Graph Minor Theory and its algorithmic consequences
MPRI Graph Algorithms

Pierre Aboulker - pierreaboulker@gmail.com
1 - Sparsity of classes of graph defined by forbidding a (topological) minor
Containment relations

We define four operations on a graph $G$:

1. **Remove a vertex** $v$ (and all its incident edges), denoted $G \setminus v$.
2. **Remove an edge** $e$ (but not its end vertices), denoted $G \setminus e$.
3. **Contract an edge** $e = xy$, denoted $G / e$:
   (i.e. remove $x$ and $y$, add a new vertex $z$ with neighbourhood $N(z) = (N(x) \cup N(y)) \setminus \{z\}$ (no loops))
4. **Topological contraction** is a contraction of edge $e$ that has an endvertex of degree 2. Its inverse is the **subdivision operation** which consists in removing an edge $xy$, adding a new vertex $z$, and adding the edges $xz$ and $zy$. 

Definition

Let $G$ and $H$ be two graphs.

- $H$ is an **induced subgraph** of $G$ if $H$ is obtained from $G$ by the repeated use of 1.
- $H$ is a **subgraph** of $G$ if $H$ is obtained from $G$ by the repeated use of 1 and 2.
- $H$ is a **topological minor** of $G$ if $H$ is obtained from $G$ by the repeated use of rule 1, 2 and 4.
- $H$ is a **minor** of $G$ if $H$ is obtained from $G$ by the repeated use of rule 1, 2 and 3.
Containment relations

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1. **Remove a vertex** $v$ (and all its incident edges), denoted $G \setminus v$.

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4. **Topological contraction** is a contraction of edge $e$ that has an endvertex of degree 2. Its inverse is the **subdivision operation** which consists in removing an edge $xy$, adding a new vertex $z$, and adding the edges $xz$ and $zy$.

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Minors

Here is an equivalent definition for minors that is often useful:

\textbf{Definition (H-model)}

Let $G$ and $H$ be two graphs, and denote $V(H) = \{v_1, \ldots, v_p\}$. Then $H$ is a minor of $G$ if and only if there exists $p$ connected and disjoint subgraphs $G_1, \ldots, G_p$ of $G$ such that for every edge $v_i v_j$ of $H$, there exists an edge between $G_i$ and $G_j$. The graphs induced by $G_1, \ldots, G_p$ is called a \textbf{H-model} of $G$. 

Exercise 1

Show that the $(3 \times 3)$-grid contains $K_4$ as a minor.
Here is an equivalent definition for minors that is often useful:

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Let $G$ and $H$ be two graphs, and denote $V(H) = \{v_1, \ldots, v_p\}$. Then $H$ is a minor of $G$ if and only if there exists $p$ connected and disjoint subgraphs $G_1, \ldots, G_p$ of $G$ such that for every edge $v_i v_j$ of $H$, there exists an edge between $G_i$ and $G_j$. The graphs induced by $G_1, \ldots, G_p$ is called a **H-model** of $G$.

**Exercice 1**

Show that the $(3 \times 3)$-grid contains $K_4$ as a minor.
A topological minor is also called **subdivision**. Here is an equivalent definition of topological minor.

**Definition**

A graph $H$ is a topological minor of a graph $G$ if there exists a injective mapping $f$ from $V(H)$ to $V(G)$ such that for each edge $uv$ of $H$, there exists in $G$ a path $P_{uv}$ connecting $f(u)$ and $f(v)$ in $G$ with the property that all these paths are internally disjoint.

Another equivalent definition.

**Definition**

A graph $H$ is a topological minor of $G$ if $G$ contains a subdivision of $H$ as a subgraph.
Exercice 2

Show that the \((3 \times 3)\)-grid contains \(K_4\) as a minor.
Average degree and sparsity

Observation: \( \sum_{v \in V(G)} d(v) = 2|E(G)| \)

The average degree of a graph \( G \) is:

\[
\frac{\sum_{v \in V(G)} d(v)}{|V(G)|} = \frac{2|E(G)|}{|V(G)|} = \frac{2m}{n}
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A class of graphs \( C \) is sparse if there exists an integer \( k \) such that for every graph \( G \in C \), \( G \) has average degree at most \( k \).

Equivalently, there exists an integer \( k \) such that every \( G \in C \) has at most \( \frac{k}{2} \cdot |V(G)| \) edges.
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Examples:
- Planar graphs are sparse: average degree < 6.
- Complete bipartite graphs are not sparse, average degree = \( \frac{n}{2} \).
From average degree to minimum degree

**Observation**

A graph $G$ with average degree $2k$ contains a subgraph $H$ with minimum degree $k + 1$. 
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**Key**: prove that deleting a vertex of degree at most $k$ from a graph with average degree $2k$ can only increase the average degree.
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Proof: by induction on $|V(G)|$.

- If $G$ has minimum degree $k + 1$, we are done.
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Proof: by induction on $|V(G)|$.
- If $G$ has minimum degree $k + 1$, we are done.
- Let $u \in V(G)$ such that $d(v) \leq k$. Then:

\[
\frac{2|E(G - v)| + 2d(v)}{|V(G - v)| + 1} = \frac{2|E(G)|}{|V(G)|} \geq 2k \geq \frac{2d(v)}{1}
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- Since $\frac{a+b}{c+d} \geq \frac{c}{d} \Rightarrow \frac{a}{c} \geq \frac{a+b}{c+d}$, we have:

$$\frac{2|E(G - v)|}{|V(G - v)|} \geq \frac{2|E(G)|}{|V(G)|} \geq 2k$$
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- If $G$ has minimum degree $k + 1$, we are done.
- Let $u \in V(G)$ such that $d(v) \leq k$. Then:

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- Since $\frac{a+b}{c+d} \geq \frac{c}{d} \Rightarrow \frac{a}{c} \geq \frac{a+b}{c+d}$, we have:

$$\frac{2|E(G - v)|}{|V(G - v)|} \geq \frac{2|E(G)|}{|V(G)|} \geq 2k$$

- By induction, $G - v$ has a subgraph with minimum degree at least $k + 1$. 

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Graph Minor Theory and its algorithmic consequences
Theorem

A graph with average degree at least $2^{k-2}$ contains $K_k$ as a minor.
Sparse classes of graphs: $K_k$-minor free graphs

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**Proof**: by induction on $k$.

- Let $G$ be a graph of average degree at least $2^{k-2}$.
- Therefore $\frac{|E(G)|}{|V(G)|} \geq 2^{k-3}$. 

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Sparse classes of graphs: $K_k$-minor free graphs

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**Proof**: by induction on $k$.

- Let $G$ be a graph of average degree at least $2^{k-2}$.
- Therefore, $\frac{|E(G)|}{|V(G)|} \geq 2^{k-3}$.
- Let $H$ be minimal amongst all minors of $G$ such that $\frac{|E(G)|}{|V(G)|} \geq 2^{k-3}$.
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- Let $H$ be minimal amongst all minors of $G$ such that $\frac{|E(G)|}{|V(G)|} \geq 2^{k-3}$.
- It implies that when one contracts an edge in $H$, one must loose at least $2^{k-3}$ edges.
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- Hence, for any $xy \in E(H)$, $x$ and $y$ have at least $2^{k-3}$ common neighbours.
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- Hence, for any $xy \in E(H)$, $x$ and $y$ have at least $2^{k-3}$ common neighbours.
- Now, for every vertex $x$ of $H$, the graph induced by $N_H(x)$ has average degree at least $2^{k-3}$.
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- It implies that when one contracts an edge in $H$, one must lose at least $2^{k-3}$ edges.
- Hence, for any $xy \in E(H)$, $x$ and $y$ have at least $2^{k-3}$ common neighbours.
- Now, for every vertex $x$ of $H$, the graph induced by $N_H(x)$ has average degree at least $2^{k-3}$.
- So, by induction $N(x)$ contains $K_{k-1}$ as a minor, which yields with $x$ the desired $K_k$-minor.
Sparse classes of graphs: $K_k$-subdivision free graphs

Theorem

A graph of average degree $2^{\left(k^{\frac{1}{2}}\right)}$ contains $K_k$ as a topological minor.
Sparse classes of graphs: $K_k$-subdivision free graphs

**Theorem**

A graph of average degree $2^{\binom{k}{2}}$ contains $K_k$ as a topological minor.

**Proof:**

- We prove by induction on $m = k - 1, \ldots, \binom{k}{2}$ that every graph with average degree at least $2^m$ contains a subdivision of a graph with $k$ vertices and at least $m$ edges.
Sparse classes of graphs: $K_k$-subdivision free graphs

**Theorem**

A graph of average degree $2^{k/2}$ contains $K_k$ as a topological minor.

**Proof:**

- We prove by induction on $m = k - 1, \ldots, \binom{k}{2}$ that every graph with average degree at least $2^m$ contains a subdivision of a graph with $k$ vertices and at least $m$ edges.
- For $m = k - 1$, we may choose a vertex $v$ of degree $2^{k-1} \geq k - 1$. 
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- Now assume the result is valid for all $m' < m \leq \binom{k}{2}$.
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- Now assume the result is valid for all $m' < m \leq \binom{k}{2}$.
- So let $G$ be a graph with average degree at least $2^m$. 

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- So let $G$ be a graph with average degree at least $2^m$.
- Choose a maximal connected subgraph $U$ of $G$ such that the contraction of $U$ result in a graph $H$ with average degree at least $2^m$. 
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- Such a $U$ exists since it can be reduced to a single vertex.
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- Now assume the result is valid for all $m' < m \leq \binom{k}{2}$.
- So let $G$ be a graph with average degree at least $2^m$.
- Choose a maximal connected subgraph $U$ of $G$ such that the contraction of $U$ results in a graph $H$ with average degree at least $2^m$.
- Such a $U$ exists since it can be reduced to a single vertex.
- Prove that the graph induced by $N_H(u)$ has minimum degree at least $2^{m-1}$.
Sparse classes of graphs: $K_k$-subdivision free graphs

**Theorem**

A graph of average degree $2^{k\choose 2}$ contains $K_k$ as a topological minor.

**Proof:**

- We prove by induction on $m = k - 1, \ldots, {k\choose 2}$ that every graph with average degree at least $2^m$ contains a subdivision of a graph with $k$ vertices and at least $m$ edges.
- For $m = k - 1$, we may choose a vertex $v$ of degree $2^{k-1} \geq k - 1$.
- The vertex $v$ together with $k - 1$ of its neighbors induces a graph with $k$ vertices and at least $k - 1$ edges.
- Now assume the result is valid for all $m' < m \leq {k\choose 2}$.
- So let $G$ be a graph with average degree at least $2^m$.
- Choose a maximal connected subgraph $U$ of $G$ such that the contraction of $U$ results in a graph $H$ with average degree at least $2^m$.
- Such a $U$ exists since it can be reduced to a single vertex.
- Prove that the graph induced by $N_H(u)$ has minimum degree at least $2^{m-1}$.
- So $N_H(u)$ contains a subdivision of a graph on $k$ vertices and $m - 1$ edges.
Sparse classes of graphs: $K_k$-subdivision free graphs

Theorem

A graph of average degree $2^\binom{k}{2}$ contains $K_k$ as a topological minor.

Proof:

- We prove by induction on $m = k - 1, \ldots, \binom{k}{2}$ that every graph with average degree at least $2^m$ contains a subdivision of a graph with $k$ vertices and at least $m$ edges.
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- So let $G$ be a graph with average degree at least $2^m$.
- Choose a maximal connected subgraph $U$ of $G$ such that the contraction of $U$ result in a graph $H$ with average degree at least $2^m$.
- Such a $U$ exists since it can be reduced to a single vertex.
- Prove that the graph induced by $N_H(u)$ has minimum degree at least $2^{m-1}$.
- So $N_H(u)$ contains a subdivision of a graph on $k$ vertices and $m - 1$ edges.
- We conclude that $G$ must contains a subdivision of a graph with $k$ vertices and $m$ edges (more details on board).
The two following bounds are tight.

**Theorem** (Kostochka, 1982)

There is a constant $c$ such that every graph with average degree $c \cdot k \sqrt{\log k}$ contains $K_k$ as a minor.

**Theorem** (Bollobás and Thomasson, 1998, Komlós and Szemerédi, 1996)

There is a constant $c$ such that every graph with average degree at least $c \cdot k^2$ contains $K_k$ as a topological minor.
### Problem (k disjoint paths problem)

**Input**: A graph $G$, an integer $k$ and two subsets of vertices $A$ and $B$

**Output**: TRUE if there exists $k$ vertex disjoint paths from $A$ to $B$. 

---

**Theorem (Menger, 1927)**

Let $x, y \subseteq V(G)$. The minimum number of vertices separating $x$ from $y$ equal the maximum number of internally disjoint $x$−$y$-paths.

See Section 3.3 of Diestel Book.
A Classic Connectivity Problem

Consider the following problem of connectivity.

**Problem** (*k* disjoint paths problem)

**Input**: A graph *G*, an integer *k* and two subsets of vertices *A* and *B*

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CLASSIC: Can be solved in time $O((k|E(G)|))$ using **Ford-Fulkerson Algorithm**.
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CLASSIC: Can be solved in time $O((k|E(G)|))$ using Ford-Fulkerson Algorithm.

From a structural point of view, the maximum number of paths linking $A$ and $B$ corresponds to a minimum vertex cut separating $A$ and $B$ and is a classical result of Menger.

**Theorem** (Menger, 1927)

Let $x, y \subseteq V(G)$. The minimum number of vertices separating $x$ from $y$ equal the maximum number of internally disjoint disjoint $x – y$-paths.

See Section 3.3 of Diestel Book.
Exercise on connectivity

Let $G$ be a graph, $x \in V(G)$ and $Y \subseteq V(G) \setminus \{x\}$. A family of $k$ internally disjoint $(x, Y)$-paths whose terminal vertices are distinct is referred to as a $k$-fan from $x$ to $Y$.

Exercice 3

Let $G$ be a $k$-connected graph.

1. Let $x$ be a vertex of $G$, and let $Y \subseteq V \setminus \{x\}$ be a set of at least $k$ vertices of $G$. Then there exists a $k$-fan in $G$ from $x$ to $Y$. (This property is known as the Fan Lemma).

2. Let $S$ be a set of $k$ vertices in a $k$-connected graph $G$, where $k \geq 2$. Then there is a cycle in $G$ which includes all the vertices of $S$. 
2 - Containement relations
Notations and containment relations

Four containment relations:

- $H$ induced subgraph of $G$: $H \subseteq_i G$
- $H$ subgraph of $G$: $H \subseteq G$
- $H$ topological minor of $G$: $H \preceq_t G$
- $H$ minor of $G$: $H \preceq_m G$
Classes of graph defined by forbidden configurations

For a set $\mathcal{F}$ of graphs, let $\text{Forb}_{\preceq}(\mathcal{F}) = \{ G : \forall F \in \mathcal{F}, F \not\preceq G \}$ i.e. the class of graphs not containing any graphs of $\mathcal{F}$ under $\preceq$-relation. We say such graph are $\mathcal{F}$-$\preceq$-free.
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Kuratowski’s Theorem, 1930

A graph is planar if and only if it does contain $K_5$ nor $K_{3,3}$ as a topological minor.
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Kuratowski’s Theorem (equivalent formulation, Wagner 1937)

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For a set $\mathcal{F}$ of graphs, let $\text{Forb}_{\preceq}(\mathcal{F}) = \{ G : \forall F \in \mathcal{F}, F \not\preceq G \}$ i.e. the class of graphs not containing any graphs of $\mathcal{F}$ under $\preceq$-relation. We say such graph are $\mathcal{F}$-$\preceq$-free.

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A graph is planar if and only if it does contain $K_5$ nor $K_{3,3}$ as a topological minor.

Kuratowski’s Theorem (equivalent formulation, Wagner 1937)

A graph is planar if and only if it does contain $K_5$ nor $K_{3,3}$ as a topological minor.

Planar Graphs = $\text{Forb}_{\preceq_t}(K_5, K_{3,3}) = \text{Forb}_{\preceq_m}(K_5, K_{3,3})$. 
Other examples

Example:

- $\text{Forb}_{\leq t}(K_5, K_{3,3}) = ??$
- $\text{Forb}_{\subseteq}(C_3, C_5, C_7, ...) = ??$
- $\text{Forb}_{\subseteq}(K_{1,2}) = ??$
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- $\text{Forb}_{\leq t}(K_5, K_{3,3}) = \text{planar graphs}$
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- $\text{Forb}_{\subseteq_i}(K_{1,2}) = \text{graphs whose connected components are cliques}$.
Class of graphs closed under $\preceq$

We say that a class of graphs $C$ is **minor-closed** (or closed under taking minor) if for every $G \in C$, if $H \preceq_m G$, then $H \in G$.

Equivalently, if $G \in C$, you can safely delete vertices, delete edges or contract edges.

Examples:

- The class of planar graphs is closed under taking minor (and thus closed under taking topological minor, subgraph and induced subgraph).
- The class of bipartite graphs is closed under taking subgraph, but not under taking topological minor.
- The class of all graphs whose connected components are cliques is under taking induce subgraphs, but not under taking subgraph.
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Obstructions

Let $C$ be a minor-closed class of graphs. The minor minimal graph not in $C$ are called the minor-obstruction of $C$.
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- $F \not\in C$ and
- for every $H \preceq_m F$, $H \in C$

- $K_5$ is a minor-obstruction for planar graphs since $K_5$ is not planar, but every proper minor of $K_5$ is.
- $K_6$ is not a minor obstruction for planar graphs since $K_5 \preceq_m K_6$ and $K_5$ is not planar.
Theorem

Let $\mathcal{C}$ be a class of graphs. Then $\mathcal{C}$ is closed under taking minor if and only if there exists a (possibly infinite) set of graphs $\mathcal{F}$ such that $\mathcal{C} = \text{Forb}_{\leq m}(\mathcal{F})$.
Let $\mathcal{C}$ be a class of graphs.

Then $\mathcal{C}$ is closed under taking minor if and only if there exists a (possibly infinite) set of graphs $\mathcal{F}$ such that $\mathcal{C} = \text{Forb}_{\leq m}(\mathcal{F})$.

Proof:

- If $\mathcal{C} = \text{Forb}_{\leq m}(\mathcal{F})$, then it is clearly closed under taking minor.

- Assume $\mathcal{C}$ is a class of graphs closed under taking minor.
- Let $\mathcal{F}$ be the set of graphs $H$ such that $H \notin \mathcal{C}$, but every proper minor $H$ belongs to $\mathcal{C}$.
  
  In other words $\mathcal{F}$ is the set of minor obstruction of $\mathcal{C}$.
- Then $\mathcal{C} = \text{Forb}_{\leq m}(\mathcal{F})$. 
3 - Detection of minors and topological minors
The disjoint path problem

Problem \((k\text{-disjoint rooted paths problem})\)

Input: A graph \(G\), an integer \(k\), and two subsets of vertices \(S = \{s_1, s_2, \ldots, s_k\}\) and \(T = \{t_1, t_2, \ldots, t_k\}\)

Output: TRUE iff there exists disjoint paths \(P_1, P_2, \ldots, P_k\), such that \(P_i\) is a path from \(s_i\) to \(t_i\).
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- With $k \geq 2$ part of the input, this problem is NP-complete, even restricted to the class of planar graphs.

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**Theorem** (Robertson-Seymour, 1995 (XIII))

The $k$-disjoint rooted path problem can be solved in time $O\left(f(k)n^3\right)$. In other words it is FPT.

(Improved to quadratic time by Kawarabayashi, Kobashi and Reed, 2012)
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The $k$-disjoint rooted path problem can be solved in time $O((f(k) \cdot n^3))$. In other words it is FPT.

(Improved to quadratic time by Kawarabayashi, Kobashi and Reed, 2012)
Problem (Topological $H$-minor detection)

**Input**: A graph $G$ and a graph $H$.

**Output**: TRUE if $H$ is a topological minor of $G$, FALSE otherwise.

- With $H$ part of the input: NP-complete.
- What about the complexity parametrized by the size of $H$?

Set $k = |E(H)|$.

Theorem (Robertson and Seymour, 1995)
Topological $H$-minor detection can be solved in time $f(k) \cdot n^k$, so it is in $XP$.

Theorem (Grohe, Kawabarayashi, Marx, and Wollan, 2010)
Topological $H$-minor detection can be solved in time $O(f(k) \cdot n^3)$, so it is $FPT$. 

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**Theorem (Robertson and Seymour, 1995)**

Let $H$ be a fixed graph with $k$ edges. One can decide whether $H$ is a topological minor of a given graph $G$ in time $O(f(k)n^k)$. In other words, topological minor detection is in XP.

**Sketch proof:**

- Let $f : V(H) \to V(G)$ be an injection.
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Topological Minor Detection II

**Theorem** (Robertson and Seymour, 1995)

Let $H$ be a fixed graph with $k$ edges. One can decide whether $H$ is a topological minor of a given graph $G$ in time $O(f(k)n^k)$. 

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**Sketch proof:**

- Let $f : V(H) \rightarrow V(G)$ be an injection.
- Observe that there is $\binom{n}{|V(H)|}$ such functions.
- We want to decide if there exists disjoint paths in $G$ between the $f(v)$ corresponding to edges of $H$. 

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Graph Minor Theory and its algorithmic consequences
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- To do that, we replace each vertex $f(v)$ by $d_H(v)$ copies of $f(v)$ (having the same neighbours).
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- To do that, we replace each vertex $f(v)$ by $d_H(v)$ copies of $f(v)$ (having the same neighbours).
- Now, for $k = |E(H)|$, solving the $k$-Rooted Disjoint Path Problem for these sources clearly solves the desired question.
In particular, the previous theorem implies that any class of graphs defined by forbidding a FINITE family of graphs as topological minors is polynomially testable.

In other words if $C = \text{Forb}_{\leq t}(\mathcal{F})$ where $\mathcal{F}$ is a finite set of graphs, then we can decide in polynomial time if a graph $G$ belongs to $C$.

Example of such class?
Consequences

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Example of such class?

\textbf{Kuratowski’s Theorem, 1930}

A graph is planar if and only if it does contain \( K_5 \) nor \( K_{3,3} \) as a topological minor.

Note that one does not need to solve \( k \) rooted paths problem to get polytime algorithms for recognizing planar graphs (there exist even linear algorithms to do that).
Minors Vs Topological Minors

- By definition: $H$ topological minor of $G \Rightarrow H$ minor of $G$
- **Exercise:** converse not true: find a pair of graphs $G$ and $H$ such that $H$ is a minor of $G$ but $H$ is not a topological minor of $G$. 

When $H$ is subcubic (maximum degree at most 3), this is nevertheless true.
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When \( H \) is subcubic (maximum degree at most 3), this is nevertheless true.
**Theorem**

Let $H$ be a graph with maximum degree at most 3. Let $G$ be a graph. Then $G$ contains $H$ as a minor if and only if it contains $H$ as a topological minor.

**Key idea:** if $H$ has maximum degree 3, the $G_i$ of an $H$-model are subdivisions of $K_{1,3}$.

**Sketch proof:**

- Assume $H$ is a minor of $G$. 

Minors Vs Topological Minors

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Sketch proof:

- Assume $H$ is a minor of $G$.
- Let $G'$ be a minimal subgraph of $G$ such that $H$ is a minor of $G'$ (i.e. $|V(G')| + |E(G')|$ is minimized).
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- By minimality of $G$, each $G_i$ is a tree with at most 3 leaves.
- Each such tree must be a subdivision of $K_{1,2}$ or $K_{1,3}$, so we get the topological minor.
A similar argument proves this more general result (proof on the next slide).

**Theorem**

For every graph $F$, there exists a finite family of graphs $\mathcal{F}$ such that:

$G$ contains $F$ as a minor if and only if it contains some graph in $\mathcal{F}$ as a topological minor.

In other words: $\text{Forb}\preceq_m(F) = \text{Forb}\preceq_t(\mathcal{F})$. 

Proof: We start the proof exactly as in the previous result, and by again choosing minimal $G_i$, we now get for each $G_i$ a tree with at most $\Delta(H)$ leaves and no vertex of degree 2 (just shrink edges with an extremity of degree 2). There is finitely many such trees (why?). So by replacing the vertices of $H$ by these trees in all possible ways, we obtain a finite collection of graphs $H$ with the desired properties.
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Minor detection

**Theorem**

For every graph $F$, there exists a finite family of graphs $\mathcal{F}$ such that:

$G$ contains $F$ as a minor if and only if it contains some graph in $\mathcal{F}$ as a topological minor. In other words: $\text{Forb}_{m}(F) = \text{Forb}_{t}(\mathcal{F})$.

This result combined with the theorem on topological minor detection implies:
Minor detection

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For every graph $F$, there exists a finite family of graphs $\mathcal{F}$ such that: 
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**Theorem**

Let $H$ be a fixed graph. There is an FPT time algorithm parametrized by the size of $H$ to decide whether $H$ is a minor of a given graph $G$. 

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Theorem

Let $H$ be a fixed graph. There is an FPT time algorithm parametrized by the size of $H$ to decide whether $H$ is a \textit{minor} of a given graph $G$.

Actually, Robertson and Seymour prove that it is FPT:

Theorem

Given a graph $G$ and a graph $H$, one can decide in time $f(H) \cdot n^3$ whether $H$ is a minor of $G$ or not.
Corollary

If $C$ is a class of graphs defined by forbidding finitely many minors, then there exists a polynomial algorithm to decide whether an input graph belongs to $C$. 
Some classes of graphs closed under taking minor

For each of these problems, the set of YES-instances is closed under taking minor.

- **$k$-Vertex Cover**
  
  **Given**: A graph $G$ and an integer $k$,
  
  **Question**: does $G$ has a set $S$ of at most $k$ vertices such that $G \setminus S$ is a edgeless.

- **$k$-Feedback vertex set**
  
  **Given**: A graph $G$ and an integer $k$,
  
  **Question**: does $G$ has a set $S$ of at most $k$ vertices such that $G \setminus S$ is a forest.

- **$k$-leaf Spanning Tree**
  
  **Given**: A graph $G$ and an integer $k$,
  
  **Question**: does $G$ contain a spanning tree $T$ with at least $k$ leaves.

If by chance they were defined by forbidding finitely many minors, we would have a proof that these problems are FPT.

Question: What are the classes of graphs defined by finitely many forbidden minors?
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**Question**

What are the classes of graphs defined by finitely many forbidden minors?
4 - Well Quasi Orders and Wagner Conjecture
Our goal in this last chapter is a single theorem, one which dwarfs any other result in graph theory and may doubtless be counted among the deepest theorems that mathematics has to offer: in every infinite set of graphs there are two such that one is a minor of the other. This graph minor theorem (or minor theorem for short), inconspicuous though it may look at first glance, has made a fundamental impact both outside graph theory and within. Its proof, due to Neil Robertson and Paul Seymour, takes well over 500 pages.

Reinhart Diestel
In this section, we first give a short introduction to Wagner Conjecture by giving equivalent statements.

We will try to understand some of the ideas behind the proof of Wagner’s conjecture by proving similar but (much) easier results.

We introduce the notion of **well quasi order** that gives an equivalent way to state Wagner Conjecture.

We prove a theorem due to Kruskal saying that trees are well quasi ordered for the minor relation.

In the next section, we will explain through the notion of treewidth why Kruskal Theorem and its proof is central in Robertson and Seymour’s proof.
Conjecture (Wagner, 1937)
In every infinite set of graphs, there are two such that one is the minor of another
Wagner’s conjecture and forbidden minors

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In every infinite set of graphs, there are two such that one is the minor of another.

- Let $C$ be a class of graphs closed under taking minor.
- So there exists a set of graphs $F$ such that $C = \text{Forb}(F)$.
- Choose such an $F$ with minimum number of graphs.
- Then there are no two graphs of $F$ such that one is the minor of another.
Wagner’s conjecture and forbidden minors

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**Conjecture (Wagner, 1937)**
Every class of graphs closed under taking minor can be defined by a finite set of forbidden minor.
Algorithmic consequences of Wagner Conjecture

If Wagner Conjecture is true:

**Theorem**

*Each of these problems is FPT.*

- **$k$-Vertex Cover**
  - **Given**: A graph $G$ and an integer $k$,
  - **Question**: does $G$ has a set $S$ of at most $k$ vertices such that $G \setminus S$ is a edgeless.

- **$k$-Feedback vertex set**
  - **Given**: A graph $G$ and an integer $k$,
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- **$k$-leaf Spanning Tree**
  - **Given**: A graph $G$ and an integer $k$,
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  **Given:** A graph $G$ and an integer $k$,
  
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**Proof:**

- for each of these problems, the set of YES-instance is closed under taking minors
- so, by the graph minor theorem, they form a class of graphs defined by forbidding finitely many graphs as minors.
- so we can recognise them in time $f(k) \cdot n^3$

**Fun fact:** we know the list of forbidden minor is finite, but we don’t know which ones it is. So all that remains very theoretical :)
The graph minor theorem

In a monumental work:

- more than 500 pages of proof,

Robertson and Seymour proved Wagner’s conjecture.
The graph minor theorem

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**Theorem** *(Graph Minor Theorem, Robertson and Seymour, XX (2004))*

*Any minor closed class of graphs is defined by a finite list of forbidden minors*
A quasi-order is a binary relation that is both reflexive \((x \preceq x)\) and transitive \((x \preceq y \text{ and } y \preceq z \Rightarrow x \preceq z)\).
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Minor containment defines a quasi-ordering on graphs:
- If $G_1 \preceq_m G_2$ and $G_2 \preceq_m G_3$, then $G_1 \preceq_m G_3$. 
Well Quasi Order

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Wagner’s Conjecture says there is no infinite antichain for the minor relation.
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**Definition (Well Quasi Order)**

A partial order \(\preceq\) defined on a set \(X\) is a well quasi order (WQO) if there is no infinite strictly decreasing sequence and no infinite antichain.

Wagner’s conjecture is equivalent to say that the class of all graphs with the minor relation is WQO.
Dealing with wqo: a first tool

**Proposition**

Let $(X, \preceq)$ be a quasi ordered set and $(x_i)_{i \in \mathbb{N}}$ be any sequence. Then this sequence has an infinite subsequence that is either

- increasing or
- strictly decreasing or
- an antichain.

This is equivalent to Ramsey Theorem for infinite graphs.

Proof on next slide.
Let \((x_i)_{i \in \mathbb{N}}\) be any sequence. Start with \(x_1\), and consider

- \(A_1 = \{i, \ i > 1 \text{ and } x_1 \succ x_i\}\)
- \(B_1 = \{i, \ i > 1 \text{ and } x_1 \preceq x_i\}\)
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If \(A_1\) is infinite we say that \(x_1\) is of type \(A\) and delete all elements that are not in \(A_1\). If not, but \(B_1\) is infinite, say that \(x_1\) is of type \(B\) and delete all elements that are not in \(B_1\). Finally in the last case, say that \(x_1\) is of type \(C\) and delete all vertices not in \(C_1\).
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Up to extracting a subsequence and renaming, we can assume no elements were deleted, so that the \(x_i\) for \(i \geq 2\) are all in \(A_1\), or all in \(B_1\), or all in \(C_1\).
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We do this sequentially for \(x_2\), then \(x_3\), ... At each step, we define \(A_i\), \(B_i\), \(C_i\) as
- \(A_i = \{i, \; k > i \text{ and } x_i \succ x_k\}\)
- \(B_i = \{i, \; k > i \text{ and } x_i \preceq x_k\}\)
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and at each step we define the type of \(x_i\) to be one of \(A\), \(B\), \(C\) depending on which is infinite. Then we extract by keeping only the elements in the infinite set.
Let \((x_i)_{i \in \mathbb{N}}\) be any sequence. Start with \(x_1\), and consider

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Up to extracting a subsequence and renaming, we can assume no elements were deleted, so that the \(x_i\) for \(i \geq 2\) are all in \(A_1\), or all in \(B_1\), or all in \(C_1\).

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and at each step we define the type of \(x_i\) to be one of \(A, B, C\) depending on which is infinite. Then we extract by keeping only the elements in the infinite set.

Eventually we have a type for each element of the sequence (which is in fact a subsequence of the original sequence). Now there must be a type with infinitely number of elements and to each type clearly corresponds one of the three possible type of infinite subsequence.
Dealing with wqo: a first tool

**Corollary**

Let \((X, \preceq)\) be a quasi ordered set. The three assertions are equivalent:

1. \((X, \preceq)\) is a wqo (no infinite strictly decreasing sequence and no infinite antichain).
2. from every sequence \((x_i)_{i \in \mathbb{N}}\) one can extract an infinite increasing subsequence.
3. from every sequence \((x_i)_{i \in \mathbb{N}}\) one can extract \(i < j\) such that \(x_i \preceq x_j\).

This will be useful:

- in order to prove that a given quasi order is wqo, we will only prove the third statement,
- but when we use the fact that a quasi order is wqo (for example in a proof by induction), we can use the second statement which is (in appearance) much stronger.
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- in order to prove that a given quasi order is wqo, we will only prove the third statement,
- but when we use the fact that a quasi order is wqo (for example in a proof by induction), we can use the second statement which is (in appearance) much stronger.
Second tool: extending a partial order

Let \((X, \preceq)\) be a partial order and \((x_n)_{n \in \mathbb{N}}\) an infinite sequence. A pair \((x_i, x_j)\) is a good pair if \(i < j\) and \(x_i \preceq x_j\). The sequence \((x_n)_{n \in \mathbb{N}}\) if a good sequence if it has a good pair. It is a bad sequence otherwise.

For finite subsets \(A, B \subset X\), write \(A \preceq B\) if there is an injective mapping \(f: A \rightarrow B\) such that \(a \leq f(a)\) for all \(a \in A\). This naturally extends \(\preceq\) to a quasi-order on \([X]_{\omega}\), the set of all finite subsets of \(X\).

Lemma
If \(X\) is WQO, then so is \([X]_{\omega}\).

Proof [see Diestel, Lemma 12.1.3]:
Main idea: start with a "minimum" infinite antichain.
Second tool: extending a partial order

Let \((X, \preceq)\) be a partial order and \((x_n)_{n \in \mathbb{N}}\) an infinite sequence. A pair \((x_i, x_j)\) is a **good pair** if \(i < j\) and \(x_i \preceq x_j\). The sequence \((x_n)_{n \in \mathbb{N}}\) is a **good sequence** if it has a good pair. It is a **bad sequence** otherwise.

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**Lemma**

*If $X$ is WQO, then so is $[X]^\omega$.***

**Proof [see Diestel, Lemma 12.1.3]:** Main idea: start with a “minimum” infinite antichain.
Proof sketch

- Assume for contradiction that $[X]^w$ has an infinite bad sequence.
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- Construct a "minimal" bad sequence $(A_n)_{n \in \mathbb{N}}$ as follows:
Proof sketch

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- Construct a "minimal" bad sequence \((A_n)_{n \in \mathbb{N}}\) as follows:

- Assume inductively that \(A_i\) has been defined for every \(i < n\), and that there exists a infinite bad sequence in \([X]^w\) starting with \(A_0, \ldots, A_{n-1}\).
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  - Choose $A_n$ such that some infinite antichain starts with $(A_0, \ldots, A_{n-1}, A_n)$ and $|A_n|$ is minimum with this property.
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  - For each $n$, pick en element $a_n \in A_n$, and set $B_n = A_n \setminus \{a_n\}$.
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  - Since $X$ is wqo, $(a_n)_{n \in \mathbb{N}}$ has an infinite increasing subsequence $(a_{n_i})_{i \in \mathbb{N}}$.
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- Since \(X\) is wqo, \((a_n)_{n \in \mathbb{N}}\) has an infinite increasing subsequence \((a_{n_i})_{i \in \mathbb{N}}\).

- Now look at the sequence \((A_0, A_1, \ldots, A_{n_0-1}, B_{n_0}, B_{n_1}, \ldots)\).
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- By the minimal choice of \(A_n\), it is not a bad sequence, i.e. it has a good pair.
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- Now look at the sequence $(A_0, A_1, \ldots, A_{n_0-1}, B_{n_0}, B_{n_1}, \ldots)$.

- By the the minimal choice of $A_n$, it is not a bad sequence, i.e. it has a good pair.

- Conclude.
The graph minor theorem for trees

**Theorem (Kruskal 1960)**

The finite trees are WQO by the topological minor relation, i.e. for every infinite sequence of trees $T_0, T_1, \ldots$, there exists $i < j$ such that $T_i \not\preceq T_j$. 
Proof

Let $T_1$ and $T_2$ be two rooted trees. We say that $T_1 \leq T_2$ if there is a subdivision of $T_1$ that can be embedded into $T_2$ in such a way that the tree order of $T_1$ is preserved.

We are going to prove that the set of trees is wqo by $\leq$ (which is slightly stronger than the announced result).

*Fig. 12.2.1.* An embedding of $T$ in $T'$ showing that $T \leq T'$

Reinhart Diestel
Sketch of proof:

- Choose a "minimal" bad sequence \((T_n)_{n \in \mathbb{N}}\) as in the previous Lemma.

For a full proof see Diestel, Chapter 12.2.
Sketch of proof:

- Choose a "minimal" bad sequence \((T_n)_{n \in \mathbb{N}}\) as in the previous Lemma.
- For \(n \in \mathbb{N}\), name \(r_n\) the root of \(T_n\).

For a full proof see Diestel, Chapter 12.2.
Sketch of proof:

- Choose a "minimal" bad sequence \((T_n)_{n \in \mathbb{N}}\) as in the previous Lemma.
- For \(n \in \mathbb{N}\), name \(r_n\) the root of \(T_n\).
- For \(n \in \mathbb{N}\), set \(A_n\) to be the set of connected components of \(T_n \setminus r_n\) made into rooted trees by choosing the neighbors of \(r_n\) as the roots.

For a full proof see Diestel, Chapter 12.2.
Sketch of proof:

1. Choose a "minimal" bad sequence \((T_n)_{n \in \mathbb{N}}\) as in the previous Lemma.
2. For \(n \in \mathbb{N}\), name \(r_n\) the root of \(T_n\).
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4. Set \(A = \bigcup_{n \in \mathbb{N}} A_n\).

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Sketch of proof:
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- We are going to prove that \(A\) is WQO:

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  - Let \((T^k)_{k \in \mathbb{N}}\) be an infinite sequence of trees from \(A\).

For a full proof see Diestel, Chapter 12.2.
Sketch of proof:

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  - Let \((T^k)_{k \in \mathbb{N}}\) be an infinite sequence of trees from \(A\).
  - For every \(k \in \mathbb{N}\), choose \(n(k)\) such that \(T_k \in A_{n(k)}\).

For a full proof see Diestel, Chapter 12.2.
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  - Pick a \(k\) with smallest \(n(k)\). Consider the following sequence:
    \[T_0, T_1, \ldots, T_{n(k)-1}, T^k, T^{k+1}, \ldots\]

For a full proof see Diestel, Chapter 12.2.
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  - Pick a $k$ with smallest $n(k)$. Consider the following sequence:
    
    $T_0, T_1, \ldots, T_{n(k)-1}, T^k, T^{k+1}, \ldots$

  - Let $(T, T')$ be a good pair of this sequence (it exists by minimality of $(T_n)_{n \in \mathbb{N}}$).

For a full proof see Diestel, Chapter 12.2.
Sketch of proof:

- Choose a "minimal" bad sequence \((T_n)_{n \in \mathbb{N}}\) as in the previous Lemma.
- For \(n \in \mathbb{N}\), name \(r_n\) the root of \(T_n\).
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  - Show that \((T, T')\) cannot be a good pair (3 cases)
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- Choose a "minimal" bad sequence \( (T_n)_{n \in \mathbb{N}} \) as in the previous Lemma.
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  - So \( A \) is WQO.
- Consider the sequence \( (A_n)_{n \in \mathbb{N}} \).
- It lives in \( [A]^\omega \) which is WQO by the previous Lemma.
- So there exists \( i < j \) such that \( A_i \preceq A_j \).
- Conclude.

For a full proof see Diestel, Chapter 12.2.
Exercises on well quasi ordering

**Exercice 4**

For each of these, say if it is a wqo.

- $(\mathbb{N}, \leq)$.
- $(\mathbb{R}, \leq)$.
- $(\mathbb{N}^2, \leq)$ where $(x, y) \leq (x', y')$ iff $(x \leq x'$ and $y \leq y')$.
- $(\mathcal{G}, \subseteq_i)$ where $\mathcal{G}$ is the class of all graphs (recall that $H \subseteq_i G$ means $H$ is an induce subgraph of $G$).
- $(\mathcal{G}, \subseteq)$ where $\mathcal{G}$ is the class of all graphs (recall that $H \subseteq G$ means $H$ is a subgraph of $G$).
- Finite trees ordered by subgraph relation.
- $(\mathcal{G}, \preceq_t)$ where $G \preceq_t H$ if $G$ is a topological minor of $H$.
5 - TreeWidth
We proved (Kruskal Theorem) that Wagner conjecture holds for trees. So maybe we can use the same ideas to prove Wagner conjecture for graphs that look like trees. So we would like a notion that measure how much a graph looks like a tree.
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Moreover, since it is easy to compute on trees, it should be easy to compute on graphs that “looks like” trees.
Treewidth

- We proved (Kruskal Theorem) that Wagner conjecture holds for trees. So maybe we can use the same ideas to prove Wagner conjecture for graphs that look like trees. So we would like a notion that **measure how much a graph looks like a tree**.

- Moreover, since it is easy to compute on trees, it should be easy to compute on graphs that “looks like” trees.

- This is achieved by the notion of **Treewidth** which is a notion of “treelikeness”. In other words it measures how much a graph looks like a tree.

  You can understand it like this: if a graph has treewidth 5, then it is at distance 5 from being a tree. Or it is a tree of width 5.
We proved (Kruskal Theorem) that Wagner conjecture holds for trees. So maybe we can use the same ideas to prove Wagner conjecture for graphs that look like trees. So we would like a notion that measure how much a graph looks like a tree.

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The goal of this section is to introduce treewidth, tree decomposition, and to extend Kruskal Theorem to graphs with bounded treewidth (no proof), and finally to look at graphs of treewidth at most 3.
Detailed content of the section

- Three equivalent definitions of three decomposition and treewidth.

- Helly Property for trees.

- As a consequence: \( \omega(G) - 1 \leq tw(G) \).

- Definition of irreducible decomposition.

- As a consequence:
  - \( \delta(G) \leq tw(G) \) and
  - A tree decomposition is on at most \( V(G) \) nodes.

- Separation property of tree decomposition (fundamental).
How much a graph look like a tree

How can we measure how much a graph looks like a tree?
How much a graph look like a tree

How can we measure how much a graph looks like a tree?

- Number of cycles is bounded

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How much a graph look like a tree

How can we measure how much a graph looks like a tree?

1. Number of cycles is bounded

2. It has bounded Feedback Vertex Set.
How much a graph look like a tree

How can we measure how much a graph looks like a tree?

1. Number of cycles is bounded
2. It has bounded Feedback Vertex Set.
Definition of a tree decomposition and of treewidth

Let $G$ be a graph. A **tree decomposition** of $G$ is a pair $(T, B)$, where $T$ is a tree and $B = (B_t)_{t \in V(T)}$ a collection of subsets of $V(G)$ indexed on $V(T)$ satisfying:

1. **(T1)** For every $v \in V(G)$, there exists $t \in V(T)$ such that $v \in B_t$
   - *every vertex is in some bag*

2. **(T2)** For every edge $uv \in E(G)$, there exists $t \in V(T)$ such that $u, v \in B_t$
   - *every edge is in a bag*

3. **(T3)** For every $u \in V(G)$, $T_u = \{t \in V(T) \mid u \in B_t\}$ induces a connected subgraph of $T$.

The width of a tree decomposition is $\max_{t \in V(T)} (|B_t| - 1)$, where $|B_t|$ is the size of the largest bag minus 1.

The tree width of a graph $G$, denoted $\text{tw}(G)$, is the minimum width of a tree decomposition of $G$. 
Definition of a tree decomposition and of treewidth

Let $G$ be a graph. A tree decomposition of $G$ is a pair $(T, \mathcal{B})$, where $T$ is a tree and $\mathcal{B} = (B_t)_{t \in V(T)}$ a collection of subsets of $V(G)$ indexed on $V(T)$ satisfying:

1. For every $v \in V(G)$, there exists $t \in V(T)$ such that $v \in B_t$ - every vertex is in some bag -

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3. For every $u \in V(G)$, $T_u = \{t \in V(T) : u \in B_t\}$ induces a connected subgraph of $T$.

The width of a tree decomposition is $\max_{t \in V(T)}(|B_t| - 1)$ - size of the largest bag minus 1 -

The tree width of a graph $G$, denoted $\text{tw}(G)$, is the minimum width of a tree decomposition of $G$. 
**Example**

(T₁) For every \( v \in V(G) \), there exists \( t \in V(T) \) such that \( v \in B_t \)
- every vertex is in some bag -

(T₂) For every edge \( uv \in E(G) \), there exists \( t \in V(T) \) such that \( u, v \in B_t \)
- every edge is in a bag -

(T₃) For every \( u \in V(G) \), \( T_u = \{ t \in V(T) \mid u \in B_t \} \) induces a connected subgraph of \( T \).
Example

\((T_1)\) For every \(v \in V(G)\), there exists \(t \in V(T)\) such that \(v \in B_t\)
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\((T_2)\) For every edge \(uv \in E(G)\), there exists \(t \in V(T)\) such that \(u, v \in B_t\)
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\((T_3)\) For every \(u \in V(G)\), \(T_u = \{t \in V(T) \mid u \in B_t\}\) induces a connected subgraph of \(T\).
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\((T_3)\) For every \(u \in V(G)\), \(T_u = \{t \in V(T) \mid u \in B_t\}\) induces a connected subgraph of \(T\).
Prove the following:

- Every tree has tree width 1.
- Show that cycles has tree width at most 2.
Equivalent definitions of tree decomposition

**Definition 2:** a tree decomposition of $G$ is a tree $T$ along with a collection of nonempty subtrees $T_v$, one for each vertex of $G$, with the condition that, for each edge $uv$, $T_u$ and $T_v$ intersect.
Equivalent definitions of tree decomposition

**Definition 2:** a tree decomposition of $G$ is a tree $T$ along with a collection of nonempty subtrees $T_v$, one for each vertex of $G$, with the condition that, for each edge $uv$, $T_u$ and $T_v$ intersect.

**Definition 3:** replace property $(T_3)$ by the following:

$(T_3)$ $B_{t_1} \cap B_{t_3} \subseteq B_{t_2}$ whenever $t_1, t_2, t_3 \in V(T)$ satisfy $t_2 \in t_1 \cdot T t_3$ (where $t_1 \cdot T t_3$ denote the unique path linking $t_1$ and $t_3$ in $T$).
Example of a tree decomposition
Example of a tree decomposition

The original graph $G$

A tree-decomposition of width 3

A tree-decomposition of width 2
**Proposition:** for every graph $G$, $tw(G) \leq |V(G)| - 1$. 
A trivial upper bound

**Proposition:** for every graph $G$, $tw(G) \leq |V(G)| - 1$.

**Proof:** The tree decomposition $(T, \{V(G)\})$ where $T$ is the tree made of a single vertex is a tree decomposition of $G$ of width $|V(G)| - 1$. 
Helly property

Here is a key lemma regarding subtrees intersection; By analogy with Helly’s Theorem on convex subsets of $\mathbb{R}^d$, this property is often called Helly property for subtrees of a tree.

**Lemma** (Helly Property for trees)

Let $T$ be a collection of pairwise intersecting subtrees of a given tree $T$. Then $\bigcap_{T \in \mathcal{F}} T \neq \emptyset$.

Proof:

If not, for each vertex $t$ of the tree, there is a subtree in $T$ that does not contain $t$, and therefore is contained in one of the components of $T \setminus t$. One edge incident to $t$ corresponds to this component, orient this edge out from $t$.

$n$ vertices $\Rightarrow n$ edges receive an orientation. But since there are less edges than vertices in a tree, and an edge cannot receive twice the same direction, there must be an edge oriented both ways. This results in two non intersecting subtrees in $\mathcal{F}$, contradiction.
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- But since there are less edges than vertices in a tree, and an edge cannot receive twice the same direction, there must be an edge oriented both ways.
- This results in two non intersecting subtrees in $\mathcal{F}$, contradiction.
A first lower bound on the treewidth

Recall that $\omega(G)$ denotes the size of a maximum clique of $G$.

**Corollary**

Let $G$ be a graph and $K$ be a complete subgraph of $G$. In any tree decomposition $(T, B)$ of $G$, there exists a vertex $t$ of $T$ such that $K \subseteq B_t$.

In particular, $\text{tw}(G) \geq \omega(G) - 1$
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- Set $K = v_1 \ldots v_k$ a clique of $G$.
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- Set $K = v_1 \ldots v_k$ a clique of $G$.
- Recall that $T_v = \{ t \in V(T) : v \in B_t \}$.
- For every $i \neq j$, $v_i v_j \in E(G)$, so $T_{v_i}$ intersects $T_{v_j}$.
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- so $T = \{T_{v_i} : 1 \leq i \leq n\}$ is a collection of pairwise intersecting subtrees of $T$. 
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- Thus, by the Helly property of trees, the $T_{v_i}$ have a common intersection.
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Let $G$ be a graph and $K$ be a complete subgraph of $G$. In any tree decomposition $(T, B)$ of $G$, there exists a vertex $t$ of $T$ such that $K \subseteq B_t$.

In particular, $\text{tw}(G) \geq \omega(G) - 1$

**Proof:**

- Set $K = v_1 \ldots v_k$ a clique of $G$.
- Recall that $T_v = \{t \in V(T) : v \in B_t\}$.
- For every $i \neq j$, $v_i v_j \in E(G)$, so $T_{v_i}$ intersects $T_{v_j}$
- so $T = \{T_{v_i} : 1 \leq i \leq n\}$ is a collection of pairwise intersecting subtrees of $T$.
- Thus, by the Helly property of trees, the $T_{v_i}$ have a common intersection.
- In other words, there is a vertex of $v$ of $T$ such that $\{v_1, \ldots, v_k\} \subseteq B_v$
Irreducible tree decomposition

- Of course tree decompositions, even optimal ones, are not unique.
- Let us try to define a way to somehow minimise an optimal decomposition.
- Assume there are two adjacent vertices \( s \) and \( t \) of \( T \) such that \( B_s \subset B_t \).
- Then we can contract the edge \( st \) of \( T \) to a new vertex \( r \), and define \( B_r = B_t \).
- It is trivial to check that this defines a valid tree decomposition with width no larger than the original one

**Proposition**

For every graph \( G \), there exists a tree decomposition of width \( \text{tw}(G) \) such that for every edge \( st \in E(T) \), \( B_s \not\subseteq B_t \) and \( B_t \not\subseteq B_s \). Such a tree decomposition is called irreducible.

**Observation:** if \( (T, \mathcal{B}) \) is an irreducible tree decomposition, then for every leaf \( f \in V(T) \), there exists a vertex \( u \in V(G) \) such that \( T_u = \{f\} \)
Another lower bound on tree-width

We denote by $\delta(G)$ the minimum degree of a graph $G$.

**Corollary**

In every graph $G$, there exists a vertex of degree at most $tw(G)$, i.e. $\delta(G) \leq tw(G)$.

**Proof**: Let $(T, B)$ be an irreducible tree decomposition of $G$ and $f$ a leaf of $T$. By the previous observation, there exists $u \in V(G)$ such that $T_u = f$. So all the neighbours of $u$ are included in $B_f$. 
Another lower bound on tree-width

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A hereditary class of graphs $C$ is *$k$-degenerated* if for every $G \in C$, $G$ has a vertex of degree at most $k$.

**Corollary**

*The class of graph with treewidth at most $k$ is $k$-degenerated. Hence, for all graphs $G$, $\chi(G) \leq tw(G) + 1$.***
Separation property of tree decompositions

The following is an easy but fundamental result. It says that a tree decomposition transfers the separation properties of the tree to the decomposed graph.

**Theorem**

Let \((T, W)\) be a tree decomposition of \(G\) and \(t_1 t_2\) be an edge of \(T\) and let \(S = B_{t_1} \cap B_{t_2}\). For \(i = 1, 2\), set

- \(T_i\) to be the connected component of \(T \setminus t_1 t_2\) containing \(t_i\) and
- \(G_i\) to be the subgraph of \(G\) induced by \(\bigcup_{t \in T_i} (B_t \setminus S)\).

Then \(S\) is a cutset of \(G\) separating \(G_1\) from \(G_2\).

**Proof**: On board, easy by definition. See Diestel, Lemma 12.3.1.

- Prove that if \(x \in V(G_1)\), then \(x \notin V(G_2)\).
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- Prove that if \(x \in V(G_1)\), then \(x \notin V(G_2)\).
- Similarly, if \(y \in V(G_2)\), then \(y \notin V(G_1)\).
- Prove that if \(x \in V(G_1)\) and \(y \in V(G_2)\), then \(xy \notin E(G)\).
6 - Tree width and minors
Detailed content of the section

- If $H$ is a minor of $G$, then $tw(H) \leq tw(G)$.

- Wagner Conjecture for bounded treewidth graphs.

- Characterization for graphs with treewidth at most 3.

- Hadwiger Conjecture.

- Wagner Theorem.
Closure property

Proposition

Let $G$ be a graph, $v$ a vertex of $G$ and $e$ an edge of $G$.

- $\text{tw}(G \setminus v) \leq \text{tw}(G)$ -deleting a vertex can only decrease the treewidth-
- $\text{tw}(G \setminus e) \leq \text{tw}(G)$ -deleting an edge can only decrease the treewidth-
- $\text{tw}(G/e) \leq \text{tw}(G)$ -contracting an edge can only decrease the treewidth-

Proof:

for $G \setminus e$, do nothing
for $G \setminus v$, just remove $v$ from every bag containing it.
for $G/e$, where $e = uv$: let $w$ be the new vertex. Add $w$ in every bag containing $u$ or $v$, and delete every occurrence of $u$ and $v$. 

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Closure property

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Graph Minor Theory and its algorithmic consequences
Treewidth and Minors

**Proposition**

*If* $H$ *is a minor of* $G$, *then* $\text{tw}(H) \leq \text{tw}(G)$

**Corollary**

*The class of graphs with treewidth at most* $k$ *is closed under taking minors.*
Wagner’s conjecture for graphs with bounded tree-width

Graphs with bounded treewidth are sufficiently similar to trees that it becomes possible to adapt the proof of Kruskal Theorem to them.

Very roughly, one has to iterate the “minimal bad sequence” used in Kruskal proof \(\text{tw}(G)\) times.

This takes us a step further towards a proof of the Graph Minor Theorem:

**Theorem** (Robertson and Seymour, IV)

*For every integer \(k\), the class of graphs with treewidth at most \(k\) is well quasi ordered by the minor relation.*
Minor obstructions for graphs with treewidth at most 2

By the previous Theorem, for every fixed $k$, the class $\{G : \text{tw}(G) \leq k\}$ has a finite number of obstructions.

Could there be equality?

Theorem $\text{tw}(G) \leq 1 \iff G \in \text{Forb}_\text{minor}(K_3)$ ($G$ is a forest).

$\text{tw}(G) \leq 2 \iff G \in \text{Forb}_\text{minor}(K_4)$.
Minor obstructions for graphs with treewidth at most 2

By the previous Theorem, for every fixed $k$, the class $\{G : \text{tw}(G) \leq k\}$ has a finite number of obstructions.

It is clear that $\{G : \text{tw}(G) \leq k\} \subseteq \text{Forb}_{\text{minor}}(K_{k+2})$: if $G$ contains $K_{k+2}$ as a minor, then $\text{tw}(G) \geq \text{tw}(K_{k+2}) = k + 1$.

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Minor obstructions for graphs with treewidth at most 2

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Could there be equality?

Theorem

\[ \text{tw}(G) \leq 1 \Leftrightarrow G \in \text{Forb}_{\text{minor}}(K_3) \quad (G \text{ is a forest}). \]

\[ \text{tw}(G) \leq 2 \Leftrightarrow G \in \text{Forb}_{\text{minor}}(K_4). \]

The first item is easy, let us prove the second.
Proof sketch:

- If $G$ contains $K_4$ as a minor, then $\text{tw}(G) \geq \text{tw}(K_4) = 3$. So $\text{tw}(G) \leq 2 \Rightarrow G \in \text{Forb}_{\approx m}(K_4)$. 
Proof sketch:

- If $G$ contains $K_4$ as a minor, then $\text{tw}(G) \geq \text{tw}(K_4) = 3$. So $\text{tw}(G) \leq 2 \Rightarrow G \in \text{Forb}_{\preceq m}(K_4)$.
- Let $G \in \text{Forb}_{\preceq m}(K_4)$ and let us prove that $\text{tw}(G) \leq 2$. We proceed by induction on $|V(G)|$. 

So every proper subgraph of $G$ has treewidth at most 2.

Prove first that every 3-connected graph contains $K_4$ as a minor (Use Menger Theorem).

So we may assume that $G$ has a cutset of size $S$ at most 2.

If $S$ is of size 1, conclude.

Assume $S = \{a, b\}$.

If $ab \notin E(G)$, then add $ab$ to $G$ and prove that this does not create a $K_4$-minor.

So now $S$ is a clique (we call that a clique cutset).

Let $C_1$ be a connected component of $G \setminus S$ and $C_2 = G \setminus (S \cup C_1)$.

For $i = 1, 2$, set $G_i = G[C_i \cup S]$ (The $G_i$ are often called block decomposition).

By minimality of $G$, $\text{tw}(G_i) \leq 2$.

Take a tree decomposition of $G_1$ and $G_2$ of width at most 2 and link a bag of $G_1$ containing $ab$ to a bag of $G_2$ containing $ab$.

Prove that this is a tree decomposition of $G$ of width 2.
Proof sketch:

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- For $i = 1, 2$, set $G_i = G[C_i \cup S]$ (The $G_i$ are often called \textbf{block decomposition}). By minimality of $G$, $\text{tw}(G_i) \leq 2$. 
Proof sketch:

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Obstructions for graphs with treewidth at most 2

**Theorem**

- $\text{tw}(G) \leq 1 \iff G \in \text{Forb}_{\text{minor}}(K_3)$
- $\text{tw}(G) \leq 2 \iff G \in \text{Forb}_{\text{minor}}(K_4)$

- The proof for $\text{tw}(G) = 2$ shows the role of separators with treewidth.
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- The proof for \( \text{tw}(G) = 2 \) shows the role of separators with treewidth.
- One could hope for a general result of the type:

  \[
  \text{tw}(G) \leq k \text{ iff } G \in \text{Forb}_{\text{minor}}(K_{k+2}) \]

  **FALSE**
Obstructions for graphs with treewidth at most 2

Theorem

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The proof for $\text{tw}(G) = 2$ shows the role of separators with treewidth.

One could hope for a general result of the type:

$$\text{tw}(G) \leq k \iff G \in \text{Forb}_{\text{minor}}(K_{k+2}) \quad \text{FALSE}$$

There exists graphs with no $K_5$ minor and with arbitrarily large treewidth. (As we will soon see, even planar graphs can have arbitrarily large treewidth).
Forbidden minors for graphs with treewidth at most 3

Theorem

\[ \text{tw}(G) \leq 3 \Leftrightarrow G \text{ does not contain one of the four following graphs as a minor: } K_5, W_8, O \text{ and } C_5 \times K_2. \]
We know that for every graph $G$:

$$\omega(G) \leq \chi(G) \leq \text{tw}(G) + 1$$

$$\omega(G) \leq \omega_m(G) \leq \text{tw}(G) + 1$$

where $\omega_m(G)$ denotes the largest integer $k$ such that $G$ has a $K_k$ minor.
Digression: Hadwiger Conjecture

We know that for every graph $G$:

$$\omega(G) \leq \chi(G) \leq tw(G) + 1$$
$$\omega(G) \leq \omega_m(G) \leq tw(G) + 1$$

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**Conjecture (Hadwiger)**

For every graph $G$, $\chi(G) \leq \omega_m(G)$. 
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where $\omega_m(G)$ denotes the largest integer $k$ such that $G$ has a $K_k$ minor.

Conjecture (Hadwiger)

For every graph $G$, $\chi(G) \leq \omega_m(G)$.

- For $k = 2$: $\omega_m(G) \leq 2 \iff G$ is a forest $\Rightarrow \chi(G) \leq 2$. 
We know that for every graph $G$:

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**Conjecture (Hadwiger)**

For every graph $G$, $\chi(G) \leq \omega_m(G)$.

- For $k = 2$: $\omega_m(G) \leq 2 \iff G$ is a forest $\Rightarrow \chi(G) \leq 2$.
- For $k = 3$: $\omega_m(G) \leq 3 \iff tw(G) \leq 2 \Rightarrow \chi(G) \leq 3$ by the above inequalities.
Digression: Hadwiger Conjecture

We know that for every graph $G$:

$$\omega(G) \leq \chi(G) \leq tw(G) + 1$$
$$\omega(G) \leq \omega_m(G) \leq tw(G) + 1$$

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**Conjecture (Hadwiger)**

For every graph $G$, $\chi(G) \leq \omega_m(G)$.

- For $k = 2$: $\omega_m(G) \leq 2 \iff G$ is a forest $\Rightarrow \chi(G) \leq 2$.
- For $k = 3$: $\omega_m(G) \leq 3 \iff tw(G) \leq 2 \Rightarrow \chi(G) \leq 3$ by the above inequalities.
- For $k = 4$: $\omega_m(G) \leq 4 \Rightarrow \chi(G) \leq 4$ contains the Four Colour Theorem since planar graphs are $K_5$-minor free. In fact it is equivalent (and hence true), thanks to a structural characterisation of graphs with no $K_5$ minor due to Wagner.
Theorem (Wagner, 1956)

\[ G \text{ is } K_5\text{-minor free if and only if } G \text{ is a subgraph of some graph built recursively by clique sums operation, starting from planar graphs and } W_8. \]

We will see later in the course that this theorem together with the 4-color theorem implies Hadwiger conjecture for \( k = 5 \), that is

\[ \omega_m(G) \leq 4 \Rightarrow \chi(G) \leq 4 \]
Exercises on treewidth

**Exercice 5**
Prove that if $H$ is a subdivision of $G$, then $\text{tw}(H) = \text{tw}(G)$

The following exercise says that classes of graphs with bounded treewidth are sparse.

**Exercice 6**
Show that graphs $G$ of treewidth at most $k$ with $k \geq 1$ have at most $k|V(G)|$ edges.

Next exercise is very important to design algorithm based on the tree decomposition.

**Exercice 7**
Show that every graph $G$ admits a tree decomposition of width $\text{tw}(G)$ with at most $|V(G)|$ bags.
Exercises on treewidth

**Exercice 8**
Determine the treewidth of a path, a tree, a complete graph, a complete bipartite graph, the cube.

**Exercice 9**
Prove that if $G$ contains (as a subgraph) a complete bipartite graph with parts $A$ and $B$, then in every tree decomposition there exists a bag that contains $A$ or a bag that contains $B$.

**Exercice 10**
Prove that if $x$ and $y$ are two vertices that are joined by $k + 1$ internally vertex disjoints paths, then in every tree decomposition of $G$ of width at most $k$, there exists a bag containing both $x$ and $y$.

**Exercice 11**
Prove that if $G$ is $K_{2,3}$-minor-free then $tw(G) \leq 3$. 
Treewidth also plays a crucial role in algorithmic, we’ll come back to it.
7 - Brambles - Duality - Cops and Robbers
In the previous section, we have seen that Wagner Conjecture holds for classes of graphs with bounded treewidth.

To make a proof of the general case, we should be able to say stuff about the graphs it does not cover, i.e. to deduce informations about a graph from the assumption it has large treewidth.

The main theorem of this section achieves that: it identifies a canonical obstruction to small treewidth, a structural phenomenon that occurs in a graph if and only if it has large treewidth.

This phenomenon is called **Bramble**.
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This phenomenon is called **Bramble**.
(In reality, it is mainly used to get certificate on the value of the treewidth of a graph, the notion of **tangle** is used as an obstruction for large treewidth, but we won't see it during this class).
Definition of a bramble.

Duality Theorem: $bn(G) = tw(G) + 1$.

Grids have treewidth $n$.

Introduction to the cops and robber game.

Cop number of trees.

Thm: Cops number is at most treewidth.

Thm: Bramble number is at most cops number.
We say that two connected subgraphs of $G$ touch if they have non-empty intersection or if they are joined by an edge.
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Definition

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- A **bramble** of $G$ is a collection $B$ of connected subgraphs that are pairwise touching.
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- A **transversal** of a bramble $\mathcal{B}$ is a set of vertices of $G$ that has non-empty intersection with each element of $\mathcal{B}$.

The **order** of a bramble $\mathcal{B}$ is the minimum size of a transversal of $\mathcal{B}$.

The **bramble number** of $G$, denoted $bn(G)$, is the maximum order of a bramble of $G$.

Note that if $G$ contains $K_p$ as a minor, then the connected subgraphs of a $K_p$-model of $G$ form a bramble (no intersection, just touching) of order $p$. 

$bn(G) = \max_{\text{all brambles } \mathcal{B}} \min(\text{transversal of } \mathcal{B})$
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\[ bn(G) = \max_{\text{all brambles } B} \min(\text{transversal of } B) \]
A Bramble

A bramble of order 4 of $G_{3,3}$:
Duality Theorem I

**Proposition**

If \((T, \mathcal{W})\) is a tree decomposition of \(G\) and \(B\) is a bramble in \(G\), then there exists \(t \in T\) such that \(W_t\) is a transversal of \(B\)

**Proof sketch:** (idea: usual "orientation of edges of the tree decomposition".)
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- If \(B \in \mathcal{B}\) does not intersect \(S\), then there is \(i \in \{1, 2\}\) such that \(B \subseteq G_i\).
- Moreover, since \(S\) is a cutset separating \(G_1\) from \(G_2\), all such \(B\) are contained in the same \(G_i\).
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- Orient the edge \(t_1t_2\) toward \(t_i\). Hence, we may assume that all edges of the tree decomposition has an orientation.
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- Orient the edge \(t_1t_2\) toward \(t_i\). Hence, we may assume that all edges of the tree decomposition has an orientation.
- Let \(t\) be the last vertex of a maximal directed path. Observe that \(t\) is a sink and thus \(W_t\) is a transversal of \(B\).
**Proposition**

If \((T,W)\) is a tree decomposition of \(G\) and \(B\) is a bramble of \(G\), then there exists \(t \in T\) such that \(W_t\) is a transversal of \(B\).

Therefore \(bn(G) \leq tw(G) + 1\).

In other words, if \(B\) is a bramble of \(G\), then \(tw(G) + 1\) is at most the order of \(B\).
Duality Theorem II

**Proposition**

If $(T, W)$ is a tree decomposition of $G$ and $B$ is a bramble of $G$, then there exists $t \in T$ such that $W_t$ is a transversal of $B$.

Therefore $b_n(G) \leq tw(G) + 1$.

In other words, if $B$ is a bramble of $G$, then $tw(G) + 1$ is at most the order of $B$.

The converse inequality is true but harder to prove.

It gives the following sort of minmax theorem (in fact maxmin=minmax).

**Theorem (Seymour and Thomas, 1993)**

*For every graph $G$, $b_n(G) = tw(G) + 1$.***
What is the treewidth of the grid?

**Proposition**

The grid $G_{n,n}$ has treewidth $n$. 

To prove that $\text{tw}(G_{n,n}) \leq n$, find a tree decomposition (actually you can find a path decomposition) of width $n$.

To prove that $\text{tw}(G_{n,n}) \geq n$, it is enough to find a bramble of order $n + 1$.

It is easy to check that the following is a bramble of order $n + 1$:

- $A = \{x_i, 1 \leq i \leq n\}$, the last row,
- $B = \{x_1, j, 1 \leq j < n\}$, the last column minus its last element,
- $C_{ij} = \{x_kj, 1 \leq k < n\} \cup \{x_ik, 1 \leq k < n\}$ (crosses minus the last element of row and column).
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- It is easy to check that the following is a bramble of order $n + 1$:
  - $A = \{x_{i,1}, 1 \leq i \leq n\}$, the last row,
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A Game of Cops and Robber

- 2 player game on a graph: one controls the Robber, the other control Cops
- Goal of the cops is to capture the robber
- Many variants exist

In our variant:

- Cops and robbers are standing on vertices of the graph
- At each turn, cops can move by helicopter and land on any vertex of the graph, or stay on its vertex.
- The robber sees an helicopter approaching and can instantly move at infinite speed to any other vertex along a path of a graph. The only constraint is that he is not permitted to run through a vertex occupied by some cop.
- The cops win if at some point they occupy all vertices adjacent to the position of the robber, and an extra cop lands by helicopter on the robber.

Definition

The cop number of a graph $G$, denoted $cn(G)$, is the smallest number of cops needed to ensure the capture of the robber.

Pierre Aboulker - pierreaboulker@gmail.com
Graph Minor Theory and its algorithmic consequences
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**Definition**

The cop number of a graph $G$, denoted $cn(G)$, is the smallest number of cops needed to ensure the capture of the robber.
**Question**: what is the cop number of a tree $T$? In other words, how many cops are needed to capture a robber in a tree?
Cop number of trees

**Question**: what is the cop number of a tree $T$? In other words, how many cops are needed to capture a robber in a tree? **Answer**: 2.

Could there be a relation between the cop number and the treewidth?!
TreeDec = strategy for the cops

Proposition

$$cn(G) \leq tw(G) + 1$$
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- Put every cop one the vertices of some bag \( W_t \).
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**Proposition**

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- Let \( t' \) the neighbour of \( t \) in \( T \) in the direction of this component.
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- \( W_t \cap W_{t'} \) separates the component containing the robber from the rest of the graph.
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- \( W_t \cap W_{t'} \) separates the component containing the robber from the rest of the graph.
- At the next move, cops in \( W_t \setminus W_{t'} \) move to occupy all of \( W_{t'} \).
- Cops apply this strategy until it reaches some leaf of the tree and the robber cannot escape.
Brambles = Strategy for the Robber

**Proposition**

\[ bn(G) \leq cn(G) \]
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- Let \( B \) be a bramble of order \( bn(G) \) and assume only \( bn(G) - 1 \) cops.
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- By definition there exists a set \( X \in B \) such that \( X \cap C = \emptyset \).
Let $B$ be a bramble of order $bn(G)$ and assume only $bn(G) - 1$ cops.

Let $C$ be the set of initial positions of the cops.

By definition there exists a set $X \in B$ such that $X \cap C = \emptyset$.

The robber moves to some vertex $x \in X$. 

Again there exists $X' \in B$ such that $X' \cap C' = \emptyset$.

During their flight the only occupied vertices are $C \cap C'$ so $X \cup X'$ is entirely free of cops.

The robber can freely move from $X$ to $X'$ and this strategy can be applied for ever.
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- The robber moves to some vertex \( x \in X \).
- After that, the game really begins, cops move so that the new set occupied by the cops is \( C' \).
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By definition there exists a set $X \in B$ such that $X \cap C = \emptyset$.

The robber moves to some vertex $x \in X$.

After that, the game really begins, cops move so that the new set occupied by the cops is $C'$.

Again there exists $X' \in B$ such that $X' \cap C' = \emptyset$.

During their flight the only occupied vertices are $C \cap C'$ so $X \cup X'$ is entirely free of cops,

The robber can freely move from $X$ to $X'$ and this strategy can be applied for ever.
8 - The grid minor Theorem
Recall that, if $H$ is a graph,

$$Forb_{\text{minor}}(H) = \{ G : H \text{ is not a minor of } G \}$$

We have already seen that:

- Graphs in $Forb_{\text{minor}}(K_3)$ have treewidth at most 1.
- Graphs in $Forb_{\text{minor}}(K_4)$ have treewidth at most 2.
- Graphs in $Forb_{\text{minor}}(K_5)$ have unbounded treewidth (because of grids).
Treewidth of minor closed class

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A natural question to ask is then: for which $H$, graphs in $\text{Forb}_{\text{minor}}(H)$ have bounded treewidth?
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A natural question to ask is then: for which $H$, graphs in $Forb_{\text{minor}}(H)$ have bounded treewidth?

One of the most important result of graph minor theory is a complete and beautiful characterization of such $H$. 
For which $H$, graphs in $\text{Forb}_{\text{minor}}(H)$ have bounded treewidth?
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First, $H$ must be planar.
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If $H$ is non-planar, then $\text{Forb}_{\text{minor}}(H)$ contains all grids (indeed, all grids and their minors are planar) and grids have arbitrarily large treewidth.
**Theorem** (Grid Minor Theorem, Robertson and Seymour, V)

Given a graph $H$, graphs in $\text{Forb}_{\text{minor}}(H)$ have bounded treewidth if and only if $H$ is planar.
Grid Minor Theorem

**Theorem** (Grid Minor Theorem, Robertson and Seymour, V)

*Given a graph $H$, graphs in $\text{Forb}_{\text{minor}}(H)$ have bounded treewidth if and only if $H$ is planar.*

We need to prove the *if part*, that is, for $H$ a planar graph, graphs in $\text{Forb}_{\text{minor}}(H)$ have bounded treewidth.

In fact, we only need to show this for the special case where $H$ is a grid, because *every planar graph is a minor of some grid*. (To see this, draw a planar graph, fatten its edges, and superimpose a sufficiently fine grid).
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**Theorem** (Grid Minor Theorem)

Let $k$ be an integer.
There exists $f(k)$ such that if $G \in \text{Forb}_{\text{minor}}(G_{k,k})$, then $\text{tw}(G) \leq f(k)$
Very (very) rough idea of the proof:
Let $G$ be a graph with very large treewidth. We want to show that $G$ contains a large grid.

- Show that $G$ contains a large family $\{A_1, \ldots, A_m\}$ of pairwise disjoint connected subgraphs such that:

  - each pair $A_i, A_j$ can be linked in $G$ by a family $P_{i,j}$ of many disjoint $A_i - A_j$ paths avoiding the other sets.

  - We then consider all the pairs $P_{i,j}, P_{i,j}'$. If we can find a pair such that many of the paths in $P_{i,j}$ meets many of the paths in $P_{i,j}'$, then we can find a large grid (this is the most difficult part of the proof because the intersections might be very messy).

  - Otherwise, for every pair $P_{i,j}, P_{i,j}'$, many of the paths in $P_{i,j}$ avoid many of the path in $P_{i,j}'$. We can then select one path $P_{i,j} \in P_{i,j}$ from each family such that these selected path are pairwise disjoint.

Contracting each of the connected subgraph will then give us a $K_m$-minor, which contains a large grid.
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Theorem (Grid Minor Theorem)

There exists \( f(k) \) such that if \( G \) is \( G_{k,k} \)-minor free then \( \text{tw}(G) < f(k) \)

- Establishing tight bounds on \( f(k) \) is an important graph-theoretical question with many applications on structural and algorithmic graph theory.
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**Theorem (Chekuri and Chuzhoy, 2019)**

If $G$ is $G_{k,k}$-minor free then $\text{tw}(G) < O(k^9 \text{poly log } k)$. 
Planar Graphs are WQO

Tentative proof of Wagner’s Conjecture: Let \((G_n)_{n \in \mathbb{N}}\) be a sequence of graphs
We want to prove that there exists \(G_i\) and \(G_j\) with \(i < j\) and \(G_i\) is a minor of \(G_j\).

Corollary
The class of planar graphs is wqo for the minor relation.

Would be nice to be able to say stuff on \(H\)-minor free graphs even when \(H\) is non-planar.
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The class of planar graphs is wqo for the minor relation. Would be nice to be able to say stuff on \(H\)-minor free graphs even when \(H\) is non-planar.
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9 - Erdős-Pósa Property
Theorem (Helly Property for trees)

Let $\mathcal{T}$ be a collection of pairwise intersecting subtrees of a given tree $T$. Then $\bigcap_{T \in \mathcal{T}} T \neq \emptyset$. 
Theorem (Helly Property for trees)

Let $T$ be a collection of pairwise intersecting subtrees of a given tree $T$. Then $\bigcap_{T \in \mathcal{F}} T \neq \emptyset$.

The following generalises Helly property for trees.

Theorem

Let $T$ be a collection of subtree of a given tree $T$. For every integer $k \geq 1$, either

- there exists $k + 1$ disjoint trees in $T$, or

- there is a set $S$ of at most $k$ vertices such that $S$ intersects each tree in $T$. 

The Erdő-Pósa theorem

Erdő-Pósa theorem, 1965

For every integer $k$, there exists a number $f(k)$ such that for every graph $G$:
- either $G$ has $k + 1$ vertex disjoint cycles, or
- a set $S$ of at most $f(k)$ vertices such that $G \setminus S$ is a forest.

**Morality:** a graph $G$ either has many disjoint cycles, or has a bounded set of vertices that hits every cycle.
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Erdős and Pósa showed that $f(k) = \Theta(k \cdot \log(k))$ and this is tight.
Erdős-Pósa Property

Definition (Erdős-Pósa Property)

A connected graph $H$ has the Erdős-Pósa Property if for every $k \geq 1$, there exist $f(k)$ such that, for every graph $G$:

- either $G$ contains $k + 1$ vertex disjoint $H$-model, or
- $G$ has a set $S$ of at most $f(k)$ vertices such that $G \setminus S$ is $H$-minor free.

Since $K_3$-models are precisely cycles:

Erdős-Pósa theorem (equivalent formulation)

$K_3$ has the Erdős-Pósa Property.

Question

Which graphs has the Erdős-Pósa Property?
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Corollary of the Grid Minor Theorem

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A graph $H$ has the *Erdős-Pósa Property* if and only if $H$ is planar.
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**Corollary** (Robertson and Seymour)

A graph $H$ has the Erdős-Pósa Property if and only if $H$ is planar.

We first prove that if $H$ is non-planar, then it does not have the Erdős-Pósa Property.
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- $G_k$ has no two vertex disjoing $H$-model and
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- Transform every crossing edge into a vertex of degree 4.
Corollary of the Grid Minor Theorem

**Corollary (Robertson and Seymour)**

A graph $H$ has the *Erdős-Pósa Property* if and only if $H$ is planar.

We first prove that if $H$ is non-planar, then it does not have the Erdős-Pósa Property.
In order to do so, we need to construct a sequence of graphs $(G_k)_{k \in \mathbb{N}}$ with the following properties:

- $G_k$ has no two vertex disjoing $H$-model and
- Deleting $k$ vertices does not make $G_k$ $H$-minor free.

**Construction:**

- Let $H$ be a non planar graph.
- Let $\Sigma$ be a surface of minimum genus in which $H$ embeds.
- This minimality ensures that any two drawings of $H$ intersects.
- Now one can construct a graph by embedding $2k + 1$ vertex disjoint copies of $H$ in $\Sigma$ such that no point of $\Sigma$ is contained in more than 2 of these copies.
- Transform every crossing edge into a vertex of degree 4.
- The resultant graph $G$ has no transversal of size $k$, but every two copies of $H$ intersect.
We now prove that every planar graph has the Erdős-Pósa Property.
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- $G$ has a set $S$ of at most $f(k)$ vertices such that $G \setminus S$ is $H$-minor free.
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**Proof:**

- Let \( H \) be a planar graph.
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- Let $H$ be a planar graph.
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- By the Grid Minor Theorem, there exists an integer $t_k$ such that $tw(G) \leq t_k$. 

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- By the Grid Minor Theorem, there exists an integer $t_k$ such that $tw(G) \leq t_k$.
- Let us define recursively the function $f$ by $f(0) = 0$ and $f(k) = 2f(k - 1) + t_k$ for any $k \geq 1$. 
We now prove that every planar graph has the Erdős-Pósa Property. That is, there exists $f : \mathbb{N} \rightarrow \mathbb{N}$, such that every graph $G$,

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- Let us define recursively the function $f$ by $f(0) = 0$ and $f(k) = 2f(k - 1) + t_k$ for any $k \geq 1$.
- We are going to prove by induction on $k$ that $f$ is the desired function.
Let \((T, B)\) be a tree decomposition of width \(t_k\).
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If a vertex \(t\) has no out-going edge, then \(G \setminus B_t\) has no \(H\)-model and we are happy since \(|B_t| \leq t_k \leq f(k)\).
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We orient each edge \(t_1t_2 \in E(T)\) toward \(t_i\) if \(G_i\) contains an \(H\)-model.

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So by induction both \(G_1\) and \(G_2\) has a set of at most \(f(k - 1)\) vertices that hits every \(H\)-model.

Together with \(B_{t_1} \cap B_{t_2}\), it gives us a set of size at most \(2f(k - 1) + t_k\) that hits every \(H\)-model.
Tight function

**Theorem** (van Batenburg, Huynh, Joret, Raymond, 2019)

For every planar graph $H$, there exists a constant $c_H$ such that: For every graph $G$, one of the following holds:

- $G$ has $k + 1$ vertex disjoint $H$-model, or
- $G$ has a set $S$ of at most $c_H k \cdot \log(k)$ vertices such that $G \setminus S$ is $H$-minor free.
10 - The Decomposition Paradigm
The decomposition paradigm have lead to many difficult and important results.

It is used to describe a fixed class of graphs, say $C$.

The key is to describe how every graph of $C$ can be constructed by gluing together certain basic graphs by a well defined composition rules.

The main result of the graph minor project is a (approximate) decomposition theorem for $\text{Forb}_{\text{minor}}(K_k)$; observe that for every fixed graph $H$,

$$\text{Forb}_{\text{minor}}(H) \subseteq \text{Forb}_{\text{minor}}(K_{|V(H)|})$$

This section can be seen as an introduction to decomposition theorem, we will show, among other things, a decomposition theorem for chordal graphs as well as for graphs of bounded treewidth.
Detailed content of the section

- Definition of clique cutset, and linked between clique cutset and treewidth.
- Decomposition theorem for chordal graphs.
- Exercises on chordal graphs and their link with treewidth.
- Definition of clique sum.
Let $G$ be a graph and $S$ a set of vertices of $G$. We denote by $G[S]$ the induced subgraphs of $G$ induced by $S$. $S$ is a cutset or separator of $G$ if $G \setminus S$ has at least two connected components. It is a clique cutset if $S$ induces a clique (i.e. $G[S]$ is a clique).
Clique cutsets and treewidth

Let $G$ be a graph and $S$ a set of vertices of $G$. We denote by $G[S]$ the induced subgraphs of $G$ induced by $S$. 

$S$ is a cutset or separator of $G$ if $G \setminus S$ has at least two connected components. It is a clique cutset if $S$ induces a clique (i.e. $G[S]$ is a clique).

The following say that we should be happy when a graph has a clique cutset.

**Proposition**

Let $G$ be a graph with a clique cutset $S$ and let $(C_i)_{i \in I}$ be the connected components of $G \setminus S$. Let $G_i = G[C_i \cup S]$. Then $\text{tw}(G) = \max_{i \in I}(\text{tw}(G_i))$.

**Proof**: easy, on board.
Exercises on clique cutsets

Exercice 12

Let $G$ be a graph with a clique cutset $S$ and let $(X_i)_{i \in I}$ be the connected components of $G \setminus S$. Let $G_i = G[C_i \cup S]$. Prove that:

1. $\chi(G) = \max_{i \in I}(\chi(G_i))$.
2. $\omega_m(G) = \max_{i \in I}(\omega_m(G_i))$.
Decomposition theorem for chordal graph

A graph $G$ is **chordal** (or **triangulated**) if it has no **induced subgraph** isomorphic to a cycle of length at least 4.
Chordal graphs is one of the oldest studied class of graphs. They have a very strong structure that permits to design efficient algorithms to compute on them.

**Theorem (Dirac, 1961)**

*If $G$ is a chordal graph, then either it is a clique or it admits a clique cutset.*
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**Theorem (Dirac, 1961)**

*If $G$ is a chordal graph, then either it is a clique or it admits a clique cutset.*

**Proof.**

Suppose that $G$ is not a clique. Let $S$ be a minimal vertex-cutset of $G$, and let $C_1$ and $C_2$ be two connected components of $G \setminus S$. Suppose for contradiction that $G[S]$ is not a clique. So $S$ contains two non-adjacent vertices $u$ and $v$. Since $S$ is minimal, both $u$ and $v$ have a neighbor in both $C_1$ and $C_2$. Hence, for $i = 1, 2$, there exists a chordless $uv$-path $P_i$ whose interior vertices are in $C_i$. Then $P_1 \cup P_2$ induces a chordless cycle, a contradiction. So $S$ is a clique-cutset of $G$. 

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Decomposition theorem for chordal graph II

It is now easy to deduce the following decomposition theorem for chordal graphs.

**Theorem**

*A graph is chordal if and only if it can be constructed recursively by pasting along complete subgraphs, starting from complete graphs.*

**Exercice 13**

1. Let $G$ be a chordal graph. Prove that $\chi(G) = \omega(G)$.
2. Find a graph $H$ such that $\chi(H) \neq \omega(H)$. 
Exercice 14

Show that the following statements are equivalent:

1. $G$ is chordal
2. $G$ admits an optimal tree decomposition such that every bag is a clique.
3. Every induced subgraph of $G$ contains a simplicial vertex, i.e. a vertex whose neighborhood is a clique.
4. $G$ admits a tree decomposition with the property that $uv \in E(G)$ if and only if $T_u$ and $T_v$ have non empty intersection (recall that $T_u$ is the set of bags containing $u$).
5. $G$ is the intersection graph of a family of subtrees of a tree.

Using one of the above characterization of chordal graphs, prove that for every graph $G$:

$$\text{tw}(G) = \min\{\omega(G') - 1, \ G \text{ subgraph of } G' \text{ and } G' \text{ is chordal}\}$$

See Diestel Book, Proposition 12.3.11 and Corollary 12.3.12
Defintion (Clique sum)

Let $G_1$ and $G_2$ be two graphs and $K_1$ a clique of $G_1$, $K_2$ a clique of $G_2$ with $|K_1| = |K_2|$. If $G$ is a graph obtained by identifying vertices of $K_1$ and $K_2$, and then removing some edges of this clique, then $G$ is a clique sum of $G_1$ and $G_2$. Similarly as for clique cutset, we have the following:

Proposition

If $G$ is a clique sum of $G_1$ and $G_2$, then $\text{tw}(G) \leq \max(\text{tw}(G_1), \text{tw}(G_2))$.

And another characterization of treewidth, that is also a decomposition theorem for classes of graphs with bounded treewidth.

Proposition

$G$ has treewidth at most $k$ if and only if it can be constructed recursively by clique sum operations starting from graphs on at most $k+1$ vertices.
Clique Sums

**Definition (Clique sum)**
Let $G_1$ and $G_2$ be two graphs and $K_1$ a clique of $G_1$, $K_2$ a clique of $G_2$ with $|K_1| = |K_2|$. If $G$ is a graph obtained by identifying vertices of $K_1$ and $K_2$, and then removing some edges of this clique, then $G$ is a **clique sum** of $G_1$ and $G_2$.

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11 - Graphs are WQO (Warning : contains major handwaving)
Starts as before: Assume $(G_n)_{n \in \mathbb{N}}$ is a counterexample.

Can we describe the structure of these graphs? It is sufficient to get a structure theorem for $\text{Forb}_{\text{minor}}(K_k)$.

For $k \leq 4$ we have seen characterizations (small treewidth).

For $k = 5$ there is one due to Wagner:
Wagner’s Conjecture: Sketch

- Starts as before: Assume \((G_n)_{n \in \mathbb{N}}\) is a counterexample.
- We can assume that no graph \(G_i\) with \(i \geq 1\) has \(G_0\) as a minor.
Wagner’s Conjecture : Sketch

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- Can we describe the structure of these graphs??
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- For \(k = 5\) there is one due to Wagner:
Theorem (Wagner - 1937)

$K_5$-minor free graphs are constructed by a sequence of 3-clique sums operations starting from $W_8$ and planar graphs.
Wagner decomposition Theorem

**Theorem (Wagner - 1937)**

$k_5$-minor free graphs are constructed by a sequence of 3-clique sums operations starting from $W_8$ and planar graphs.

- Assume $(G_i)_{i \in \mathbb{N}}$ is an infinite antichain.

Assume there exists $n \in \mathbb{N}$ such that $|V(G_n)| \leq 5$. Then $(G_i)_{i > n}$ are $K_5$-minor-free.

Then we can use Wagner Theorem: the graphs $G_i$, $i > n$ have some kind of a 2-layer structure:

- Outside we have a tree-like structure, which can be handled with similar methods used to handle trees (and graphs with bounded treewidth).
- Inside (that is in the “bag” of the tree decomposition given by Wagner Theorem), graphs are planar or $W_8$, and we already know they are WQO.

Hence, all we need is a generalisation of Wagner decomposition Theorem for all complete graphs.
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Wagner decomposition Theorem

**Theorem (Wagner - 1937)**

*K*₅-minor free graphs are constructed by a sequence of 3-clique sums operations starting from *W*₈ and planar graphs.

- Assume \((G_i)_{i \in \mathbb{N}}\) is an infinite antichain.
- Assume there exists \(n \in \mathbb{N}\) such that \(|V(G_n)| \leq 5\).
- Then \((G_i)_{i > n}\) are *K*₅-minor-free.
- Then we can use Wagner Theorem: the graphs \(G_i, i > n\) have some kind of a 2-layer structure:
  - Outside we have a tree-like structure, which can be handled with similar methods used to handles trees (and graphs with bounded treewidth).
  - Inside (that is in the "bag" of the tree decomposition given by Wagner Theorem), graphs are planar or *W*₈, and we already know they are WQO.
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Hence, all we need is a generalisation of Wagner decomposition Theorem for all complete graphs.
Vortices and Fringes

Let us start with a technical definition. If $C$ is a cycle, a **vortex** on $C$ is defined as follows:

- Select a collection of arcs $A_1, A_2, \ldots, A_l$ on $C$ so that each vertex is in at most $k$ arcs.
- For each arc we add a vertex $v_i$ that is linked to some vertices of $A_i$.
- We can also add edges $v_i v_j$ if $A_i \cap A_j \neq \emptyset$.
- We call this **adding a fringe** of width $k$ to $C$.

![Diagrams showing vortices and fringes](image)

**Figure 4.** A fringe of width 2 and a fringe of width 3.

*Laszló Lovász*
Almost $k$-embeddable

Now let us define a class $\mathcal{G}_k$ of almost $k$-embeddable graphs

i. Start with a surface of genus at most $k$ with a graph $G$ embedded in it so that each face is homeomorphic to a disc.

ii. Add at most $k$ vortices (local perturbation of a face of the embedding)

iii. Add at most $k$ apexes (vertices linked arbitrarily to the rest of the graph)
**Theorem** (Robertson and Seymour Theorem, XX)

For every graph $H$, there exists an integer $k$ such that all $H$-minor free graphs can be obtained by a sequence of $k$-clique sum operations starting from almost $k$-embeddable graphs.

Felix Reidl
"Proof" of Wagner Conjecture

Very (very) roughly, the proof that graphs are WQO for minor ordering is

- Show that graphs of bounded genus are WQO by induction on the genus (very hard).
- Almost $k$-embedable graphs are taken care to the cost of more very hard work.
- Kruskal’s Theorem’s proof is adapted to deal with the tree structure given by the clique sums operations.
Algorithmic implications

**General message:**

- if something works for planar graphs,
- then we might generalize it to bounded genus graphs,
- then we might generalize it to $H$-minor-free graphs.
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**What next?**
Algorithmic implications

**General message:**

- if something works for planar graphs,
- then we might generalize it to bounded genus graphs,
- then we might generalize it to $H$-minor-free graphs.

What next?

What about $Forb_{≤t}(H)$?
$H$-topological minor free graphs

$H$-topological minor free graphs look like that (Grohe and Marx, 2012)
Theorem (Grohe and Marx, 2012)

For every $H$, there is an integer $k$ such that every $H$-subdivision-free graph has a tree decomposition where the torso of every bag is either:

- $k$-almost embeddable in a surface of genus at most $k$ or
- has degree at most $k$ with the exception of at most $k$ vertices ("almost bounded degree").

General message:
If a problem can be solved both
- on (almost-)embeddable graphs and
- on (almost-)bounded degree graphs,
then these results can be raised to $H$-subdivision-free graphs without too much extra effort.