - Wagner Conjecture and minor closed classes

Wagner Conjecture (now *Graph Minor Theorem*)

Our goal in this last chapter is a single theorem, one which dwarfs any other result in graph theory and may doubtless be counted among the deepest theorems that mathematics has to offer: *in every infinite set of graphs there are two such that one is a minor of the other.* This *graph minor theorem* (or *minor theorem* for short), inconspicuous though it may look at first glance, has made a fundamental impact both outside graph theory and within. Its proof, due to Neil Robertson and Paul Seymour, takes well over 500 pages.

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Minors Vs Topological Minors

- By definition: H topological minor of $G \Rightarrow H$ minor of G
- **Exercise:** converse not true: find a pair of graphs G and H such that H is a minor of G but H is not a topological minor of G.
  Solution: Set H to be two disjoint $K_{1,2}$ and link their vertices of degree 2 by an edge. Then H is a minor of $K_{1,4}$, but not a topological minor.

When H is subcubic (maximum degree at most 3), this is nevertheless true.

Theorem

Let H be a graph with maximum degree at most 3. Then a graph G has an H-minor if and only if it contains an H-subdivision.

Proof on the next slide.

Theorem

Let H be a graph with maximum degree at most 3. Then a graph G has an H-minor if and only if it contains an H-subdivision.

Sketch proof:

- Assume H is a minor of G
- Let G' be a **minimal** topological minor of G such that H is a minor of G' ($[V(G)| + |E(G)|$ is minimized).
- Note that G' is a topological minor of G means that G' is btained from G by deleting vertices, edges, a contracting edges with at least one extremity of degree at most 2.
- Look at an H-model $(G_1, \dots, G_p)$ (where $p = |V(H)|$) of G.
- By minimality of G, each G_i is a tree with at most 3 leaves and no vertex of degree 2 (at most three leaves because if it has 4, then one is not used to connect G_i to another G_j, and if there is a vertex of degree 2 (resp. a cycle) we can contract an edge (resp. delete an edge) and have a smaller topological minor of G that still contains an H-model).
- Each such tree must be a star, so we get the topological minor.

Minors Vs Topological Minors

A similar argument proves this more general result.

Theorem

For every graph F, there exists a **finite** *family of graphs $\mathcal{F}$ such that:*
G contains F as a minor if and only if it contains some graph in $\mathcal{F}$ as a topological minor.
In other words: $Forb_{\preccurlyeq_m}(F) = Forb_{\preccurlyeq_t}(\mathcal{F})$.

Proof: We start the proof exactly as in the previous result, and by again choosing minimal G_i, we now get for each G_i a tree with at most $|H|$ leaves and no vertex of degree 2. There is finitely many such trees (why?). So by replacing the vertices of H by these trees in all possible ways, we obtain a finite collection of graphs $\mathcal{H}$ with the desired properties.

Minor detection

This result combined with the theorem on topological minor detection now clearly implies the following theorem.

Theorem (Robertson and Seymour, 1995)

Let H be a fixed graph. There exists a polynomial time algorithm to decide whether H is a minor of a given graph G.

Corollary

If $\mathcal{C}$ is a class of graphs defined by forbidding finitely many minors, then there exists a polynomial algorithm to decide wether an input graph belongs to $\mathcal{C}$

Wagner Conjecture

Question

What are the families defined by finitely many forbidden minors?

Examples :

Graph Class	Minor minimal graphs
Forests	triangle
Union of Paths	triangle, claw
Planar	K_5 , $K_{3,3}$
Toric	$\geq$ 17523 (but finite)

Exercice 9

For each of the following classes, decide if it is minor closed or not. If not, try to describe the smallest minor closed class containing it: cliques, paths, cycles, graphs of max degree k?

Minor Closed Classes

Question

What are the families defined by finitely many forbidden minors?

- A trivial fact is that such families are **closed under minors** (every minor of a graph in the family is in the family).
- In a monumental work ($>$700 pages, Graph Minors I-II-III-...-XXV), Robertson and Seymour solved a conjecture of Wagner from 1937 saying that this is sufficient.

Theorem (Graph Minor Theorem, Robertson and Seymour, XX)

Any minor closed class of graphs is defined by a finite list of forbidden minors

With an important consequence (among many others):

Corollary

If $\mathcal{C}$ is a minor closed class, then there exists a polynomial time algorithm to decide if an input graph belongs to $\mathcal{C}$.

Exercises

Exercice 10

Prove that the following problems are solvable in time $O\big(f(k)n^3\big)$.

- k-**Vertex Cover**
  Input : A graph G.
  Output : TRUE if there exists a set S of at most k vertices such that $G \setminus S$ has no edge.
- k-**Feedback vertex set**
  Input : A graph G.
  Output : TRUE if there exists a set S of at most k vertices such that $G \setminus S$ has no cycle.
- k-**leaf Spanning Tree**
  Input : A graph G.
  Output : TRUE if there exists in G a spanning tree T with at least k leaves.

HINT: Observe that for each of these problems, the set of TRUE instances is closed under taking minor.

The idea is that any propery closed under taking minor is testable in time $O(f(k))\,n^3$

4 - Well Quasi Orders and Wagner Conjecture

Introduction

In this section we will try to understand some of the ideas behind the proof of Wagner's conjecture by proving similar but (much) easier results.

We first introduce the notion of well quasi order that gives an equivalent way to state Wagner Conjecture.

Then we will prove a theorem due to Kruskal saying that trees are well quasi ordered for the minor relation.

In the next section, we will explain through the notion of treewidth why Kruskal Theorem and its proof is central in Robertson and Seymour's proof.

Wagner's Conjecture

Definition (Obstructions)

For a given minor closed class $\mathcal{C}$, a graph H is said to be an **obstruction** of $\mathcal{C}$ if H is not in $\mathcal{C}$ but every strict minor of H is.

Proposition

Let $\mathcal{C}$ be a minor closed class, and $\mathcal{O}$ be its (possibly infinite) set of obstructions. Then $G \in \mathcal{C}$ if and only if G does not contain any graph of $\mathcal{O}$ as a minor
In particular $\mathcal{C} = Forb_{\preccurlyeq_m}(\mathcal{O})$. Moreover, $\mathcal{O}$ is the smallest set of graphs with this property.

- So Wagner's conjecture is to prove that a set of obstructions is always finite.
- Observe that by definition the set of obstructions forms an **antichain** of the minor partial order: no obstruction is a minor of another obstruction.
- Is it true that **every antichain is finite?**

Wagner's Conjecture - continued

Proposition

The following are equivalent :

- *Every minor closed class has a finite set of obstructions.*
- *There is no infinite antichain for the minor relation.*

Definition

A partial order $\preccurlyeq$ defined on a set X is a **well quasi order** (WQO) if there is no infinite strictly decreasing sequence and no infinite antichain.

Wagner's conjecture is equivalent to say that the class of all graphs with the minor relation is a WQO.

Exercises on well quasi ordering

Exercice 11

For each of these, say if it is a wqo.

- $(\mathbb{N}, \leqslant)$.
- $(\mathbb{R}, \leqslant)$.
- $(\mathbb{N}^2, \preccurlyeq)$ where $(x, y) \preccurlyeq (x'y')$ iff $(x \preccurlyeq x'$ and $y \preccurlyeq y')$,
- $(\mathcal{G}, \subseteq_i)$ where $\mathcal{G}$ is the class of all graphs (recall that $H \subseteq_i G$ means H is an induce subgraph of G).
- Finite trees ordered by subgraph relation.
- $(\mathcal{G}, \preccurlyeq)$ where $G \preccurlyeq H$ if G topological minor of H

Some solution one the next slide.

Finite trees ordered by subgraph relation. No: take double broom: paths with end vertices of degree 3.

$(\mathcal{G}, \preccurlyeq)$ **where** $G \preccurlyeq H$ **if** G **topological minor of** H: NO, take the family of thick cycle, where a thick cycle is a cycle where each edge is doubles (parallel edges).

Dealing with WQO: a first tool

Proposition: Let $(X, \preccurlyeq)$ be a partially ordered set and $(x_i)_{i \in \mathbb{N}}$ be any sequence. Then $(x_i)_{i \in \mathbb{N}}$ has an infinite subsequence that is either strictly increasing, or strictly decreasing or an antichain.

Proof: By Ramsey, or: Let (x_i) be any sequence. Starts with x_1, and consider

- $A_1 = \{j,\ j > 1 \text{ and } x_1 \preccurlyeq x_j\}$
- $B_1 = \{j,\ j > 1 \text{ and } x_1 \preccurlyeq x_i\}$
- $C_1 = \{j,\ j > 1 \text{ and } x_1 \text{ and } x_j \text{ are incomparable}\}$

If A_1 is infinite we say that x_1 is of type A and delete all elements that are not in A_1. If not, but B_1 is infinite, say that x_1 is of type B and delete all elements that are not in B_1. Finally in the last case, say that x_1 is of type C and delete all vertices not in C_1. Up to extracting a subsequence and renaming, we can assume no element were deleted, so that the x_i with $i \geq 2$ were all in A_1, or all in B_1, or all in C_1. We do this sequentially for x_2, then x_3, ... I.e., at each step, we define A_i, B_i, C_i as

- $A_i = \{j,\ j > i \text{ and } x_i \preccurlyeq x_j\}$
- $B_i = \{j,\ j > i \text{ and } x_j \preccurlyeq x_i\}$
- $C_i = \{j,\ j > i \text{ and } x_i \text{ and } x_j \text{ are incomparable}\}$

and at each step we define the type of x_i to be one of A, B, C depending on which is infinite. Then we extract by keeping only the elements in the infinite set.
Eventually we have a type for each element of the sequence (which is in fact a subsequence of the original sequence). Now there must be a type with infinitely number of elements and to each type clearly corresponds one of the three possible type of infinite subsequence.

Dealing with WQO: a first tool

Corollary

Let $(X, \preccurlyeq)$ be a partially ordered set. The three assertions are equivalent

1. $(X, \preccurlyeq)$ *is a wqo.*
2. *from every sequence $(x_i)_{i \in \mathbb{N}}$ one can extract an infinite increasing subsequence.*
3. *from every sequence $(x_i)_{i \in \mathbb{N}}$ one can extract $i < j$ such that $x_i \preccurlyeq x_j$.*

This will be useful: in order to prove that a given partial order is a WQO, we will only prove the third statement, but when we use the fact that an order is a WQO (for example in a proof by induction), we can use the second statement which is (in appearance) much stronger.

(x_i, x_j) is a **good pair** if $i < j$ and $x_i \preccurlyeq x_j$.

Hence, $(X, \preccurlyeq)$ is a WQO if and only if every sequence $(x_i)_{i \in \mathbb{N}}$ has a good pair.

Second tool: extending a partial order

Let $(X, \leq)$ be a partial order. For finite subsets $A, B \subset X$, write $A \preccurlyeq B$ if there is an injective mapping $f : A \to B$ such that $a \leq f(a)$ for all $a \in A$.

This naturally extends $\leq$ to a partial order on $[X]^\omega$, the set of all finite subsets of X.

Lemma

If X is a WQO, then so is $[X]^\omega$.

Proof [see Diestel, Lemma 12.1.3]: Main idea: start with a "minimum" infinite antichain.

Proof sketch

- Assume for contradiction that $[X]^w$ has a bad sequence, i.e. an infinite sequence with no good pair.
- We construct a "minimal" bad sequence $(A_n)_{n\in\mathbb{N}}$ as follows:
- Assume inductively that A_i has been defined for every $i < n$, and that there exists a bad sequence in $[X]^w$ starting with $A_0, \ldots, A_{n-1}$.
- Choose A_n such that some bad sequence starts with $(A_0, \ldots, A_{n-1}, A_n)$ and $|A_n|$ is minimum with this property.
- For each n, pick en element $a_n \in A_n$, and set $B_n = A_n \setminus \{a_n\}$.
- Since X is WQO, $(a_n)_{n\in\mathbb{N}}$ has an infinite increasing subsequence $(a_{n_i})_{i\in\mathbb{N}}$
- Now look at sequence $(A_0, \ldots, A_{n_0-1}, B_{n_0}, B_{n_1}, \ldots)$.
- By the the minimal choice of A_n, it is not a bad sequence, i.e. there exists a good pair (X, Y), i.e. $X \prec Y$.
- If $X = A_i$ and $Y = A_j$ where $i < j < n_0$, contradiction since $(A_0, A_1, \ldots)$ has no good pair.
- If $X = A_i$ and $Y = B_{n_j}$, then $A_i \prec B_{n_j} \prec A_{n_j}$ so $(A_i, A_{n_j}$ is a good pair of $(A_0, A_1, \ldots)$, contradiction.
- If $X = B_{n_i}$ and $Y = B_{n_j}$ with $i < j$, then, since $a_i \prec a_j$, we again have $A_{n_i} \prec A_{n_j}$, again the same contradiction.

The graph minor theorem for trees

Theorem (Kruskal 1960)

The finite trees are WQO by the topological minor relation, i.e. for every infinite sequence of trees $T_0, T_1, \ldots,$ there exists $i < j$ such that $T_i \preccurlyeq_t T_j$.

Proof: See Theorem 12.2.1 In Diestel's book.

Proof

Let T_1 and T_2 be two rooted trees. We say that $T_1 \leq T_2$ if there is a subdivision of T_1 that can be embedded into T_2 so that the root of T_1 is mapped onto the root of T_2.

We are going to prove (on board) that the set of tree is WQO by $\leq$ (which is slightly stronger than the announced result).

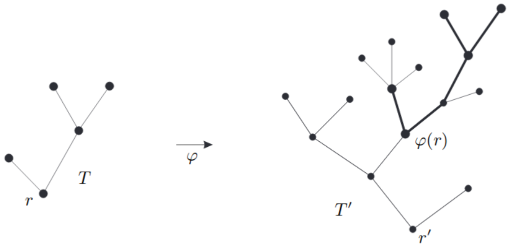

Fig. 12.2.1. An embedding of T in T' showing that $T \leqslant T'$

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5 - TreeWidth

Treewidth

- We proved (Kruskal Theorem) that Wagner conjecture holds for trees. So maybe we can use the same ideas to prove Wagner conjecture for graphs that look like trees. So we would like a notion that **measure how much a graph looks like a tree**.

- Moreover, since it is easy to compute on trees, it should be easy to compute on graphs that "looks like" trees.

- This is achieved by the notion of **Treewidth** which is a notion of "treelikeness". In other words it measures how much a graph look like a tree.

  You can understand it like this: if a graph has treewidth 5, then it is at distance 5 from being a tree. Or it is a tree of width 5.

- The goal of this section is to introduce treewidth, tree decomposition, and to extend Kruskal Theorem to graphs with bounded treewidth (no proof), look at graphs of treewidth at most 3.

Definition of a tree decomposition and of treewidth

Let G be a graph. A **tree decomposition** of G is a pair (T, W), where T is a tree and $W = (W_t)_{t \in V(T)}$ a collection of subsets of $V(G)$ indexed on $V(T)$ satisfying :

(T_1) For every $v \in V(G)$, there exists $t \in V(T)$ such that $v \in W_t$
- every vertex is in some bag -

(T_2) For every edge $uv \in E(G)$, there exists $t \in V(T)$ such that $u, v \in W_t$
- every edge is in a bag -

(T_3) For every $u \in V(G)$, $T_u = \{t \in V(T) \ , \ u \in W_t\}$ induces a connected subgraph of T.

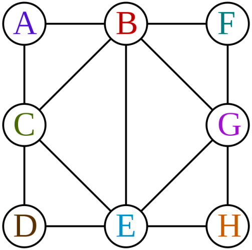
The **width** of a tree decomposition is $\max_{t \in V(T)}(|W_t| - 1)$

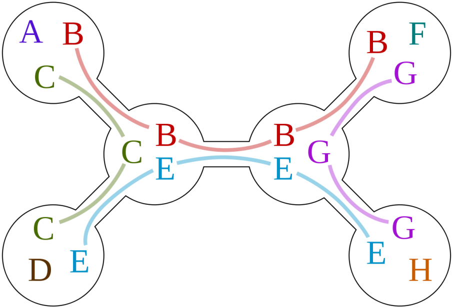
The **tree width** of a graph G, denoted $\mathbf{tw(G)}$, is the **minimum width of a tree decomposition of G**.

Equivalent definitions of tree decomposition

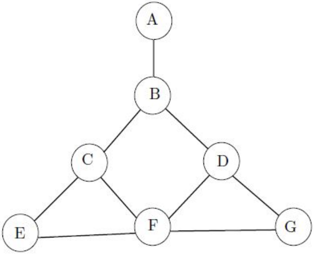
Equivalent definition of tree decomposition: a tree decomposition of G is a tree T along with a collection of subtrees T_v if T, one for each vertex of G, with the condition that T_u and T_v intersect **if** uv is an edge of G.

Example of a tree decomposition

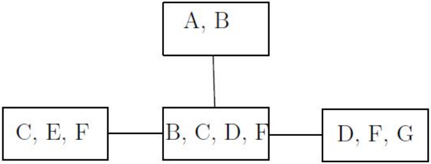




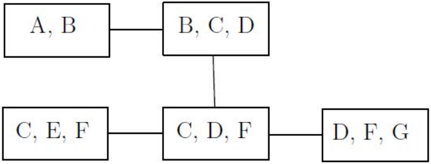
Example of a tree decomposition



The original graph G



A tree-decompostion of width 3



A tree-decompostion of width 2

Helly Property

Here is a key lemma regarding subtrees intersection; by analogy with Helly's Theorem on convex subsets of $\mathbb{R}^d$, this property is often called **Helly property for subtrees of a tree**.

Lemma

Let $\mathcal{T}$ be a collection of pairwise intersecting subtrees of a given tree T. Then $\cap_{T \in \mathcal{F}} T \neq \emptyset$.

Proof:

- Assume not. So for each vertex x of T, there is a subtree T_x in $\mathcal{T}$ that does not contain x.
- Therefore T_x is contained in one of the components of $T \setminus x$.
- One edge incident to x corresponds to this component, orient this edge out from x.
- This way, we get an orientation of some edges of T such that each vertex has one outgoing edge.
- Since there are less edges than vertices in a tree, there must be an edge oriented both ways, which results in two non intersecting subtrees in $\mathcal{T}$. Contradiction.

Helly Property

Lemma

Let $\mathcal{T}$ be a collection of pairwise intersecting subtrees of a given tree T. Then $\cap_{T \in \mathcal{F}} T \neq \emptyset$.

Corollary

Let G be a graph and K be a complete subgraph of G. In any tree decomposition (T, W) of G, there exists a vertex t of T such that $K \subseteq W_t$.
in particular, $\mathrm{tw}(G) \geq \omega(G) - 1$

A first lower bound on tree-width

Proposition

For every graph G, there exists a tree decomposition of width $\mathrm{tw}(G)$ such that for every edge $st \in E(T)$, $W_s \not\subset W_t$ and $W_t \not\subset W_s$. In particular, for every leaf f of T, there exists a vertex $u \in V(G)$ such that $T_u = \{f\}$.
Such a tree decomposition is called **irreducible**.

Theorem

In every graph G, there exists a vertex of degree at most $\mathrm{tw}(G)$, i.e. $\delta(G) \leq \mathrm{tw}(G)$.

Corollary

The class of graph with treewidth at most k is k-degenerated.
Hence, for all graphs G, $\chi(G) \leq \mathrm{tw}(G) + 1$.

Separation property of tree decompositions

The following is an easy but fundamental result. It says that a tree decomposition transfers the separation properties of the tree to the decomposed graph.

Proposition

Let (T, W) be a tree decomposition of G and $t_1 t_2$ be an edge of T and let $S = W_{t_1} \cap W_{t_2}$. For $i = 1, 2$, denote by T_i the connected component of $T \setminus t_1 t_2$ containing t_i, and G_i the subgraph of G induced by $\cup_{t \in T_i}(W_t \setminus S)$. Then S is a cutset of G separating G_1 from G_2.

Closure property

Proposition

Let G be a graph, v a vertex of G and e an edge of G.

- $\mathrm{tw}(G \setminus e) \leq \mathrm{tw}(G)$
- $\mathrm{tw}(G \setminus v) \leq \mathrm{tw}(G)$
- $\mathrm{tw}(G/e) \leq \mathrm{tw}(G)$

Proof:

- for $G \setminus e$, do nothing
- for $G \setminus v$, just remove v from every bag containing it.
- for G/e, where $e = uv$: let w be the new vertex. Add w in every bag containing u or v, and delete every occurrence of u and v.

Treewidth and Minors

Proposition

If H is a minor of G, then $\mathrm{tw}(H) \leq \mathrm{tw}(G)$

Corollary

The class of graphs of treewidth at most k is closed under taking minors.

Wagner's conjecture for graphs with bounded tree-width

Graphs with bounded treewidth are sufficiently similar to trees that it becomes possible to adapt the proof of Kruskal Theorem to them.

Very roughly, one has to iterate the "minimal bad sequence" used in Kruskal proof $\mathrm{tw}(G)$ times.

This takes us a step further towards a proof of the Graph Minor Theorem:

Theorem (Robertson and Seymour, IV)

For every integer k, the class of graphs with treewidth at most k is WQO by the minor relation.

Obstructions for graphs with treewidth at most 2

So, for every fixed k, the class $\{G : \mathrm{tw}(G) \preccurlyeq k\}$ has a finite number of obstructions.

Let us try to describe the obstructions for small values of k.

Theorem

- $\mathrm{tw}(G) \leq 1 \Leftrightarrow$ *G is a forest* $\Leftrightarrow$ *G does not contain K_3 as a minor* $\Leftrightarrow G \in \mathit{Forb}_{\preccurlyeq_m}(K_3)$
- $\mathrm{tw}(G) \leq 2 \Leftrightarrow$ *G does not contain K_4 as a minor* $\Leftrightarrow G \in \mathit{Forb}_{\preccurlyeq_m}(K_4)$

The first item is easy, let us prove the second.

Proof sketch:

- If G contains K_4 as a minor, then $\text{tw}(G) \geq \text{tw}(K_4) = 3$. So $\text{tw}(G) \leq 2 \Rightarrow G \in Forb_{\preccurlyeq_m}(K_4)$.
- Let $G \in Forb_{\preccurlyeq_m}(K_4)$ and let us prove that $\text{tw}(G) \leq 2$. We proceed by induction on $V(G)$.
- So every proper induced subgraph of G has treewidth at most 2.
- Prove first that every 3-connected graph contains K_4 as a minor (Use Menger Theorem).
- So we may assume that G has a cutset of size S at most 2.
- If $S = \{a\}$ is of size 1: let C_1 be a connected component of $G \setminus x$, $C_2 = G - (C_1 \cup \{x\})$.
- Set $G_1 = G[C_1 \cup \{x\}]$ and $G_2 = G[C_2 \cup \{x\}]$.
- By induction $tw(G_1) \leq 2$ and $tw(G_2) \leq 2$.
- Let (T_1, W_1) and (T_2, W_2) be tree decomposition of G_1 and G_2.
- Let $t_1 \in V(T_1)$ such that $x \in W_{t_1}$ and $t_2 \in V(T_2)$ such that $x \in W_{t_2}$.
- Take the disjoint union of T_1 and T_2 and add an edge between t_1 and t_2, and don't change the bags.
- Check that this gives a tree decomposition of G (i.e. check that the three axioms of the definition of tree decomposition are still satisfied).

- Assume now that $S = \{a, b\}$.
- If $ab \notin E(G)$, then add ab to G and prove that this does not create a K_4-minor.
- To do it, assume that $G + ab$ has a K_4 model, then prove that you can choose it such that it is included in $G[C \cup S]$ for some connected component of $G \setminus S$. Then observe that you can replace the edge ab by a path linking a and b that has interior vertices in a connected component $C' \neq C$. Conclude that this gives a K_4-model in G, contradiction.
- So now S is a clique (we call that a **clique cutset**).
- Let C_1 be a connected component of $G \setminus S$ and $C_2 = G \setminus (S \cup C_1)$.
- For $i = 1, 2$, set $G_i = G[C_i \cup S]$ (The G_i are often called **block decomposition**). By minimality of G, $\mathrm{tw}(G_i) \leq 2$.
- Take a tree decomposition of G_1 and G_2 of width at most 2 and link a bag of G_1 containing ab to a bag of G_2 containing ab.
- Prove that this is a tree decomposition of G of width 2.

Bounds for graphs with treewidth at most 2

Theorem

- $\mathrm{tw}(G) \leq 1 \Leftrightarrow G \in Forb_{\preccurlyeq_m}(K_3)$
- $\mathrm{tw}(G) \leq 2 \Leftrightarrow G \in Forb_{\preccurlyeq_m}(K_4)$

- The proof for $\mathrm{tw}(G) = 2$ shows the role of separators with treewidth.
- One could hope for a general result of the type:

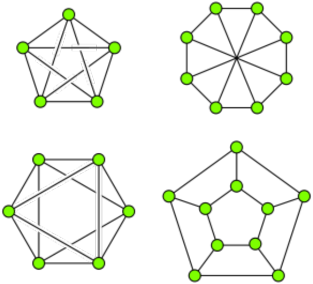
$$\mathrm{tw}(G) \leq k \text{ iff } G \in Forb_{\preccurlyeq_m}(K_{k+2}) \quad \textbf{FALSE}$$

- There exists graph with no K_5 minor with arbitrarily large treewidth.
  (As we will soon see, even planar graphs can have arbitrarily large treewidth).

Obstructions for graphs with treewidth at most 3

Theorem

$\mathrm{tw}(G) \leq 3 \Leftrightarrow G$ *does not contain one of the four following graphs as as a minor :* K_5,W_8, O *and* $C_5 \times K_2$.



We know that for every graph G:

$$\begin{aligned} \omega(G) \leq &\quad \chi(G) &\leq \mathrm{tw}(G) + 1 \\ \omega(G) \leq &\quad \omega_m(G) &\leq \mathrm{tw}(G) + 1 \end{aligned}$$

where $\omega_m(G)$ denotes the largest integer k such that G has a K_k minor.

Conjecture (Hadwiger)

For every graph G, $\chi(G) \leq \omega_m(G)$.

- For $k = 2$: $\omega_m(G) \leq 2 \Leftrightarrow G$ is a forest $\Rightarrow \chi(G) \leq 2$.
- For $k = 3$: $\omega_m(G) \leq 3 \Leftrightarrow \mathrm{tw}(G) \leq 2 \Rightarrow \chi(G) \leq 3$ by the above inequalities.
- For $k = 4$: $\omega_m(G) \leq 4 \Rightarrow \chi(G) \leq 4$ contains the Four Colour Theorem since planar graphs are K_5-minor free. In fact it is equivalent (and hence true), thanks to a structural characterisation of graphs with no K_5 minor due to Wagner.

Theorem (Wagner, 1956)

G is K_5-minor free if and only G is a subgraph of some graph built recursively by clique sums operation, starting from planar graphs and W_8.

We will see later in the course that this theorem together with the 4-color theorem implies Hadwiger conjecture for $k = 5$, that is

$$\omega_m(G) \leq 4 \Rightarrow \chi(G) \leq 4$$

Exercises on treewidth

Exercice 12

Prove that if H is a subdivision of G, then $\mathrm{tw}(H) = \mathrm{tw}(G)$

Solution: H is a subdivision of G means that H can be obtained from G be replacing some edges by path.

If G is a tree then H is also a tree and we have $\mathrm{tw}(H) = \mathrm{tw}(G) = 1$. Otherwise $tw(G) \geq 2$. Then for each bag W containing both a and b, add a a new bag $\{a, x, b\}$ adjacent to it.

The following exercise says that classes of graphs with bounded treewidth are sparse.

Exercice 13

Show that graphs G of treewidth at most k with $k \geq 1$ have strictly less than $k|V(G)|$ edges.

Next exercise is very important to design algorithm based on the tree decomposition.

Exercice 14

Show that every graph G admits a tree decomposition of width $\mathrm{tw}(G)$ with at most $|V(G)|$ bags.

Hint: prove the stronger statement that a irreducible tree decomposition has at most n bags.

Exercises on treewidth

Exercice 15

Determine the treewidth of a path, a tree, a complete graph, a complete bipartite graph, the cube.

Solution: For complete bipartite with part A and B: suppose $|A| = a \leq b = |B|$. Min degree is a, so $\mathrm{tw}(G) \geq a$. Here is a decomposition of width a: A path where nodes are $B \cup a$ for each $a \in A$.

Exercice 16

Prove that if G contains (as a subgraph) a complete bipartite graph with parts A and B, then in every tree decomposition there exists a bag that contains A or a bag that contains B.

Hint: Delete all vertices but the vertices of the complete bipartite graph. We have a tree decomposition of the complete bipartite. A bag that is not a leaf must be a cutset, and thus contains A or B.

You should be able to do this exercise, moreover the fact it proves is quite important.

Exercice 17

Prove that if x and y are two vertices that are joined by $k+1$ internally vertex disjoints paths, then in every tree decomposition of G of width at most k, there exists a bag containing both x and y.

Hint: Use the separation property of tree decomposition.

Solution on the next slide.

Solution Let (T, W) an irreducible (no bag is included in another one) tree decomposition of width $k = tw(G)$. Assume for contradiction that no bag contains both x and y. Let t and t' two nodes of T such that $x \in W_t$, $y \in W_{t'}$. Let uv be an edge on the unique path linking t and t' in T. Then, by the separation property, $W_u \cap W_v$ is a cutset of G, of size at most k (because the tree decomposition is irreducible) that separates x and y. By Menger Theorem, it contradicts the fact that x and y are linked by $k+1$ internally vertex disjoint paths.

As you have already seen, treewidth also plays a crucial role in algorithmic.
We'll come back to it.

6 - Brambles - Duality - Cops and Robbers

In the previous section, we have seen that Wagner Conjecture holds for class of graphs with bounded treewidth.

To make a proof of the general case, we should be able to say stuff about the graphs it does not cover, i.e. to deduce informations about a graph from the assumption it has large treewidth.

The main theorem of this section achieves that: it identifies a canonical obstruction to small treewidth, a structural phenomenon that occurs in a graph if and only if it has large treewidth.

This phenomenon is called **Bramble**.
(In reality, it is mainly used to get certificate on the value of the treewidth of a graph, the notion of tangle is used as an obstruction for large treewidth, but we won't see it during this class).

Bramble

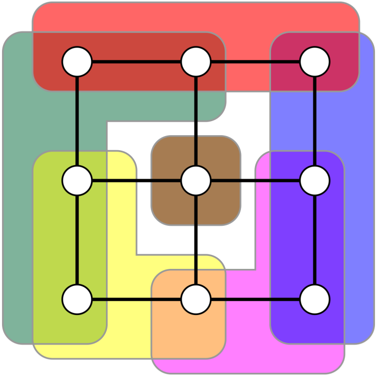
Definition (Bramble)

- We say that two connected subgraphs of G **touch** if they have non empty intersection or if they are joined by an edge.
- A **bramble** of G is a collection $\mathcal{B}$ of connected subgraphs that are pairwise touching.
- A **transversal** of a bramble $\mathcal{B}$ is a set of vertices of G that has non empty intersection with each element of $\mathcal{B}$.
- The **order** of a bramble $\mathcal{B}$ is the minimum size of a transversal of $\mathcal{B}$.
- The **bramble number** of G, denoted $\mathrm{bn}(G)$, is the maximum order of a bramble of G.

Note that if G contains K_p as a minor, then the connected subgraphs of a K_p-model of G form a bramble (no intersection, just touching) of order p.

A Bramble

A bramble of order 4 of $G_{3,3}$:



Duality Theorem I

Proposition: If (T, W) is a tree decomposition of G and $\mathcal{B}$ is a bramble in G, then there exists $t \in T$ such that W_t is a transversal of $\mathcal{B}$

Proof sketch: (main idea: "orientation of edges of the tree decomposition".)

- For each edge $t_1 t_2$, if $S = W_{t_1} \cap W_{t_2}$ intersects all sets of the bramble, we are done.
- Otherwise, for $i = 1, 2$, denote by T_i the connected component of $T \setminus t_1 t_2$ containing t_i, and G_i the subgraph of G induced by $\cup_{t \in T_i}(W_t \setminus S)$.
- We know that S is a cutset of G separating G_1 from G_2.
- If every $B \in \mathcal{B}$ intersect S, we are done. So there is $B \in \mathcal{B}$ such that B is included in G_i for some $i \in \{1, 2\}$, say $i = 1$.
- Hence no $B' \in \mathcal{B}$ is included in G_2, otherwise it does not touch B.
- This implies that every $B \in \mathcal{B}$ intersects G_1.
- Orient the edge $t_1 t_2$ toward t_1. Hence, we may assume that all edges of the tree T has an orientation.
- Hence, we get an orientation of every edge of T such that each vertex has ne outgoing edge.
- But the last vertex of a maximal directed path has no outgoing edge, contradiction.

Duality Theorem II

Proposition

If (T, W) is a tree decomposition of G and $\mathcal{B}$ is a bramble in G, then there exists $t \in T$ such that W_t is a transversal of $\mathcal{B}$

Therefore

$$\mathrm{bn}(G) \leq tw(G) + 1$$

The converse inequality is true but harder to prove.
It gives the following sort of minmax theorem (in fact maxmin=minmax).

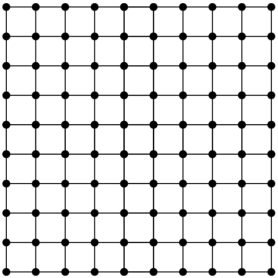
Theorem (Seymour and Thomas, 1993)

For every graph G, $\mathrm{bn}(G) = \mathrm{tw}(G) + 1$

Grids

Now, we know that if a graph has large treewidth, then it also has a large brumble. But is it so usefull?

We are going to see later that it also has a large grid (as a minor), which is often way more useful. For the moment, let us just prove that grids have large treewidth.



What is the treewidth of the grid?

Proposition

Prove that the treewidth of the grid $G_{n,n}$ is equal to n.

- To prove that $\mathrm{tw}(G_{n,n}) \leq n$, find a tree decomposition (actually you can find a path decomposition) of width n.
- To prove that $\mathrm{tw}(G_{n,n}) \geq n$, it is enough to find a bramble of order $n+1$.
- It is easy to check that the following is a bramble of order $n+1$:
  - $A = \{x_{i,1},\, 1 \leq i \leq n\}$, the last row,
  - $B = \{x_{1,j},\, 1 \leq j < n\}$ the last column minus its last element,
  - $C_{ij} = \{x_{kj},\ 1 \leq k < n\} \cup \{x_{ik},\ 1 \leq k < n,\}$ (crosses minus the last element of row and column).

A Game of Cops and Robber

- 2 player game on a graph: one controls the Robber, the other control Cops
- Goal of the cops is to capture the robber
- Many variants exist

In our variant :

- cops and robbers are standing on vertices of the graph
- at each turn a fraction of the cops can move by helicopter and land on any vertex of the graph.
- The robber sees an helicopter approaching and can instantly move at infinite speed to any other vertex along a path of a graph. The only constraint is that he is not permitted to run through a vertex occupied by some cop.

The cops win if at some point they occupy all vertices adjacent to the position of the robber, and an extra cop lands by helicopter on the robber.

Definition

The **cop number** of a graph G, denoted $\mathrm{cn}(G)$, is the smallest number of cops needed to ensure the capture of the robber.

TreeDec = strategy for the cops

Proposition

$$\mathrm{cn}(G) \leq \mathrm{tw}(G) + 1$$

- Put every cop on the vertices of some bag W_t.
- The robber, if it escapes has to be in some vertex appearing only in the bags of some component of $T \setminus t$.
- Let t' the neighbour of t in T in the direction of this component.
- $W_t \cap W_{t'}$ separates the component containing he robber form the rest of the graph.
- At the next move, cops in $W_t \setminus W_{t'}$ move to occupy all of $W_{t'}$.
- Cops apply this strategy until it reaches some leaf of the tree and the robber cannot escape.

Brambles = Strategy for the Robber

Proposition

$$\mathrm{bn}(G) \leq \mathrm{cn}(G)$$

- Let $\mathcal{B}$ be a bramble of order $\mathrm{bn}(G)$ and assume only $\mathrm{bn}(G) - 1$ cops.
- Let C be the set of initial positions of the cops.
- By definition there exists a set $X \in \mathcal{B}$ such that $X \cap C = \emptyset$.
- The robber moves to some vertex $x \in X$.
- After that, the game really begins, cops move so that the new set occupied by the cops is C'.
- Again there exists $X' \in \mathcal{B}$ such that $X' \cap C' = \emptyset$.
- During their flight the only occupied vertices are $C \cap C'$ so $X \cup X'$ is entirely free of cops,
- The robber can freely move from X to Y and this strategy can be applied for ever.

7 - Treewidth, Forbidden Minor and planar graphs

How to prove Wagner Conjecture

Wagner Conjecture: in every infinite sequence of graphs $(G_1, G_2, \dots)$, one is the minor of another.

How to prove Wagner Conjecture? Well, the natural way is:

- Assume $(G_n)_{n\in\mathbb{N}}$ is a counterexample.
- We can assume that no graph G_i with $i \geq 1$ has G_0 as a minor.
- Hence $G_i \in Forb_{\preccurlyeq_m}(G_0)$ for $i \geq 1$.
- More generally, we may assume that for every $i < j$, $G_j \in Forb_{\preccurlyeq_m} G_i)$.

Hence, understanding graphs in $Forb_{\preccurlyeq_m}(H)$ for any fixed graph H would help a lot.

For example, if for some i, $Forb_{\preccurlyeq_m}(G_i)$ has bounded treewidth, then we may assume that for every $j \geq i$, G_j has bounded treewidth and since we know that bounded tree width graphs satisfy Wagner Conjecture, we are done.

Moreover, since $Forb_{\preccurlyeq_m}(H) \subseteq Forb_{\preccurlyeq_m}(K_{|V(H)|})$, it is enough to understand K_t-minor-free graphs.

Treewidth of minor closed class

Recall that, if H is a graph, $Forb_{\preccurlyeq_m}(H) = \{G : H$ **is not a minor of** $G\}$
We have already seen that:

- Graphs in $Forb_{\preccurlyeq_m}(K_3)$ have treewidth at most 1.
- Graphs in $Forb_{\preccurlyeq_m}(K_4)$ have treewidth at most 2.
- Graphs in $Forb_{\preccurlyeq_m}(K_5)$ have unbounded treewidth (because of grids).

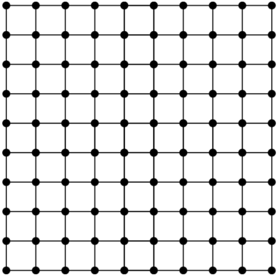
A natural question to ask is then: **for which H, graphs in $Forb_{\preccurlyeq_m}(H)$ have bounded treewidth?** i.e. there exists a number t such that for every $G \in Forb_{\preccurlyeq_m}(H)$, $tw(G) \leq t$.

One of the most important result of graph minor theory is a complete and beautiful characterization of such H.

Question: For which H, graphs in $Forb_{\preccurlyeq_m}(H)$ has bounded treewidth?

First, H **must be planar.**

- Indeed, all grids and their minor are planar (why?).
- And grids can have arbitrarily large treewidth.
- Hence, if H is non-planar, then $Forb_{\preccurlyeq_m}(H)$ contains all grids, and thus $Forb_{\preccurlyeq_m}(H)$ does not have bounded treewidth?



Grid Minor Theorem

Theorem (Grid Minor Theorem, Robertson and Seymour, V)

Given a graph H, graphs in $Forb_{\preccurlyeq_m}(H)$ have bounded treewidth if and only if H is planar.

We need to prove the *if part*, that is, for H a planar graph, graphs in $Forb_{\preccurlyeq_m}(H)$ have bounded treewidth.

In fact, we only need to show this for the special case where H is a grid, because **every planar graph is a minor of some grid**. (To see this, draw a planar graph and superimpose a sufficiently fine grid, then fatten vertices and edges of the planar graph).

We denote by $G_{k,k}$ the $k \times k$ grid.

Theorem (Grid Minor Theorem)

Let k be an integer.
There exists $f(k)$ such that if $G \in Forb_{\preccurlyeq_m}(G_{k,k})$, then $tw(G) \leq f(k)$

Very (very) rough idea of the proof:
Let G be a graph with very large treewidth. We want to show that G contains a large grid as minor.

- Show that G contains a large family $\{A_1, \ldots, A_m\}$ of pairwise disjoint connected subgraphs such that:
- each pair A_i, A_j can be linked in G by a family $\mathcal{P}_{i,j}$ of many disjoint $A_i - A_j$ paths avoiding the other sets.
- We then consider all the pairs $\mathcal{P}_{i,j}, \mathcal{P}_{i,'j'}$
- If we can find such a pair such that many of the paths in $\mathcal{P}_{i,j}$ meets many of the path in $\mathcal{P}_{i',j'}$, then we can find a large grid (this is the most difficult part of the proof because the intersections might be very messy).
- Otherwise, for every pair $\mathcal{P}_{i,j}, \mathcal{P}_{i,'j'}$, many of the paths in $\mathcal{P}_{i,j}$ avoid many of the path in $\mathcal{P}_{i',j'}$.
- We can then select one path $P_{i,j} \in \mathcal{P}_{i,j}$ from each family such that these selected path are pairwise disjoint.
- Contracting each of the connected subgraph will then give us a large clique minor, which contains a large grid.

For a full proof, see section 12.4 of the book Graph Theort of Diestel.

Theorem (Grid Minor Theorem)

There exists $f(k)$ such that if G is $G_{k,k}$-minor free then $tw(G) < f(k)$

- Establishing tight bounds on $f(k)$ is an important graph-theoretical question with many applications on structural and algorithmic graph theory.
- Robertson and Seymour showed that $f(k) = \Omega(k^2 \log k)$ must hold.
- For a long time, the best known upper bounds on $f(k)$ were super-exponential in k.
- The first polynomial upperbound of $f(k) = O(k^{98} poly \log k)$ was proved by Chekuri and Chuzhoy in 2013.
- Since then, many ameliorations have been proved, the best one is:

Theorem (Chekuri and Chuzhoy, 2019)

If G is $G_{k,k}$-minor free then $tw(G) < O(k^9 \, poly \log k)$.

Planar Graphs are WQO

Tentative proof of Wagner's Conjecture : Let $(G_n)_{n \in \mathbb{N}}$ be a sequence of graphs
We want to prove that there exists G_i and G_j with $i < j$ and G_i is a minor of G_j.

- If there exists $i \geq 1$ such that $G_0 \preccurlyeq_m G_i$, WIN
- So we may assume that, for every $i \geq 1$, G_i is G_0-minor free.
- If G_0 is planar, then G_0-minor free graphs have bounded treewidth.
- In this case $(G_1, G_2, G_3, \dots)$ is WQO, WIN.

Corollary

The class of planar graphs is WQO for the minor relation.

Would be nice to be able to say stuff on H-minor free graphs even when H is non-planar.

More precisely, Robertson and Seymour find a way to describe graphs that are H-minor-free graphs for ayny fixed graphs. To do it, they us what can be called the decomposition paradigm.

8 - The Decomposition Paradigm

Introduction

The decomposition Paradigm have lead to many difficult and important results.

It is used to describe a fixed class of graphs, say $\mathcal{C}$ (in graph minor theory, classes of the form $Forb_{\preccurlyeq_m}(H)$).

The key is to describe how every graph of $\mathcal{C}$ can be constructed by gluing together certain basic graphs by a well defined composition rules.

The main result of the graph minor project is a (approximate) decomposition theorem for $Forb_{\preccurlyeq_m}(K_k)$.

This section can be seen as an introduction to the decomposition paradigm, we will show, among other things, a decomposition theorem for chordal graphs as well as for graphs of bounded treewidth.

The next section will be dedicated to perfect graphs, an illustration of what is perhaps the most dramatic success of the decomposition method (we will skip this section, no time).

Clique cutsets and treewidth

Let G be a graph and S a set of vertices of G. $G[S]$ denotes the induced subgraphs of G induced by S.
S is a **cutset** or **separator** of G if $G \setminus S$ has at least two connected components.
It is a **clique cutset** if S induces a clique (i.e. $G[S]$ is a clique).

The following say that we should be happy when a graph has a clique cutset.

Proposition

Let G be a graph with a clique cutset S and let $(X_i)_{i \in I}$ be the connected components of $G \setminus S$. Let $G_i = G[C_i \cup S]$. Then $\mathrm{tw}(G) = max_{i \in I}(\mathrm{tw}(G_i))$.

Proof sketch: Take a tree decomposition (T_i, W_i) of G_i. S induces a clique, so it is contained in a bag B_i of each (T_i, W_i). Take the disjoint union of the (T_i, W_i), and add a new bag containing S and adjacent to each B_i. Prove that this is a tree decomposition of G of width $\mathrm{tw}(G) = max_{i \in I}(\mathrm{tw}(G_i))$.

Exercises on clique cutsets

Recall that $\omega(G)$ is the size of a largest clique of G.

We denote by $\omega_m(G)$ the largest k such that G contains K_k as a minor.

Exercice 18

Let G be a graph with a clique cutset S and let $(X_i)_{i \in I}$ be the connected components of $G \setminus S$. Let $G_i = G[C_i \cup S]$. Prove that:

1. $\chi(G) = max_{i \in I}(\chi(G_i))$.
2. $\omega_m(G) = max_{i \in I}(\omega_m(G_i))$

The graphs $G[C_i \cup S]$ are called the block of decomposition of G. The proposition in the previous slide and this exercise show the importance of block of decomposition.

Decomposition theorem for chordal graphs I

A graph G is **chordal** if it has no **induced subgraph** isomorphic to a cycle of length at least 4.

Chordal graphs is one of the oldest studied class of graphs. They have a very strong structure that permits to design efficient algorithms to compute on them.

Decomposition theorem for chordal graphs [Dirac, 1961]: If G is a chordal graph, then:

- either G is a complete graph, or
- G has a clique cutset.

Proof of Dirac theorem

Decomposition theorem for chordal graphs [Dirac, 1961]: If G is a chordal graph, then:

- either G is a complete graph, or
- G has a clique cutset.

Proof:

- Suppose that G is not a complete graph.
- Let x and y be two non-adjacent vertices. Then $V(G) \setminus \{x, y\}$ is a cutset of G separating u and v. This to say that G has some cutsets.
- Let S be a minimal vertex-cutset of G, and let C_1 and C_2 be two connected components of $G \setminus S$.
- The fact that S is minimal implies that every vertex of S has a neighbour in both C_1 and C_2.
- Suppose that $G[S]$ is not a clique.
- So S contains two non-adjacent vertices u and v.
- Since S is minimal, both u and v have a neighbor in both C_1 and C_2.
- Hence, for $i = 1, 2$, there exists an induced uv-path P_i whose interior vertices are in C_i.
- Then $P_1 \cup P_2$ induces a cycle of length at least 4, a contradiction.
- So S is a clique-cutset of G.

Decomposition theorem for chordal graph II

It is now easy to deduce the following decomposition theorem for chordal graphs.

Decomposition theorem for chordal graphs
A graph is chordal if and only if it can be constructed recursively by pasting along complete subgraphs, starting from complete graphs.

Exercice 19

Let G be a chordal graph. Prove that $\chi(G) = \omega(G)$.

Exercice 20

Let G be a chordal graph. Prove that G has a *simplicial vertex*, that is a vertex x such that $N(x) \cup \{x\}$ is a complete graph.

Hint: Among all clique cutsets S, choose one that minimize the the size of the smallest connected component C of $G \setminus S$. Prove that $G[S \cup C]$ is a complete graph, and thus all vertices in C are simplicial.

Treewidth and Chordal graphs

Given a family $\mathcal{T} = \{T_1, \dots, T_n\}$ of trees, the intersection graph of $\mathcal{T}$ is the graph with vertices $\{v_1, \dots, v_n\}$ such that v_i is adjacent to v_j if $V(T_i) \cap V(T_k) \neq \emptyset$.

Exercice 21

Show that the following statement are equivalent:

1. G is chordal
2. G admits a tree decomposition such that every bag is a clique.
3. G admits a tree decomposition with the property that $uv \in E(G)$ **if and only if** T_u and T_v have non-empty intersection[a] (and equivalently if and only if a bag contains both u and v).
4. G is the intersection graph of a family of subtrees of a tree[b].

Finally, use the second characterization to prove that for every graph H:

$$\mathrm{tw}(H) = \min\{\omega(G) - 1 \mid H \text{ subgraph of } G \text{ and } G \text{ is chordal}\}$$

Above, you may assume that G is obtained from H by adding some edges.

[a] recall that T_u is the subgraph of T induced by the node x such that $u \in W_x$ where W_x is the bag associated with x.

[b] So chordal graphs can be seen as generalisation of interval graphs

Hints for the exercise

$1 \Rightarrow 2$: Proceed by induction and use the decomposition theorem for chordal graphs.

$2 \Rightarrow 3$: If every bag is a clique, then a bag does not contain two non-adjacent vertices.

$3 \Rightarrow 4$: Let G be a graph and let $(T, W))$ a tree decomposition of G satisfying 3. For every $v \in V(G)$, let T_v the subtree of T induced by the nodes x of T such that the bag associated to x contains v. Let H be the intersection graph of $\{T_u \mid u \in V(G)\}$. We claim that $G = H$. They have the same set of vertices and, $uv \in E(G)$ if and obly if T_u and T_v intersect.

$4 \Rightarrow 1$ the intersection graphs of a family of trees cannot contain induced cycle of length at least 4 (do it when all trees are paths, it is kind of the same).

Fot the last question, see Corollary 12.3.12 of the book Graph Theory of Diestel.

Clique Sums

Definition

Let G_1 and G_2 be two graphs and K_1 a clique of G_1, K_2 a clique of G_2 with $|K_1| = |K_2|$. If G is a graph obtained by identifying vertices of K_1 and K_2, and then removing some edges of this clique, then G is a **clique sum** of G_1 and G_2.

Similarly as for clique cutset, we have the following:

Proposition

If G is a clique sum of G_1 and G_2, then $\mathrm{tw}(G) \leq \max(\mathrm{tw}(G_1), \mathrm{tw}(G_2))$.

And another characterization of treewidth, that is also a decomposition theorem for classes of graphs with bounded treewidth.

Theorem

G has treewidth at most k if and only if it can be constructed recursively by clique sum operations starting from graphs on at most $k+1$ vertices.

9 - Graphs are WQO (Warning: contains major handwaving)

Proof of Wagner's Conjecture: general strategy

- Starts as before: Assume $(G_n)_{n\in\mathbb{N}}$ is a counterexample.
- We can assume that no graph G_i with $i \geq 1$ has G_0 as a minor.
- Hence $G_i \in Forb_{\preccurlyeq_m}(G_0)$ for $i \geq 1$
- Can we describe the structure of these graphs??
- It is sufficient to get a structure theorem for $Forb_{\preccurlyeq_m}(K_k)$.
- For $k \leq 4$ we have seen characterizations (small treewidth).
- For $k = 5$ there is one due to Wagner:

Wagner decomposition Theorem

Theorem (Wagner - 1937)

K_5-minor free graphs are constructed by a sequence of 3-clique sums operations starting from W_8 and planar graphs.

How to use that to prove Wagner Conjecture?
Like that:

- Assume $(G_i)_{i \in \mathbb{N}}$ is bad sequence (for every $i < j$, G_i is not a minor of G_j).
- Assume there exists $n \in \mathbb{N}$ such that such that $|V(G_n)| \leq 5$.
- Then $(G_i)_{i \geq n}$ are K_5-minor-free.
- Then we can use Wagner Theorem: the graphs G_i, $i > n$ have some kind of a 2-layer structure:
  - Outside we have a tree-like structure, which can be handled with similar methods used to handles trees (and graphs with bounded treewidth).
  - Inside (that is in the "bag" of the tree decomposition given by Wagner Theorem), graphs are planar or W_8, and we already now they are WQO.

Hence, all we need is a generalisation of Wagner decomposition Theorem for all complete graphs.

Vortices and Fringes

Let us start with a technical definition. If C is a cycle, a **vortex** on C is defined the following way :

- Select a collection of arcs $A_1, A_2, \ldots, A_l$ on C so that each vertex is in at most k arcs.
- For each arc we add a vertex v_i that is linked to some vertices of A_i.
- We can also add edges $v_i v_j$ if $A_i \cap A_j \neq \emptyset$.
- We call this **adding a fringe** of width k to C.

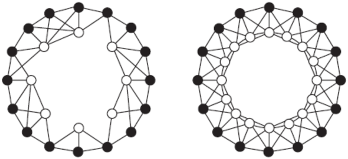


FIGURE 4. A fringe of width 2 and a fringe of width 3.

Figure of Laszló Lovász

Almost k-embeddable

Now let us define a class $\mathcal{G}_k$ of **almost k-embeddable** graphs

i Start with a surface of genus at most k and a graph G embedded in it so that each face is homeomorphic to a disc.

ii Add at most k **vortices** (local perturbation of a face of the embedding)

iii Add at most k **apexes** (vertices linked arbitrarily to the rest of the graph)

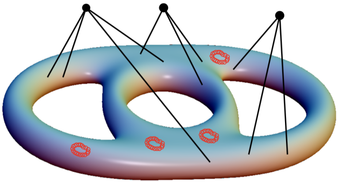
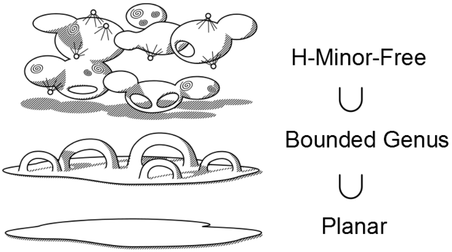


Figure by Daniel Marx

Structure Theorem

Theorem (Robertson and Seymour Theorem, XX)

For every graph H, there exists an integer k such that all H-minor free graphs can be obtained by a sequence of k-clique sum operations starting from almost k-embeddable graphs.



Felix Reidl

"Proof" of Wagner Conjecture

Very (very) roughly, the proof that graphs are WQO for minor ordering is

- Show that graphs of bounded genus are WQO by induction on the genus (very hard).
- Almost k-embedable graphs are taken care to the cost of more very hard work.
- Kruskal's Theorem's proof is adapted to deal with the tree structure given by the clique sums operations.

General message:

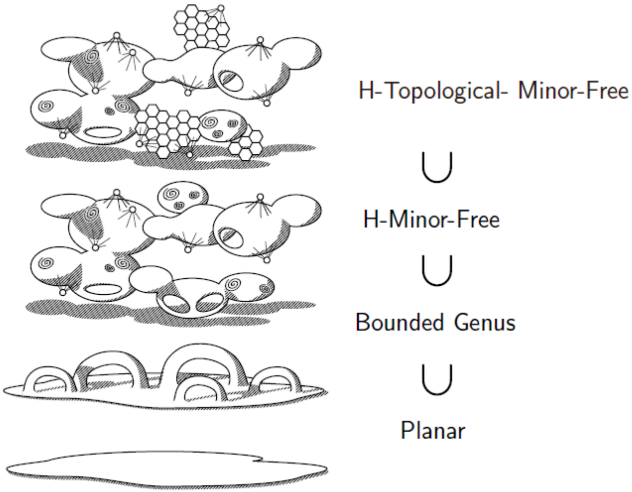
- if something works for planar graphs,
- then we might generalize it to bounded genus graphs,
- then we might generalize it to H-minor-free graphs.

What next?

What about $Forb_{\preccurlyeq_t}(H)$?

H-topological minor free graphs

H-topological minor free graphs look like that (Grohe and Marx, 2012)



Felix Reidl

Decomposition theorem for H-topological minor free graphs

Theorem (Grohe and Marx, 2012)

For every H, there is an integer k such that every H-subdivision-free graph has a tree decomposition where the torso of every bag is either:

- *k-almost embeddable in a surface of genus at most k or*
- *has degree at most k with the exception of at most k vertices ("almost bounded degree").*

General message:
If a problem can be solved both

- on (almost-)embeddable graphs and
- on (almost-)bounded degree graphs,
- then these results can be raised to H-subdivision-free graphs without too much extra effort.

10 - FPT algorithm via the Graph Minor Theorem

Graph modification problem

Problem (Graph modification problem for $\mathcal{C}$)

Given: (G, k)
Question: Is there a set S of at most k vertices such that $G \setminus S \in \mathcal{C}$?

- Vertex Cover: $\mathcal{C}$ is the class of edgeless graphs.
- Feedback vertex set: $\mathcal{C}$ is the set of forest (forbidden (induce) subgraphs is the set of all cycles).
- You can take any class of graphs for $\mathcal{C}$: planar graphs, bipartite graphs, chordal graphs etc etc.

We have seen that, using the branching method:

Theorem
If $\mathcal{C}$ is closed under taking induced subgraph and can be characterized by a finite set $\mathcal{F}$ of forbidden induced subgraphs (i.e. $\mathcal{C} = Forb_{\subseteq_i}(\mathcal{F})$, then the graph modification problem for $\mathcal{C}$ is FPT.

Graph modification problem

The following is one of the main algorithmic consequence of the graph minor theorem:

Theorem
If $\mathcal{C}$ is closed under taking minor, then the graph modification problem for $\mathcal{C}$ is FPT.

Proof: Assume $\mathcal{C}$ is closed under taking minor, then:

- The set of YES-instance for a fixed k is also closed under taking minor (why?).
- So, for each k, there exists a set of graphs $\mathcal{F}_k$ such that the question is: does G in $Forb_{minor}(\mathcal{F}_k)$?
- By The Graph Minor Theorem, $\mathcal{F}_k$ is finite.
- There is a $f(|F|)O(n^3)$ algorithm to decide if a graph contains a given graph F as a minor.

Caveats:

- This is just an existential proof, we do not know how to get the set $\mathcal{F}$ of forbidden minor (the Graph Minor Theorem is not constructive)
- It is not uniform: not the same algorithm for different values of k.

11 - FPT Algorithms parametrized by treewidth

Max Weighted Independent Set

Problem (Maximum Weighted Independent Set - MWIS)

Input : A graph G with weight function $\omega : V(G) \to \mathbb{R}$
Output : an Independent set of maximum weight.

- NP-complete for general graphs.
- Polynomial for trees by Dynamic programming

MWIS for Trees with dynamic Programming

Fix a root r arbitrarily.

Denote by $ch(v)$ the set of children of v, by $T(v)$ the subtree rooted at v (hence $T(r) = T$) and set:

- $f(v)$ denotes the maximum weight of an independent set of $T(v)$,
- $f^+(v)$ denotes the maximum weight of an independent set of $T(v)$ containing v
- $f^-(v)$ denotes the maximum weight of an independent set of $T(v)$ not containing v

The value of a maximum weight independent set of T is precisely $f(r)$.

MWIS for Trees with dynamic Programming

Let v be a vertex of T, and let $ch(v)$ be the set of children of v. We have:

$$f^+(v) = \Sigma_{x \in ch(v)} f^-(x) + \omega(v)$$
$$f^-(v) = \Sigma_{x \in ch(v)} f(x)$$
$$f(v) = \max(f^+(v), f^-(v))$$

It only remains to compute these three functions in a bottom-up fashion (that is starting from the leaves and computing layer after layer until we rich the root), which take $O(|V(T)|)$ time.

Many NP-hard problems are solvable in polytime on trees, using dynamic programming. We are going to see that the same strategy stands when applied on tree decomposition.

MWIS parametrized by treewidth

Theorem

Given a tree decomposition of width k, Maximum Weighted Independant Set can be computed in time $O(2^k \cdot k^{O(1)} \cdot n)$.

For each vertex $t \in V(T)$, set:
$W_t \subseteq V(G)$: vertices appearing in node t
$V_t \subseteq V(G)$: vertices appearing in the subtree rooted at t.

Generalizing the strategy used for tree:
Instead of computing two values $f^+(t)$ and $f^-(t)$, we compute $2^{|W_t|} \leq 2^k$ values for each bag W_t.

For each node t and each subset S of W_t:
$M[t,S]$ = max weighted independant I such that $I \subseteq V_t$ and $I \cap W_t = S$.

It is easy to compute $M[t,S]$ if the values are known for the children of t. But we are going to define a tree decomposition with a particular structure to ease it even more.

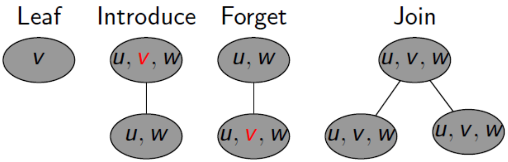
Nice Tree Decompositions

To design algorithms parametrized by treewidth, it is convenient to use the following particular tree decompositions.

Definition

A nice tree decomposition of G is a tree decomposition where T is a rooted binary tree with bags $(W_t)_{t \in V(T)}$ and each inner node t is of three possible kind :

- Leaf: t has no child and $|W_t| = 1$.
- Introduce: t has one child t' and $W_t = W_{t'} \setminus \{v\}$ for some $v \notin W_{t'}$
- Forget: t has one child t' and $W_t = W_t \cup \{v\}$ for some $v \notin W_t$.
- Join: t has two children t_1 and t_2 and $W_t = W_{t_1} = W_{t_2}$.



From tree decomposition to nice tree decomposition

Theorem

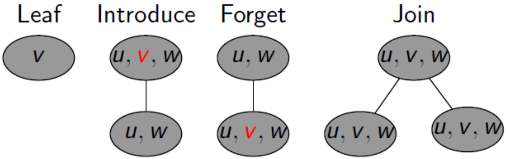
A tree decomposition of width k and n nodes can be turned into a nice tree decomposition of width k and $O(k \cdot n)$ nodes in time $O(k^2 \cdot n)$.

Proof Sketch:

- Root the decomposition arbitrarily.
- For each internal node with p children, it is possible to add $2p$ new join nodes to make it binary.
- For each edge $t_1 t_2$ replace $t_1 t_2$ by a path with at most k forget nodes and at most k introduce nodes.

Using nice decomposition, it becomes super easy to compute $M[t, S]$ in a bottom-up fashion.

For each node $t \in V(T)$ and each independant subset S of W_t:
$M[t,S] =$ max weighted independant I such that $I \subseteq V_t$ and $I \cap W_t = S$.



- **Leaf**: $|W_t| = 1$, trivial
- **Introduce**: one child t' with $W_t = W_{t'} \cup v$:

$$\begin{aligned} M[t,S] &= M[t',S] && \text{if } v \notin S \\ &= M[t', S \setminus \{v\}] + \omega(v) && \text{if } v \in S \end{aligned}$$

- **Forget**: one child t' with $W_t = W_{t'} \setminus v$:

$$M[t,S] = \max(M[t',S], M[t', S \cup \{v\}])$$

- **Join**: t has two children t_1 and t_2 such that $W_t = W_{t_1} = W_{t_2}$:

$$M[t,S] = M[t_1,S] + M[t_2,S] - \omega(S)$$

Other Problems that are FPT by treewidth

Here is a list of results one can prove similarly using a tree decomposition of treewidth k.

Theorem

Let G be given with a tree decomposition of width at most k.

1. *Computing $vc(G)$ can be done in time $O\left(2^k \cdot k^{O(1)} \cdot n\right)$.*
2. *Computing $\chi(G)$ can be done in time $O(f(k) \cdot n)$.*
3. *Computing $\omega(G)$ can be done in time $O(2^k \cdot k^k \cdot n)$.*
4. *Computing $\gamma(G) := \min\{|X| : X \cup N(X) = V(G)\}$, can be done in time $O\left(4^k \cdot k^{O(1)} n\right)$. (dominating set).*
5. *Deciding if G has a hamiltonian cycle can be done in time $O\left(k^{O(k)} \cdot n\right)$.*

12 - Courcelle's Theorem

Monadic Second Order Logic

A celebrated algorithmic meta-theorem of Courcelle generalises all the previous results to monadic second order formulas.

Logical formulas on graphs are constructed inductively using

- atomic formulas : $x = y$, $v \in X$, $e \in F$ for subsets of vertices or edges.
- the binary relation $Inc(x, e)$ which is satisfied if $x \in V$ and x is incident with $e \in E$.
- logical operators $\vee$ and $\wedge$ and $\neg$
- quantifiers $\forall$ and $\exists$

- First Order formulas (FO): quantifiers over vertices and edges ($\forall v \in V(G)$; $\exists e \in E(G)$)
- $\text{MSO}_1 = \text{FO} +$ quantify over sets of vertices,
- $\text{MSO}_2 = \text{MSO}_1 +$ quantify over sets of edges.

Formula for 3-colorability

This is a second order formula for 3 colourability :

$$\begin{aligned}
& \exists X_1 \subset V \ \exists X_2 \subset V \ \exists X_3 \subset V \\
(\forall x \in V \quad & (x \in X_1 \vee x \in X_2 \vee x \in X_3) \\
& \wedge \neg(x \in X_1 \wedge x \in X_2) \wedge \neg(x \in X_1 \wedge x \in X_3) \wedge \neg(x \in X_2 \wedge x \in X_3)) \\
\wedge \ (\forall xy \in E \quad & \neg(x \in X_1 \wedge y \in X_1) \wedge \neg(x \in X_2 \wedge y \in X_2) \wedge \neg(x \in X_3 \wedge y \in X_3))
\end{aligned}$$

Courcelle Theorem

The theorem of Courcelle asserts that every such property is easy to decide for bounded treewidth graphs.

Theorem (Courcelle, 1990)

Let G be a graph and ϕ a formula of MSO_2. Assume that we are given a tree decomposition of G of width at most k. Then there is an algorithm that verify if φ is satisfied in G in time $f(|\varphi|, k) \cdot n$ for some computable function f.

Note: The dependance on k can be very large (double, triple exponential etc.), therefore a direct dynamic programming algorithm can be more efficient.

If we can express a property in MSO_2, then we immediately get that testing this property is FPT parameterized by the treewidth of the input graph, as soon as we have a tree decomposition of graph at hand.

But can we compute a tree decomposition???

13 - Computing Tree Decomposition

Computing tree width

Problem

Input : A graph G and an integer k
Output : TRUE if and only if $\mathrm{tw}(G) \leq k$

- **NP-Hard**: Arnborg, Corneil, Proskurowski '87 (note that polytime open for planar)

Anyway, we want to use tree decomposition to design *FPT* algorithm with parameter $\mathrm{tw}(G)$, so we would be happy with time $O\left(f(\mathrm{tw}(G)) \cdot n^{0(1)}\right)$.

- **FPT** : $O\left(\mathrm{tw}(G)^{\mathrm{tw}(G)^3} n\right)$ algorithm (Bodlaender, 96)

Approximate the treewidth

We don't really need to compute an *optimal tree decomposition, the following is enough.*

Theorem (Robertson and Seymour)

Given a graph G and an integer k, there is an algorithm running in time $O\big(f(k).n^2\big)$ that output:

- *either a small certificate showing that $\mathrm{tw}(G) \geq k$*
- *or a tree decomposition of width at most $4k+1$.*

This is enough for our FPT algorithms seen before : simply run this for $k=1$, $k=2$, $k=3$,... one is guaranteed to find a tree decomposition of G of width at most $4\mathrm{tw}(G)$ in time $O\big(f(\mathrm{tw}(G)).n^2\big)$.

Computing the treewidth: state of the art

Approximation	$f(k)$	$g(n)$	reference
exact	$O(1)$	$O(n^{k+2})$	Arnborg, Corneil & Proskurowski (1987)
$4k+3$	$O(3^{3k})$	$O(n^2)$	Robertson & Seymour (1995)
$8k+7$	$2^{O(k \log k)}$	$n \log^2 n$	Lagergren (1996)
$5k+4$ (or $7k+6$)	$2^{O(k \log k)}$	$n \log n$	Reed (1992)
exact	$2^{O(k^3)}$	$O(n)$	Bodlaender (1996)
$O\left(k \cdot \sqrt{\log k}\right)$	$O(1)$	$n^{O(1)}$	Feige, Hajiaghayi & Lee (2008)
$4.5k+4$	2^{3k}	n^2	Amir (2010)
$\frac{11}{3}k+4$	$2^{3.6982k}$	$n^3 \log^4 n$	Amir (2010)
exact	$O(1)$	$O(1.7347^n)$	Fomin, Todinca & Villanger (2015)
$3k+2$	$2^{O(k)}$	$O(n \log n)$	Bodlaender et al. (2016)
$5k+4$	$2^{O(k)}$	$O(n)$	Bodlaender et al. (2016)
k^2	$O(k^7)$	$O(n \log n)$	Fomin et al. (2018)
$5k+4$	$2^{8.765k}$	$O(n \log n)$	Belbasi & Fürer (2021a)
$2k+1$	$2^{O(k)}$	$O(n)$	Korhonen (2021)
$5k+4$	$2^{6.755k}$	$O(n \log n)$	Belbasi & Fürer (2021b)
exact	$2^{O(k^2)}$	n^4	Korhonen & Lokshtanov (2022)
$(1+\varepsilon)k$	$k^{O(k/\varepsilon)}$	n^4	Korhonen & Lokshtanov (2022)

picture from wikipedia

Approximate the treewidth

Theorem

There exists an algorithm with input a graph G and an integer k and that outputs in time $O\big(f(k).n^2\big)$:

- *either a small certificate that $\mathrm{tw}(G) \geq k$*
- *or a tree decomposition of width at most $4k+3$.*

The rest of this section is dedicated to design this algorithm.

See Section 7.6 of the book Parametrized Algorithms.

Good separator

Let G be a graph. A set of vertices S is a separator (or vertex cutset) if S disconnects G, that is $G \setminus S$ has at least two connected components.

Let S, X be two sets of vertices, S is a good separator with respect to X if:

- S disconnects G into two parts V_1 and V_2 both intersecting X.
- For $i = 1, 2$, V_i contains at most $2|X|/3$ vertices of X.

(Such a separator is sometime called a 2/3-separator w.r.t. X).

Certificate that $\mathrm{tw}(G) \geq k$

lemma: If $\mathrm{tw}(G) < k$, then every $X \subseteq V(G)$ of size at least $2k+1$ admits a good separator of size at most k

Proof Ideas:

- Take a tree decomposition (T, W) of width $k-1$ where T has maximum degree 3.
- For each node $t \in T$, the bag W_t separates G into 2 or 3 connected components.
- If one of the connected components contains more than half of the vertices of X, then orient the corresponding edge out from t.
- Prove that there exists an internal node t with no outgoing edge (observe that any edge incident with a leaf is oriented out from the leaf).
- Show that W_t is a good separator with respect to X:
  - Let A, B, C be the three connected components of $G \setminus W_x$.
  - We now that $|A|, |B|, |C| \leq 1/2|X|$.
  - If $|A \cup B| \leq \frac{2}{3}|X|$, then we win (take $V_1 = A \cup B$ and $V_2 = C$).
  - Same if $|A \cup C| \leq \frac{2}{3}|X|$ or $|B \cup C| \leq \frac{2}{3}|X|$.
  - Simple calculation show that one of $|A \cup B|, |A \cup C|, |B \cup C|$ has at most $\frac{2}{3}|X|$ vertices.

Certificate that $\mathrm{tw}(G) \geq k$:
If G contains a set X of at least $2k+1$ vertices that do not admit a good separator of size a most k, then $\mathrm{tw}(G) \geq k$.

FPT algorithm to aprrox the tw

We prove by induction on the number of vertices of G the following algorithm (apply it with $X = \emptyset$ to get the desired algorithm).

Problem

Input : A graph G, an integer k and a set $X \subseteq V(G)$ such that $|X| \leq 3k$
Output : A certificate that $tw(G) \geq k$ or a rooted tree decomposition T of G of width at most $4k+1$ where $X \subseteq root(G)$

- If G has at most $4k$ vertices then put all vertices in a single bag.
- If X has less than $2k+1$ vertices, then augment X arbitrarily by adding vertices until its size is at least $2k+1$.
- Assume for the moment that we know how to compute a good separator of size at most k w.r.t. X.
- If X admits no good separator of size at most k, then by what precedes it is a certificate that $tw(G) \geq k$.
- So we may assume that X has a good separator S of size at most k.
- We are going to use it to compute the tree decomposition of width at most $4k+1$.

- Let S be a good separator for X (Recall $2k+1 \leq |X| \leq 3k$ and $|S| \leq k$)
- $G \setminus S$ disconnects G into two non-empty parts V_1 and V_2 such that $X \cap V_i \leq 2|X|/3$ for $i = 1, 2$
- Define $X_i = S \cup (X \cap V_i)$. Then $|X_i| \leq k + \frac{2}{3}3k = 3k$
- Set $G_i = G[V_i \cup S]$, and apply induction on (G_i, X_i) for $i = 1, 2$
- Observe that $|V(G_i)| < |V(G)|$.
- Apply the algorithm on $(G_i,\ k,\ X_i)$.
- Either certificate that $tw(G_i) \geq k$ for some i and therefore $tw(G) \geq k$
- Or get two rooted decompositions T_1, T_2 of G_1 and G_2 with $X_i \subseteq root(T_i)$
- Add a root bag containing all vertices in $X \cup S$ (note that $|X \cup S| \leq 4k$) attached to the roots of T_1 and T_2.
- Check that it is indeed a tree decomposition of the desired width.

Computing the good separator

Here is how to compute a good separator S of size at most k wrt X:

- S exists if and only if one can partition X into three subsets X_1, X_2, X_0 such that
  - X_1 and X_2 have size at most $2|X|/3$,
  - X_0 is a subset of a separator of size at most k separating V_1 and V_2 where $X_i \subseteq V_i$
- Equivalently if and only if in $G \setminus X_0$, there are at most $k - |X_0|$ disjoint paths from X_1 to X_2.
- Ford Fulkerson : $O\left(k^2 n\right)$ [1]
- 3^{3k} ways of defining the partition X_0, X_1, X_2 so $O\left(27^k . k^2 n\right)$ for this step

So the **total complexity** for the algorithm is $O\left(27^k . k^2 n^2\right)$ since the tree decomposition has at most n nodes.

[1] FF runs a number of iterations; each iteration takes $O(n + m)$ time and either concludes that the currently found flow is maximum, or augments it by 1. Since we are interested only in situations when the maximum flow is of size at most $k + 1$, we may terminate the computation after $k + 2$ iterations. Moreover $m \leq kn$, otherwise $\mathrm{tw}(G) > k$

14 - Win/Win approach and planar graph problems

Vertex Cover via treewidth

Observation

If $vc(G) \leq k$, then $tw(G) \leq k$

Indeed, if $G - S$ is edgeless and $|S| \leq k$, then we have a path decomposition where the set of bags is $\{S \cup \{x\} \mid x \in V(G)\}$.

FPT algorithm for Vertex Cover (parametrized by the size of the solution):

- Run our algorithm to compute tree decomposition on (G, k).
- If it ouputs that $tw(G) \geq k$, then (G, k) is a NO-instance.
- Otherwise we have a tree decomposition of width at most $4k + 3$ at hand.
- Use Dynamic Programming to compute a minimum vertex cover.

Subexponential FPT algorithm for planar graphs

See Section 7.6 of the book Parametrized Algorithms.

Grid Minor for planar graphs

We denote by $\boxplus_t$ the $t \times t$ grid.

Planar grid minor Theorem

- Every planar graph G with $tw(G) \geq 9t/2$ contains $\boxplus_t$ as a minor.
- Moreover, there is a $O(n^2)$-time algorithm that, given a planar graph, either output a tree-decomposition of width $9t/2$, or constructs a $\boxplus_t$-model.

Corollary

Let G a planar graph on n vertices. Then:

- $tw(G) \leq \frac{9}{2}\sqrt{n+1}$ *and*
- *a tree decomposition of width* $\frac{9}{2}\sqrt{n+1}$ *can be constructed in* $O(n^2)$ *time.*

We want to solve k-Vertex Cover for an instance (G, k) where G is planar.

- Observe that $vc(\boxplus_t) = \lceil \frac{t^2}{2} \rceil$ (because it has a matching of size $\lceil \frac{t^2}{2} \rceil$).
- So if G contains $\boxplus_t$ as a minor for some $t \geq \sqrt{2k+2}$, it has no vertex cover of size k.
- So, by the Planar Grid Minor Theorem, if $vc(G) \leq k$, then $tw(G) \leq \frac{9}{2}\sqrt{2k+2}$.

We now have the following algorithm:

- In $O(n^2)$, we get either a $\boxplus_{\sqrt{2k+2}}$-model, and in this cas we output NO.
- Or we get a tree decomposition of width at most $\frac{9}{2}\sqrt{2k+2}$.
- Then we use dynamic programming to computer the minimum vertex cover in time $2^{\sqrt{2k+2}} \cdot k^{0(1)} \cdot n$.
- In total, we get an algorithm in $2^{0(\sqrt{k})} \cdot n \cdot O(n^2)$.

Subexponential parameterized algorithm

Any problem satisfying the following properties has a subexponential time FPT algorithm:

- The size of a solution in $\boxplus_k$ is of order $\Omega(k^2)$.
- Given a tree decomposition of width $O(k)$, the problem can be solved in time $O(2^k) \cdot n^{0(1)}$.
- If G has a solution of size at most k, then every minor of G too.

Dominating set

A vertex set S of a graph G is a dominating set if $S \cup N(S) = V(G)$.
In other words, every vertex has a neighbour in S or is in S.

Problem (Dominating set parametrized by the size of the solution)

Question: Given (G, k), does G have a dominating set of size at most k?

Exercice 22

Can you use the graph minor theorem to prove that Dominating set parametrized by the size of the solution of FPT?

Question: Does the subexponantial strategy used for vertex cover in planar graph works?

- A dominating set of $\boxplus_k$ has size at least $\frac{k^2}{4}$.
- Given a tree decomposition of width $O(k)$, we can compute a minimum dominating set in time $O(2^{O(k)}) \cdot n^{0(1)}$.
- But it might be that G has a smaller dominating set then one of its minor.

Indeed, deleting a vertex or even an edge, might increase a lot the size of a smallest dominating set.

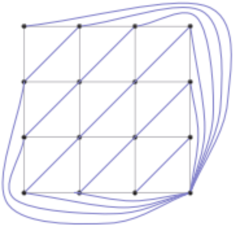
Solution: Observe that contracting an edge can only decrease the size of a smallest dominating set, and modify the grod minor theorem!

Planar grid minor theorem for edge contraction

Given two graphs G and H, we say that G contains H as a contraction, if H can be obtained from G by contracting some edges.

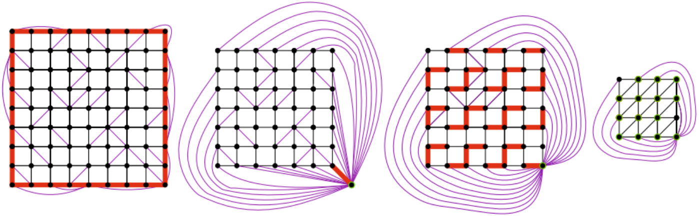
Planar grid minor theorem for edge contraction

Let G be a planar graph. If $tw(G) \geq 9t + 5$, then G contain Γ_t as a contraction. Moreover, there is an algorithm running in time $O(n^2)$ that either output a tree decomposition of width $9t + 5$, or output a set of edges whose contraction resuts in Γ_t.



Proof:

- If $tw(G) \geq 9t + 5$, then G contain a $\boxplus_{2t+1}$ model.
- Hence, after a sequence of vertex deletion, edge deletion and edge contraction, we get $\boxplus_{2t+1}$.
- Instead of deleting the vertices, contract them with one of their neighbor and omit edge deletion.
- This way we get $\boxplus_{2t+1}$ plus some edges. And we get Γ_t by doing the following contradction:



- Finally, the obtained Γ_t has no extra edge, since adding an edge to Γ_t spoils its planarity.

Subexponential parameterized algorithm for dominating set

Observation

A minimum dominating set of Γ_k has size $\Omega(k^2)$.

So the strategy works again, and we get a subexponential FPT time algorithm for dominating set in planar graphs.

General strategy:

- The size of a solution in $\boxplus_k$ is of order $\Omega(k^2)$.
- Given a tree decomposition of width $O(k)$, the problem can be solved in time $O(2^k) \cdot n^{0(1)}$.
- Contracting edges can only decrease the size of the solution.