Étude de l’algorithme glouton pour résoudre le problème du stable maximum

Mathieu Mari
Conférence ROADEF - Février 2018 - Lorient

Joint work with Pr. Piotr Krysta (U. Liverpool) and Nan Zhi (U. Liverpool)
Introduction: Greedys

Follow the way of best **local** choices in order to reach the best **global** solution.

- simple
- Low processing time
- Efficient
Introduction: Greedys

Follow the way of best **local** choices in order to reach the best **global** solution.

- simple
- Low processing time
- Efficient

► Limited to exact solutions!
Theorem (Hastad 96’)

\(\text{MIS} \) is hard to approximate within \( n^{1-\epsilon} \) for any \( \epsilon > 0 \)
Theorem (Hastad 96’)
MIS is hard to approximate within $n^{1-\epsilon}$ for any $\epsilon > 0$
Theorem (Hastad 96’)
MIS is hard to approximate within $n^{1-\epsilon}$ for any $\epsilon > 0$
Theorem (Hastad 96’)
MIS is hard to approximate within $n^{1-\epsilon}$ for any $\epsilon > 0$
Theorem (Hastad 96’)
MIS is hard to approximate within $n^{1-\epsilon}$ for any $\epsilon > 0$
Theorem (Hastad '96')

**MIS** is hard to approximate within $n^{1-\epsilon}$ for any $\epsilon > 0$. 

**Greedy** ($G$)

While $G \neq \emptyset$:

• Find $v \in G$ with minimum degree.
• Remove $v$ and its neighbours from $G$. 

Not deterministic! How to guide Greedy to best possible solution?
**Theorem (Hastad 96')**

MIS is hard to approximate within $n^{1-\epsilon}$ for any $\epsilon > 0$

**Greedy($G$)**

While $G \neq \emptyset$:
- Find $v \in G$ with minimum degree.
- Remove $v$ and its neighbours from $G$. 
Maximum Independent Set (MIS)

Theorem (Hastad 96')
MIS is hard to approximate within $n^{1-\epsilon}$ for any $\epsilon > 0$

**Greedy**($G$)

While $G \neq \emptyset$:
- Find $v \in G$ with **minimum degree**.
- Remove $v$ and its neighbours from $G$.

Not deterministic!

How to guide **Greedy** to best possible solution?
How to measure the performance of an advised Greedy algorithm?

We compare the size of the solution output by an **advised-Greedy** with the size of the **best greedy set**!
Negative results
There is no good advise for large class of graphs …

**Theorem (Bodlaender et al.) [BTY97]**

MaxGreedy is NP-complete

▶ Bodlaender, Thilikos, Yamazaki: *It is hard to know when greedy is good for finding maximum independent set*, 1996

<table>
<thead>
<tr>
<th></th>
<th>Lower bound (MIS)</th>
<th>Apx. ratio (Greedy)</th>
<th>Lower bound (MaxGreedy)</th>
</tr>
</thead>
<tbody>
<tr>
<td>General graphs</td>
<td>$n^{1-\epsilon}$</td>
<td>$n$</td>
<td>$n^{1-\epsilon}$</td>
</tr>
<tr>
<td>Bounded degree $\Delta$</td>
<td>$\Delta^{1-\epsilon}$</td>
<td>$\frac{\Delta + 2}{3}$</td>
<td>$\frac{\Delta + 2}{3} - O(1/\Delta)$</td>
</tr>
<tr>
<td>[Alon et al. 95]</td>
<td></td>
<td>[Hall. et al. 97]</td>
<td></td>
</tr>
<tr>
<td>Bipartite</td>
<td>MIS $\in P$</td>
<td>$\sqrt{n}$</td>
<td>$n^{1/2-\epsilon}$</td>
</tr>
</tbody>
</table>
How to prove inapproximability results for MaxGreedy?

Goal: There are no $\rho$-approximation algorithm $\mathcal{A}$ for MaxGreedy in $G$.

- Let $\phi$ be a SAT formula with variables $x_1, \ldots, x_n$ and build graph $G_\phi$.

Check that:

- $\phi$ satisfiable $\Rightarrow \alpha^-(G_\phi) \geq A$
- $\phi$ not satisfiable $\Rightarrow \alpha^+(G_\phi) \leq B$
- $A/B > \rho$
- $G_\phi \in G$
- $G_\phi$ has polynomial size

$\phi$ satisfiable iff $\mathcal{A}(G_\phi) \geq A$

Figure 1: The graph $G_\phi$
Simulating $\text{SAT}$ with Greedy

First phase: Choice of a valuation $\nu$ such that $x \in S$ iff $\nu(x) = 1$ → Not deterministic

Second phase: Build a greedy set $S$ with size depending only on $\nu$ → Deterministic
General case

- Hastad: MIS is hard to approximate within $n^{1-\epsilon}$.
  - $\alpha^+(G_n) = 2$
  - $\alpha(G_n) = n$
  - $|G_n| = 2n + 1$

\[
\frac{\alpha(G_n)}{\alpha^+(G_n)} = \Omega(|G_n|)
\]

Figure 2: The graph $G_n$

Theorem

MaxGreedy is hard to approximate within $n^{1-\epsilon}$, for any $\epsilon > 0$. 
Hard bipartite graphs

Proposition
The Greedy algorithm achieves a $\sqrt{n}$-approximation for MIS.

Theorem
There are no $(n^{1/2-\epsilon})$-approximation algorithms for MaxGreedy in bipartite graphs.

Monotone SAT: $x y w \land \overline{x} \overline{u} \land u w \land \overline{y} \overline{x} \overline{z}$
One potential application

**Single-minded bidders** auction

- $n$ bidders, each interested in **one** bundle of items
- Find the best allocation
- **Incentive-compatible** mechanism

- Best Incentive-compatible mechanism $\approx$ **Greedy**
Graphs with maximum degree $\Delta \geq 3$

- Any *maximal* independent set in $G$ has size $\geq \frac{n}{\Delta(G) + 1}$

**Theorem (Halldorsson et al.) [HR97]**

**Greedy** achieves an $\frac{\Delta + 2}{3}$-approximation of $\text{MIS}$ (**tight**)

- Halldorsson, Radhakrishnan: *Greed is good: Approximating independent sets in sparse and bounded-degree graphs*, 1994

**Theorem**

MaxGreedy is hard to approximate within $\frac{\Delta + 2}{3} - O(1/\Delta)$

- Alon et al.: $\text{MIS}$ is hard to approximate within $\Delta^{1-\epsilon}$
\( \Delta = 3 \)

► \( \Delta(G) = 2 : \text{Optimal} \)
Theorem

MaxGreedy is NP-hard on cubic planar graphs

→ Reduction from \textsc{MIS} (NP-hard)
Positive results
Positive result for small degree graphs

Theorem (Halldorsson et al.) [HR97]

**Greedy** achieves a approximation ratio of $\frac{3+2}{3} = 1.666...$ (tight) for graphs where $\Delta \leq 3$.

→ Can we help **Greedy** in its choices to get a better guarantee?
Positive result for small degree graphs

Theorem (Halldorsson et al.) [HR97]

**Greedy** achieves a approximation ratio of \( \frac{3+2}{3} = 1.666... \) (tight) for graphs where \( \Delta \leq 3 \).

\[ \rightarrow \] Can we help **Greedy** in its choices to get a **better guarantee**?

**YES** : If you can choose between two nodes with degree two, pick the one which has a neighbour with degree three (**MoreEdges**)

Theorem [HY95]

**MoreEdges** has an approximation ratio of 1.5 for **MIS**

\[ \text{Halldorsson, Yoshihara: Greedy approximations of independent sets in low degree graphs, 1995} \]

**Theorem**

**MoreEdges** achieves an \( 9/5 \)-approximation for **MIS** when \( \Delta \leq 4 \).
Better advises when $\Delta \leq 3$

**Good reductions**

- $(2, 5)$
- $(2, 6)$

**Bad reductions**

- $(2, 6)$
- $(2, 5)$
- Reduction $(2, 4)$

**Smart-Greedy**($G$)

While $G \neq \emptyset$:

- Let $S$ be the set of $v \in G$ with minimum degree.
- Pick $v \in S$ with the following order of preference:
  1. **Good** reductions
  2. **Bad** reduction $(2, 6)$
  3. **Bad** reduction $(2, 5)$
  4. Reduction $(2, 4)$
**Theorem**

**Smart-Greedy** achieves a 14/11-approximation of \( \text{MIS} \) \((\Delta \leq 3)\)

- \(\frac{14}{11} \approx 1.272\)

**Hard examples:**

\[
\frac{\alpha(H_{2p})}{\alpha^+(H_{2p})} \rightarrow 1.25
\]
Conclusion and future work

► **Negative results**

- Adapt the method to other optimisation problems
  - Set cover, dominating sets, coloring problems, machine scheduling…
- Extend to other heuristics
  - Local searches

► **Positive results**

- Get closer to 1.25
- Find a general algorithm to larger degrees
Conclusion and future work

► Negative results

• Adapt the method to other optimisation problems
  • Set cover, dominating sets, coloring problems, machine scheduling…

• Extend to other heuristics
  • local searches

► Positive results

• Get closer to 1.25

• Find a general algorithm to larger degrees

Merci pour votre attention !
Hans L Bodlaender, Dimitrios M Thilikos, and Koichi Yamazaki. *It is hard to know when greedy is good for finding independent sets.*

M. M. Halldórsson and J. Radhakrishnan. *Greed is good: Approximating independent sets in sparse and bounded-degree graphs.*