MPRI

The Arithmetic-Geometric Progression Abstract Domain

VMCAI 2005

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Overview

Issue

- In automatically generated programs using floating point arithmetics, some computations may diverge because of rounding errors.

- We prove the absence of floating point number overflows:

  we bound rounding errors at each loop iteration by a linear combination of the loop inputs; we get bounds on the values that depends exponentially on the program execution time.

- We use non polynomial constraints. Our domain is both precise (no false alarm) and efficient (linear in memory / $n\ln(n)$ in time).

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Running example (in $\mathbb{R}$)

```
X := 0; k := 0;
while (k < 1000) {
    if (?) {X ∈ [−10; 10]};
    X := X/3;
    X := 3 × X;
    k := k + 1;
}
```

Interval analysis: first loop iteration

$X := 0; k := 0;$	
	$X = 0$
$\text{while } (k < 1000) \, \{$	
	$X = 0$
$\quad \text{if } (?) \, \{X \in [-10; 10]\};$	
	$\lvert X \rvert \leq 10$
$\quad X := X/3;$	
	$\lvert X \rvert \leq \frac{10}{3}$
$\quad X := 3 \times X;$	
	$\lvert X \rvert \leq 10$
$\quad k := k + 1;$	
$\quad \}$	

Interval analysis: Invariant

$1: \; X := 0; k := 0;$

    $X = 0$

$2: \; \textsf{while}\ (k < 1000)\ \{$

    $|X| \leq 10$

$3: \quad \textsf{if}\ (?)\ \{X \in [-10; 10]\};$

    $|X| \leq 10$

$4: \quad X := X/3;$

    $|X| \leq \frac{10}{3}$

$5: \quad X := 3 \times X;$

    $|X| \leq 10$

$6: \quad k := k + 1;$

$7: \quad \}$

    $|X| \leq 10$

Including rounding errors [Miné–ESOP'04]

$$X := 0; k := 0;$$

$$\textsf{while}\ (k < 1000)\ \{$$

$$\quad \textsf{if}\ (?)\ \{X \in [-10; 10]\};$$

$$\quad X := X/3 + [-\varepsilon_1; \varepsilon_1].X + [-\varepsilon_2; \varepsilon_2];$$

$$\quad X := 3 \times X + [-\varepsilon_3; \varepsilon_3].X + [-\varepsilon_4; \varepsilon_4];$$

$$\quad k := k + 1;$$

$$\}$$

The constants ε_1, ε_2, ε_3, and ε_4 (≥ 0) are computed by other domains.

Interval analysis

Let $M \geq 0$ be a bound:

$$X := 0;\ k := 0;$$

$$X = 0$$

$$\textbf{while}\ (k < 1000)\ \{$$

$$|X| \leq M$$

$$\quad \textbf{if}\ (?)\ \{X \in [-10; 10]\};$$

$$|X| \leq max(M, 10)$$

$$\quad X := X/3 + [-\varepsilon_1; \varepsilon_1].X + [-\varepsilon_2; \varepsilon_2];$$

$$|X| \leq (\varepsilon_1 + \tfrac{1}{3}) \times max(M, 10) + \varepsilon_2$$

$$\quad X := 3 \times X + [-\varepsilon_3; \varepsilon_3].X + [-\varepsilon_4; \varepsilon_4];$$

$$|X| \leq (1 + a) \times max(M, 10) + b$$

$$\quad k := k + 1;$$

$$\}$$

with $a = 3 \times \varepsilon_1 + \frac{\varepsilon_3}{3} + \varepsilon_1 \times \varepsilon_3$ and $b = \varepsilon_2 \times (3 + \varepsilon_3) + \varepsilon_4$.

Ari.-geo. analysis: first iteration

$$1: \quad X := 0; k := 0;$$

$$X = 0,\ k = 0$$

$$2: \quad \text{while } (k < 1000)\ \{$$

$$X = 0$$

$$3: \qquad \text{if } (?)\ \{X \in [-10; 10]\};$$

$$|X| \leq 10$$

$$4: \qquad X := X/3 + [-\varepsilon_1; \varepsilon_1].X + [-\varepsilon_2; \varepsilon_2];$$

$$|X| \leq \left[v \mapsto \left(\frac{1}{3} + \varepsilon_1\right) \times v + \varepsilon_2\right](10)$$

$$5: \qquad X := 3 \times X + [-\varepsilon_3; \varepsilon_3].X + [-\varepsilon_4; \varepsilon_4];$$

$$|X| \leq f^{(1)}(10)$$

$$6: \qquad k := k + 1;$$

$$|X| \leq f^{(k)}(10),\ k = 1$$

$$7: \qquad \}$$

with $f = \left[v \mapsto \left(1 + 3 \times \varepsilon_1 + \frac{\varepsilon_3}{3} + \varepsilon_1 \times \varepsilon_3\right) \times v + \varepsilon_2 \times (3 + \varepsilon_3) + \varepsilon_4\right].$

Ari.-geo. analysis: Invariant

$X := 0; k := 0;$

$$X = 0,\ k = 0$$

$\text{while}\ (k < 1000)\ \{$

$$|X| \leq f^{(k)}(10)$$

$\quad \text{if}\ (?)\ \{X \in [-10; 10]\};$

$$|X| \leq f^{(k)}(10)$$

$\quad X := X/3 + [-\varepsilon_1; \varepsilon_1].X + [-\varepsilon_2; \varepsilon_2];$

$$|X| \leq (\tfrac{1}{3} + \varepsilon_1) \times \left(f^{(k)}(10)\right) + \varepsilon_2$$

$\quad X := 3 \times X + [-\varepsilon_3; \varepsilon_3].X + [-\varepsilon_4; \varepsilon_4];$

$$|X| \leq f\left(f^{(k)}(10)\right)$$

$\quad k := k + 1;$

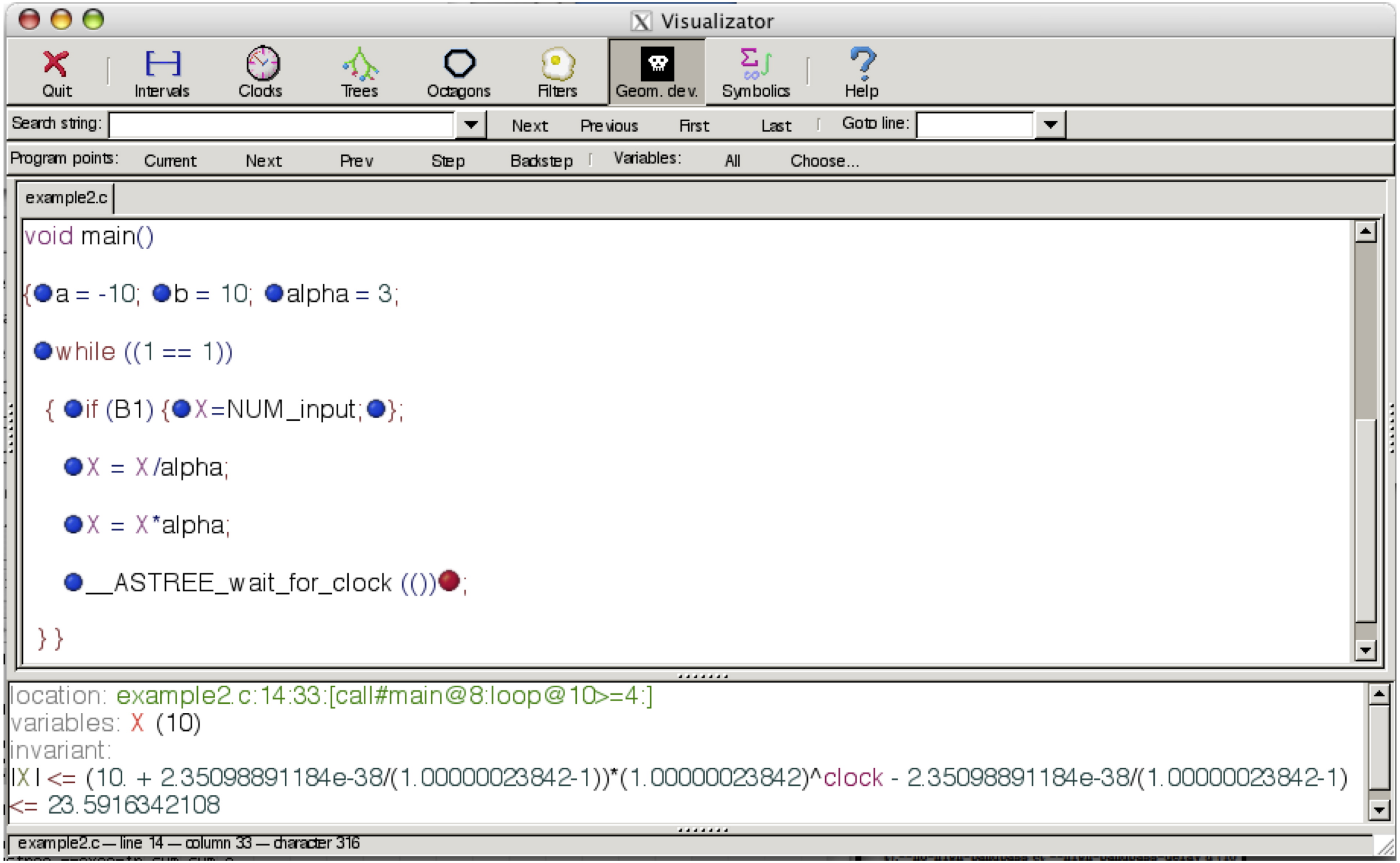
$$|X| \leq f^{(k)}(10)$$

$\}$

$$|X| \leq f^{(1000)}(10)$$

with $f = \left[v \mapsto \left(1 + 3 \times \varepsilon_1 + \frac{\varepsilon_3}{3} + \varepsilon_1 \times \varepsilon_3\right) \times v + \varepsilon_2 \times (3 + \varepsilon_3) + \varepsilon_4\right]$.

Analysis session



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Arithmetic-geometric progressions (in $\mathbb{R}$)

An arithmetic-geometric progression is a 5-tuple in $(\mathbb{R}^+)^5$.
An arithmetic-geometric progression denotes a function in $\mathbb{N} \to \mathbb{R}^+$:

$$\beta_{\mathbb{R}}(M, a, b, a', b')(k) \triangleq \left[v \mapsto a \times v + b\right] \left(\left[v \mapsto a' \times v + b'\right]^{(k)} (M) \right)$$

Thus,

- k is the loop counter;
- M is an initial value;
- $\left[v \mapsto a \times v + b\right]$ describes the current iteration;
- $\left[v \mapsto a' \times v + b'\right]^{(k)}$ describes the first k iterations.

A concretization $\gamma_{\mathbb{R}}$ maps each element $d \in (\mathbb{R}^+)^5$ to a set $\gamma_{\mathbb{R}}(d) \subseteq (\mathbb{N} \to \mathbb{R}^+)$ defined as:

$$\{f \mid \forall k \in \mathbb{N},\ |f(k)| \leq \beta_{\mathbb{R}}(d)(k)\}$$

Monotonicity

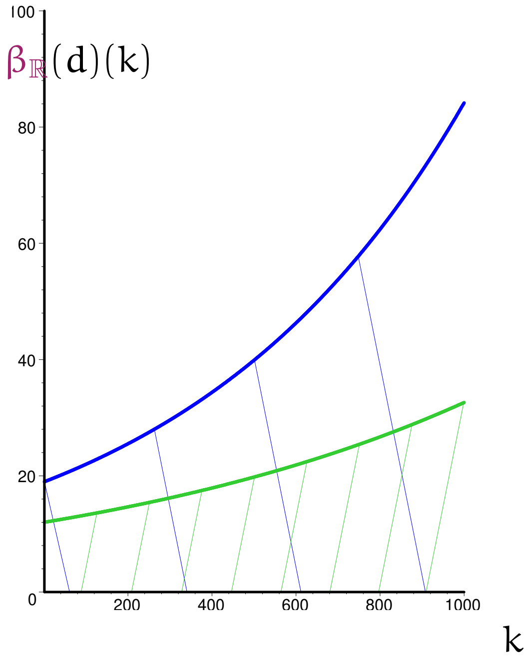
Let $d = (M, a, b, a', b')$ and $d = (M, a, b, a', b')$ be two arithmetic-geometric progressions.

If:

- $M \leq M$,
- $a \leq a$, $a' \leq a'$,
- $b \leq b$, $b' \leq b'$.

Then:

$$\forall k \in \mathbb{N},\ \beta_{\mathbb{R}}(d)(k) \leq \beta_{\mathbb{R}}(d)(k).$$



Disjunction

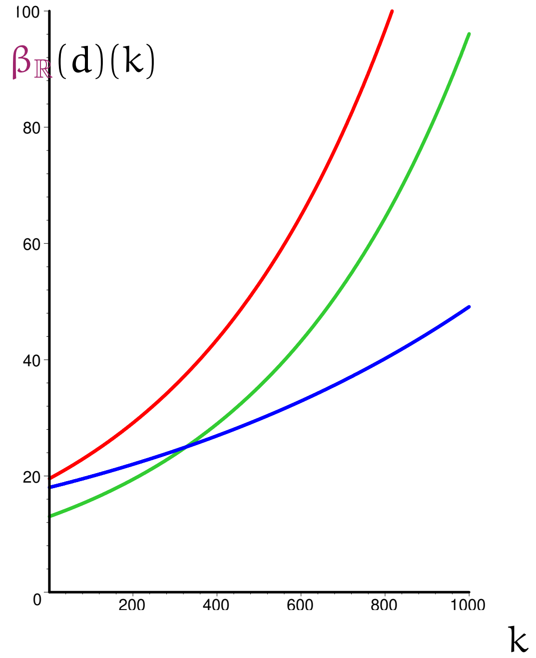
Let $d = (M, a, b, a', b')$ and $d = (M, a, b, a', b')$ be two arithmetic-geometric progressions.

We define:

$$d \sqcup_{\mathbb{R}} d \triangleq (M, a, b, a', b')$$

where:

- $M \triangleq \mathit{max}(M, M)$,
- $a \triangleq \mathit{max}(a, a)$, $a' \triangleq \mathit{max}(a', a')$,
- $b \triangleq \mathit{max}(b, b)$, $b' \triangleq \mathit{max}(b', b')$,



For any $k \in \mathbb{N}$, $\beta_{\mathbb{R}}(d \sqcup_{\mathbb{R}} d)(k) \geq \mathit{max}(\beta_{\mathbb{R}}(d)(k), \beta_{\mathbb{R}}(d)(k))$.

Conjunction

Let d and d be two arithmetic-geometric progressions.

1. If d and d are comparable (component-wise), we take the smaller one:

$$d \sqcap_{\mathbb{R}} d \triangleq \mathit{Inf}_{\dot{\leq}}\{d; d\}.$$

2. Otherwise, we use a parametric strategy:

$$d \sqcap_{\mathbb{R}} d \in \{d; d\}.$$

For any $k \in \mathbb{N}$, $\beta_{\mathbb{R}}(d \sqcap_{\mathbb{R}} d)(k) \geq \mathit{min}(\beta_{\mathbb{R}}(d)(k), \beta_{\mathbb{R}}(d)(k))$.

Assignment (I/III)

We have:

$$\beta_{\mathbb{R}}(M, a, b, a', b')(k) = a \times (M + b' \times k) + b \quad \text{when } a' = 1$$
$$\beta_{\mathbb{R}}(M, a, b, a', b')(k) = a \times \left((a')^k \times \left(M - \frac{b'}{1-a'}\right) + \frac{b'}{1-a'}\right) + b \quad \text{when } a' \neq 1.$$

Thus:

1. for any $a, a', M, b, b', \lambda \in \mathbb{R}^+$,

$$\lambda \times \left(\beta_{\mathbb{R}}\left(M, a, b, a', b'\right)(k)\right) = \beta_{\mathbb{R}}\left(\lambda \times M, a, \lambda \times b, a', \lambda \times b'\right)(k);$$

2. for any $a, a', M, b, b', M, b, b' \in \mathbb{R}^+$, for any $k \in \mathbb{N}$,

$$\beta_{\mathbb{R}}\left(M, a, b, a', b'\right)(k) + \beta_{\mathbb{R}}\left(M, a, b, a', b\right)(k) = \beta_{\mathbb{R}}\left(M + M, a, b + b, a', b' + b'\right)(k).$$

Assignment (II/III)

For $k \in \mathbb{N}$, if:

$$|X_i| \leq \beta_{\mathbb{R}}\left(M_i, a_i, b_i, a_i', b_i'\right)(k)$$

then:

$$\frac{|B + \sum \alpha_i \times X_i| - |B|}{\sum |\alpha_i|} \leq \beta_{\mathbb{R}}\left(\frac{\sum |\alpha_i| \times M_i}{\sum |\alpha_i|}, \mathit{Max}(a_i), \frac{\sum |\alpha_i| \times b_i}{\sum |\alpha_i|}, \mathit{Max}(a_i'), \frac{\sum |\alpha_i| \times b_i'}{\sum |\alpha_i|}\right)(k)$$

so:

$$\left|B + \sum \alpha_i \times X_i\right| \leq \beta_{\mathbb{R}}\left(\frac{\sum |\alpha_i| \times M_i}{\sum |\alpha_i|}, \sum |\alpha_i| \times \mathit{Max}(a_i), \frac{\sum |\alpha_i| \times b_i}{\sum |\alpha_i|} + |B|, \mathit{Max}(a_i'), \frac{\sum |\alpha_i| \times b_i'}{\sum |\alpha_i|}\right)(k)$$

Assignment (III/III)

If for $k \in \mathbb{N}$, $|X| \leq \beta_{\mathbb{R}}(M_X, a_X, b_X, a'_X, b'_X)(k)$ and $|Y| \leq \beta_{\mathbb{R}}(M_Y, a_Y, b_Y, a'_Y, b'_Y)(k)$, then:

1. increment:

$$|X + 3| \leq \beta_{\mathbb{R}}(M_X, a_X, b_X + 3, a'_X, b'_X)(k)$$

2. multiplication:

$$|3 \times X| \leq \beta_{\mathbb{R}}(M_X, 3 \times a_X, b_X, a'_X, b'_X)(k)$$

3. barycentric mean:

$$\left|\frac{X+Y}{2}\right| \leq \beta_{\mathbb{R}}\left(\frac{M_X + M_Y}{2}, \mathit{Max}(a_X, a_Y), \frac{b_X + b_Y}{2}, \mathit{Max}(a'_X, a'_Y), \frac{b'_X + b'_Y}{2}\right)(k)$$

Parametric strategies can be used to transform expressions.

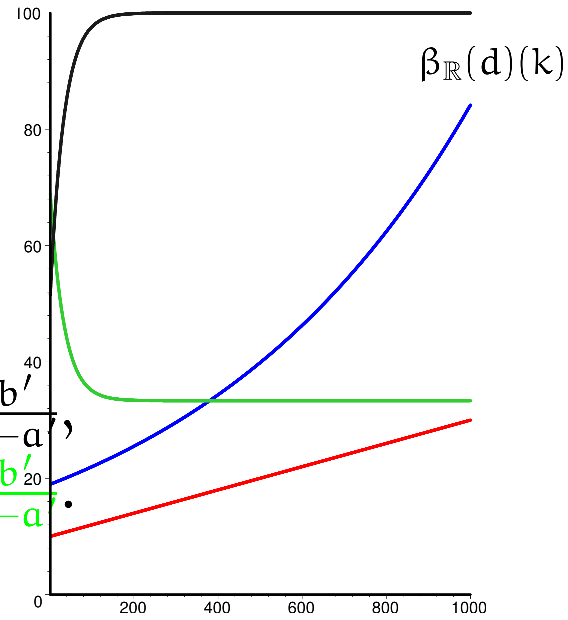
Projection I

$$\beta_{\mathbb{R}}(M,a,b,a',b')(k) = a \times (M + b' \times k) + b \quad \text{when } a' = 1$$
$$\beta_{\mathbb{R}}(M,a,b,a',b')(k) = a \times \left((a')^k \times \left(M - \frac{b'}{1-a'}\right) + \frac{b'}{1-a'}\right) + b \quad \text{when } a' \neq 1.$$

Thus, for any $d \in (\mathbb{R}^+)^5$,
the function $\left[k \mapsto \beta_{\mathbb{R}}(d)(k)\right]$ is:

- either monotonic,
- or anti-monotonic.

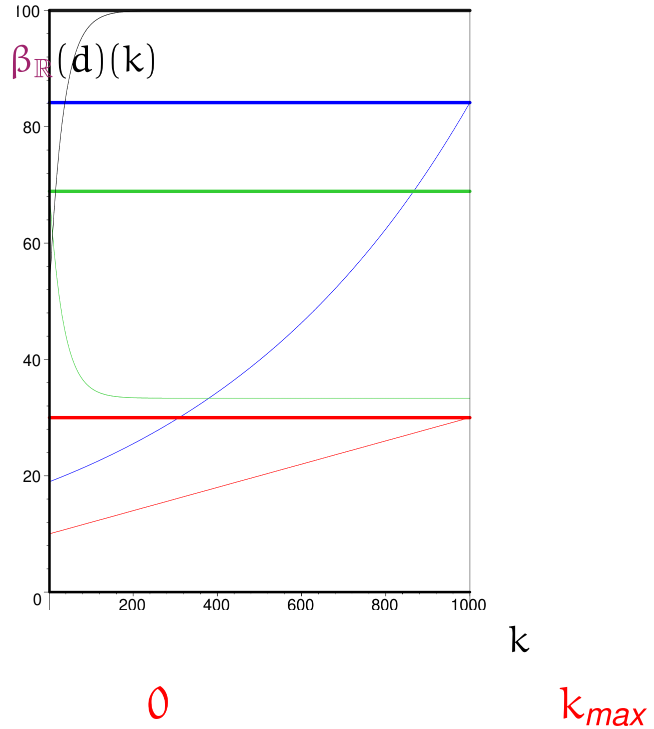
$$\begin{cases} a' > 1, \\ a' = 1, \\ a' < 1 \text{ and } M < \frac{b'}{1-a'}, \\ a' < 1 \text{ and } M > \frac{b'}{1-a'}. \end{cases}$$



Projection II

Let $d \in (\mathbb{R}^+)^5$ and $k_{max} \in \mathbb{N}$.

$$\mathit{bound}(d, k_{max}) \triangleq \mathit{max}(\beta_{\mathbb{R}}(d)(0), \beta_{\mathbb{R}}(d)(k_{max}))$$



For any $k \in \mathbb{N}$ such that $0 \leq k \leq k_{max}$:

$$\beta(d)(k) \leq \mathit{bound}(d, k_{max}).$$

Incrementing the loop counter

We integrate the current iteration into the first k iterations:

- the first $k+1$ iterations are chosen as the worst case among the first k iterations and the current iteration;
- the current iteration is reset.

Thus:

$$\textit{next}_{\mathbb{R}}(M, a, b, a', b') \stackrel{\Delta}{=} (M, 1, 0, \textit{max}(a, a'), \textit{max}(b, b')).$$

For any $k \in \mathbb{N}, d \in \left(\mathbb{R}^+\right)^5, \ \beta_{\mathbb{R}}(d)(k) \leq \beta_{\mathbb{R}}(\textit{next}_{\mathbb{R}}(d))(k+1)$.

About floating point numbers

Floating point numbers occur:

1. in the concrete semantics:

   Floating point expressions are translated into real expressions with interval coefficients [Miné—ESOP'04].

   In other abstract domains, we handle real numbers.

2. in the abstract domain implementation:

   For efficiency purpose, we implement each primitive in floating point arithmetics: each real is safely approximated by an interval with floating point number bounds.

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Applications

Arithmetic-geometric progressions provide bounds for :

1. division by α followed by a multiplication by α:

   $\Longrightarrow$ our running example;

2. barycentric means:

   $\Longrightarrow$ at each loop iteration, the value of a variable X is computed as a barycentric mean of some previous values of X

   (not necessarily the last values);

3. bounded incremented variables:

   $\Longrightarrow$ it replaces the former domain that bounds the difference and the sum between each variable and the loop counter.

Benchmarks

We analyze three programs in the same family on a AMD Opteron 248, 8 Gb of RAM (analyses use only 2 Gb of RAM).

lines of C	70,000			216,000			379,000		
global variables	13,400			7,500			9,000		
iterations	80	63	**37**	229	223	**53**	253	286	**74**
time/iteration	1mn14s	1mn21s	**1mn16s**	4mn04s	5mn13s	**4mn40s**	7mn33s	9mn42s	**8mn17s**
analysis time	2h18mn	2h05mn	**47mn**	15h34mn	19h24mn	**4h08mn**	31h53mn	43h51mn	**10h14mn**
false alarms	625	24	**0**	769	64	**0**	1482	188	**0**

1. without using computation time;
2. with the former loop counter domain,
   (without the arithmetic-geometric domain);
3. with the arithmetic-geometric domain,
   (without the former loop counter domain).

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A new abstract domain

- non polynomial constraints;
- sound with respect to rounding errors
  (both in the concrete semantics and in the domain implementation);
- accurate
  (we infer bounds on the values that depend on the execution time of the program);
- efficient:
  - in time: $\mathcal{O}(n \times \ln(n))$ per abstract iteration
    (n denotes the program size),
  - in memory: at most 5 coefficients per variable in the program,
  - sparse implementation.

http://www.astree.ens.fr