Parametrized Complexity and Graph Minor Theory

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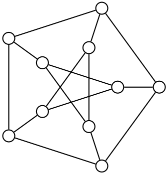
6 Hours' Programm

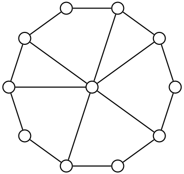
- Definitions of parametrized complexity (FPT, XP, W[1])
- Branching method
  - VERTEX COVER in time $O(1.46^k n^{0(1)})$
  - Branching vector
  - GRAPH MODIFICATION PROBLEM
  - FEEDBACK VERTEX SET in time $(3k)^k \cdot n^{O(1)}$
- Kernelization
  - k-VERTEX COVER has a $k^2 + k$ kernel
  - VERTEX COVER has a $3k$ kernel (crown decomposition)
  - VERTEX COVER has a $2k$ kernel (Linear Programming)
  - d-HITTING SET PROBLEM has a $d!k^d d^2$ kernel (Sunflower Lemma)
- Color Coding
  - LONGEST PATH in time $2^k n^{0(1)}$
- Iterative Compression
  - FEEDBACK VERTEX SET in time $5^k n^{0(1)}$

Graphs

A graph $G = (V, E)$:

- V is the set of vertices
- $E \subseteq V \times V$ is the set of edges.





All along the course, particularly for complexity analysis,

- n is the number of vertices,
- m is the number of edges.

An algorithm going in time $O(n + m)$ is said to be linear.

Basic Definitions and Terminology

In this course, all graphs are simple (no parallel edges) and without loop, unless expressly stated.

If G is a graph, we denote $V(G)$ its set of vertices and $E(G)$ its set of edges.

A vertex v is adjacent with a vertex u if $uv \in E(G)$. The neighbourhood of u, denoted $N(u)$ is the set of neighbours of u.
Its degree, denoted tcdarkred$d(u)$ is the cardinality of its neighbourhood. The maximum degree of a graph is denoted $\Delta(G)$. Given a set of vertices X, $N(X)$ is the set of vertices not in X that have at least one neighbour in X.

A graph with no edge is a stable set, or independent set, and a graph with all possible edges ($\binom{n}{2}$) is a clique, or complete graph. The complete graph on n vertices is denoted K_n. The complete bipartite graph with parts of size a and b is denoted $K_{a,b}$.

The path P_k is a graph with $V(P_k) = \{x_1, x_2, \ldots, x_k\}$ and $E(P_k) = \{x_i x_{i+1},\ 1 \leq i \leq k-1\}$. The vertices x_1 and x_k are called the endpoints of the path. If we add the edge $x_k x_1$ to P_k, then the resulting graph is the cycle on k vertices, denoted C_k.

Some graph parameters

- $\delta(G)$: minimum degree.
- $\Delta(G)$: maximum degree.
- $\omega(G)$: clique number.
- $\alpha(G)$: size of a maximum independent set.
- $\chi(G)$: chromatic number.
- $\tau(G)$: vertex cover.
- $\kappa(G)$: vertex connectivity.
- $tw(G)$: treewidth, measure how much a graph looks like a tree.

Parametrized Complexity and FPT Algorithms

Slides are inspired by a course of Daniel Marx, and another course of Marcin Pilipczuk.

Classical Complexity

A brief review:

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Can we say anything nontrivial about NP-hard problems?

What can you do in front of a hard problem

If a problem is NP-hard, then there is no algorithm that solves

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But why is a problem hard to solve?

It is certainly easy to solve on some easy instances.

But how to capture the notion of easy instances?

Maybe some parameter of the input play an important role, and if this parameter is small we can solve the problem efficiently.

How to cheat in front of a hard problem?

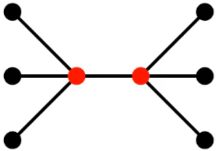
The size of the input is never the only thing that affects the running time of an algorithm.

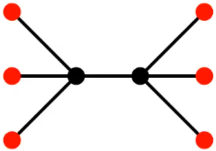
Main idea: measure the complexity in term of the input size and something else.

Formally: Instead of expressing the running time by a function $T(n)$ of the input size n, express it by a function $T(n, k)$ of the input size n and of a parameter k of the input.

Parametrized complexity

Problem:	VERTEX COVER	INDEPENDENT SET
Input:	Graph G, integer k	Graph G, integer k
Question:	Is it possible to cover the edges with k vertices?	Is it possible to find k independent vertices?





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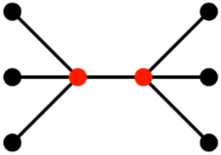

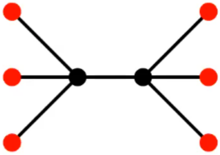

Complexity:	NP-complete	NP-complete
Brute force:	$O(n^k)$ possibilities	$O(n^k)$ possibilities
	$O(2^k n^2)$ **algorithm exists**	**No** $n^{o(k)}$ **algorithm known**

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Parametrized complexity, definitions

- A **parametrized algorithmic** problem is a problem where a certain parameter k is given in addition to the input (of size n).
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- Or it is **Fixed Parameter Tractable** (**FPT**) for k: Algorithm in time $O\left(f(k) \cdot n^{O(1)}\right)$

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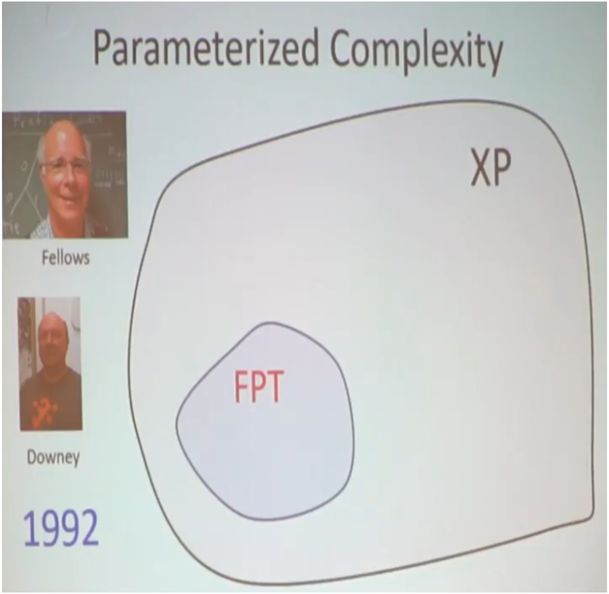
For example, the set of tuples $\{(G, k) \in \mathcal{G} \times \mathbb{N} : vc(G) \leq k\}$ is the problem VERTEX-COVER parametrized by the size of the solution.

Parametrized Complexity



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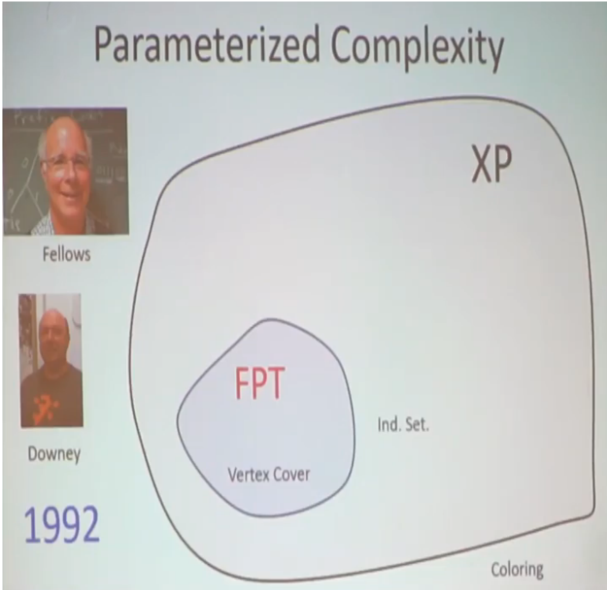

figure by Daniel Lokshtanov

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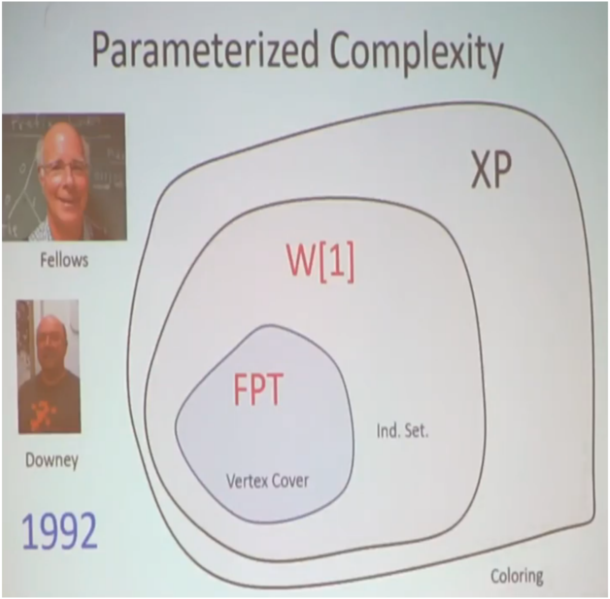

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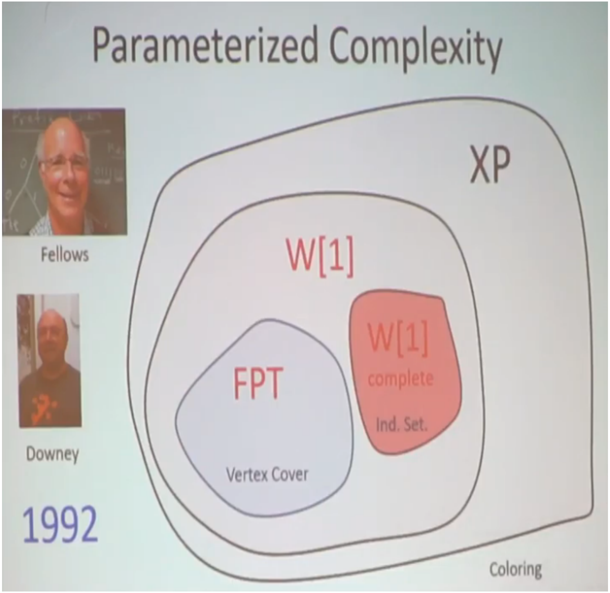

$W[1]$-hardness

Negative evidence similar to NP-completeness: if a (parametrized) problem is $W[1]$-hard, then the problem is not FPT unless $FPT = W[1]$.

Some $W[1]$-hard problem:

- Find a clique/stable set of size k.
- Find a dominating set of size k
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Exponential Time Hypothesis (ETH):
n-variable 3-SAT cannot be solved in time $2^{o(n)}$.

Clique parametrized by maximum degree

Problem (CLIQUE parametrized by Δ)

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Running time: $O(2^{\Delta} n)$, FPT!!

So CLIQUE parametrized by $\Delta(G)$ is FPT.

But CLIQUE parametrized by solution size k is $W[1]$-hard. That is, probably no algorithm in time $f(k) \cdot n^{O(1)}$.

Parametrized Complexity

Rod G. Downey
Michael R. Fellows

Parameterized Complexity

Springer 1999

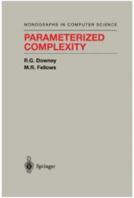
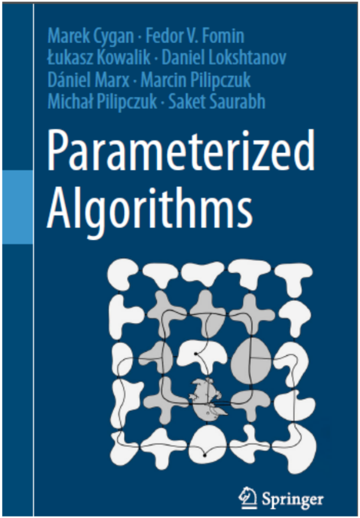


figure by Daniel Marx

- The study of parameterized complexity was initiated by Downey and Fellows in the early 90s.
- First monograph in 1999.
- By now, strong presence in most algorithmic conferences.

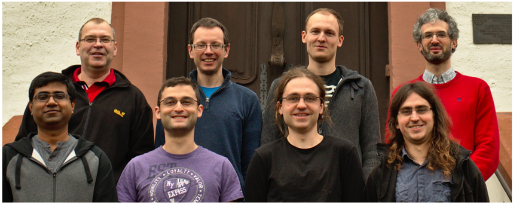
Source for this class



Parameterized Algorithms

Marek Cygan, Fedor V. Fomin,
Lukasz Kowalik, Daniel Lokshtanov,
Dániel Marx, Marcin Pilipczuk,
Michał Pilipczuk, Saket Saurabh

Springer 2015



Algorihtmic techniques to design FPT algorithm

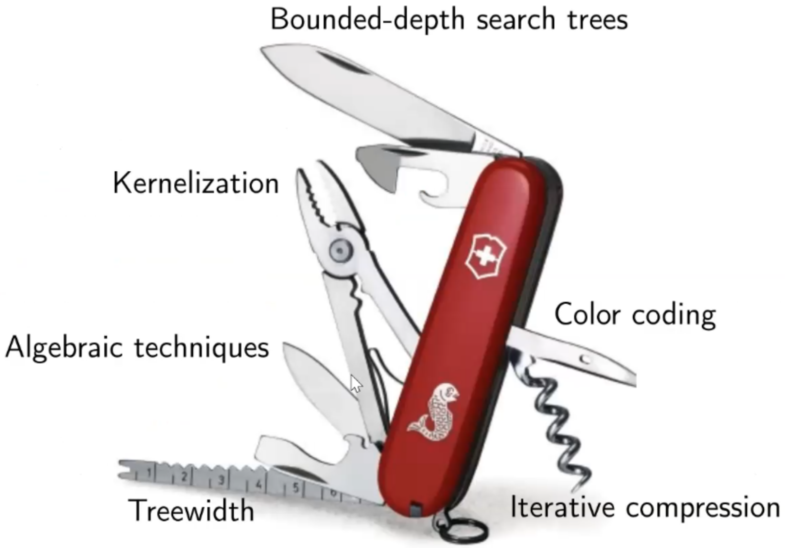


Figure by Daniel Marx

1 - Branching Method

First problem:

Vertex Cover

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Problem (VERTEX COVER parametrized by the size of the solution)

Question: Given (G, k), does G have a vertex cover of size at most k?

Brute force: For every set S of k vertices, check if $G \setminus S$ is edgeless.
Running time: $O(n^k \cdot n^2) = O(n^{k+2})$.

So VERTEX COVER parametrized by the size of the solution is in *XP*.

But is it in *FPT*?

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- So G has a VC of size at most k if and only if $G \setminus \{u\}$ or $G \setminus \{v\}$ has a VC of size at most $k-1$.

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- So G has a VC of size at most k if and only if $G \setminus \{u\}$ or $G \setminus \{v\}$ has a VC of size at most $k-1$.
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- So the running time: $O(2^k \cdot n^{O(1)})$.

Branching method, size of the search tree and complexity

To solve instance (G, k) of Vertex Cover:

- **Main idea**: reduce the problem to solving a bounded number of problems with paramater $k' < k$.
- We need to be able to solve instance (G, k) in poly-time knowing the solution of the new instances.
- Since the parameter decrease in every recursive call, the depth of the search tree is at most k.
- Size of the seach tree:
  - If we branch into c directions: c^k
  - If we branch into k directions: $k^k = 2^{k \log(k)}$
  - If we branch into $\log(n)$ directions: $n + 2^{k \log(k)}$

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We are now going to solve VERTEX COVER in time $1.46^k \cdot n^{O(1)}$!

Notation: $1.46^k \cdot n^{O(1)} = O^*(1.46^k)$

More thinking about the problem

Idea: instead of branching on edges, we are going to branch on vertices of degree at least 3. It is going to work faster because in some of the branches, the parameter is going to decrease faster.

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Algebraic resolution

Let $T(k)$ be the number of leaves in the search tree, and $T(k) = 0$ if $k \leq 1$. Then:

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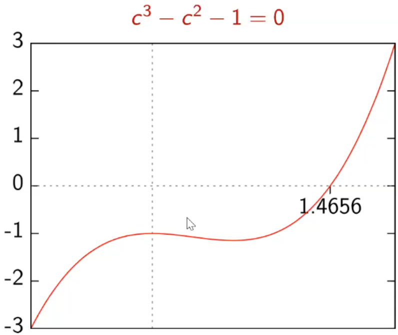
and in particular:

$$c^3 - c^2 - 1 \geq 0$$

So we want to find the smallest positive root of this equation.
Actually, such equations have a unique postive root.

Solving the equation



$c = 1.4656$ is a good value, so we get $T(k) \leq 1.4656^k$.
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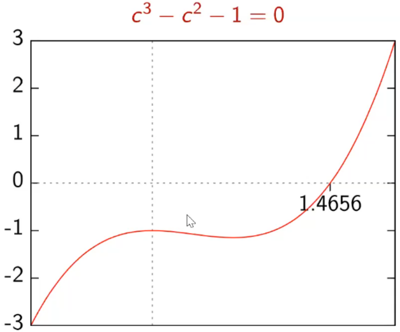

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Best known FPT algorithm: $O^*(1.2738^k)$, by J. Chen, I. A. Kanj and G. Xia, *Simplicity is beauty: improved upper bounds for Vertex Cover*.

Branching method

The branching vector of our $O^*(1.4656k)$ Vertex Cover algorithm was $(1, 3)$.

Example: Let us bound the search tree for the branching vector $(2, 5, 6, 6, 7, 7)$. (2 out of the 6 branches decrease the parameter by 7, etc.).

The value $c > 1$ has to satisfy:

$$c^k \geq c^{k-2} + c^{k-5} + 2c^{k-6} + 2c^{k-7}$$

And thus c satisfies:

$$c^7 - c^5 - c^2 - 2c - 2 \geq 0$$

Unique positive root of the characteristic equation: 1.4483, so $T(k) \leq 1.4483^k$.

In general, it is hard to compare branching vectors intuitively.

Next problem:

GRAPH MODIFICATION PROBLEM

Definition: Given a graph property $\mathcal{P}$, find a set of vertices S such that $G \setminus S$ satisfies $\mathcal{P}$.

If $\mathcal{P}$ is the property of being edgeless, we recover vertex cover.

Triangle-free deletion problem

Problem (Triangle-free deletion)

Given: a graph G and an integer k,
Question: is there a set of at most k vertices such that $G \setminus S$ is triangle-free?

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Key idea showing that the branching method is going to work:

If $v_1 v_2 v_3$ is a triangle of G, then:

$$(G, k) \text{ is a YES instance}$$
$$\Leftrightarrow$$
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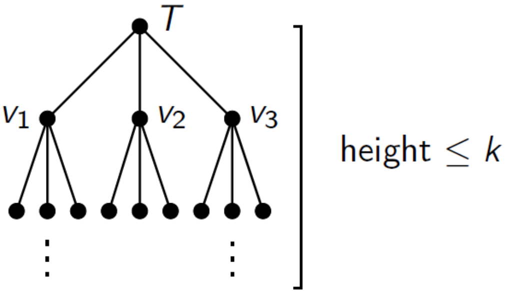
Algo:

- Find a triangle $v_1 v_2 v_3$ (time: $O(n^3)$)
- Solve the instance $(G \setminus v_i, k-1)$ for $i = 1, 2, 3$.

Complexity analysis



The search tree has depth at most k and thus has at most 3^{k+1} vertices.

Find a triangle or check if a graph is triangle-free: n^3,

Running time: $O(3^k \cdot n^3)$.

Graph modification problem

Problem (Graph modification problem)

Given: (G, k)
Question: do at most k allowed operation on G can make G to have property $\mathcal{P}$?

- Allowed operations: vertex deletion, edge deletion, edge contraction, edge addition...
- Property $\mathcal{P}$: edgeless, no triangle, no cycles, disconnected...

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Examples:

- VERTEX COVER: delete k vertices to make G edgeless,
- TRIANGLE-FREE DELETION: delete k vertices to make G triangle-free,
- FEEDBACK VERTEX SET: delete k vertices to make G a forest.
- CHORDAL COMPLETION: add k edges to make the graph chordal.

Subgraphs and induce subgraph

1. Remove a vertex v (and all its incident edges), denoted $G \setminus v$.
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- H is an induced subgraph of G if H obtained from G by the repeated use of 1.
- H is a subgraph of G if H obtained from G by the repeated use of 1 and 2.

Hereditary property

Definition: a graph property $\mathcal{P}$ is hereditary or closed under taking induced subgraph if whenever $G \in \mathcal{P}$, every induced subgraph H of G are also in $\mathcal{P}$.

small-*Deleting vertices do not ruin the property*-

Examples: edgeless, triangle-free, bipartite, planar...

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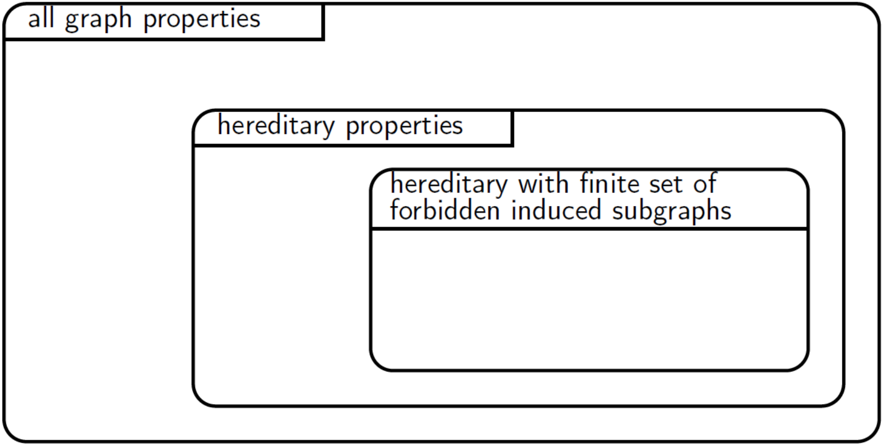
Examples: edgeless, triangle-free, bipartite, planar...

Observation: Every hereditary property $\mathcal{P}$ can be characterized by a (finite or infinite) set $\mathcal{F}$ of minimal obstructions or forbidden induced subgraphs: $G \in \mathcal{P}$ if and only if G does not have an induced subgraph isomorphic to a member of $\mathcal{F}$.

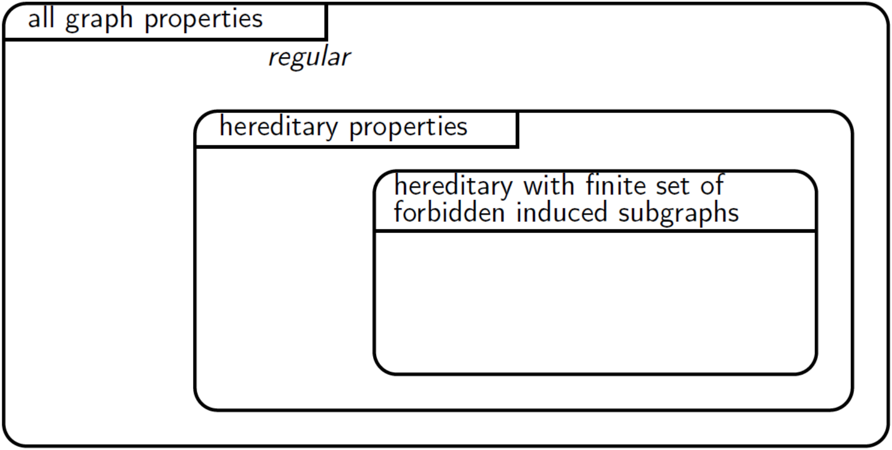
Example: a graph is bipartite if and only if it does not contain odd cycles as induced subgraph.

Graph properties



regular *bipartite* *triangle free* *connected*
planar *empty* *complete* *acyclic*

Graph properties



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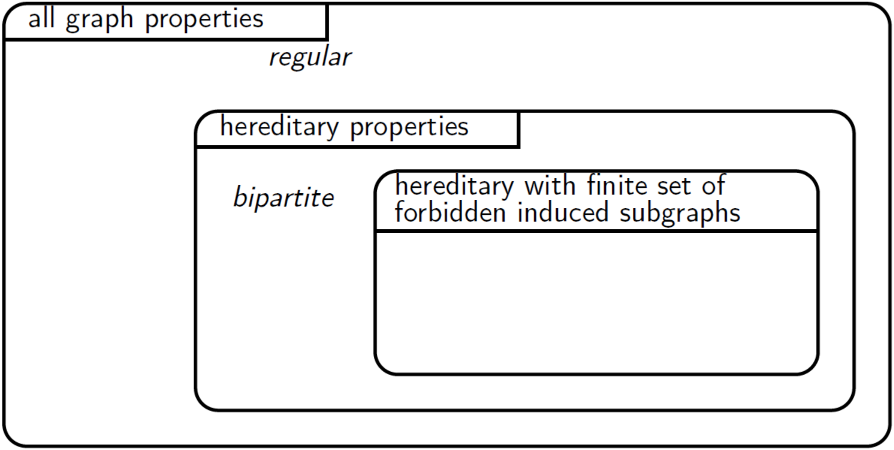
Graph properties



triangle free *connected*

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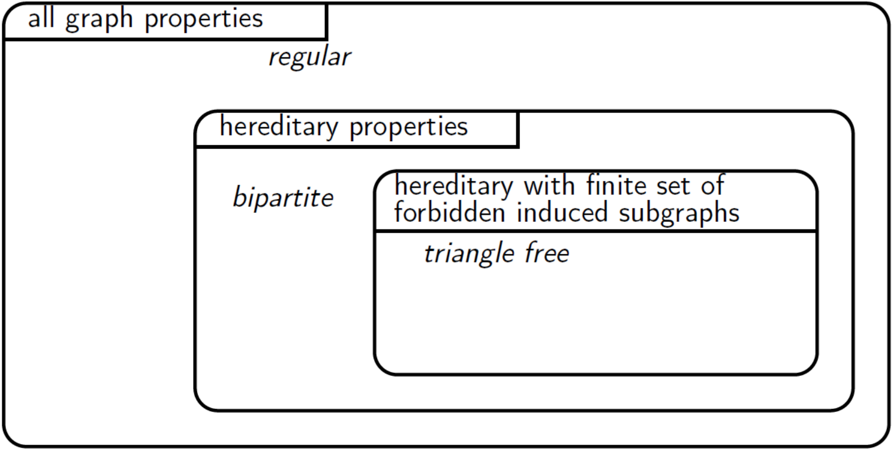
Graph properties



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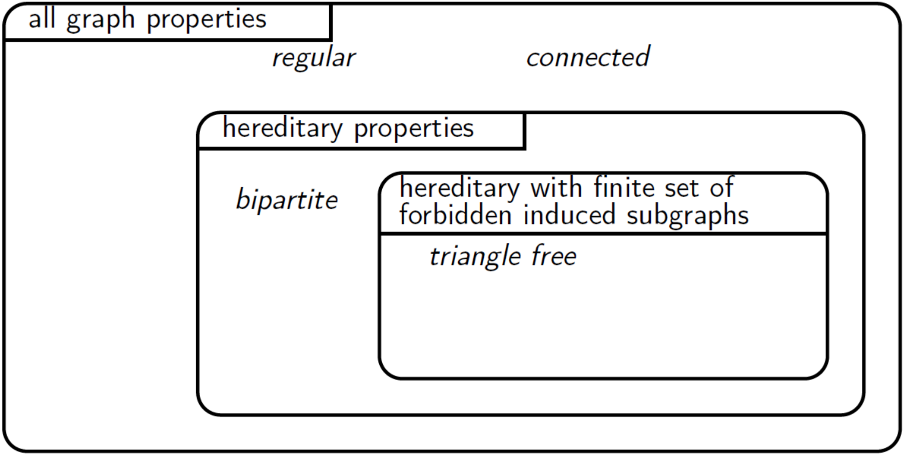

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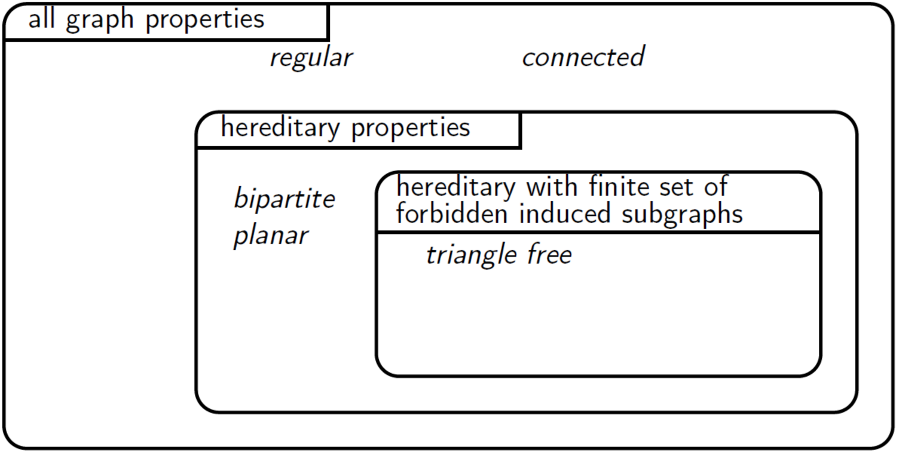

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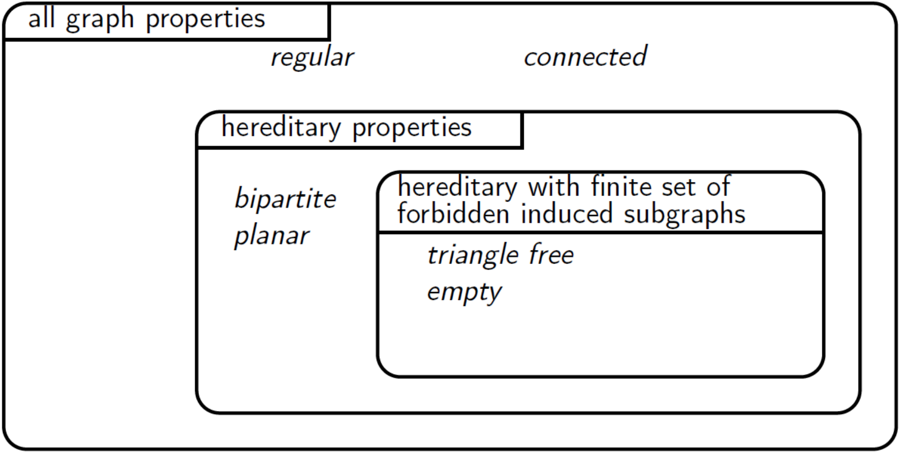

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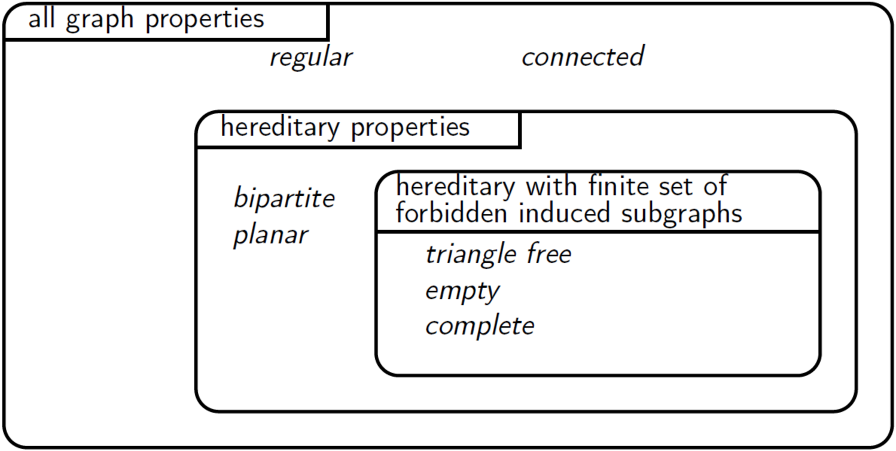

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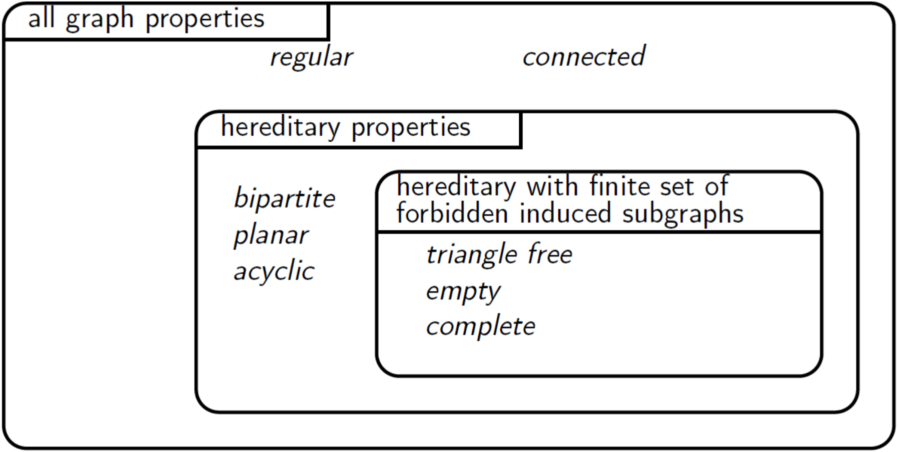

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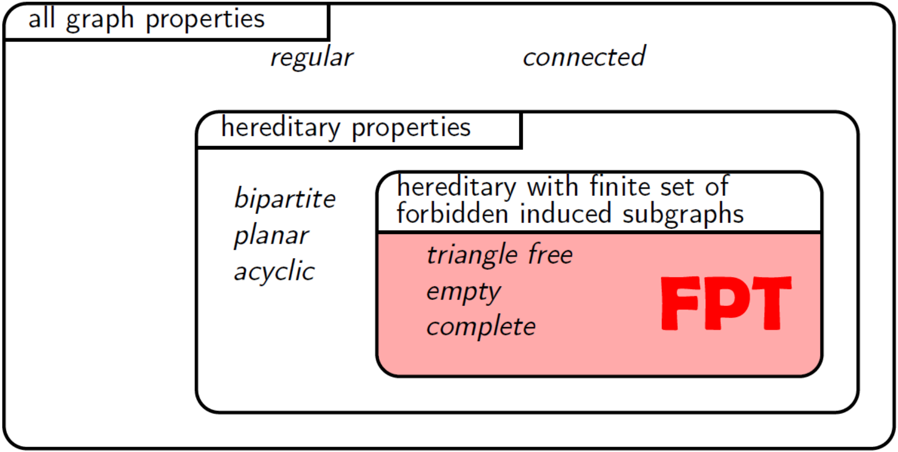

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- The tree has at most r^{k+1} vertices, and the work to be done at each vertex is $O(n^r)$.
- Total running time: $O(r^{k+1} \cdot n^r)$.

An active area of research

Graph modification problem is a very wide and active research area in parameterized algorithms.

- If the set of forbidden subgraphs is finite, then the problem is immediately FPT (e.g., Vertex Cover, Triangle Free Deletion). Here the challange is improving the naive running time.
- If the set of forbidden subgraphs is infinite, then very different techniques are needed to show that the problem is FPT (e.g., Feedback Vertex Set, Bipartite Deletion, Planar Deletion).

Next problem:

FEEDBACK VERTEX SET

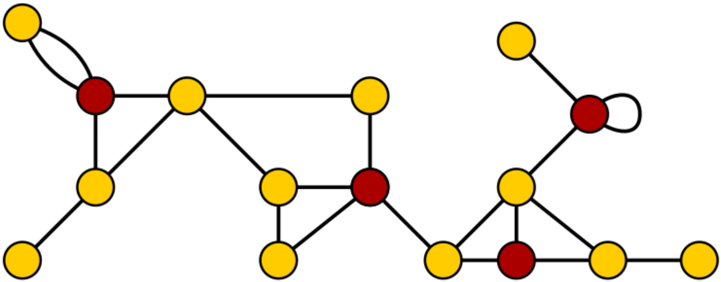
A **Feedback Vertex Set** (**FVS**) of a graph G is a set S of vertices such that $G \setminus S$ is a forest.
In other words S hits all cycles.

Feeback Vertex set

Problem (Feedback Vertex set (FVS))

Question: Given (G, k), find a set S of at most k vertices such that $G \setminus S$ has no cycle (i.e. $G \setminus S$ is a forest).

- We allow **loop**, and **multiple edges** (G is a **multigraph**).
- A Feedback Vertex Set is a set of vertices that hits every cycle of the graph.



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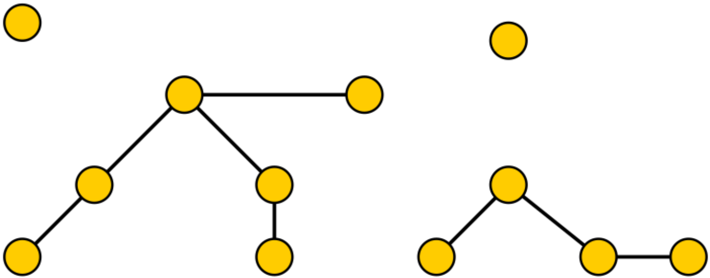

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Link with vertex cover: a vertex cover is a set of vertices that hits every edge of the graph.

Thinking about the problem

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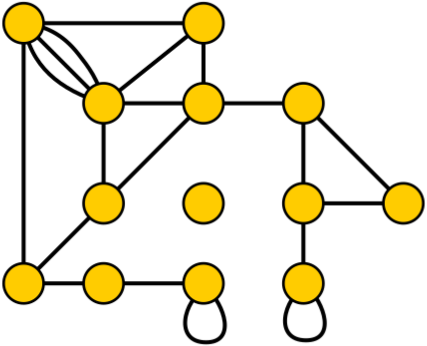
- We are going to: identify a set of $O(k)$ vertices such that any size-k feedback vertex set has to contain one of these vertices, and branch on it.
- But first, as often, some reduction rules.

  The reduction rules are here to simplify the input in such a way that the new input is a YES-instance if and only if the orginal one is.

Reduction rules for FVS

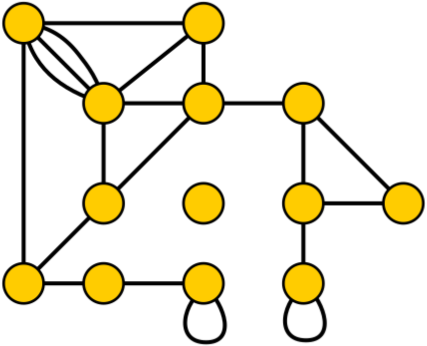
(R1) If there is a loop at v, then delete v and decrease k by one.



Reduction rules for FVS

(R1) If there is a loop at v, then delete v and decrease k by one.

(R2) If there is an edge of multiplicity larger than 2, then reduce its multiplicity to 2.



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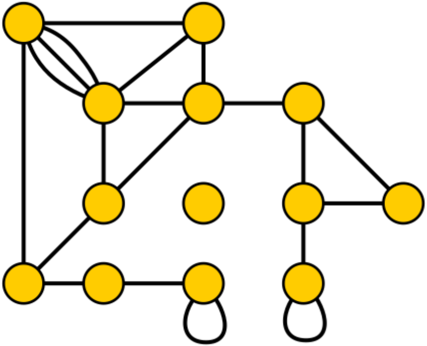
(R3) If there is a vertex v of degree 0 or 1, then delete v.



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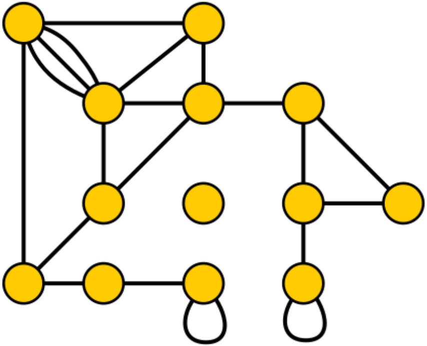

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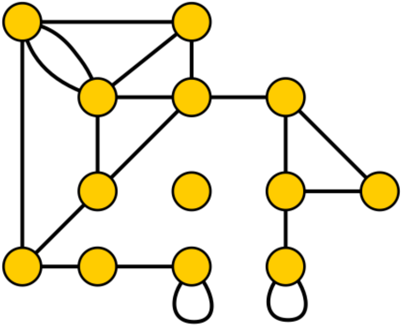

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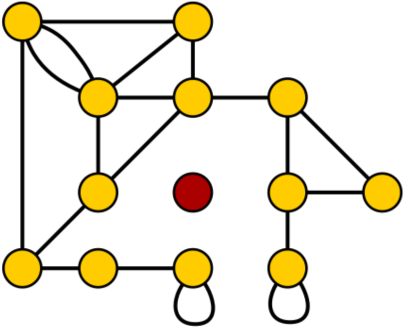

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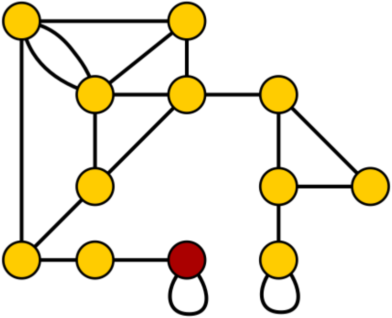

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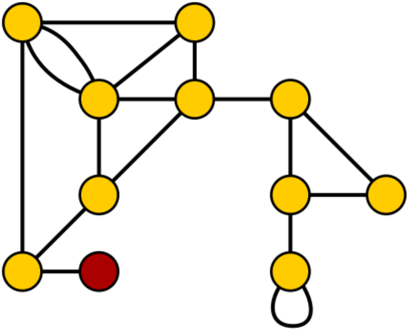

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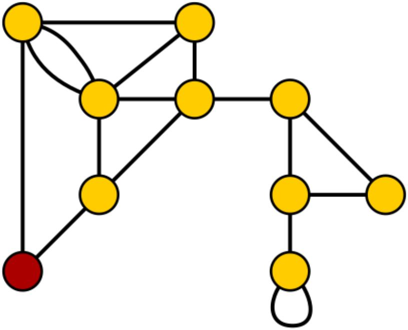

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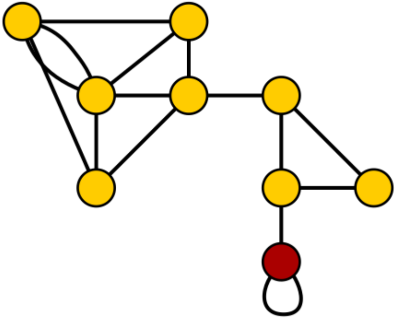

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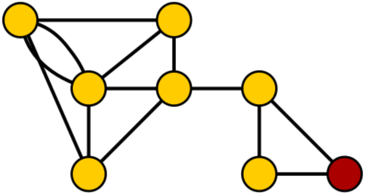

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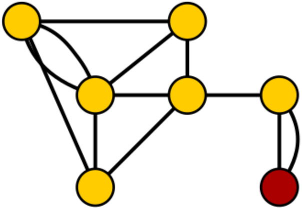

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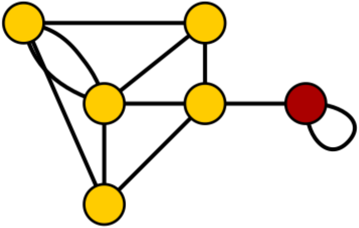

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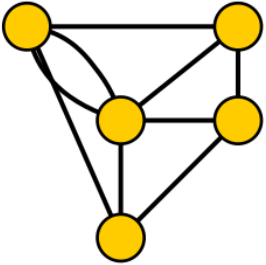

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After exhaustively applying these reduction rules, the resulting graph G satisfies:

- no loop,
- edge multiplicity is 1 or 2,
- minimum degree 3

Key property of reduction rules

Key Property of the reduction rules:
If (G, k) is an instance of FVS graph and (G', k') is the instance obtained after applying the reduction rules as much as we can, then

- G has a FVS of size at most k if and only if G' has a FVS of size at most k' and
- If S is a FVS of G', then it is a FVS of G together with the vertices deletes by R1. (not necessary if we don't care about the set and just want a YES/NO answer).

In other words, we can safely apply the reduction rules and work on the resulting graph.

Branching

Lemma: Let G be a graph with minimum degree 3, and let V_{3k} be the $3k$ largest degree vertices. Then every Feedback Vertex set of size at most k contains at least one vertex of V_{3k}.

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Assuming the Lemma we can easily design our FPT algorithm:

- Apply reduction rules to obtain G' and compute V_{3k}.
- Branch on each vertex $x \in V_{3k}$, that is solve the problems for the k instances: $(G' \setminus \{x\}, k-1)$.
- Branching into $3k$ directions $\Rightarrow$ $O^*((3k)^k)$
- Applying reduction rules and finding the $3k$ largest degree vertices can easily be done in poly-time.

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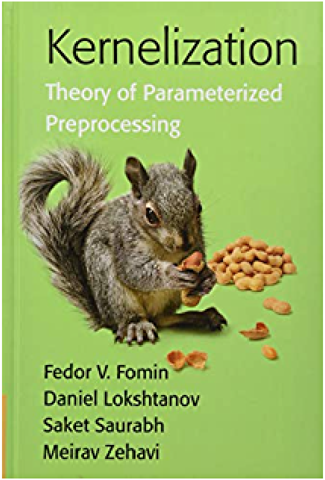
  - $\geq 3kd + 3|X| - (2|X| + 6k - 2) > 3kd - 6k$
  - $\leq dk$ because S has k vertices, each of degree at most d.
- So $3kd - 6k < kd \Leftrightarrow 2kd - 6k < 0$ which is false because $d \geq 3$.

2 - Kernelization


Kernelization
Theory of Parameterized Preprocessing
Fedor V. Fomin
Daniel Lokshtanov
Saket Saurabh
Meirav Zehavi

Data reduction

- Kernelization is a method for parameterized preprocessing:
  We want to efficiently reduce the size of the instance (x, k) to an equivalent instance with size bounded by $f(k)$.

- A basic way of obtaining FPT algorithms:
  Reduce the size of the instance to $f(k)$ in polynomial time and then apply any brute force algorithm to the shrunk instance.

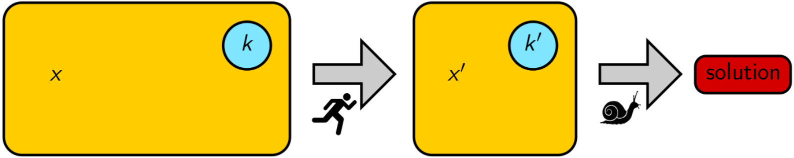


Figure by Daniel Marx

Kernelization: formal definition

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Question: which problem has a kernel??

A crazy equivalence

Theorem: A parametrized problem is FPT if and only if it is decidable and has a kernel (of arbitrary size).

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• If the problem can be solved in time $f(k) \cdot |x|^c$:

- If $|x| \leq f(k)$, then we already have our kernel.
- If $|x| \geq f(k)$, then we can solve the problem in time $f(k) \cdot |x|^c \leq |x|^{c+1}$ (which is polynomial in $|x|$) and then output a trivial YES or NO answer.

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- So asking if there is a kernel is the same question as asking for an FPT algorithm.
- The important question: is there a polynomial kernel?

Back to vertex cover

Let us prove that Vertex Cover has a polynomial kernel.

A **vertex cover** of a graph G is a set S of vertices such that $G \setminus S$ is edgeless. In other words S hits all edges.

Thinking about the problem

Observe that if a vertex v has degree 0, then:
G has a vertex cover of size k if and only if $G - \{v\}$ has a vertex cover of size k.

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This leads us to define the two following reduction rules:

(R1) If v has degree 0, then reduce to $(G - v, k)$

(R2) If v has degree at least $k+1$, then reduce to $(G - v, k-1)$.

Now, if (G, k) is an instance of VERTEX COVER and (G', k') is the instance obtained after an exhaustive application of $R1$ and $R2$, then:

(G, k) is a YES-instance if and only if (G', k') is a YES-instance.

Kernel for vertex cover

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Proof:

- Let S be a vertex cover of G of size at most k.
- Each vertex hits at most k edges because (R2) does not apply. So there is at most k^2 edges.
- Each vertex is either in S, or is one of the k neighbors of a vertex in S. So $|V(G)| \leq k^2 + k$.

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Kernelization for VERTEX COVER:

- Apply rules $(R1)$ and $(R2)$ exhaustively. We get a new instance (G', k') with $k' \leq k$ and such that (G, k) is a YES-instance if and only if (G', k') is.
- If $|E(G')| > k'^2$ or $|V(G)| > k'^2 + k'$, output NO.
- Otherwise we have a kernel of size $O(2k^2 + k)$.

Crown decomposition

Theorem: VERTEX COVER has a kernel with at most $3k$ vertices.

Crown decomposition

A crown decomposition of a graph G is a partitioning of $V(G)$ into three parts C, H and R such that:

1. C is a nonempty independent set;
2. There are no edge between C and R;
3. There is a matching between C and H of size $|H|$.

C is the crown, H the head, and R the rest.

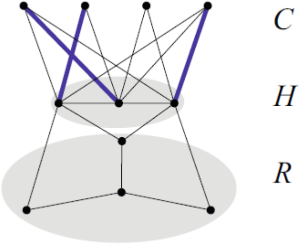

Figure from Parametrized Algorithm by CFKLMPPS

Matching in bipartite graphs

Let G be a bipartite graph with partition (V_1, V_2).

Kőnig's Theorem: The size of a maximum matching of G equal the size of a minimum vertex cover.

Hall's Theorem: G has a matching saturating V_1 if and only if for all $X \subseteq V_1$, $|N(X)| \geq |X|$.

Hopcroft-Karp algorithm: There is a $O(m\sqrt{n})$-time algorithm that finds a maximum matching as well as a minimum vertex cover in G. It furthermore finds a matching saturating V_1, or a inclusion-wise minimal set $X \subseteq V_1$ such that $|N(X)| < |X|$.

Crown Algorithm: Let G be a graph with no isolated vertex and with at least $3k+1$ vertices. There is a poly-time algorithm that either:

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Corollary: Vertex Cover has a kernel with at most $3k$ vertices.

Proof: Consider a Vertex Cover instance (G, k). By an exhaustive application of (R1), we may assume G has no isolated vertex. If $|V(G)| \geq 3k+1$, by the crown lemma applied to (G, k), either G has a $(k+1)$ matching, or a crown decomposition (C, H, R). In a former case, output NO. In the latter case, let M be a matching between H and C of size $|H|$. Observe that the matching M witnesses that, for every vertex cover X of G, X contains at least $|M| = |H|$ vertices of $H \cup C$ to cover the edges of M. On the other hand, H covers all edges of G that are incident to $H \cup C$. Consequently, there exists a minimum vertex cover of G that contains H. Moreover, vertices in C are isolated in $G - H$. Hence, (G, k) is a YES-instance if and only if $(G - (C \cup H), k - |H|)$ is. As $H \neq \emptyset$, we can run the crown algorithm until it outputs a matching of size $k+1$ or until $|V(G)| \leq 3k$. $\square$

Proof of the crown lemma

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- Set $V_M = V(M)$ and $I = V(G) \setminus V(M)$. Note that I is an independent set.

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Crown Algorithm: Let G be a graph with no isolated vertex and with at least $3k+1$ vertices. There is a poly-time algorithm that either:

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Kernels based on linear programming

Theorem: VERTEX COVER has a kernel with at most $2k$ vertices.

Integer Linear Programming

Many combinatorial problems can be expressed in the language of Integer Linear Programming (ILP).

In an ILP instance, we are given a set of integer-valued variables, a set of linear inequalities (called constraints) and a linear cost function.
The goal is to minimize or maximize the value of the cost function respecting the constraints.

$$\begin{array}{llll} \text{Minimise :} & \sum_{j=1}^{n} c_j \cdot x_j & & \\ \text{Subject to:} & \sum_{j=1}^{n} a_{ij} \cdot x_j & \leq b_i & \text{for } 1 \leq i \leq m \\ & x_j & \in \mathbb{Z} & \text{for } 1 \leq j \leq m \end{array}$$

The a_{ij}, b_i and c_i are constants, the x_i are the variables.

Encode VERTEX COVER as an ILP

Introduce a variable $x_v \in \{0,1\}$ for each $v \in V(G)$.
Setting $x_v = 0$ means that x_v is not in the solution, while $x_v = 1$ means it is.

$$
\begin{array}{lrll}
\text{Minimise :} & \sum_{v \in V(G)} x_v & & \\
\text{Subject to:} & x_u + x_v & \geq 1 & \text{for all } uv \in E(G) \\
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Beautifull, but how is it helpful? ILP is extremely hard to solve.

Fractional relaxation

Linear Programming is famously known for being solvable in (weakly) poly-time, so let us relax our problem. Call it $LPVC(G)$.

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$x_v = \frac{1}{3}$ is understood as we take one third of the vertex.
A solution to $LPVC(G)$ is a called a fractional vertex cover of G. Its size if dentoed by $VC_f(G)$.

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and for example, if G is a triangle, $VC_f(G) = \frac{3}{2} < 2 = VC(G)$.

Let $(x_v)_{v \in V(G)}$ be a minimum fractional vertex cover, i.e. an optimal solution to:

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Partition the vertices with respect to their value as follows:

- $V_0 = \{v : x_v < \frac{1}{2}\}$
- $V_{\frac{1}{2}} = \{v : x_v = \frac{1}{2}\}$
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- V_0 is an independent set, and
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Theorem (Nemhauser-Trotter, 1975)
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Proof:

- Let S^* be a minimum vertex cover of G.
- Set $S = V_1 \cup (V_{\frac{1}{2}} \cap S^*)$, and observe that $V_1 \subseteq S \subseteq V_{\frac{1}{2}} \cup V_1$.
- Since there is no edge between V_0 and $V_{\frac{1}{2}}$, S is a VC of G.
- It remains to prove that S is a minimal VC. Assume $|S| > |S^*|$.
- So

$$|V_0 \cap S^*| < |V_1 \setminus S^*| \tag{1}$$

- Set $\varepsilon = \min(|x_v - \frac{1}{2}| : v \in V_0 \cup V_1)$ and define:

$$y_v = \begin{cases} x_v - \varepsilon & \text{if } v \in V_1 \setminus S^* \\ x_v + \varepsilon & \text{if } v \in V_0 \cap S^* \\ x_v & \text{otherwise} \end{cases}$$

- It is easy to check that $(y_v)_{v \in V(G)}$ is a fractional vertex cover.
- But by (1), $\sum_{v \in V(G)} y_v < \sum_{v \in V(G)} x_v$, a contradiction.

Nemhauser-Trotter's theorem allows the following reduction rule:

(R3) Given an minimum fractional vertex cover $(x_v)_{v \in V(G)}$ and the partition $(V_0, V_{\frac{1}{2}}, V_1)$:

- if $\sum_{v \in V(G)} x_v > k$, output NO.
- Otherwise, solve $(G[V_{\frac{1}{2}}], k - |V_1|)$.

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This is a safe rule in the sense that:

- if $\sum_{v \in V(G)} x_v > k$, then (G, k) is indeed a NO-instance.
- $(G[V_{\frac{1}{2}}], k - |V_1|)$ is a YES-instance if and only (G, k) is.

Moreover, if (G, k) is a YES-instance, then
$|V_{\frac{1}{2}}| = \sum_{v \in V_{\frac{1}{2}}} 2x_v \leq 2 \sum_{v \in V(G)} x_v \leq 2k$.

Theorem: Vertex Cover has a kernel with at most $2k$ vertices.

Lemma: An minimum fractional vertex cover with each weight in $\{0, \frac{1}{2}, 1\}$ can be found in time $O(m\sqrt(n)$

Proof: We reduce fractional vertex cover to $\textsc{Vertex Cover}$ in the following bipartite graph H: take two copies V_1 and V_2 of $V(G)$ (if $u \in V(G)$, there is a copy u_1 of u in V_1 and a copy u_2 of u in V_2.) and if $uv \in E(G)$, then $u_1v_2, v_1u_2 \in E(G)$.

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Since S is a vertex cover of H, at least two of the vertices u_1, v_1, u_2, v_2 are in S, and thus, for every edge uv, $x_u + x_v \geq 1$. So $(x_v)_{v \in V(G)}$ is a fractional vertex cover G. Let us prove it is minimum.

Let $(y_v)_{v \in V(G)}$ be a minimum fractional vertex cover G.
We define a weight on $V(H)$ as follows:
For every $v \in V(G)$, $w(v_1) = w(v_2) = y_v$.
This weight assignment is a fractionnal vertex cover of H, i.e., for every edge $u_1 v_2$ of H, we have $w(u_1) + w(v_2) \geq 1$. Hence, $\sum_{v \in V(H)} w(v)$ is at least the size of a maximum matching M of H.
Now, by Kőnig Theorem, $|M| = |S|$, so:

$$\sum_{v \in V(G)} y_v = \frac{1}{2} \sum_{v \in V(G)} (w(v_1) + w(v_2)) = \frac{1}{2} \sum_{v \in V(H)} w(v) \geq \frac{|S|}{2} = \sum_{v \in V(G)} x_v$$

The sunflower Lemma

Theorem: d-HITTING SET has a kernel with at most $d!k^d$ hyperedges and $d!k^d d^2$ vertices.

The d-HITTING SET PROBLEM

Let V be a finite set. A set system $\mathcal{F}$ on V is a collection of subsets of X. We call $\mathcal{F}$ a d-set system if each set has size at most d. A hitting set of $\mathcal{F}$ is a set of vertices that intersects (hits) every set of $\mathcal{F}$.

Problem (d-HITTING SET PROBLEM)

Given: a d-set system $\mathcal{F}$ and a an integer k.
Question: does $\mathcal{F}$ admits a hitting set of size at most k?.

Note that when $d = 2$, it is vertex cover!

Sunflower

A collection of sets $S_1, \ldots S_k$ is a k-sunflower if

$$S_i \cap S_j = S_1 \cap S_2 \cap \ldots S_k \quad \forall\, i \neq j$$

The set $K = S_1 \cap S_2 \cap \ldots S_k$ is the core of the sun flower and the sets $S_i \setminus K$ are its petals.

Note that a set of k pairwise disjoint sets is a sunflower with k petals and an empty core.

The Sunflower Lemma, or Erdős-Rado Lemma

Lemma [The Sunflower Lemma, or Erdős-Rado Lemma, 1960]
Let $\mathcal{F}$ be a d-set system on a set V. If $|\mathcal{F}| > d!(k-1)^d$, then $\mathcal{F}$ has a sunflower with k petals.
Moreover, it can be found in time polynomial in $|V| + |\mathcal{F}| + k$.

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Moreover, it can be found in time polynomial in $|V| + |\mathcal{F}| + k$.

Proof: We proceed by induction on d. For $d = 1$ it is trivial. Assume $d \geq 2$. Let $\mathcal{M} = \{S_1, \dots, S_\ell\}$ be a maximal collection of pairwise disjoint sets of $\mathcal{F}$. If $\ell \geq k$ we are done, we may assume $k < \ell$. Set $S = S_1 \cup \dots \cup S_\ell$ and observe $|S| \leq d(k-1)$. Moreover, every set of $\mathcal{F}$ intersects S. Hence, there is $u \in S$ that belongs to at least

$$\frac{d!(k-1)^d}{d(k-1)} = (d-1)!(k-1)^{d-1}$$

sets of $\mathcal{F}$. Construct a $(d-1)$-set system by taking all these sets and removing u from each of them. By induction it has a k-sunflower and thus, puting u back in, we get a k-sunflower in $\mathcal{F}$.

The Sunflower Conjecture

Sunflower Conjecture (Erdős-Rado, 1960)
Let $k \geq 3$. There exists $c = c(k)$ such that every d-set system $\mathcal{F}$ with $|\mathcal{F}| \geq c^d$ contains a d-sunflower.

Theorem (Alweiss, Lovett, Wu and Zhang, 2021):
Let $k \geq 3$. There exists c such that every d-set system $\mathcal{F}$ with $|\mathcal{F}| \geq (ck^3 \log d \log\log d)^d$ contains a k-sunflower.

Trendy topic:
Blog of Terry Tao
Polymath10

Kernel for d-HITTING SET

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Given: a d-set system $\mathcal{F}$ and an integer k.
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Theorem: d-HITTING SET has a kernel with at most $d!k^d$ sets and $d!k^d d^2$ vertices.

Crucial Observation: If $\mathcal{F}$ has a $(k+1)$-sunflower with core K, then every hitting set of $\mathcal{F}$ intersects K.

Reduction rule: Given an instance $(V, \mathcal{F}, k)$, if $\mathcal{F}$ has a $(k+1)$-sunflower $S = \{S_1, \dots, S_{k+1}\}$ with core K, return $(V', \mathcal{F}', k)$ where:

- $\mathcal{F}' = (\mathcal{F} \setminus S) \cup K$ and
- $V' = \cup_{F \in \mathcal{F}'} F$

3 - Color coding

k-PATH PROBLEM

Problem (k-PATH)

Given (G, k), decide if G contains a (simple) path on k vertices as a subgraph.

A long history:

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- Bjorklund, Husfeldt, Kaski, Koivisto 2010: $1.66^k n^{O(1)}$

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  - Standard derandomization techniques exist.

Monte-carlo algorithm

A typical situation in randomized algorithm is the so-called Monte-Carlo algorithm with one-sided error:

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Morality: any constant probability is ok.

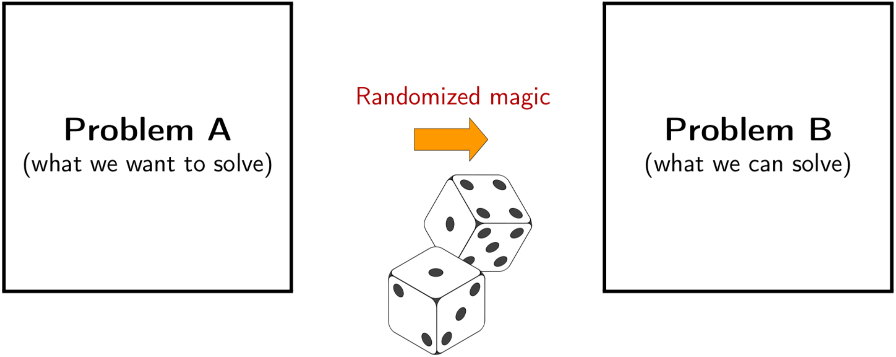

Figure by Daniel Marx

Color coding

Surprising idea: transform the problem into the following:

- Assume the vertices are colored randomly with $\{1, \ldots, k\}$
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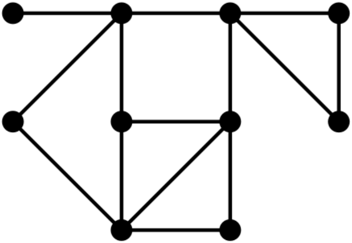

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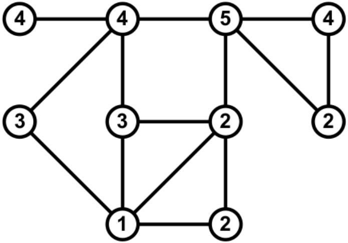

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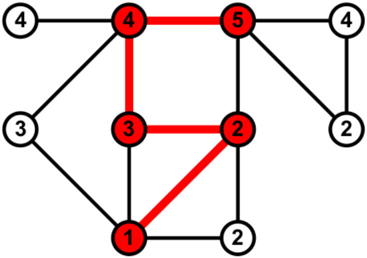

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- So if G is a YES instance, the algo output YES with probability at least $1/k^k$
- And if it is a NO instance, the algorithm output NO.
- This looks very bad, but since k is considered as a constant maybe it is not that bad!

Brillant idea: do it a lot of times

Useful fact

If the probability of success of a (Monte-Carlo) algorithm is at least p, then the probability that, given a YES-instance, the algorithm return NO $1/p$ times in a row is at most:

$$(1-p)^{1/p} < (e^{-p})^{1/p} = 1/e \approx 0.38$$

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Thus if $p \geq \frac{1}{k^k}$, then after k^k repetitions error probability is at most $1/e$:

$$\left(1 - \frac{1}{k^k}\right)^k < \frac{1}{e}$$

Hence, by trying $100 \cdot k^k$ random colorings, the probability of a wrong answer is at most $1/e^{100}$.

Find a $1 - 2 - \cdots - k$ colored path

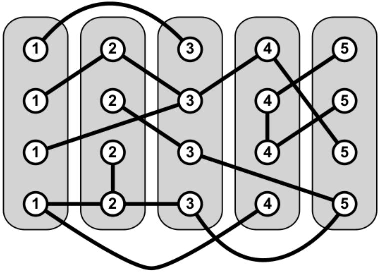

Figure by Daniel Marx

- Let V_i be the set of vertices colored i (color class)
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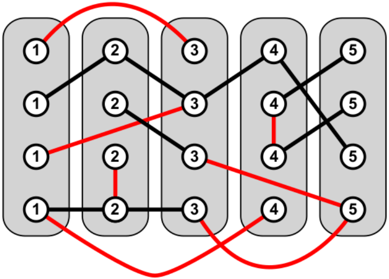

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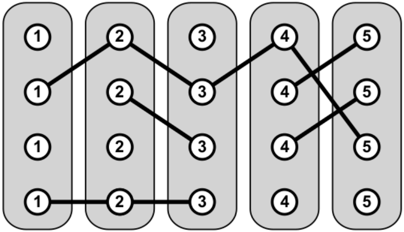

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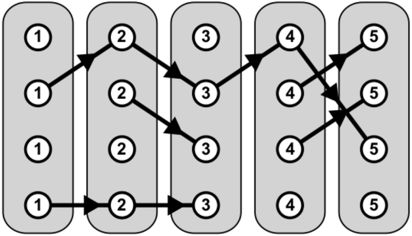

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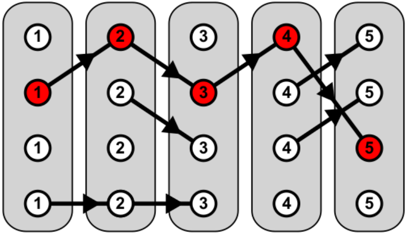

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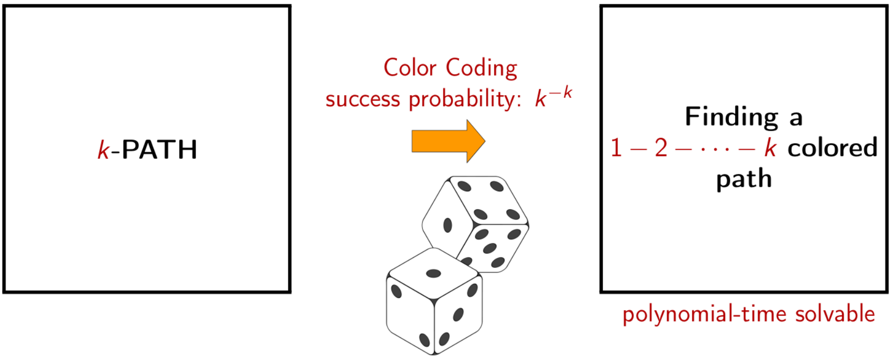

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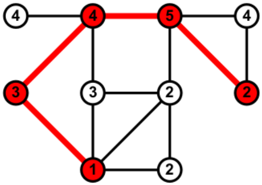
Complexity: $O(c \cdot k^k \cdot (n + m))$.
Probability of sucess: $1/e^c$

Improved color coding

- Assign colors from $[k]$ to the vertices uniformly and independantly at random.



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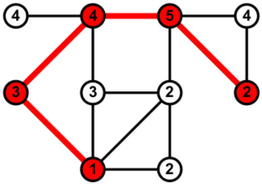

- Output YES if there is a colorfull k-path.
  - If there is no k-path, no colorfull path exist, and the algo output NO.
  - If there is a k-path, probability that it is colorfull is

$$\frac{k!}{k^k} > \frac{\left(\frac{k}{e}\right)^k}{k^k} = e^{-k}$$

- Repeat the algorithm $100e^k$ times decrease the error probability to e^{-100}.

Improved color coding

So replacing the problem "Find a k-path colored $1 - 2 - \cdots - k$?" by "Is there a k-path coloured with k colours?" allowed us to go from probability of sucess of $1/k^k$ to $1/e^k$.

Recall that this means that we need to solve the problem e^k times instead of k^k.

But how hard is it to solve colorfull path problem?

Find a colorfullpath with dynamic programming

Subproblem: For each vertex v and each set of color $C \subseteq [k]$, define:

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Now, we can solve this DP in time $2^k \cdot |E|$

Recap

The algorithm: Repeat e^k times:

1. Sample a coloring $c : V \leftarrow \{1, \ldots, k\}$
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- Total running time: $O((2e)^k \cdot |E|)$.

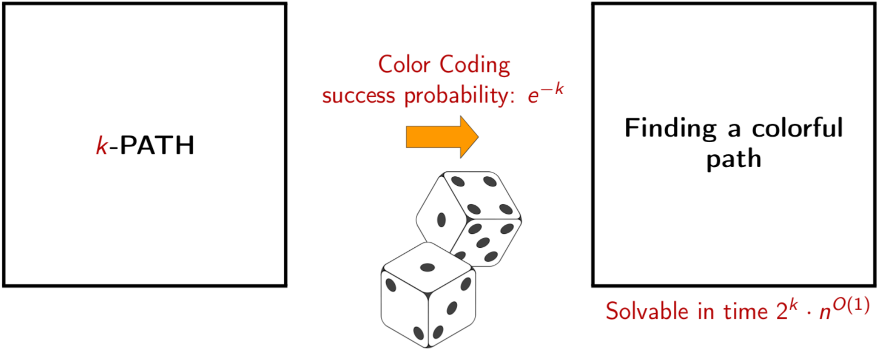

Figure by Daniel Marx

Derandomization

Definition:
A family $\mathcal{H}$ of functions $[n] \to [k]$ is a **k-perfect** family of hash functions if for every $S \subseteq [n]$ with $|S| = k$, there is an $h \in \mathcal{H}$ such that $h(x) \neq h(y)$ for any $x, y \in S$, $x \neq y$

Theorem: There is a k-perfect family of functions $[n] \to [k]$ having size $2^{O(k)} \log n$ (and can be constructed in time polynomial in the size of the family).

Instead of trying $O(e^k)$ random colorings, we go through a k-perfect family $\mathcal{H}$ of functions $V(G) \to [k]$. If there is a solution S

- $\Rightarrow$ The vertices of S are colorful for at least one $h \in \mathcal{H}$
- $\Rightarrow$ Algorithm outputs "YES".
- $\Rightarrow$ k-Path can be solved in deterministic time $2^{O(k)} \cdot n^{O(1)}$

k-PATH

k-perfect family

$2^{O(k)} \log n$ functions

Finding a colorful path

Solvable in time $2^k \cdot n^{O(1)}$

Figure by Daniel Marx

4 - Iterative Compression

Iterative compression

General technique used for graph modification problems: Find a set S of k vertices/edges such that $G \setminus S$ has a particular property.

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We'll do it for Feedback Vertex Set:

- Goal: find a set S of at most k vertices such that $G \setminus S$ is a forest.
- Running time: $5^k \cdot n^{O(1)}$.

Recall that we have seen an algorithm runing in $(3k)^k n^{O(1)}$ using the branching method.

Best known algorithm: $2.7^k \cdot n^{O(1)}$, Li and Nederlof, 2020.

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Main idea: introduce vertices one by one and maintain a solution

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- Key: $S_{i-1} \cup \{v_i\}$ is a solution for G_i but potentially of size $k+1$.

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- Key: $S_{i-1} \cup \{v_i\}$ is a solution for G_i but potentially of size $k + 1$.

General idea of iterative compression

Main idea: introduce vertices one by one and maintain a solution

- Order the vertices: $v_1, v_2, \ldots, v_n$.
- Set $G_i = G[\{v_1, v_2, \ldots, v_i\}]$.
- Oberve that if G is a YES-instance, then each G_i is a YES-instance.
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So we can focus on the following problem:

Problem (FVS COMPRESSION)

Input: (G, k) and a vertex set S with $|S| \leq k+1$ and $G \setminus S$ is a forest.
Output: A FVS of size at most k (if it exists).

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Observation: if we can solve FVS COMPRESSION in time $f(k) \cdot n^c$, then we can solve FVS in time $f(k) \cdot n^{c+1}$.

So we can assume that we have a FVS of size $k + 1$ essentially for free

This FVS of size $k + 1$ gives us a lot of structure that will help us to find a smaller FVS.

Solve FVS COMPRESSION with Branching

Branching: 'guess' a set $X_S \subseteq S$ (2^{k+1} choices) that goes into the solution X.

- Delete X_S from G.
- Set $W = S - X_S$ and $\ell = |W| = k + 1 - |X_S|$
- It remains to solve the following:

Problem (Disjoint FVS)

Input: G, $W \subseteq V(G)$ such that $G \setminus W$ is a forest.
Output: a FVS X such that $|X| \leq |W| - 1$ and $X \cap W = \emptyset$.
Parameter: $|W| = \ell$.

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Computation: If $f(\ell) = c^\ell$, then $\sum_{\ell=0}^{k} \binom{k+1}{\ell} f(\ell) = (c+1)^{k+1}$

Goal: Disjoint FVS in $4^\ell \cdot n^{0(1)}$ ($\Rightarrow 5^k \cdot n^{0(1)}$ for FVS compression
$\Rightarrow 5^{k+1} \cdot n^{0(1)}$ for FVS).

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Case 3: u has at least 2 neighbors in W:

- if there exists $w_1, w_2 \in N(u) \cap W$ such that w_1 and w_2 are in the same connected component of W, then u must be in the solution. So we may delete u, and solve Disjoint FVS on $(G \setminus u, \ell - 1)$

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- otherwise, branch on u:
  - u is in the solution, solve $(G - u, \ell - 1)$, or
  - u is not in the solution, add u into W.
    Then the number of connected components of W decreseases, which make us happy.
    Also observe that at the beginning, W has at most ℓ connected components.

Solving Disjoint FVS

Branch on u:

- u is in the solution, solve $(G - u, \ell - 1)$, or
- u is not in the solution, add u into W.
  Then the number of connected components of W decreseases by 1.

Formally: for an instance $I = (G, W, \ell)$, define a potential function

$$\mu(I) = \ell + \text{number of connected components of } G[W]$$

At the beginning: $\mu(I) \leq 2\ell$.
In each branch, μ decreases strictly in both branches,
So the tree has depth at most 2ℓ, and thus has at most $2^{2\ell} = 4^\ell$ vertices.
So the running time is $4^\ell \cdot n^{O(1)}$.

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If $\mathcal{C} = \{\text{edgeless graphs}\} \Rightarrow$ Vertex Cover
If $\mathcal{C} = \{\text{forest graphs}\} \Rightarrow$ Feedback Vertex Set

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Disjoint $\mathcal{C}$-vertex deletion in FPT time $\Rightarrow$

$\mathcal{C}$-vertex deletion Compression in FPT time $\Rightarrow$

$\mathcal{C}$-vertex deletion in FPT time

If we are only interested to know if the problem is FPT or not, this is for free!