You can answer in any order that you like, but indicate the question numbers in your answers. The questions are roughly sorted by difficulty except for ⋆ questions, which are somewhat harder.

1 Universal Languages and Time Bounds

Place in as low a complexity class as you can.

(a) \( L := \{ (M, x) : M(x) \text{ halts} \} \)
(b) \( L := \{ M : \exists x \ M(x) \text{ halts} \} \)
(c) \( L := \{ (M, x, t) : M(x) \text{ halts within time } t \} \)
(d) \( L := \{ (M, t) : \exists x \ M(x) \text{ halts within time } t \} \)
(e) \( L := \{ (M, t) : \forall x \ M(x) \text{ halts within time } t \} \)
(f) \( L := \{ (M, 1^t) : \forall x \ M(x) \text{ halts within time } t \} \)

Which languages are complete for their classes? Briefly justify your answers.

2 Relations between Classes

Assume \( P = \text{PSPACE} \). Now consider each of the following statements and say whether it is true, false, or an open question. In case you say true or false, also give a short justification of your answer.

(a) \( P = \text{NP} \)
(b) \( \text{NP} = \text{coNP} \)
(c) \( P = L \)
(d) \( \text{NP} = \text{EXP} \)
(e) \( \text{NP} = L \)
(f) \( \text{NPSpace} = P \)

3 Kleene Star

The Kleene star of a language \( L \) is the language

\[ L^* := \{ x_1 \cdots x_k : k \geq 0 \text{ and } x_1, \ldots, x_k \in L \} \, . \]

That is, \( L^* \) consists of strings formed by concatenating a finite number of elements of \( L \).

(a) Show that \( \text{NP} \) is closed under Kleene star.
(b) Show that \( \text{P} \) is closed under Kleene star.

4 PCP Variants

Determine which of the following variants of PCP are decidable:

(a) PCP over a unary alphabet.
(b) PCP over a binary alphabet.
(c) PCP (over an arbitrary alphabet, as usual) but now the goal is determine whether or not there is an infinite sequence \( i_1, i_2, \ldots \) such that \( t_{i_1} t_{i_2} \cdots = b_{i_1} b_{i_2} \cdots \), where \( \{(t_i, b_i)\}_{i=1}^n \) is a finite set of tiles.
5 Randomization and Nondeterminism

A language $L \in \text{BP} \cdot \text{NP}$ if there exists a polynomial-time deterministic Turing machine $M$ such that

\[ x \in L \implies \Pr_{r \in \{0,1\}^m(n)} \left[ \exists y \in \{0,1\}^{k(n)} M(x, y; r) = 1 \right] \geq \frac{2}{3} \]

\[ x \notin L \implies \Pr_{r \in \{0,1\}^m(n)} \left[ \exists y \in \{0,1\}^{k(n)} M(x, y; r) = 1 \right] \leq \frac{1}{3} \]

where $m(n), k(n) \leq \text{poly}(n)$. A language $L \in \text{NP} \cdot \text{BP}$ if there exists a polynomial-time deterministic Turing machine $M$ such that

\[ x \in L \implies \exists y \in \{0,1\}^{k(n)} \Pr_{r \in \{0,1\}^m(n)} \left[ M(x, y; r) = 1 \right] \geq \frac{2}{3} \]

\[ x \notin L \implies \forall y \in \{0,1\}^{k(n)} \Pr_{r \in \{0,1\}^m(n)} \left[ M(x, y; r) = 1 \right] \leq \frac{1}{3} \]

where $m(n), k(n) \leq \text{poly}(n)$. Show that $\text{NP} \cdot \text{BP} \subseteq \text{BP} \cdot \text{NP}$.

6 Separation via Closure

For a language $L \subseteq \Sigma^*$ and a function $f : \Sigma^* \rightarrow \Sigma^*$, let

\[ L_f := \{ x \in \Sigma^* : f(x) \in L \} \]

We say that a complexity class is closed under (polynomial) composition if for any $L$ in the class and any polynomial-time computable function $f$ it is the case that language $L_f$ is also in that class.

For a language $A$, let $A^\sharp := \{ x0^{|x|^2 - |x|} : x \in A \}$ be the language consisting of elements $x$ of $A$ appended with $|x|^2 - |x|$ zeros.

(a) Show that $\text{NP}$ is closed under composition.

(b) Show that if $A \in \text{SPACE}(n^2)$ then $A^\sharp \in \text{SPACE}(n)$.

(c) Show that $\text{SPACE}(n)$ is not closed under composition.

(d) Conclude that $\text{NP} \neq \text{SPACE}(n)$.

7 NP-Completeness

CLIQUE-COVER is the following problem:

INPUT: A graph $G = (V, E)$ and a positive integer $K \leq |V|$;

QUESTION: Can the vertices of $G$ be partitioned into $k \leq K$ disjoint sets $V_1, \ldots, V_k$ such that for every $i = 1, \ldots, k$ the subgraph induced by $V_i$ is a complete graph?

Show that CLIQUE-COVER is $\text{NP}$-complete.

8 Bonus Question: Sometimes Space = Time

Show that there exists a function $T(n) \geq n$ such that $\text{DTIME}(T(n)) = \text{DSPACE}(T(n))$. 