Étude de l’algorithme glouton pour résoudre le problème du stable maximum

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Conférence ROADEF - Février 2018 - Lorient

Joint work with Pr. Piotr Krysta (U. Liverpool) and Nan Zhi (U. Liverpool)

Introduction : Greedys

Follow the way of best **local** choices in order to reach the best **global** solution.

- simple
- Low processing time
- Efficient

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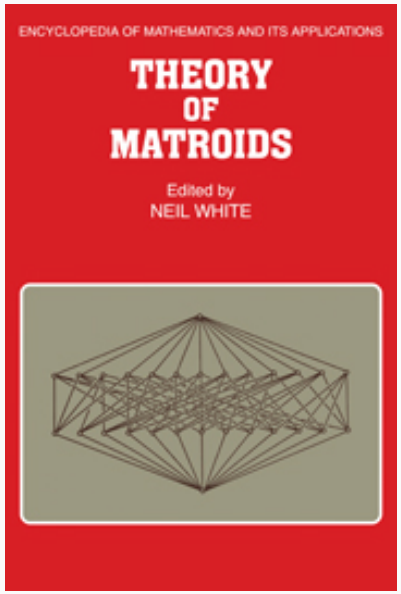
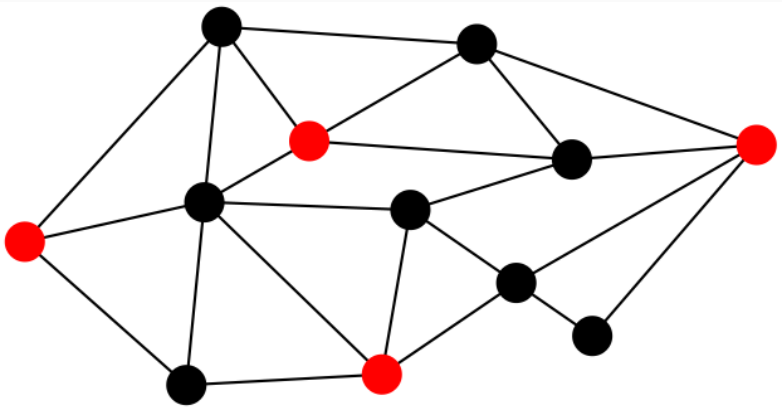


- simple
- Low processing time
- Efficient

▶ Limited to exact solutions !

Context : `Maximum Independent Set` (`MIS`)



Theorem (Hastad 96')

`MIS` is hard to approximate within $n^{1-\epsilon}$ for any $\epsilon > 0$

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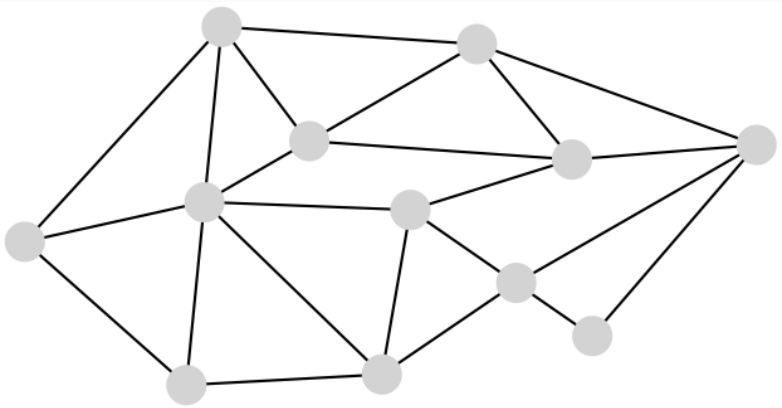

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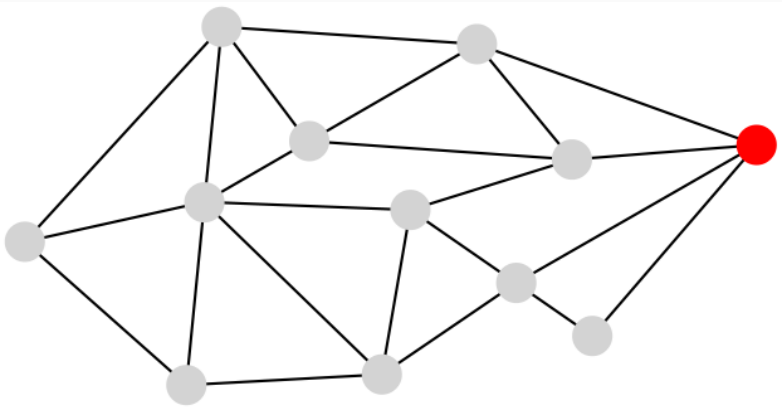

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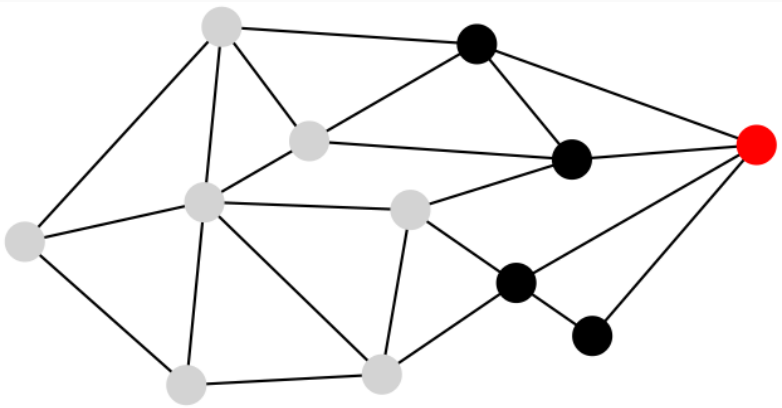

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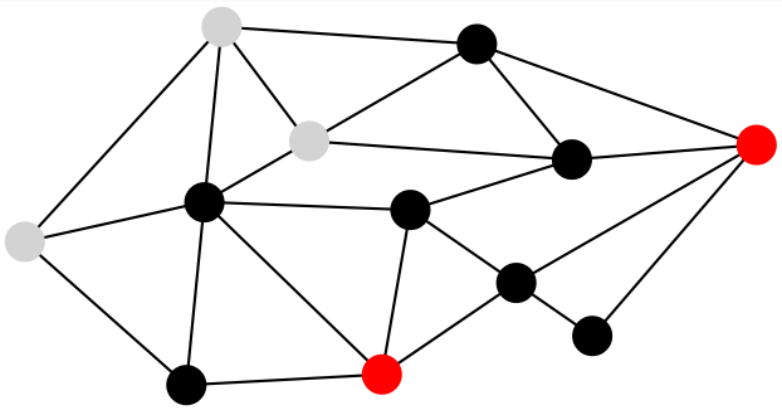

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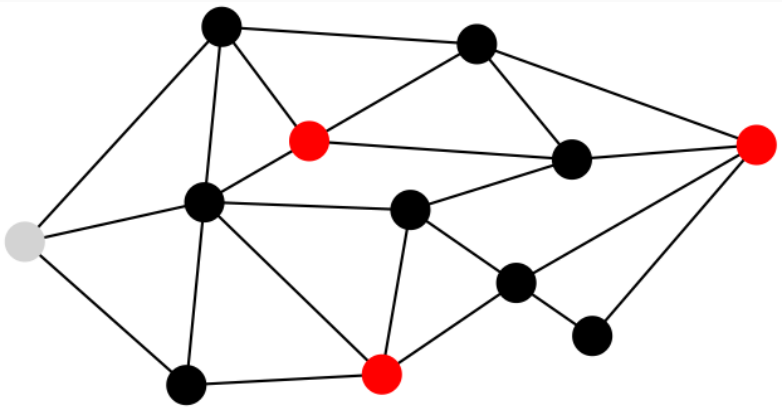

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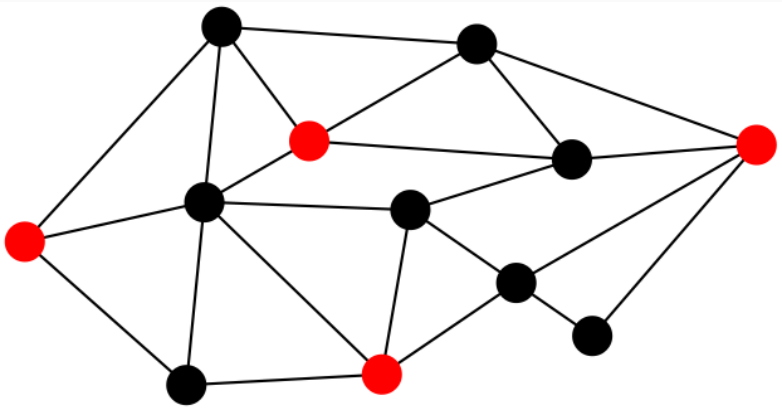

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Greedy(G)

While $G \neq \emptyset$:

- *Find* $v \in G$ with **minimum degree**.
- Remove v and its neighbours from G.

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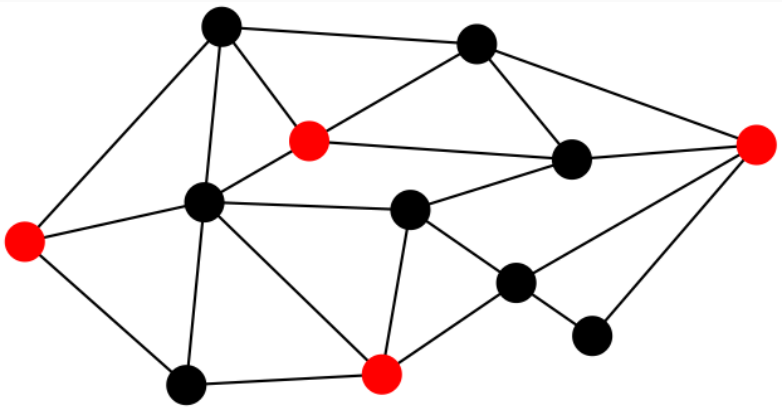

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Not deterministic !

How to guide **Greedy** to best possible solution ?

How to measure the performance of an advised Greedy algorithm ?

Independent set	Greedy set
$\alpha(G)$	Max. Greedy set : $\alpha^+(G)$ Min. Greedy set : $\alpha^-(G)$
`MIS` : $\alpha(G) \geq k$?	`MaxGreedy` : $\alpha^+(G) \geq k$?

- We compare the size of the solution output by an **advised-Greedy** with the size of the **best greedy set** !

Negative results

There is no good advise for large class of graphs ...

Theorem (Bodlaender *et al.*) [BTY97]

`MaxGreedy` is NP-complete

- Bodlaender, Thilikos, Yamazaki : *It is hard to know when greedy is good for finding maximum independent set*, 1996

	Lower bound (`MIS`)	Apx. ratio **Greedy**	Lower bound (`MaxGreedy`)
General graphs	$n^{1-\epsilon}$	n	$n^{1-\epsilon}$
Bounded degree Δ	$\Delta^{1-\epsilon}$ [Alon et al. 95]	$\frac{\Delta+2}{3}$ [Hall. et al. 97]	$\frac{\Delta+2}{3} - O(1/\Delta)$
Bipartite	`MIS` $\in P$	$\sqrt{n}$	$n^{1/2-\epsilon}$

How to prove inapproximability results for `MaxGreedy` ?

Goal : There are no ρ-approximation algorithm $\mathcal{A}$ for `MaxGreedy` in $\mathcal{G}$

- Let ϕ be a SAT formula with variables $x_1, \ldots, x_n$ and build graph G_ϕ.

Check that :

- ϕ satisfiable $\Rightarrow \alpha^-(G_\phi) \geq A$
- ϕ not satisfiable $\Rightarrow \alpha^+(G_\phi) \leq B$
- $A/B > \rho$
- $G_\phi \in \mathcal{G}$
- G_ϕ has polynomial size

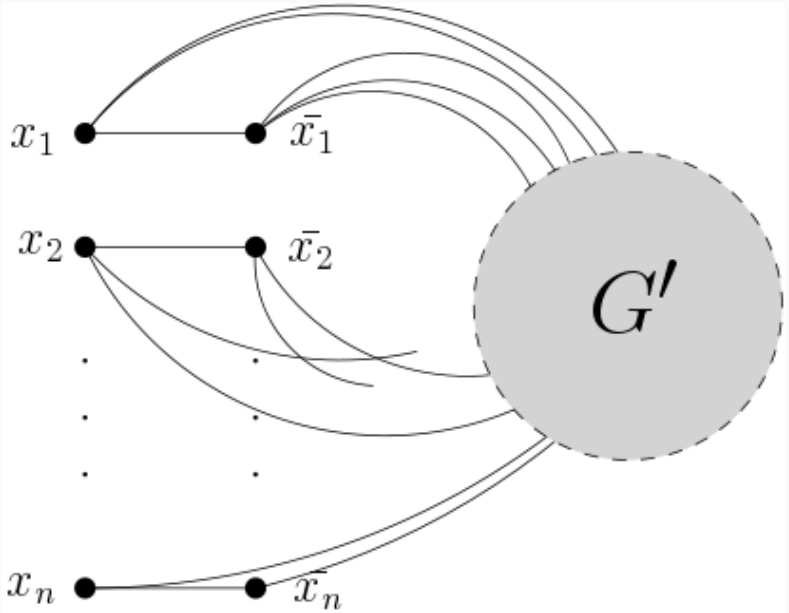
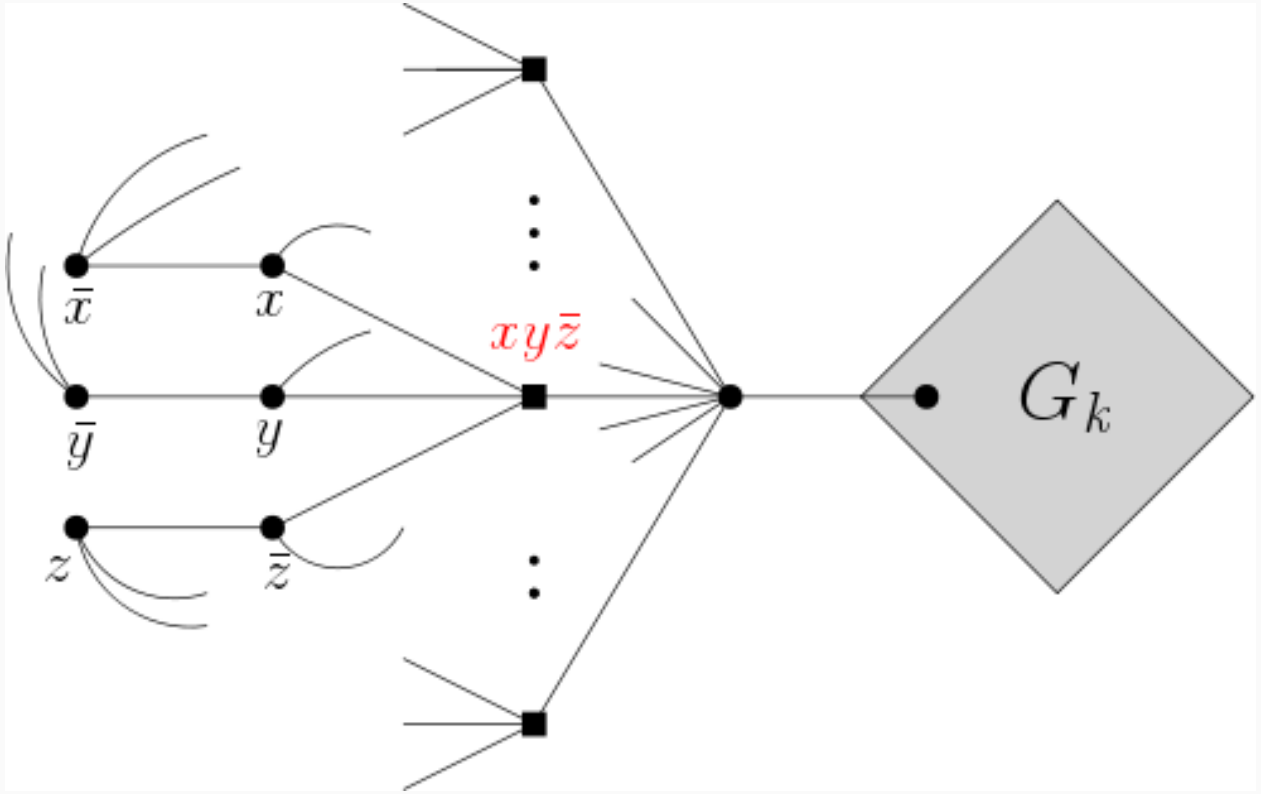


Figure 1: *The graph G_ϕ*

ϕ satisfiable iff $\mathcal{A}(G_\phi) \geq A$

Simulating SAT with Greedy



First phase : Choice of a valuation ν such that $x \in S$ iff $\nu(x) = 1$

$\longrightarrow$ **Not deterministic**

Second phase : Build a greedy set S with size depending only on ν

$\longrightarrow$ **Deterministic**

General case

- Hastad : `MIS` is hard to approximate within $n^{1-\epsilon}$.

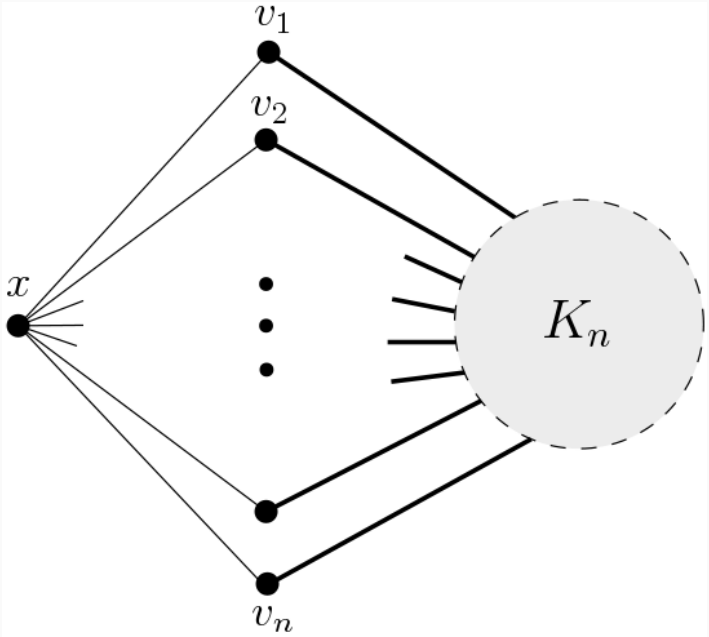


Figure 2: *The graph G_n*

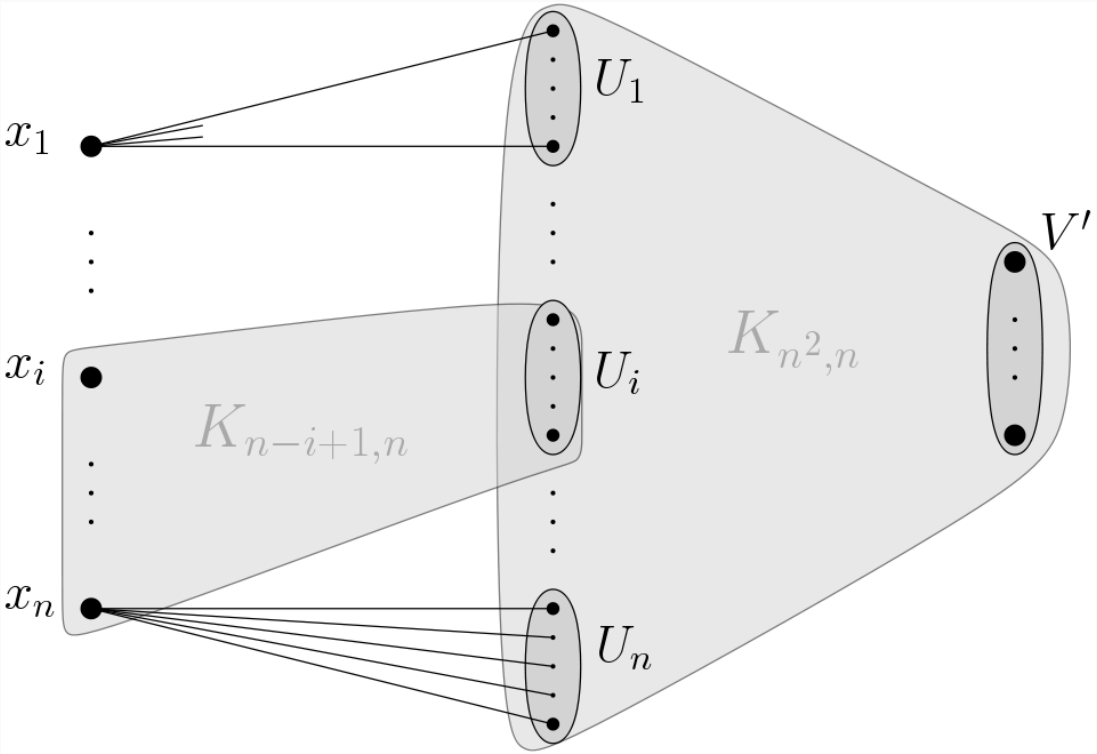
- $\alpha^+(G_n) = 2$
- $\alpha(G_n) = n$
- $|G_n| = 2n + 1$

$$\frac{\alpha(G_n)}{\alpha^+(G_n)} = \Omega(|G_n|)$$

Theorem

`MaxGreedy` is hard to approximate within $n^{1-\epsilon}$, for any $\epsilon > 0$.

Hard bipartite graphs



Proposition

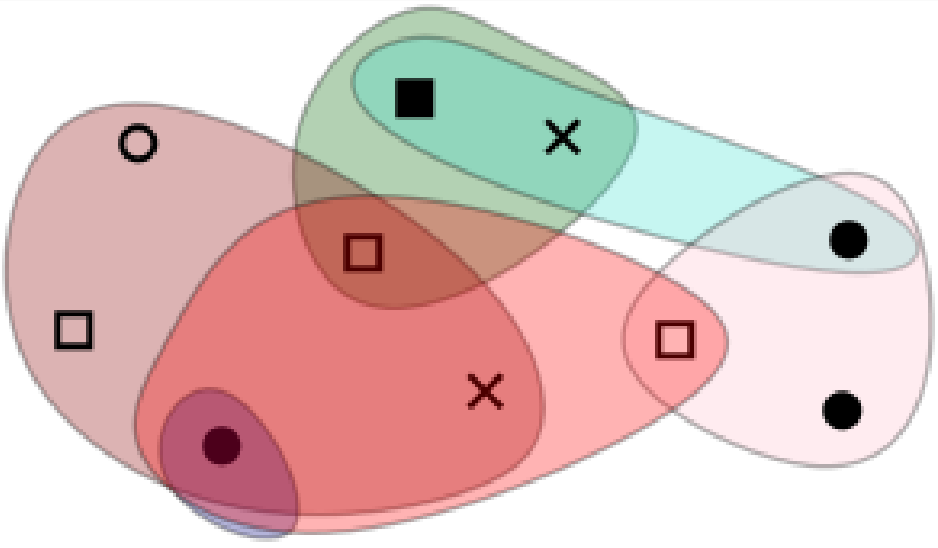
Greedy achieves a $\sqrt{n}$-approximation for `MIS`

Theorem

There are no $(n^{1/2-\epsilon})$-approximation algorithms for `MaxGreedy` in bipartite graphs.

- Monotone `SAT` : $xyw \wedge \bar{x}\bar{u} \wedge uw \wedge \bar{y}\bar{x}\bar{z}$

One potential application



Single-minded bidders auction

- *n* bidders, each interested in **one** bundle of items
- Find the best allocation
- **Incentive-compatible** mecanism

- Best Incentive-compatible mecanism ≈ **Greedy**

Graphs with maximum degree $\Delta \geq 3$

- Any *maximal* indendent set in G has size $\geq \dfrac{n}{\Delta(G)+1}$

Theorem (Halldorsson *et al.*) [HR97]

Greedy achieves an $\frac{\Delta+2}{3}$-approximation of `MIS` (**tight**)

▶ Halldorsson, Radhakrishnan: *Greed is good : Approximating independent sets in sparse and bounded-degree graphs*, 1994

Theorem

`MaxGreedy` is hard to approximate within $\dfrac{\Delta+2}{3} - O(1/\Delta)$

▶ Alon *et al.* : `MIS` is hard to approximate within $\Delta^{1-\epsilon}$

$\Delta = 3$

- $\Delta(G) = 2$: Optimal

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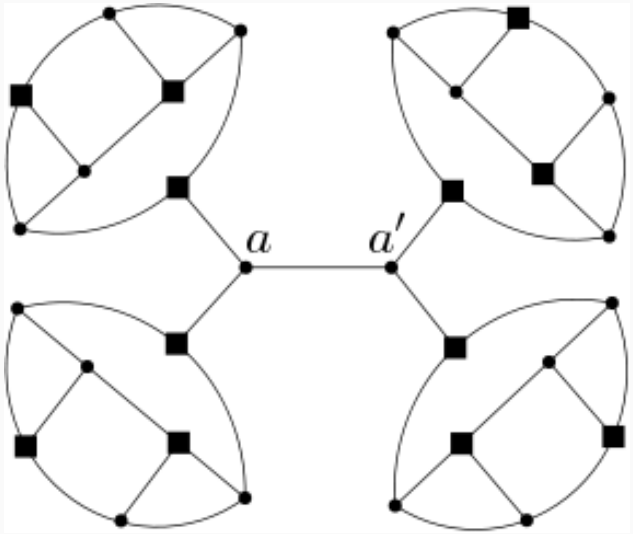
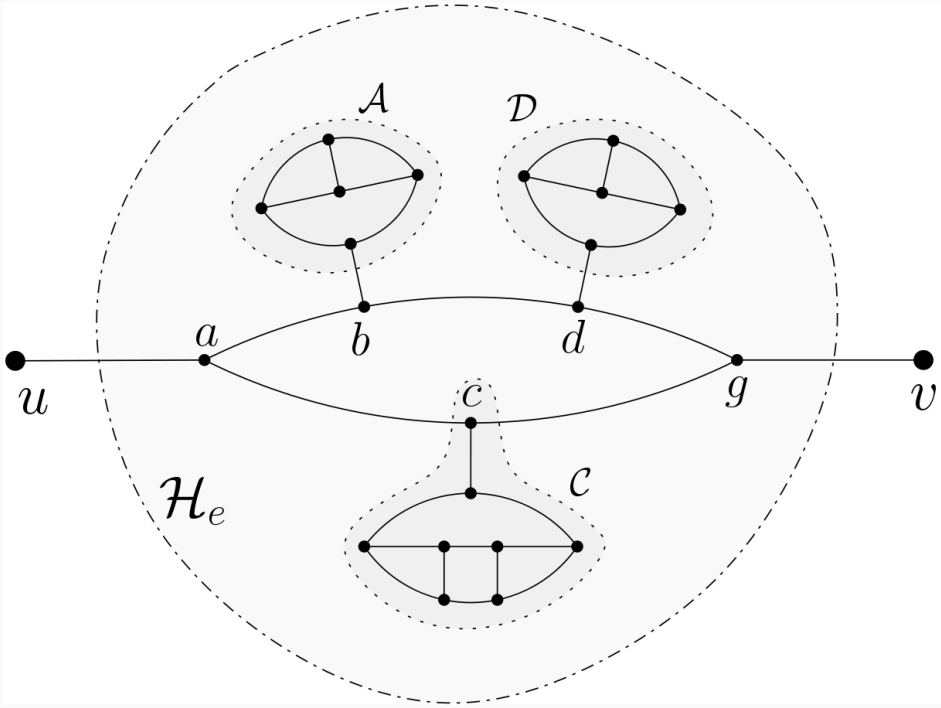
Theorem

`MaxGreedy` is NP-hard on cubic planar graphs

$\longrightarrow$ **Reduction from `MIS` (NP-hard)**





Positive results

Positive result for small degree graphs

Theorem (Halldorsson *et al.*) [HR97]

Greedy achieves a approximation ratio of $\frac{3+2}{3}$ =1.666... (tight) for graphs where $\Delta \leq 3$.

$\longrightarrow$ Can we help **Greedy** in its choices to get a **better guarantee** ?

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$\longrightarrow$ Can we help **Greedy** in its choices to get a **better guarantee** ?

YES : If you can choose between two nodes with degree *two*, pick the one which has a neighbour with degree *three* (**MoreEdges**)

Theorem [HY95]

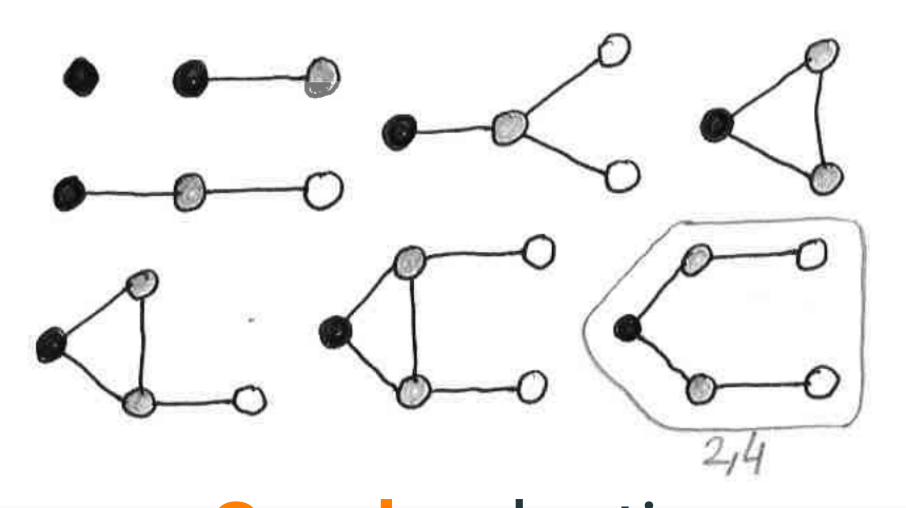
MoreEdges has an approximation ratio of 1.5 for `MIS`

▶ Halldorsson, Yoshihara: *Greedy approximations of independent sets in low degree graphs*, 1995

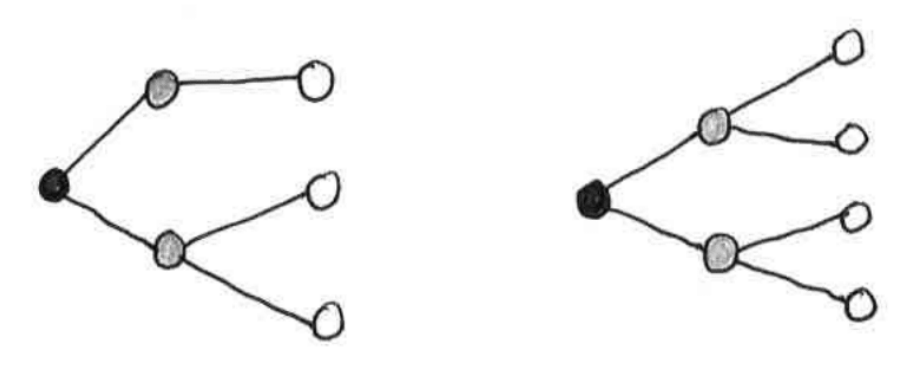
Theorem

MoreEdges achieves an $9/5$-approximation for `MIS` when $\Delta \leq 4$.

Better advises when $\Delta \leq 3$



Good reductions



$(2,5)$ and $(2,6)$
Bad reductions

Smart-Greedy(G)

While $G \neq \emptyset$:

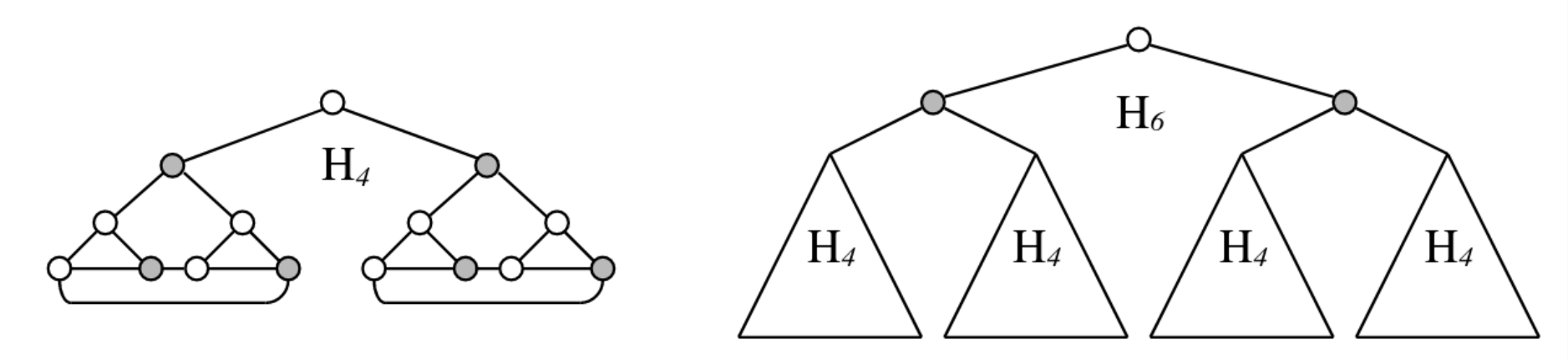
- Let S be the set of $v \in G$ with minimum degree.
- Pick $v \in S$ with the following order of preference :
  1. **Good** reductions
  2. **Bad** reduction $(2,6)$
  3. **Bad** reduction $(2,5)$
  4. Reduction $(2,4)$

Theorem

Smart-Greedy achieves a 14/11-approximation of MIS ($\Delta \leq 3$)

- $\frac{14}{11} \approx 1.272$

▶ Hard examples :



$$\frac{\alpha(H_{2p})}{\alpha^{+}(H_{2p})} \longrightarrow 1.25$$

Conclusion and future work

- **Negative results**
  - Adapt the method to other optimisation problems
    - Set cover, dominating sets, coloring problems, machine scheduling…
  - Extend to other heuristics
    - local searchs
- **Positive results**
  - Get closer to 1.25
  - Find a general algorithm to larger degrees

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Merci pour votre attention !

Bibliography

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