V

Frames

A signal representation may provide "analysis" coefficients which are inner products with a family of vectors, or "synthesis" coefficients that compute an approximation by recombining these vectors. Frames are families of vectors where these representations are stable, and computed with a dual frame. Frames are potentially redundant and thus more general than bases, with a redundancy measured by frame bounds. They provide the flexibility needed to build signal representations with unstructured families of vectors.

Complete and stable wavelet and windowed Fourier transforms are constructed with frames of wavelets and windowed Fourier atoms. In two dimensions, frames of directional wavelets and curvelets are introduced to analyze and process multiscale image structures.

5.1 Frames and Riesz Bases

5.1.1 Stable Analysis and Synthesis Operators

The frame theory was originally developed by Duffin and Schaeffer [234] to reconstruct band-limited signals from irregularly spaced samples. They established general conditions to recover a vector f in a Hilbert space $\mathbf{H}$ from its inner products with a family of vectors $\{\phi_n\}_{n\in\Gamma}$. The index set Γ might be finite or infinite. The following frame definition gives an energy equivalence to invert the operator Φ defined by

$$\forall n \in \Gamma \quad , \quad \Phi f[n] = \langle f, \phi_n \rangle. \tag{5.1}$$

Definition 5.1 (Frame and Riesz Basis)**.** *The sequence $\{\phi_n\}_{n\in\Gamma}$ is a frame of $\mathbf{H}$ if there exist two constants $B \geqslant A > 0$ such that*

$$\forall f \in \mathbf{H} \quad , \quad A\,\|f\|^2 \leqslant \sum_{n\in\Gamma} |\langle f, \phi_n\rangle|^2 \leqslant B\,\|f\|^2. \tag{5.2}$$

When $A = B$ the frame is said to be tight. If the $\{\phi_n\}_{n\in\Gamma}$ are linearly independant then the frame is not redundant and is called a Riesz basis.

If the frame condition is satisfied then Φ is called a frame analysis operator. Section 5.1.2 proves that (5.2) is a necessary and sufficient condition guaranteeing that Φ is invertible on its image space, with a bounded inverse. A frame thus defines a complete and stable signal representation, which may also be redundant.

Frame Synthesis Let us consider the space of finite energy coefficients

$$\boldsymbol{\ell}^2(\Gamma) = \{a \;:\; \|a\|^2 = \sum_{n\in\Gamma} |a[n]|^2 < +\infty\} \;.$$

The adjoint Φ^* of Φ is defined over $\boldsymbol{\ell}^2(\Gamma)$ and satisfies for any $f \in \mathbf{H}$ and $a \in \boldsymbol{\ell}^2(\Gamma)$

$$\langle \Phi^* a, f \rangle \ = \ \langle a, \Phi f \rangle \ = \ \sum_{n\in\Gamma} a[n] \, \langle f, \phi_n \rangle^* .$$

It is therefore the synthesis operator

$$\Phi^* a = \sum_{n\in\Gamma} a[n] \, \phi_n \ . \tag{5.3}$$

The frame condition (5.2) can be rewritten

$$\forall f \in \mathbf{H} \quad , \quad A \, \|f\|^2 \leqslant \|\Phi f\|^2 = \langle \Phi^* \Phi f, f \rangle \leqslant B \, \|f\|^2, \tag{5.4}$$

with

$$\Phi^* \Phi f = \sum_{m\in\Gamma} \langle f, \phi_n \rangle \, \phi_n \ .$$

It results that A and B are the infimum and supremum values of the spectrum of the symmetric operator $\Phi^*\Phi$, which correspond to the smallest and largest eigenvalues in finite dimension. The eigenvalues are also called *singular values* of Φ or *singular spectrum*. The following theorem derives that the frame synthesis operator is also stable.

Theorem 5.1. *The family* $\{\phi_n\}_{n\in\Gamma}$ *is a frame with bounds* $0 < A \leqslant B$ *if and only if*

$$\forall a \in \mathbf{Im}\Phi \quad , \quad A \, \|a\|^2 \leqslant \| \sum_{n\in\Gamma} a[n] \, \phi_n \|^2 \leqslant B \, \|a\|^2 \, . \tag{5.5}$$

Proof. Since $\Phi^* a = \sum_{n\in\Gamma} a[n] \, \phi_n$, it results that

$$\| \sum_{n\in\Gamma} a[n] \, \phi_n \|^2 = \langle \Phi \Phi^* a, a \rangle \ .$$

The operator Φ is a frame if and only if the spectrum of $\Phi^*\Phi$ is bound by A and B. The inequality (5.5) states that the the spectrum of $\Phi\Phi^*$ over $\mathbf{Im}\Phi$ is also bounded by A and B. Both statements are proved to be equivalent by verifying that sup and inf of the spectrum of $\Phi^*\Phi$ is equal to the sup and inf of the spectrum of $\Phi^*\Phi$.

In finite dimension, if λ is an eigenvalue of $\Phi^*\Phi$ with eigenvector f then λ is also an eigenvalue of $\Phi^*\Phi$ with eigenvector Φf. Indeed, $\Phi^*\Phi f = \lambda f$ so $\Phi\Phi^*(\Phi f) = \lambda \Phi f$ and $\Phi f \neq 0$ because the left frame inequality (5.2) implies that $\|\Phi f\|^2 \leqslant A \, \|f\|^2$. It results that the maximum and minimum eigenvectors of $\Phi^*\Phi$ and $\Phi\Phi^*$ on $\mathbf{Im}\Phi$ are identical.

In a Hilbert space of infinite dimension, we prove that the sup and inf of the spectrum of both operators remain identical by growing the space dimension, and computing the limit of the largest and smallest eigenvalues when the space dimension tends to infinity. ■

This theorem proves that linear combination of frame vectors define a stable signal representation. Section 5.1.2 proves that synthesis coefficients are computed with a dual frame. The operator $\Phi\Phi^*$ is the *Gram* matrix, whose coefficients are $\{\langle \phi_n, \phi_p \rangle\}_{(m,p)\in\boldsymbol{\ell}^2(\Gamma)}$:

$$\Phi\Phi^* a[p] = \sum_{m\in\Gamma} a[m] \, \langle \phi_n, \phi_p \rangle \, . \tag{5.6}$$

One must be careful because (5.5) is only valid for $a \in \mathbf{Im}\Phi$. If it is valid for all $a \in \boldsymbol{\ell}^2(\Gamma)$ with $A > 0$ then the family is linearly independant and is thus a Riesz basis.

Redundancy When the frame vectors are normalized $\|\phi_n\| = 1$, the following theorem shows that the frame redundancy is measured by the frame bounds A and B.

Theorem 5.2. *In a space of finite dimension N, a frame of $P \geqslant N$ normalized vectors has frame bounds A and B which satisfy*

$$A \leqslant \frac{P}{N} \leqslant B. \tag{5.7}$$

For a tight frame $A = B = P/N$.

Proof. It results from (5.4) that all eigenvalues of $\Phi^*\Phi$ are between A and B. The trace of $\Phi^*\Phi$ thus satisfies

$$A\,N \leqslant \mathrm{tr}(\Phi^*\Phi) \leqslant B\,N\ .$$

But since the trace is not modified by commuting matrices (Exercise 5.4), and $\|\phi_n\| = 1$

$$A\,N \leqslant \mathrm{tr}(\Phi^*\Phi) = \mathrm{tr}(\Phi\Phi^*) = \sum_{n=1}^{P} |\langle \phi_n, \phi_n \rangle|^2 = P \leqslant B\,N\ ,$$

which implies (5.7). ■

If $\{\phi_n\}_{n\in\Gamma}$ is a normalized Riesz basis and is therefore linearly independent then (5.7) proves that $A \leqslant 1 \leqslant B$. This result remains valid in infinite dimension. Inserting $f = \phi_n$ in the frame inequality (5.2) proves that the frame is orthonormal if and only if $B = 1$ in which case $A = 1$.

Example 5.1. *Let $\{g_1, g_2\}$ be an orthonormal basis of an $N = 2$ two-dimensional plane* **H**. *The $P = 3$ normalized vectors*

$$\phi_1 = g_1 \quad , \quad \phi_2 = -\frac{g_1}{2} + \frac{\sqrt{3}}{2}\, g_2 \quad , \quad \phi_3 = -\frac{g_1}{2} - \frac{\sqrt{3}}{2}\, g_2$$

have equal angles of $2\pi/3$ between each other. For any $f \in \mathbf{H}$

$$\sum_{n=1}^{3} |\langle f, \phi_n \rangle|^2 = \frac{3}{2}\, \|f\|^2.$$

These three vectors thus define a tight frame with $A = B = 3/2$.

Example 5.2. *For any $0 \leqslant k < K$, suppose that $\{\phi_{k,n}\}_{n\in\Gamma}$ is an orthonormal basis of* **H**. *The union of these K orthonormal bases $\{\phi_{k,n}\}_{n\in\Gamma, 0\leqslant k<K}$ is a tight frame with $A = B = K$. Indeed, the energy conservation in an orthonormal basis implies that for any $f \in \mathbf{H}$,*

$$\sum_{n\in\mathbb{Z}} |\langle f, \phi_{k,n} \rangle|^2 = \|f\|^2,$$

hence

$$\sum_{k=0}^{K-1} \sum_{n\in\mathbb{Z}} |\langle f, \phi_{k,n} \rangle|^2 = K\, \|f\|^2.$$

One can verify (Exercice 5.3) that a finite set of N vectors $\{\phi_n\}_{1\leqslant n\leqslant N}$ is always a frame of the space **V** generated by linear combinations of these vectors. When N increases, the frame bounds A and B may go respectively to 0 and $+\infty$. This illustrates the fact that in infinite dimensional spaces, a family of vectors may be complete and not yield a stable signal representation.

Irregular Sampling Let $\mathbf{U}_s$ be the space of $\mathbf{L^2}(\mathbb{R})$ functions whose Fourier transforms have a support included in $[-\pi/s, \pi/s]$. For a uniform sampling, $t_n = ns$, Theorem 3.5 proves that if $\phi_s(t) = s^{1/2} \sin(\pi s^{-1} t)/(\pi t)$ then $\{\phi_s(t-ns)\}_{n\in\mathbb{Z}}$ is an orthonormal basis of $\mathbf{U}_s$. The reconstruction of f from its samples is then given by the sampling Theorem 3.2.

The irregular sampling conditions of Duffin and Schaeffer [234] for constructing a frame were later refined by several researchers [100, 500, 79]. Grochenig proved [284] that if $\lim_{n\to+\infty} t_n = +\infty$ and $\lim_{n\to-\infty} t_n = -\infty$, and if the maximum sampling distance δ satisfies

$$\delta = \sup_{n\in\mathbb{Z}} |t_{n+1} - t_n| < s \ , \tag{5.8}$$

then

$$\{\lambda_n \, \phi_s(t - t_n)\}_{n\in\mathbb{Z}} \quad \text{with} \quad \lambda_n = \sqrt{\frac{t_{n+1} - t_{n-1}}{2s}}$$

is a frame with frame bounds $A \geqslant (1 - \delta/s)^2$ and $B \leqslant (1 + \delta/s)^2$. The amplitude factor λ_n compensates for the increase of sample density relatively to s. The reconstruction of f requires inverting the frame operator $\Phi f[n] = \langle f(u), \lambda_n, \phi_s(u - t_n)\rangle$.

5.1.2 Dual Frame and Pseudo Inverse

The reconstruction of f from its frame coefficients $\Phi f[n]$ is calculated with a pseudo inverse also called Moore-Penrose pseudo-inverse. This pseudo inverse is a bounded operator that is implements a dual frame reconstruction. For Riesz bases, this dual frame is a biorthogonal basis.

For any operator U, we denote by **ImU** the image space of all Uf and by **NullU** the null space of all h such that $Uh = 0$.

Theorem 5.3. *If $\{\phi_n\}_{n\in\Gamma}$ is a frame but not a Riesz basis then Φ admits an infinite number of left inverses.*

Proof. We know that $\mathbf{Null}\Phi^* = (\mathbf{Im}\Phi)^\perp$ is the orthogonal complement of $\mathbf{Im}\Phi$ in $\boldsymbol{\ell}^2(\Gamma)$ (Exercice 5.7). If Φ is a frame and not a Riesz basis then $\{\phi_n\}_{n\in\Gamma}$ is linearly dependent so there exists $a \in \mathbf{Null}\Phi^* = (\mathbf{Im}\Phi)^\perp$ with $a \neq 0$.

A frame operator Φ is injective (one to one). Indeed, the frame inequality (5.2) guarantees that $\Phi f = 0$ implies $f = 0$. Its restriction to $\mathbf{Im}\Phi$ is thus invertible, which means that Φ admits a left inverse. There is an infinite number of left inverses since the restriction of a left inverse to $(\mathbf{Im}\Phi)^\perp \neq \{0\}$ may be any arbitrary linear operator. ■ ■

The more redundant the frame $\{\phi_n\}_{n\in\Gamma}$, the larger the orthogonal complement $(\mathbf{Im}\Phi)^\perp$ of $\mathbf{Im}\Phi$ in $\boldsymbol{\ell}^2(\Gamma)$. The pseudo inverse, that we write Φ^+, is defined as the left inverse that is zero on $(\mathbf{Im}\Phi)^\perp$:

$$\forall f \in \mathbf{H} \quad , \quad \Phi^+ \Phi f = f \quad \text{and} \quad \forall a \in (\mathbf{Im}\Phi)^\perp \quad , \quad \Phi^+ a = 0. \tag{5.9}$$

The following theorem computes this pseudo-inverse.

Theorem 5.4 (Pseudo inverse)**.** *If Φ is a frame operator then $\Phi^*\Phi$ is invertible and the pseudo inverse satisfies*

$$\Phi^+ = (\Phi^*\Phi)^{-1}\Phi^*. \tag{5.10}$$

Proof. The frame condition in (5.4) is rewritten

$$\forall f \in \mathbf{H} \quad , \quad A\, \|f\|^2 \leqslant \langle \Phi^*\Phi f, f\rangle \leqslant B\, \|f\|^2.$$

It result that $\Phi^*\Phi$ is an injective self-adoint opeator: $\Phi^*\Phi f = 0$ if and only if $f = 0$. It is therefore invertible. For all $f \in \mathbf{H}$

$$\Phi^+\Phi f = (\Phi^*\Phi)^{-1}\Phi^*\Phi f = f \ ,$$

so Φ^+ is a left inverse. Since $(\mathbf{Im}\Phi)^\perp = \mathbf{Null}\Phi^*$ it results that $\Phi^+ a = 0$ for any $a \in (\mathbf{Im}\Phi)^\perp = \mathbf{Null}\Phi^*$. Since this left inverse vanishes on $(\mathbf{Im}\Phi)^\perp$, it is the pseudo-inverse. ■ ■

Dual Frame The pseudo inverse of a frame operator implements a reconstruction with a dual frame, which is specified by the following theorem.

Theorem 5.5. *Let $\{\phi_n\}_{n\in\Gamma}$ be a frame with bounds $0 < A \leqslant B$. The dual operator defined by*

$$\forall n \in \Gamma \quad , \quad \tilde{\Phi} f[n] = \langle f, \tilde{\phi}_n \rangle \quad \text{with} \quad \tilde{\phi}_n = (\Phi^* \Phi)^{-1} \phi_n \tag{5.11}$$

satisfies $\tilde{\Phi}^* = \Phi^+$ and hence

$$f = \sum_{n\in\Gamma} \langle f, \phi_n \rangle \, \tilde{\phi}_n = \sum_{n\in\Gamma} \langle f, \tilde{\phi}_n \rangle \, \phi_n. \tag{5.12}$$

It defines a dual frame

$$\forall f \in \mathbf{H} \quad , \quad \frac{1}{B} \|f\|^2 \leqslant \sum_{n\in\Gamma} |\langle f, \tilde{\phi}_n \rangle|^2 \leqslant \frac{1}{A} \|f\|^2 . \tag{5.13}$$

If the frame is tight (i.e., $A = B$), then $\tilde{\phi}_n = A^{-1} \, \phi_n$.

Proof. The dual operator can be written $\tilde{\Phi} = \Phi(\Phi^*\Phi)^{-1}$. Indeed,

$$\tilde{\Phi} f[n] = \langle f, \tilde{\phi}_n \rangle = \langle f, (\Phi^*\Phi)^{-1} \phi_n \rangle = \langle (\Phi^*\Phi)^{-1} f, \phi_n \rangle = \Phi(\Phi^*\Phi)^{-1} f.$$

We thus derive from (5.10) that its adjoint is the pseudo-inverse of Φ:

$$\tilde{\Phi}^* = (\Phi^*\Phi)^{-1} \Phi^* = \Phi^+$$

It results that $\Phi^+ \Phi = \tilde{\Phi}^* \Phi = \mathrm{Id}$ and hence that $\Phi^* \tilde{\Phi} = \mathrm{Id}$, which proves (5.12).

Let us now prove the frame bounds (5.13). Frame conditions are rewritten in (5.4):

$$\forall f \in \mathbf{H} \quad , \quad A \|f\|^2 \leqslant \langle \Phi^*\Phi f, f \rangle \leqslant B \|f\|^2. \tag{5.14}$$

The following lemma applied to $L = \Phi^*\Phi$ proves that

$$\forall f \in \mathbf{H} \quad , \quad B^{-1} \|f\|^2 \leqslant \langle (\Phi^*\Phi)^{-1} f, f \rangle \leqslant A^{-1} \|f\|^2. \tag{5.15}$$

Since for any $f \in \mathbf{H}$

$$\|\tilde{\Phi} f\|^2 = \langle \Phi(\Phi^*\Phi)^{-1} f, \Phi(\Phi^*\Phi)^{-1} f \rangle = \langle f, (\Phi^*\Phi)^{-1} f \rangle \, ,$$

the dual frame bounds (5.13) are derived from (5.15).

If $A = B$ then $\langle \Phi^*\Phi f, f \rangle = A \|f\|^2$. The spectrum of $\Phi^*\Phi$ is thus reduced to A and hence $\Phi^*\Phi = A \, \mathrm{Id}$. As a result $\tilde{\phi}_n = (\Phi^*\Phi)^{-1} \phi_n = A^{-1} \phi_n$.

Lemma 5.1. *If L is a self-adjoint operator such that there exist $B \geqslant A > 0$ satisfying*

$$\forall f \in \mathbf{H} \quad , \quad A \|f\|^2 \leqslant \langle Lf, f \rangle \leqslant B \|f\|^2 \tag{5.16}$$

then L is invertible and

$$\forall f \in \mathbf{H} \quad , \quad \frac{1}{B} \|f\|^2 \leqslant \langle L^{-1} f, f \rangle \leqslant \frac{1}{A} \|f\|^2. \tag{5.17}$$

In finite dimensions, since L is self-adjoint we know that it is diagonalized in an orthonormal basis. The inequality (5.16) proves that its eigenvalues are between A and B. It is therefore invertible with eigenvalues between B^{-1} and A^{-1}, which proves (5.17). In a Hilbert space of infinite dimension, we prove that same result on the sup and inf of the spectrum by growing the space dimension, and computing the limit of the largest and smallest eigenvalues when the space dimension tends to infinity. ■ ■

This theorem proves that f is reconstructed from frame coefficients $\Phi f[n] = \langle f, \phi_n \rangle$ with the dual frame $\{\tilde{\phi}_n\}_{n\in\Gamma}$. The synthesis coefficients of f in $\{\phi_n\}_{n\in\Gamma}$ are the dual frame coefficients $\tilde{\Phi} f[n] = \langle f, \phi_n \rangle$. If the frame is tight then both decompositions are identical

$$f = \frac{1}{A} \sum_{n\in\Gamma} \langle f, \phi_n \rangle \, \phi_n. \tag{5.18}$$

Biorthogonal Bases A Riesz basis is a frame of vectors that are linearly independent, which implies that $\mathbf{Im}\Phi = \ell^2(\Gamma)$, then its dual frame is also linearly independent. Inserting $f = \phi_p$ in (5.12) yields

$$\phi_p = \sum_{n\in\Gamma} \langle \phi_p, \tilde{\phi}_n \rangle \, \phi_n,$$

and the linear independence implies that

$$\langle \phi_p, \tilde{\phi}_n \rangle = \delta[p-n].$$

Dual Riesz bases are thus biorthogonal families of vectors. If the basis is normalized (i.e., $\|\phi_n\| = 1$), then

$$A \leqslant 1 \leqslant B. \tag{5.19}$$

This is proved by inserting $f = \phi_p$ in the frame inequality (5.13):

$$\frac{1}{B}\,\|\phi_p\|^2 \leqslant \sum_{n\in\Gamma} |\langle \phi_p, \tilde{\phi}_n \rangle|^2 = 1 \leqslant \frac{1}{A}\,\|\phi_p\|^2.$$

5.1.3 Dual Frame Analysis and Synthesis Computations

Suppose that $\{\phi_n\}_{n\in\Gamma}$ is a frame of a subspace $\mathbf{V}$ of the whole signal space. The best linear approximation of f in $\mathbf{V}$ is the orthogonal projection of f in $\mathbf{V}$. The following theorem shows that this orthogonal projection is computed with the dual frame. Two iterative numerical algorithms are described to implement such computations.

Theorem 5.6. *Let $\{\phi_n\}_{n\in\Gamma}$ be a frame of $\mathbf{V}$ and $\{\tilde{\phi}_n\}_{n\in\Gamma}$ it's dual frame in $\mathbf{V}$. The orthogonal projection of $f \in \mathbf{H}$ in $\mathbf{V}$ is*

$$P_{\mathbf{V}} f = \sum_{n\in\Gamma} \langle f, \phi_n \rangle \, \tilde{\phi}_n = \sum_{n\in\Gamma} \langle f, \tilde{\phi}_n \rangle \, \phi_n \ . \tag{5.20}$$

Proof. Since both frames are dual in $\mathbf{V}$, if $f \in \mathbf{V}$ then (5.12) proves that the operator $P_{\mathbf{V}}$ defined in (5.20) satisfies $P_{\mathbf{V}} f = f$. To prove that it is an orthogonal projection it is sufficient to verify that for all $f \in \mathbf{H}$ $\langle f - P_{\mathbf{V}} f, \phi_p \rangle = 0$ for all $p \in \Gamma$. Indeed,

$$\langle f - P_{\mathbf{V}} f, \phi_p \rangle = \langle f, \phi_p \rangle - \sum_{n\in\Gamma} \langle f, \phi_n \rangle \, \langle \tilde{\phi}_n, \phi_p \rangle = 0$$

because the dual frame property implies that $\sum_{n\in\Gamma} \langle \tilde{\phi}_n, \phi_p \rangle \, \phi_n = \phi_p$. ■

If Γ is finite then $\{\phi_n\}_{n\in\Gamma}$ is necessarily a frame of the space $\mathbf{V}$ it generates, and (5.20) reconstructs the best linear approximation of f in $\mathbf{V}$. This result is particularly important for approximating signals from a finite set of vectors.

Since Φ is not a frame of the whole signal space $\mathbf{H}$ but of a subspace $\mathbf{V}$ then Φ is only invertible on this subspace and the pseudo inverse definition becomes:

$$\forall f \in \mathbf{V} \quad , \quad \Phi^+ \Phi f = f \quad \text{and} \quad \forall a \in (\mathbf{Im}\Phi)^\perp \quad , \quad \Phi^+ a = 0. \tag{5.21}$$

Let $\Phi_{\mathbf{V}}$ be the restriction of Φ to $\mathbf{V}$. The operator $\Phi^* \Phi_{\mathbf{V}}$ is invertible on $\mathbf{V}$ and we write $(\Phi^* \Phi_{\mathbf{V}})^{-1}$ its inverse. Similarly to (5.10), we verify that $\Phi^+ = (\Phi^* \Phi_{\mathbf{V}})^{-1} \Phi^*$.

Dual Synthesis In a dual synthesis problem, the orthogonal projection is computed from the frame coefficients $\{\Phi f[n] = \langle f, \phi_n \rangle\}_{n\in\Gamma}$ with the dual frame synthesis operator:

$$P_{\mathbf{V}} f = \tilde{\Phi}^* \, \Phi f = \sum_{n\in\Gamma} \langle f, \phi_n \rangle \, \tilde{\phi}_n \ . \tag{5.22}$$

If the frame $\{\phi_n\}_{n\in\Gamma}$ does not depend upon the signal f then the dual frame vectors are precomputed with (5.11):

$$\forall n \in \Gamma \quad , \quad \tilde{\phi}_n = (\Phi^* \Phi_{\mathbf{V}})^{-1} \phi_n, \tag{5.23}$$

and the dual synthesis is solved directly with (5.22). In many applications, the frame vectors $\{\phi_n\}_{n\in\Gamma}$ depend on the signal f, in which case the dual frame vectors $\tilde{\phi}_n$ cannot be computed in advance, and it is highy inefficient to compute them. This is the case when coefficients $\{\langle f, \phi_n\rangle\}_{n\in\Gamma}$ are selected in a redundant transform, to build a sparse signal representation. For example, the time-frequency ridge vectors in Sections 4.4.1 and 4.4.2, are selected from the local maxima of f in highly redundant windowed Fourier or wavelet transforms.

The transform coefficients Φf are known and we must compute

$$P_{\mathbf{V}} f = \tilde{\Phi}\Phi f = (\Phi^* \Phi_{\mathbf{V}})^{-1} \Phi^* \Phi f \ .$$

A dual synthesis algorithm computes first

$$y = \Phi^* \Phi f = \sum_{n\in\Gamma} \langle f, \phi_n\rangle \, \phi_n \in \mathbf{V}$$

and then derives $P_{\mathbf{V}} f = L^{-1} y = z$ by applying the inverse of the symmetric operator $L = \Phi^* \Phi_{\mathbf{V}}$ to y, with

$$\forall h \in \mathbf{V} \quad , \quad Lh = \sum_{n\in\Gamma} \langle h, \phi_n\rangle \, \phi_n \ . \tag{5.24}$$

The eigenvalues of L are between A and B.

Dual Analysis In a dual analysis, the orthogonal projection $P_{\mathbf{V}} f$ is computed from the frame vectors $\{\phi_n\}_{n\in\Gamma}$ with the dual frame analysis operator $\tilde{\Phi} f[n] = \langle f, \tilde{\phi}_n\rangle$:

$$P_{\mathbf{V}} f = \Phi^* \, \tilde{\Phi} f = \sum_{n\in\Gamma} \langle f, \tilde{\phi}_n\rangle \, \phi_n \ . \tag{5.25}$$

If $\{\phi_n\}_{n\in\Gamma}$ does not depend upon f then $\{\tilde{\phi}_n\}_{n\in\Gamma}$ is precomputed with (5.23). The $\{\phi_n\}_{n\in\Gamma}$ may also be selected adaptively from a larger dictionary, to provide a sparse approximation of f. Computing the orthogonal projection $P_{\mathbf{V}} f$ is called a *back projection*. In Section 12.3, matching pursuits implement this back projection.

When $\{\phi_n\}_{n\in\Gamma}$ depends on f, computing the dual frame is inefficient. The dual coefficient $a[n] = \tilde{\Phi} f[n]$ are calculated directly, as well as

$$P_{\mathbf{V}} f = \Phi^* a = \sum_{n\in\Gamma} a[n] \, \phi_n \ . \tag{5.26}$$

Since $\Phi P_{\mathbf{V}} f = \Phi f$, we have $\Phi\Phi^* a = \Phi f$. Let $\Phi^*_{\mathbf{Im}\Phi}$ be the restriction of Φ^* to $\mathbf{Im}\Phi$. Since $\Phi\Phi^*_{\mathbf{Im}\Phi}$ is invertible on $\mathbf{Im}\Phi$

$$a = (\Phi\Phi^*_{\mathbf{Im}\Phi})^{-1} \Phi \, f \ .$$

The dual analysis algorithm thus computes $y = \Phi f = \{\langle f, \phi_n\rangle\}_{n\in\Gamma}$ and derives the dual coefficients $a = L^{-1} y = z$ by applying the inverse of the Gram operator $L = \Phi\Phi^*_{\mathbf{Im}\Phi}$ to y, with

$$Lh[n] = \sum_{p\in\Gamma} h[n] \, \langle \phi_n, \phi_p\rangle \ . \tag{5.27}$$

The eigenvalues of L are also between A and B. The orthogonal projection of f is recovered with (5.26).

Richardson Inversion of Symmetric Operators The key computational step of a dual analysis or a dual synthesis problem is to compute $z = L^{-1}y$, where L is a symmetric operator whose eigenvalues are between A and B. Theorems 5.7 and 5.8 describe two iterative algorithms with exponential convergence. The *Richardson iteration procedure* is simpler but requires knowing the frame bounds A and B. *Conjugate gradient* iterations converge more quickly when B/A is large, and do not require knowing the values of A and B.

Theorem 5.7. *To compute $z = L^{-1}y$, let z_0 be an initial value and $\gamma > 0$ be a relaxation parameter. For any $k > 0$, define*

$$z_k = z_{k-1} + \gamma\,(y - Lz_{k-1}). \tag{5.28}$$

If

$$\delta = \max\{|1-\gamma A|, |1-\gamma B|\} < 1, \tag{5.29}$$

then

$$\|z - z_k\| \leqslant \delta^k\, \|z - z_0\|, \tag{5.30}$$

and hence $\lim_{k\to+\infty} z_k = z$.

Proof. The induction equation (5.28) can be rewritten

$$z - z_k = z - z_{k-1} - \gamma\, L(z - z_{k-1}).$$

Let

$$R = Id - \gamma\, L,$$

$$z - z_k = R(z - z_{k-1}) = R^k(z - z_0). \tag{5.31}$$

Since the eigenvalues of L are between A and B

$$A\,\|z\|^2 \leqslant \langle Lz, z\rangle \leqslant B\,\|z\|^2.$$

This implies that $R = I - \gamma L$ satisfies

$$|\langle Rz, z\rangle| \leqslant \delta\,\|z\|^2,$$

where δ is given by (5.29). Since R is symmetric, this inequality proves that $\|R\| \leqslant \delta$. We thus derive (5.30) from (5.31). The error $\|z - z_k\|$ clearly converges to zero if $\delta < 1$. ■ ■

The convergence is guaranteed for all initial values z_0. If an estimation z_0 of the solution z is known then this estimation can be chosen, otherwise z_0 is often set to 0. For frame inversion, the Richardson iteration algorithm is sometimes called the *frame algorithm* [18]. The convergence rate is maximized when δ is minimum:

$$\delta = \frac{B-A}{B+A} = \frac{1-A/B}{1+A/B},$$

which corresponds to the relaxation parameter

$$\gamma = \frac{2}{A+B}. \tag{5.32}$$

The algorithm converges quickly if A/B is close to 1. If A/B is small then

$$\delta \approx 1 - 2\,\frac{A}{B}. \tag{5.33}$$

The inequality (5.30) proves that we obtain an error smaller than ε for a number n of iterations, which satisfies:

$$\frac{\|z - z_k\|}{\|z - z_0\|} \leqslant \delta^k = \varepsilon.$$

Inserting (5.33) gives

$$k \approx \frac{\log_e \varepsilon}{\log_e(1 - 2A/B)} \approx \frac{-B}{2A}\,\log_e \varepsilon. \tag{5.34}$$

The number of iterations thus increases proportionally to the frame bound ratio B/A.

The exact values of A and B are often not known, and A is generally more difficult to compute. The upper frame bound is $B = \|\Phi\,\Phi^*\|_S = \|\Phi^*\,\Phi\|_S$. If we choose

$$\gamma < 2\,\|\Phi\,\Phi^*\|_S^{-1} \tag{5.35}$$

then (5.29) shows that the algorithm is guaranteed to converge, but the convergence rate depends on A. Since $0 < A \leqslant B$, the optimal relaxation parameter γ in (5.32) is in the range: $\|\Phi\,\Phi^*\|_S^{-1} \leqslant \gamma < 2\,\|\Phi\,\Phi^*\|_S^{-1}$.

Conjugate-Gradient Inversion The conjugate gradient algorithm computes $z = L^{-1}y$ with a gradient descent along orthogonal directions with respect to the norm induced by the symmetric operator L:

$$\|z\|_L^2 = \|Lz\|^2. \tag{5.36}$$

This L norm is used to estimate the error. Grochenig's [286] implementation of the conjugate gradient algorithm is given by the following theorem.

Theorem 5.8 (Conjugate gradient)**.** *To compute $z = L^{-1}y$, we initialize*

$$z_0 = 0 \quad , \quad r_0 = p_0 = y \quad , \quad p_{-1} = 0. \tag{5.37}$$

For any $k \geqslant 0$, we define by induction

$$\lambda_k = \frac{\langle r_k, p_k\rangle}{\langle p_k, Lp_k\rangle} \tag{5.38}$$

$$z_{k+1} = z_k + \lambda_k\, p_k \tag{5.39}$$

$$r_{k+1} = r_k - \lambda_k\, Lp_k \tag{5.40}$$

$$p_{k+1} = Lp_k - \frac{\langle Lp_k, Lp_k\rangle}{\langle p_k, Lp_k\rangle}\,p_k - \frac{\langle Lp_k, Lp_{k-1}\rangle}{\langle p_{k-1}, Lp_{k-1}\rangle}\,p_{k-1}. \tag{5.41}$$

If $\sigma = \frac{\sqrt{B}-\sqrt{A}}{\sqrt{B}+\sqrt{A}}$ then

$$\|z - z_k\|_L \leqslant \frac{2\sigma^k}{1+\sigma^{2k}}\,\|z\|_L, \tag{5.42}$$

and hence $\lim\limits_{k\to+\infty} z_k = z$.

Proof. We give the main steps of the proof as outlined by Grochenig [286].

Step 1: Let $\mathbf{U}_k$ be the subspace generated by $\{L^j z\}_{1\leqslant j\leqslant k}$. By induction on k, we derive from (5.41) that $p_j \in \mathbf{U}_k$, for $j < k$.

Step 2: We prove by induction that $\{p_j\}_{0\leqslant j<k}$ is an orthogonal basis of $\mathbf{U}_k$ with respect to the inner product $\langle z, h\rangle_L = \langle z, Lh\rangle$. Assuming that $\langle p_k, Lp_j\rangle = 0$, for $j \leqslant k-1$, it can be shown that $\langle p_{k+1}, Lp_j\rangle = 0$, for $j \leqslant k$.

Step 3: We verify that z_k is the orthogonal projection of z onto $\mathbf{U}_k$ with respect to $\langle .,.\rangle_L$ which means that

$$\forall g \in \mathbf{U}_k \quad , \quad \|z - g\|_L \geqslant \|z - z_k\|_L.$$

Since $z_k \in \mathbf{U}_k$, this requires proving that $\langle z - z_k, p_j\rangle_L = 0$, for $j < k$.

Step 4: We compute the orthogonal projection of z in embedded spaces $\mathbf{U}_k$ of dimension k, and one can verify that $\lim_{k\to+\infty}\|z - z_k\|_L = 0$. The exponential convergence (5.42) is proved in [286]. ■

■

As opposed to the Richardson alorithm, the initial value z_0 must be set to 0. As in the Richardson iteration algorithm, the convergence is slower when A/B is small. In this case

$$\sigma = \frac{1-\sqrt{A/B}}{1+\sqrt{A/B}} \approx 1 - 2\sqrt{\frac{A}{B}}.$$

The upper bound (5.42) proves that we obtain a relative error

$$\frac{\|z - z_k\|_L}{\|z\|_L} \leqslant \varepsilon$$

for a number of iterations

$$k \approx \frac{\log_e \frac{\varepsilon}{2}}{\log_e \sigma} \approx \frac{-\sqrt{B}}{2\sqrt{A}} \log_e \frac{\varepsilon}{2}.$$

Comparing this result with (5.34) shows that when A/B is small, the conjugate gradient algorithm needs many fewer iterations than the Richardson iteration algorithm to compute $z = L^{-1}y$ at a fixed precision.

5.1.4 Frame Projector and Reproducing Kernel

Frame redundancy is useful in reducing noise added to the frame coefficients. The vector computed with noisy frame coefficients is projected on the image of Φ to reduce the amplitude of the noise. This technique is used for high precision analog to digital conversion based on oversampling. The following theorem specifies the orthogonal projector on $\mathbf{Im}\Phi$.

Theorem 5.9 (Reproducing kernel)**.** *Let $\{\phi_n\}_{n\in\Gamma}$ be a frame of $\mathbf{H}$ or of a subspace $\mathbf{V}$. The orthogonal projection from $\boldsymbol{\ell}^2(\Gamma)$ onto $\mathbf{Im}\Phi$ is*

$$Pa[n] = \Phi\Phi^+ a[n] = \sum_{p\in\Gamma} a[p] \, \langle \tilde{\phi}_p, \phi_n \rangle \ . \tag{5.43}$$

Coefficients $a \in \boldsymbol{\ell}^2(\Gamma)$ are frame coefficients $a \in \mathbf{Im}\Phi$ if and only if they satisfy the reproducing kernel equation

$$a[n] = \Phi\Phi^+ a[n] = \sum_{p\in\Gamma} a[p] \, \langle \tilde{\phi}_p, \phi_n \rangle \ . \tag{5.44}$$

Proof. If $a \in \mathbf{Im}\Phi$ then $a = \Phi f$ and

$$Pa = \Phi\Phi^+\Phi f = \Phi f = a.$$

If $a \in (\mathbf{Im}\Phi)^\perp$ then $Pa = 0$ because $\Phi^+ a = 0$. This proves that P is an orthogonal projector on $\mathbf{Im}\Phi$. Since $\Phi f[n] = \langle f, \phi_n \rangle$ and $\Phi^+ a = \sum_{p\in\Gamma} a[p] \, \tilde{\phi}_p$, we derive (5.43).

A vector $a \in \boldsymbol{\ell}^2(\Gamma)$ belongs to $\mathbf{Im}\Phi$ if and only if $a = Pa$, which proves (5.44). ■ ■

The reproducing kernel equation (5.44) expresses the redundancy of frame coefficients. If the frame is not redundant and is a Riesz basis then $\langle \tilde{\phi}_p, \phi_n \rangle = 0$, so this equation vanishes.

Noise Reduction Suppose that each frame coefficient $\Phi f[n]$ is contaminated by an additive noise $W[n]$, which is a random variable. Applying the projector P gives

$$P(\Phi f + W) = \Phi f + PW,$$

with

$$PW[n] = \sum_{p\in\Gamma} W[p] \, \langle \tilde{\phi}_p, \phi_n \rangle.$$

Since P is an orthogonal projector, $\|PW\| \leqslant \|W\|$. This projector removes the component of W that is in $(\mathbf{Im}\Phi)^\perp$. Increasing the redundancy of the frame reduces the size of $\mathbf{Im}\Phi$ and thus increases $(\mathbf{Im}\Phi)^\perp$, so a larger portion of the noise is removed. If W is a white noise, its energy is uniformly distributed in the space $\boldsymbol{\ell}^2(\Gamma)$. The following theorem proves that its energy is reduced by at least A if the frame vectors are normalized.

Theorem 5.10. *Suppose that $\|\phi_n\| = C$, for all $n \in \Gamma$. If W is a zero-mean white noise of variance $\mathsf{E}\{|W[n]|^2\} = \sigma^2$, then*

$$\mathsf{E}\{|PW[n]|^2\} \leqslant \frac{\sigma^2 C^2}{A}. \tag{5.45}$$

If the frame is tight then this inequality is an equality.

Proof. Let us compute

$$\mathsf{E}\{|PW[n]|^2\} = \mathsf{E}\left\{\left(\sum_{p\in\Gamma} W[p]\,\langle\tilde{\phi}_p, \phi_n\rangle\right)\left(\sum_{l\in\Gamma} W^*[l]\,\langle\tilde{\phi}_l, \phi_n\rangle^*\right)\right\}.$$

Since W is white,

$$\mathsf{E}\{W[p]\,W^*[l]\} = \sigma^2\,\delta[p-l],$$

and therefore

$$\mathsf{E}\{|PW[n]|^2\} = \sigma^2 \sum_{p\in\Gamma} |\langle\tilde{\phi}_p, \phi_n\rangle|^2 \leqslant \frac{\sigma^2\,\|\phi_n\|^2}{A} = \frac{\sigma^2 C^2}{A}.$$

The last inequality is an equality if the frame is tight. ■ ■

Oversampling This noise reduction strategy is used by high precision analog to digital converters. After a low-pass filter, a band-limited analog signal $f(t)$ is uniformly sampled and quantized. In hardware, it is often easier to increase the sampling rate rather than the quantization precision. Increasing the sampling rate introduces a redundancy between the sample values of the band-limited signal. These samples can thus be interpreted as frame coefficients. For a wide range of signals, it has been shown that the quantization error is nearly a white noise [276]. It can thus be significantly reduced by a frame projector, which in this case is a low-pass convolution operator (Exercise 5.16).

The noise can be further reduced if it is not white and if its energy is better concentrated in $(\mathbf{Im}\Phi)^\perp$. This can be done by transforming the quantization noise into a noise whose energy is mostly concentrated at high frequencies. Sigma-Delta modulators produce such quantization noises by integrating the signal before its quantization [87]. To compensate for the integration, the quantized signal is differentiated. This differentiation increases the energy of the quantized noise at high frequencies and reduces its energy at low frequencies [456].

5.1.5 Translation Invariant Frames

To construct translation invariant signal representations, Section 4.1 introduces translation invariant dictionaries obtained by translating a family of generators $\{\phi_n\}_{n\in\Gamma}$. In multiple dimensions for $\phi_n \in \mathbf{L^2}(\mathbb{R}^d)$, the resulting dictionary can be written $\mathcal{D} = \{\phi_{u,n}(x)\}_{n\in\Gamma, u\in\mathbb{R}^d}$, with $\phi_{u,n}(x) = \lambda_{u,n}\,\phi_n(x-u)$. In a translation invariant wavelet dictionary, the generators are obtained by dilating a wavelet $\psi(t)$ with scales s_n: $\phi_n(t) = s_n^{-1/2}\psi(x/s_n)$. In a window Fourier dictionary, the generators are obtained by modulating a window $g(x)$ at frequencies ξ_n: $\phi_n(x) = \mathrm{e}^{i\xi_n x} g(x)$.

The decomposition coefficients of f in $\mathcal{D}$ are convolution products

$$\Phi f(u,n) = \langle f, \phi_{u,n}\rangle = \lambda_{u,n}\, f \star \bar{\phi}_n(u) \quad \text{with} \quad \bar{\phi}_n(x) = \phi_n^*(-x)\ . \tag{5.46}$$

Suppose that Γ is a countable set. The overall index set $\mathbb{R}^d \times \Gamma$ is not countable so the dictionary $\mathcal{D}$ can not strictly speaking be considered as a frame. However, if we consider the overall energy of dictionary coefficients, calculated with a sum and a multidimensional integral

$$\sum_{n\in\Gamma} \|\Phi f(u,n)\|^2 = \sum_{n\in\Gamma} \int |\Phi f(u,n)|\,du,$$

and if there exist two constants $A > 0$ and $B > 0$ such that for all $f \in \mathbf{L^2}(\mathbb{R})$,

$$A\,\|f\|^2 \leqslant \sum_{n\in\Gamma} \|\Phi f(u,n)\|^2 \leqslant B\,\|f\|^2 \tag{5.47}$$

then the frame theory results of the previous section apply. With an abuse of language, such translation invariant dictionaries will thus also be called frames. The following theorem proves that the frame condition (5.47) is equivalent to a condition on the Fourier transform $\hat{\phi}_n(\omega)$ of the generators.

Theorem 5.11. *If there exist two constants $B \geqslant A > 0$ such that for almost all ω in $\mathbb{R}^d$*

$$A \leqslant \sum_{n\in\Gamma} |\hat{\phi}_n(\omega)|^2 \leqslant B, \tag{5.48}$$

then the frame inequality (5.47) is valid for all $f \in \mathbf{L^2}(\mathbb{R}^d)$. Any $\{\tilde{\phi}_n\}_{n\in\Gamma}$ which satisfies for almost all ω in $\mathbb{R}^d$

$$\sum_{n\in\Gamma} \hat{\phi}_n^*(\omega)\, \hat{\tilde{\phi}}_n(\omega) = 1, \tag{5.49}$$

defines a left inverse

$$f(t) = \sum_{n\in\Gamma} \Phi f(.,n) \star \tilde{\phi}_n(t). \tag{5.50}$$

The pseudo-invese (dual frame) is implemented by

$$\widehat{\tilde{\phi}_n}(\omega) = \frac{\hat{\phi}_n(\omega)}{\sum_{n\in\Gamma} |\hat{\phi}_n(\omega)|^2}. \tag{5.51}$$

Proof. The frame condition (5.47) means that $\Phi^*\Phi$ has a spectrum bounded by A and B. It results from (5.46) that

$$\Phi^*\Phi f(x) = f \star (\sum_{n\in\Gamma} \phi_n \star \bar{\phi}_n)(x)\ . \tag{5.52}$$

The spectrum of this convolution operator is given by the Fourier transform of $\sum_{n\in\Gamma} \phi_n \star \bar{\phi}_n(x)$, which is $\sum_{n\in\Gamma} |\hat{\phi}_n(\omega)|^2$. The frame inequality (5.47) is thus equivalent to condition (5.48).

Equation (5.50) is proved by taking the Fourier transform on both sides and inserting (5.49).

Theorem 5.5 proves that the dual frame vectors implementing the pseudo-inverse are $\tilde{\phi}_{n,u} = (\Phi^*\Phi)^{-1}\phi_{n,u}$. Since $\Phi^*\Phi$ is the convolution operator (5.52), its inverse is calculated by inverting its tranfer function, which yields (5.51). ■ ■

For wavelet or windowed Fourier translation invariant dictionaries, the theorem condition (5.48) becomes a condition on the Fourier transform of the wavelet $\hat{\psi}(\omega)$ or on the Fourier transform of the window $\hat{g}(\omega)$. As explained in Section 5.3 and 5.4, more conditions are needed to obtain a frame by discretizing the translation parameter u.

Discrete Translation Invariant Frames For finite dimensional signals $f[n] \in \mathbb{C}^N$ a circular translation invariant frame is obtained with a periodic shift modulo N of a finite number of generators $\{\phi_m[n]\}_{0\leqslant m<M}$:

$$\mathcal{D} = \{\phi_{m,p}[n] = \phi_m[(n-p)\mathrm{mod}N]\}_{0\leqslant m<M, 0\leqslant p<N}\ .$$

Such translation invariant frames appear in Section 11.2.3 to define translation invariant thresholding estimators for noise removal. Similarly to Theorem 5.11, the following theorem gives a necessary and sufficient condition on the discrete Fourier transform $\hat{\phi}_m[k] = \sum_{n=0}^{N-1} \phi_m[n]\mathrm{e}^{-i2\pi k/N}$ of the generators $\phi_m[n]$ to obtain a frame.

Theorem 5.12. *A circular translation invariant dictionary $\mathcal{D} = \{\phi_{m,p}[n]\}_{0\leqslant m<M, 0\leqslant n<N}$ is a frame with frame bounds $0 < A \leqslant B$ if and only if*

$$\forall 0 \leqslant k < N \quad A \leqslant \sum_{m=0}^{M-1} |\hat{\phi}_m[k]|^2 \leqslant B. \tag{5.53}$$

The proof proceeds essentially like the proof of Theorem 5.11, and is left in Exercise 5.8.

5.2 Translation Invariant Dyadic Wavelet Transform

The continuous wavelet transform of Section 4.3 decomposes one-dimensional signals $f \in \mathbf{L^2}(\mathbb{R})$ over a dictionary of translated and dilated wavelets

$$\psi_{u,s}(t) = \frac{1}{\sqrt{s}}\,\psi\left(\frac{t-u}{s}\right) .$$

Translation invariant wavelet dictionaries are constructed by sampling the scale parameter s along an exponential sequence $\{\nu^j\}_{j\in\mathbb{Z}}$, while keeping all translation parameters u. We choose $\nu = 2$ to simplifiy computer implementations:

$$\mathcal{D} = \left\{ \psi_{u,2^j}(t) = \frac{1}{\sqrt{2^j}}\,\psi\left(\frac{t-u}{2^j}\right)\right\}_{u\in\mathbb{R},j\in\mathbb{Z}} .$$

The resulting dyadic wavelet transform of $f \in \mathbf{L^2}(\mathbb{R})$ is defined by

$$Wf(u,2^j) = \langle f, \psi_{u,2^j}\rangle = \int_{-\infty}^{+\infty} f(t)\,\frac{1}{\sqrt{2^j}}\,\psi\left(\frac{t-u}{2^j}\right) dt = f \star \bar{\psi}_{2^j}(u), \tag{5.54}$$

with

$$\bar{\psi}_{2^j}(t) = \psi_{2^j}(-t) = \frac{1}{2^j}\,\psi\left(\frac{-t}{2^j}\right).$$

Translation invariant dyadic wavelet transforms are used in pattern recognition applications and for denoising with translation invariant wavelet thresholding estimators, as explained in Section 11.3.1. Fast computations with filter banks are presented in the next two sections.

Theorem 5.11 on translation invariant dictionaries can be applied to the multiscale wavelet generators $\phi_n(t) = 2^{-j/2}\,\psi_{2^j}(t)$. Since $\hat{\phi}_n(\omega) = \hat{\psi}(2^j\omega)$, the Fourier condition (5.48) means that there exist two constants $A > 0$ and $B > 0$ such that

$$\forall \omega \in \mathbb{R} - \{0\} \quad , \quad A \leqslant \sum_{j=-\infty}^{+\infty} |\hat{\psi}(2^j\omega)|^2 \leqslant B, \tag{5.55}$$

and since $\Phi f(u,n) = 2^{-j/2} Wf(u,n)$, Theorem 5.11 proves the frame inequality

$$A\,\|f\|^2 \leqslant \sum_{j=-\infty}^{+\infty} \frac{1}{2^j}\,\|Wf(u,2^j)\|^2 \leqslant B\,\|f\|^2. \tag{5.56}$$

This shows that if the frequency axis is completely covered by dilated dyadic wavelets, as illustrated by Figure 5.1, then it a dyadic wavelet transform defines a complete and stable representation.

Moreover, if $\tilde{\psi}$ satisfies

$$\forall \omega \in \mathbb{R} - \{0\} \quad , \quad \sum_{j=-\infty}^{+\infty} \hat{\psi}^*(2^j\omega)\,\widehat{\tilde{\psi}}(2^j\omega) = 1, \tag{5.57}$$

then (5.50) applied to $\tilde{\phi}_n(t) = 2^{-j}\tilde{\psi}(2^{-j}t)$ proves that

$$f(t) = \sum_{j=-\infty}^{+\infty} \frac{1}{2^j}\,Wf(.,2^j) \star \tilde{\psi}_{2^j}(t). \tag{5.58}$$

Figure 5.2 gives a dyadic wavelet transform computed over 5 scales with the quadratic spline wavelet shown in Figure 5.3.

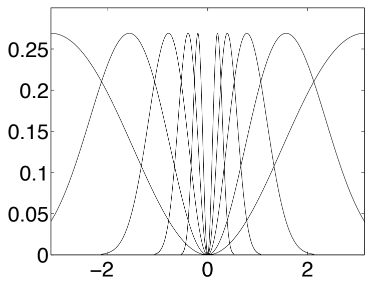

Figure 5.1: Scaled Fourier transforms $|\hat{\psi}(2^j\omega)|^2$ computed with (5.69), for $1 \leqslant j \leqslant 5$ and $\omega \in [-\pi, \pi]$.

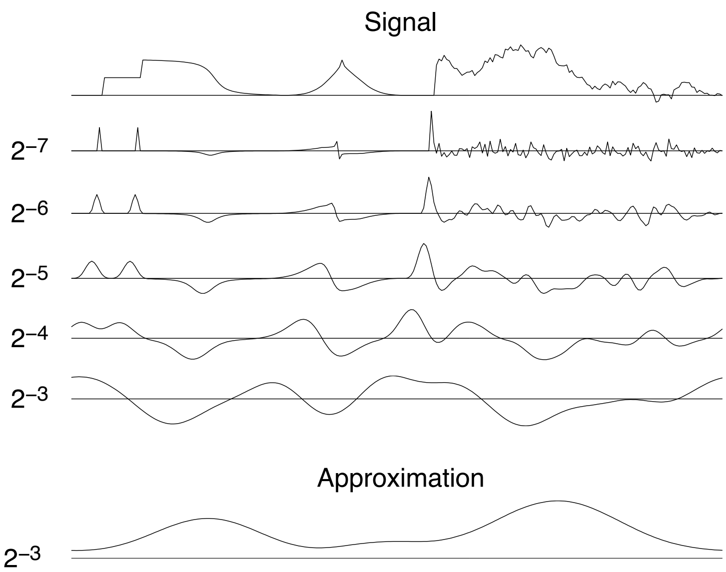

Figure 5.2: Dyadic wavelet transform $Wf(u, 2^j)$ computed at scales $2^{-7} \leqslant 2^j \leqslant 2^{-3}$ with the filter bank algorithm of Section 5.2.2, for a signal defined over $[0, 1]$. The bottom curve carries the lower frequencies corresponding to scales larger than 2^{-3}.

5.2.1 Dyadic Wavelet Design

A discrete dyadic wavelet transform can be computed with a fast filter bank algorithm if the wavelet is appropriately designed. The synthesis of these dyadic wavelets is similar to the construction of biorthogonal wavelet bases, explained in Section 7.4. All technical issues related to the convergence of infinite cascades of filters are avoided in this section. Reading Chapter 7 first is necessary for understanding the main results.

Let h and g be a pair of finite impulse response filters. Suppose that h is a low-pass filter whose transfer function satisfies $\hat{h}(0) = \sqrt{2}$. As in the case of orthogonal and biorthogonal wavelet bases, we construct a scaling function whose Fourier transform is

$$\hat{\phi}(\omega) = \prod_{p=1}^{+\infty} \frac{\hat{h}(2^{-p}\omega)}{\sqrt{2}} = \frac{1}{\sqrt{2}}\, \hat{h}\left(\frac{\omega}{2}\right) \hat{\phi}\left(\frac{\omega}{2}\right). \tag{5.59}$$

We suppose here that this Fourier transform is a finite energy function so that $\phi \in \mathbf{L^2}(\mathbb{R})$. The corresponding wavelet ψ has a Fourier transform defined by

$$\hat{\psi}(\omega) = \frac{1}{\sqrt{2}}\, \hat{g}\left(\frac{\omega}{2}\right) \hat{\phi}\left(\frac{\omega}{2}\right). \tag{5.60}$$

Theorem 7.5 proves that both ϕ and ψ have a compact support because h and g have a finite number of non-zero coefficients. The number of vanishing moments of ψ is equal to the number of zeroes of $\hat{\psi}(\omega)$ at $\omega = 0$. Since $\hat{\phi}(0) = 1$, (5.60) implies that it is also equal to the number of zeros of $\hat{g}(\omega)$ at $\omega = 0$.

Reconstructing Wavelets Reconstructing wavelets that satisfy (5.49) are calculated with a pair of finite impulse response dual filters $\tilde{h}$ and $\tilde{g}$. We suppose that the following Fourier transform has a finite energy:

$$\widehat{\tilde{\phi}}(\omega) = \prod_{p=1}^{+\infty} \frac{\widehat{\tilde{h}}(2^{-p}\omega)}{\sqrt{2}} = \frac{1}{\sqrt{2}}\, \widehat{\tilde{h}}\left(\frac{\omega}{2}\right) \widehat{\tilde{\phi}}\left(\frac{\omega}{2}\right). \tag{5.61}$$

Let us define

$$\widehat{\tilde{\psi}}(\omega) = \frac{1}{\sqrt{2}}\, \widehat{\tilde{g}}\left(\frac{\omega}{2}\right) \widehat{\tilde{\phi}}\left(\frac{\omega}{2}\right). \tag{5.62}$$

The following theorem gives a sufficient condition to guarantee that $\widehat{\tilde{\psi}}$ is the Fourier transform of a reconstruction wavelet.

Theorem 5.13. *If the filters satisfy*

$$\forall \omega \in [-\pi, \pi] \quad , \quad \widehat{\tilde{h}}(\omega)\, \hat{h}^*(\omega) + \widehat{\tilde{g}}(\omega)\, \hat{g}^*(\omega) = 2 \tag{5.63}$$

then

$$\forall \omega \in \mathbb{R} - \{0\} \quad , \quad \sum_{j=-\infty}^{+\infty} \hat{\psi}^*(2^j\omega)\, \widehat{\tilde{\psi}}(2^j\omega) = 1. \tag{5.64}$$

Proof. The Fourier transform expressions (5.60) and (5.62) prove that

$$\widehat{\tilde{\psi}}(\omega)\, \hat{\psi}^*(\omega) = \frac{1}{2}\, \widehat{\tilde{g}}\left(\frac{\omega}{2}\right) \hat{g}^*\left(\frac{\omega}{2}\right) \widehat{\tilde{\phi}}\left(\frac{\omega}{2}\right) \hat{\phi}^*\left(\frac{\omega}{2}\right).$$

Equation (5.63) implies

$$\begin{aligned} \widehat{\tilde{\psi}}(\omega)\, \hat{\psi}^*(\omega) &= \frac{1}{2}\left[2 - \widehat{\tilde{h}}\left(\frac{\omega}{2}\right) \hat{h}^*\left(\frac{\omega}{2}\right)\right] \widehat{\tilde{\phi}}\left(\frac{\omega}{2}\right) \hat{\phi}^*\left(\frac{\omega}{2}\right) \\ &= \widehat{\tilde{\phi}}\left(\frac{\omega}{2}\right) \hat{\phi}^*\left(\frac{\omega}{2}\right) - \widehat{\tilde{\phi}}(\omega)\, \hat{\phi}^*(\omega). \end{aligned}$$

Hence

$$\sum_{j=-l}^{k} \widehat{\tilde{\psi}}(2^j\omega)\, \hat{\psi}^*(2^j\omega) = \hat{\phi}^*(2^{-l}\omega)\, \widehat{\tilde{\phi}}(2^{-l}\omega) - \hat{\phi}^*(2^k\omega)\, \widehat{\tilde{\phi}}(2^k\omega).$$

Since $\hat{g}(0) = 0$, (5.63) implies $\widehat{\tilde{h}}(0)\,\hat{h}^*(0) = 2$. We also impose that $\hat{h}(0) = \sqrt{2}$ so one can derive from (5.59) and (5.61) that $\widehat{\tilde{\phi}}(0) = \hat{\phi}^*(0) = 1$. Since ϕ and $\tilde{\phi}$ belong to $\mathbf{L^1}(\mathbb{R})$, $\hat{\phi}$ and $\widehat{\tilde{\phi}}$ are continuous, and the Riemann-Lebesgue lemma (Exercise 2.8) proves that $|\hat{\phi}(\omega)|$ and $|\widehat{\tilde{\phi}}(\omega)|$ decrease to zero when ω goes to ∞. For $\omega \neq 0$, letting k and l go to $+\infty$ yields (5.64). ■ ■

Observe that (5.63) is the same as the unit gain condition (7.117) for biorthogonal wavelets. The aliasing cancellation condition (7.116) of biorthogonal wavelets is not required because the wavelet transform is not sampled in time.

Finite Impulse Response Solution Let us shift h and g to obtain causal filters. The resulting transfer functions $\hat{h}(\omega)$ and $\hat{g}(\omega)$ are polynomials in $\mathrm{e}^{-i\omega}$. We suppose that these polynomials have no common zeros. The Bezout Theorem 7.8 on polynomials proves that if $P(z)$ and $Q(z)$ are two polynomials of degree n and l, with no common zeros, then there exists a unique pair of polynomials $\tilde{P}(z)$ and $\tilde{Q}(z)$ of degree $l-1$ and $n-1$ such that

$$P(z)\,\tilde{P}(z) + Q(z)\,\tilde{Q}(z) = 1. \tag{5.65}$$

This guarantees the existence of $\widehat{\tilde{h}}(\omega)$ and $\widehat{\tilde{g}}(\omega)$ that are polynomials in $\mathrm{e}^{-i\omega}$ and satisfy (5.63). These are the Fourier transforms of the finite impulse response filters $\tilde{h}$ and $\tilde{g}$. One must however be careful because the resulting scaling function $\widehat{\tilde{\phi}}$ in (5.61) does not necessarily have a finite energy.

Spline Dyadic Wavelets A *box spline* of degree m is a translation of $m+1$ convolutions of $\mathbf{1}_{[0,1]}$ with itself. It is centered at $t = 1/2$ if m is even and at $t = 0$ if m is odd. Its Fourier transform is

$$\hat{\phi}(\omega) = \left(\frac{\sin(\omega/2)}{\omega/2}\right)^{m+1} \exp\left(\frac{-i\varepsilon\omega}{2}\right) \quad \text{with} \quad \varepsilon = \begin{cases} 1 & \text{if } m \text{ is even} \\ 0 & \text{if } m \text{ is odd} \end{cases}, \tag{5.66}$$

so

$$\hat{h}(\omega) = \sqrt{2}\,\frac{\hat{\phi}(2\omega)}{\hat{\phi}(\omega)} = \sqrt{2}\left(\cos\frac{\omega}{2}\right)^{m+1} \exp\left(\frac{-i\varepsilon\omega}{2}\right). \tag{5.67}$$

We construct a wavelet that has one vanishing moment by choosing $\hat{g}(\omega) = O(\omega)$ in the neighborhood of $\omega = 0$. For example

$$\hat{g}(\omega) = -i\,\sqrt{2}\,\sin\frac{\omega}{2}\,\exp\left(\frac{-i\varepsilon\omega}{2}\right). \tag{5.68}$$

The Fourier transform of the resulting wavelet is

$$\hat{\psi}(\omega) = \frac{1}{\sqrt{2}}\,\hat{g}\left(\frac{\omega}{2}\right)\hat{\phi}\left(\frac{\omega}{2}\right) = \frac{-i\omega}{4}\left(\frac{\sin(\omega/4)}{\omega/4}\right)^{m+2} \exp\left(\frac{-i\omega(1+\varepsilon)}{4}\right). \tag{5.69}$$

It is the first derivative of a box spline of degree $m+1$ centered at $t = (1+\varepsilon)/4$. For $m = 2$, Figure 5.3 shows the resulting quadratic splines ϕ and ψ. The dyadic admissibility condition (5.48) is verified numerically for $A = 0.505$ and $B = 0.522$.

To design dual scaling functions $\tilde{\phi}$ and wavelets $\tilde{\psi}$ which are splines, we choose $\widehat{\tilde{h}} = \hat{h}$. As a consequence, $\phi = \tilde{\phi}$ and the reconstruction condition (5.63) implies that

$$\widehat{\tilde{g}}(\omega) = \frac{2 - |\hat{h}(\omega)|^2}{\hat{g}^*(\omega)} = -i\,\sqrt{2}\,\exp\left(\frac{-i\omega}{2}\right)\sin\frac{\omega}{2}\sum_{n=0}^{m}\left(\cos\frac{\omega}{2}\right)^{2n}. \tag{5.70}$$

Table 5.1 gives the corresponding filters for $m = 2$.

5.2.2 “Algorithme à Trous”

Suppose that the scaling functions and wavelets ϕ, ψ, $\tilde{\phi}$ and $\tilde{\psi}$ are designed with the filters h, g, $\tilde{h}$ and $\tilde{g}$. A fast dyadic wavelet transform is calculated with a filter bank algorithm called in French the *algorithme à trous*, introduced by Holschneider, Kronland-Martinet, Morlet and Tchamitchian [302]. It is similar to a fast biorthogonal wavelet transform, without subsampling [432, 366].

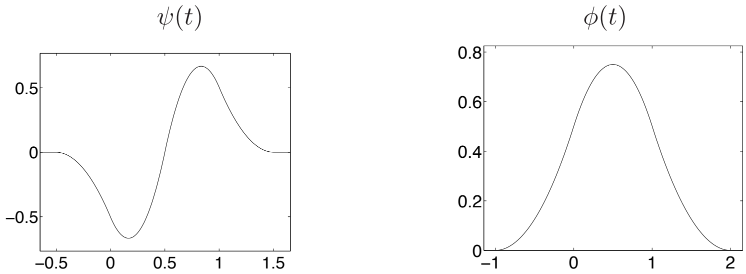

Figure 5.3: Quadratic spline wavelet and scaling function.

n	$h[n]/\sqrt{2}$	$\tilde{h}[n]/\sqrt{2}$	$g[n]/\sqrt{2}$	$\tilde{g}[n]/\sqrt{2}$
-2				-0.03125
-1	0.125	0.125		-0.21875
0	0.375	0.375	-0.5	-0.6875
1	0.375	0.375	0.5	0.6875
2	0.125	0.125		0.21875
3				0.03125

Table 5.1: Coefficients of the filters computed from their transfer functions (5.67, 5.68, 5.70) for $m = 2$. These filters generate the quadratic spline scaling functions and wavelets shown in Figure 5.3.

Fast Dyadic Transform The samples $a_0[n]$ of the input discrete signal are written as a low-pass filtering with ϕ of an analog signal f, in the neighborhood of $t = n$:

$$a_0[n] = f \star \bar{\phi}(n) = \langle f(t), \phi(t-n) \rangle = \int_{-\infty}^{+\infty} f(t)\, \phi(t-n)\, dt.$$

This is further justified in Section 7.3.1. For any $j \geqslant 0$, we denote

$$a_j[n] = \langle f(t), \phi_{2^j}(t-n) \rangle \quad \text{with} \quad \phi_{2^j}(t) = \frac{1}{\sqrt{2^j}} \phi\Big(\frac{t}{2^j}\Big) .$$

The dyadic wavelet coefficients are computed for $j > 0$ over the integer grid

$$d_j[n] = Wf(n, 2^j) = \langle f(t), \psi_{2^j}(t-n) \rangle.$$

For any filter $x[n]$, we denote by $x_j[n]$ the filters obtained by inserting $2^j - 1$ zeros between each sample of $x[n]$. Its Fourier transform is $\hat{x}(2^j\omega)$. Inserting zeros in the filters creates holes (*trous* in French). Let $\bar{x}_j[n] = x_j[-n]$. The next theorem gives convolution formulas that are cascaded to compute a dyadic wavelet transform and its inverse.

Theorem 5.14. *For any* $j \geqslant 0$*,*

$$a_{j+1}[n] = a_j \star \bar{h}_j[n] \quad , \quad d_{j+1}[n] = a_j \star \bar{g}_j[n] \quad , \tag{5.71}$$

and

$$a_j[n] = \frac{1}{2} \Big(a_{j+1} \star \tilde{h}_j[n] + d_{j+1} \star \tilde{g}_j[n] \Big) . \tag{5.72}$$

Proof. Proof of (5.71). Since

$$a_{j+1}[n] = f \star \bar{\phi}_{2^{j+1}}(n) \quad \text{and} \quad d_{j+1}[n] = f \star \bar{\psi}_{2^{j+1}}(n),$$

we verify with (3.3) that their Fourier transforms are respectively

$$\hat{a}_{j+1}(\omega) = \sum_{k=-\infty}^{+\infty} \hat{f}(\omega + 2k\pi)\, \hat{\phi}^*_{2^{j+1}}(\omega + 2k\pi)$$

and

$$\hat{d}_{j+1}(\omega) = \sum_{k=-\infty}^{+\infty} \hat{f}(\omega + 2k\pi)\, \hat{\psi}^*_{2^{j+1}}(\omega + 2k\pi).$$

The properties (5.61) and (5.62) imply that

$$\hat{\phi}_{2^{j+1}}(\omega) = \sqrt{2^{j+1}}\, \hat{\phi}(2^{j+1}\omega) = \hat{h}(2^j\omega)\, \sqrt{2^j}\, \hat{\phi}(2^j\omega),$$

$$\hat{\psi}_{2^{j+1}}(\omega) = \sqrt{2^{j+1}}\, \hat{\psi}(2^{j+1}\omega) = \hat{g}(2^j\omega)\, \sqrt{2^j}\, \hat{\phi}(2^j\omega).$$

Since $j \geqslant 0$, both $\hat{h}(2^j\omega)$ and $\hat{g}(2^j\omega)$ are 2π periodic, so

$$\hat{a}_{j+1}(\omega) = \hat{h}^*(2^j\omega)\, \hat{a}_j(\omega) \quad \text{and} \quad \hat{d}_{j+1}(\omega) = \hat{g}^*(2^j\omega)\, \hat{a}_j(\omega). \tag{5.73}$$

These two equations are the Fourier transforms of (5.71).

Proof of (5.72). Equations (5.73) imply

$$\begin{aligned} \hat{a}_{j+1}(\omega)\, \hat{\tilde{h}}(2^j\omega) + \hat{d}_{j+1}(\omega)\, \hat{\tilde{g}}(2^j\omega) \quad &= \\ \hat{a}_j(\omega)\, \hat{h}^*(2^j\omega)\, \hat{\tilde{h}}(2^j\omega) \quad &+ \quad \hat{a}_j(\omega)\, \hat{g}^*(2^j\omega)\, \hat{\tilde{g}}(2^j\omega). \end{aligned}$$

Inserting the reconstruction condition (5.63) proves that

$$\hat{a}_{j+1}(\omega)\, \hat{\tilde{h}}(2^j\omega) + \hat{d}_{j+1}(\omega)\, \hat{\tilde{g}}(2^j\omega) = 2\, \hat{a}_j(\omega),$$

which is the Fourier transform of (5.72). ■ ■

The dyadic wavelet representation of a_0 is defined as the set of wavelet coefficients up to a scale 2^J plus the remaining low-frequency information a_J:

$$\left[\{d_j\}_{1 \leqslant j \leqslant J} \ , \ a_J \right]. \tag{5.74}$$

It is computed from a_0 by cascading the convolutions (5.71) for $0 \leqslant j < J$, as illustrated in Figure 5.4(a). The dyadic wavelet transform of Figure 5.2 is calculated with this filter bank algorithm. The original signal a_0 is recovered from its wavelet representation (5.74) by iterating (5.72) for $J > j \geqslant 0$, as illustrated in Figure 5.4(b).

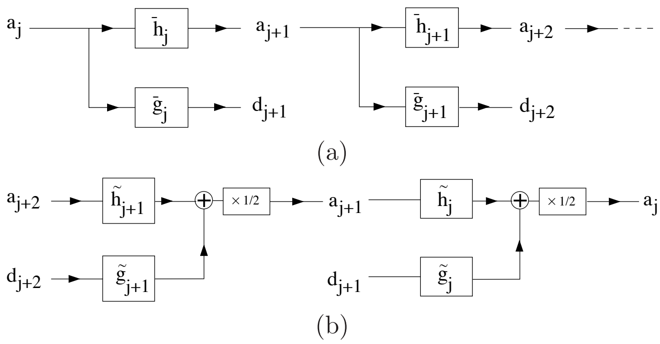

Figure 5.4: (a): The dyadic wavelet coefficients are computed by cascading convolutions with dilated filters $\bar{h}_j$ and $\bar{g}_j$. (b): The original signal is reconstructed through convolutions with $\tilde{h}_j$ and $\tilde{g}_j$. A multiplication by 1/2 is necessary to recover the next finer scale signal a_j.

If the input signal $a_0[n]$ has a finite size of N samples, the convolutions (5.71) are replaced by circular convolutions. The maximum scale 2^J is then limited to N, and for $J = \log_2 N$ one can verify that $a_J[n]$ is constant and equal to $N^{-1/2} \sum_{n=0}^{N-1} a_0[n]$. Suppose that h and g have respectively K_h and K_g non-zero samples. The "dilated" filters h_j and g_j have the same number of non-zero coefficients. The number of multiplications needed to compute a_{j+1} and d_{j+1} from a_j or the reverse is thus equal to $(K_h + K_g)N$. For $J = \log_2 N$, the dyadic wavelet representation (5.74) and its inverse are thus calculated with $(K_h + K_g)N \log_2 N$ multiplications and additions.

5.3 Subsampled Wavelet Frames

Wavelet frames are constructed by sampling the scale parameter but also the translation parameter of a wavelet dictionary. A real continuous wavelet transform of $f \in \mathbf{L^2}(\mathbb{R})$ is defined in Section 4.3 by

$$Wf(u,s) = \langle f, \psi_{u,s} \rangle \quad \text{with} \quad \psi_{u,s}(t) = \frac{1}{\sqrt{s}}\,\psi\left(\frac{t-u}{s}\right)$$

where ψ is a real wavelet. Imposing $\|\psi\| = 1$ implies that $\|\psi_{u,s}\| = 1$.

Intuitively, to construct a frame we need to cover the time-frequency plane with the Heisenberg boxes of the corresponding discrete wavelet family. A wavelet $\psi_{u,s}$ has an energy in time that is centered at u over a domain proportional to s. Over positive frequencies, its Fourier transform $\hat{\psi}_{u,s}$ has a support centered at a frequency η/s, with a spread proportional to $1/s$. To obtain a full cover, we sample s along an exponential sequence $\{a^j\}_{j\in\mathbb{Z}}$, with a sufficiently small dilation step $a > 1$. The time translation u is sampled uniformly at intervals proportional to the scale a^j, as illustrated in Figure 5.5. Let us denote

$$\psi_{j,n}(t) = \frac{1}{\sqrt{a^j}}\,\psi\left(\frac{t - nu_0a^j}{a^j}\right).$$

In the following, we give without proofs some necessary conditions and sufficient conditions on ψ, a and u_0 so that $\{\psi_{j,n}\}_{(j,n)\in\mathbb{Z}^2}$ is a frame of $\mathbf{L^2}(\mathbb{R})$.

Necessary Conditions We suppose that ψ is real, normalized, and satisfies the admissibility condition of Theorem 4.4:

$$C_\psi = \int_0^{+\infty} \frac{|\hat{\psi}(\omega)|^2}{\omega}\,d\omega < +\infty. \tag{5.75}$$

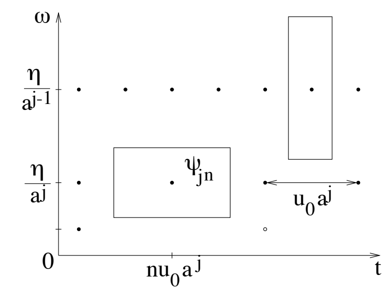

Figure 5.5: The Heisenberg box of a wavelet $\psi_{j,n}$ scaled by $s = a^j$ has a time and frequency width proportional respectively to a^j and a^{-j}. The time-frequency plane is covered by these boxes if u_0 and a are sufficiently small.

Theorem 5.15 (Daubechies)**.** *If $\{\psi_{j,n}\}_{(j,n)\in\mathbb{Z}^2}$ is a frame of $\mathbf{L^2}(\mathbb{R})$ then the frame bounds satisfy*

$$A \leqslant \frac{C_\psi}{u_0 \log_e a} \leqslant B, \tag{5.76}$$

$$\forall \omega \in \mathbb{R} - \{0\} \quad , \quad A \leqslant \frac{1}{u_0} \sum_{j=-\infty}^{+\infty} |\hat{\psi}(a^j\omega)|^2 \leqslant B. \tag{5.77}$$

This theoroem is proved in [162, 18]. Condition (5.77) is equivalent to the frame condition (5.55) for a translation invariant dyadic wavelet transform, for which the parameter u is not sampled. It requires that the Fourier axis is covered by wavelets dilated by $\{a^j\}_{j\in\mathbb{Z}}$. The inequality (5.76), which relates the sampling density $u_0 \log_e a$ to the frame bounds, is proved in [18]. It shows that the frame is an orthonormal basis if and only if

$$A = B = \frac{C_\psi}{u_0 \log_e a} = 1.$$

Chapter 7 constructs wavelet orthonormal bases of $\mathbf{L^2}(\mathbb{R})$ with regular wavelets of compact support.

Sufficient Conditions The following theorem proved by Daubechies [18] provides a lower and upper bound for the frame bounds A and B, depending on ψ, u_0 and a.

Theorem 5.16 (Daubechies)**.** *Let us define*

$$\theta(\xi) = \sup_{1 \leqslant |\omega| \leqslant a} \sum_{j=-\infty}^{+\infty} |\hat{\psi}(a^j \omega)| \, |\hat{\psi}(a^j \omega + \xi)| \tag{5.78}$$

and

$$\Delta = \sum_{\substack{k=-\infty \\ k \neq 0}}^{+\infty} \left[\theta\left(\frac{2\pi k}{u_0}\right) \theta\left(\frac{-2\pi k}{u_0}\right) \right]^{1/2}.$$

If u_0 and a are such that

$$A_0 = \frac{1}{u_0} \left(\inf_{1 \leqslant |\omega| \leqslant a} \sum_{j=-\infty}^{+\infty} |\hat{\psi}(a^j \omega)|^2 - \Delta \right) > 0, \tag{5.79}$$

and

$$B_0 = \frac{1}{u_0} \left(\sup_{1 \leqslant |\omega| \leqslant a} \sum_{j=-\infty}^{+\infty} |\hat{\psi}(a^j \omega)|^2 + \Delta \right) < +\infty, \tag{5.80}$$

then $\{\psi_{j,n}\}_{(j,n) \in \mathbb{Z}^2}$ is a frame of $\mathbf{L^2}(\mathbb{R})$. The constants A_0 and B_0 are respectively lower and upper bounds of the frame bounds A and B.

The sufficient conditions (5.79) and (5.80) are similar to the necessary condition (5.77). If Δ is small relative to $\inf_{1 \leqslant |\omega| \leqslant a} \sum_{j=-\infty}^{+\infty} |\hat{\psi}(a^j \omega)|^2$ then A_0 and B_0 are close to the optimal frame bounds A and B. For a fixed dilation step a, the value of Δ decreases when the time sampling interval u_0 decreases.

Dual Frame Theorem 5.5 gives a general formula for computing the dual wavelet frame vectors

$$\tilde{\psi}_{j,n} = (\Phi^* \Phi)^{-1} \psi_{j,n}. \tag{5.81}$$

One could reasonably hope that the dual functions $\tilde{\psi}_{j,n}$ would be obtained by scaling and translating a dual wavelet $\tilde{\psi}$. The sad reality is that this is generally not true. In general the operator $\Phi^* \Phi$ does not commute with dilations by a^j, so $(\Phi^* \Phi)^{-1}$ does not commute with these dilations either. On the other hand, one can prove that $(\Phi^* \Phi)^{-1}$ commutes with translations by $n a^j u_0$, which means that

$$\tilde{\psi}_{j,n}(t) = \tilde{\psi}_{j,0}(t - n a^j u_0). \tag{5.82}$$

The dual frame $\{\tilde{\psi}_{j,n}\}_{(j,n) \in \mathbb{Z}^2}$ is thus obtained by calculating each elementary function $\tilde{\psi}_{j,0}$ with (5.81), and translating them with (5.82). The situation is much simpler for tight frames, where the dual frame is equal to the original wavelet frame.

Mexican Hat Wavelet The normalized second derivative of a Gaussian is

$$\psi(t) = \frac{2}{\sqrt{3}} \pi^{-1/4} (t^2 - 1) \exp\left(\frac{-t^2}{2}\right). \tag{5.83}$$

Its Fourier transform is

$$\hat{\psi}(\omega) = -\frac{\sqrt{8}\, \pi^{1/4} \omega^2}{\sqrt{3}} \exp\left(\frac{-\omega^2}{2}\right).$$

The graph of these functions is shown in Figure 4.6.

The dilation step a is generally set to be $a = 2^{1/v}$ where v is the number of intermediate scales (voices) for each octave. Table 5.2 gives the estimated frame bounds A_0 and B_0 computed by Daubechies [18] with the formula of Theorem 5.16. For $v \geqslant 2$ voices per octave, the frame is nearly

a	u_0	A_0	B_0	B_0/A_0
2	0.25	13.091	14.183	1.083
2	0.5	6.546	7.092	1.083
2	1.0	3.223	3.596	1.116
2	1.5	0.325	4.221	12.986
$2^{\frac{1}{2}}$	0.25	27.273	27.278	1.0002
$2^{\frac{1}{2}}$	0.5	13.673	13.639	1.0002
$2^{\frac{1}{2}}$	1.0	6.768	6.870	1.015
$2^{\frac{1}{2}}$	1.75	0.517	7.276	14.061
$2^{\frac{1}{4}}$	0.25	54.552	54.552	1.0000
$2^{\frac{1}{4}}$	0.5	27.276	27.276	1.0000
$2^{\frac{1}{4}}$	1.0	13.586	13.690	1.007
$2^{\frac{1}{4}}$	1.75	2.928	12.659	4.324

Table 5.2: Estimated frame bounds for the Mexican hat wavelet computed with Theorem 5.16 [18].

tight when $u_0 \leqslant 0.5$, in which case the dual frame can be approximated by the original wavelet frame. As expected from (5.76), when $A \approx B$

$$A \approx B \approx \frac{C_\psi}{u_0 \log_e a} = \frac{v}{u_0} C_\psi \log_2 \mathrm{e}.$$

The frame bounds increase proportionally to v/u_0. For $a = 2$, we see that A_0 decreases brutally from $u_0 = 1$ to $u_0 = 1.5$. For $u_0 = 1.75$ the wavelet family is not a frame anymore. For $a = 2^{1/2}$, the same transition appears for a larger u_0.

5.4 Windowed Fourier Frames

Frame theory gives conditions for discretizing the windowed Fourier transform while retaining a complete and stable representation. The windowed Fourier transform of $f \in \mathbf{L^2}(\mathbb{R})$ is defined in Section 4.2 by

$$Sf(u,\xi) = \langle f, g_{u,\xi} \rangle,$$

with

$$g_{u,\xi}(t) = g(t-u)\,\mathrm{e}^{i\xi t}.$$

Setting $\|g\| = 1$ implies that $\|g_{u,\xi}\| = 1$. A discrete windowed Fourier transform representation

$$\{Sf(u_n,\xi_k) = \langle f, g_{u_n,\xi_k} \rangle\}_{(n,k)\in\mathbb{Z}^2}$$

is complete and stable if $\{g_{u,n,\xi_k}\}_{(n,k)\in\mathbb{Z}^2}$ is a frame of $\mathbf{L^2}(\mathbb{R})$.

Intuitively, one can expect that the discrete windowed Fourier transform is complete if the Heisenberg boxes of all atoms $\{g_{u,n,\xi_k}\}_{(n,k)\in\mathbb{Z}^2}$ fully cover the time-frequency plane. Section 4.2 shows that the Heisenberg box of g_{u_n,ξ_k} is centered in the time-frequency plane at (u_n, ξ_k). Its size is independent of u_n and ξ_k. It depends on the time-frequency spread of the window g. A complete cover of the plane is thus obtained by translating these boxes over a uniform rectangular grid, as illustrated in Figure 5.6. The time and frequency parameters (u,ξ) are discretized over a rectangular grid with time and frequency intervals of size u_0 and ξ_0. Let us denote

$$g_{n,k}(t) = g(t - nu_0)\,\exp(ik\xi_0 t).$$

The sampling intervals (u_0, ξ_0) must be adjusted to the time-frequency spread of g.

Window Scaling Suppose that $\{g_{n,k}\}_{(n,k)\in\mathbb{Z}^2}$ is a frame of $\mathbf{L^2}(\mathbb{R})$ with frame bounds A and B. Let us dilate the window $g_s(t) = s^{-1/2} g(t/s)$. It increases by s the time width of the Heisenberg box of g and reduces by s its frequency width. We thus obtain the same cover of the time-frequency plane by increasing u_0 by s and reducing ξ_0 by s. Let

$$g_{s,n,k}(t) = g_s(t - nsu_0)\, \exp\left(ik\frac{\xi_0}{s}t\right). \tag{5.84}$$

We prove that $\{g_{s,n,k}\}_{(n,k)\in\mathbb{Z}^2}$ satisfies the same frame inequalities as $\{g_{n,k}\}_{(n,k)\in\mathbb{Z}^2}$, with the same frame bounds A and B, by a change of variable $t' = ts$ in the inner product integrals.

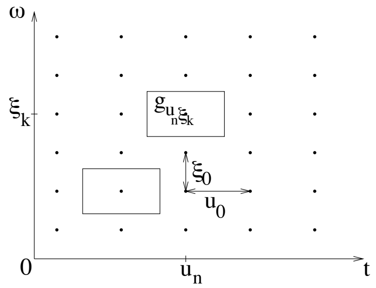

Figure 5.6: A windowed Fourier frame is obtained by covering the time-frequency plane with a regular grid of windowed Fourier atoms, translated by $u_n = n\,u_0$ in time and by $\xi_k = k\,\xi_0$ in frequency.

Tight Frames Tight frames are easier to manipulate numerically since the dual frame is equal to the original frame. Daubechies, Grossmann and Meyer [196] give sufficient conditions for building a window of compact support that generates a tight frame.

Theorem 5.17 (Daubechies, Grossmann, Meyer)**.** *Let g be a window whose support is included in $[-\pi/\xi_0, \pi/\xi_0]$. If*

$$\forall t \in \mathbb{R} \quad , \quad \frac{2\pi}{\xi_0} \sum_{n=-\infty}^{+\infty} |g(t - nu_0)|^2 = A > 0 \tag{5.85}$$

then $\{g_{n,k}(t) = g(t - nu_0)\, e^{ik\xi_0 t}\}_{(n,k)\in\mathbb{Z}^2}$ is a tight frame $\mathbf{L^2}(\mathbb{R})$ with a frame bound equal to A.

Proof. The function $g(t - nu_0)\, f(t)$ has a support in $[nu_0 - \pi/\xi_0, nu_0 + \pi/\xi_0]$. Since $\{e^{ik\xi_0 t}\}_{k\in\mathbb{Z}}$ is an orthogonal basis of this space we have

$$\int_{-\infty}^{+\infty} |g(t - nu_0)|^2\, |f(t)|^2 dt = \int_{nu_0-\pi/\xi_0}^{nu_0+\pi/\xi_0} |g(t - nu_0)|^2\, |f(t)|^2 dt = \frac{\xi_0}{2\pi} \sum_{k=-\infty}^{+\infty} |\langle g(u - nu_0)\, f(u), e^{ik\xi_0 u}\rangle|^2.$$

Since $g_{n,k}(t) = g(t - nu_0)\, e^{ik\xi_0 t}$, we get

$$\int_{-\infty}^{+\infty} |g(t - nu_0)|^2\, |f(t)|^2 dt = \frac{\xi_0}{2\pi} \sum_{k=-\infty}^{+\infty} |\langle f, g_{n,k}\rangle|^2.$$

Summing over n and inserting (5.85) proves that $A\,\|f\|^2 = \sum_{k,n=-\infty}^{+\infty} |\langle f, g_{n,k}\rangle|^2$ and hence that $\{g_{n,k}\}_{(n,k)\in\mathbb{Z}^2}$ is a tight frame of $\mathbf{L^2}(\mathbb{R})$. ■

Since g has a support in $[-\pi/\xi_0, \pi/\xi_0]$ the condition (5.85) implies that

$$\frac{2\pi}{u_0\,\xi_0} \geqslant 1$$

so that there is no whole between consecutive windows $g(t-nu_0)$ and $g(t-(n+1)u_0)$. If we impose that $1 \leqslant 2\pi/(u_0\xi_0) \leqslant 2$ then only consecutive windows have supports that overlap. The square root of a Hanning window

$$g(t) = \sqrt{\frac{\xi_0}{\pi}} \cos\left(\frac{\xi_0 t}{2}\right) \mathbf{1}_{[-\pi/\xi_0,\pi/\xi_0]}(t)$$

is positive normalized window that satisfies (5.85) with $u_0 = \pi/\xi_0$ and a redundancy factor $A = 2$. The design of other windows is studied in Section 8.4.2 for local cosine bases.

Discrete Window Fourier Tight Frames To construct a windowed Fourier tight frame of $\mathbb{C}^N$, the Fourier basis $\{e^{ik\xi_0 t}\}_{k\in\mathbb{Z}}$ of $\mathbf{L^2}[-\pi/\xi_0,\pi/\xi_0]$ is replaced by the discrete Fourier basis $\{e^{i2\pi kn/K}\}_{0\leqslant k<K}$ of $\mathbb{C}^K$. The following theorem is a discrete equivalent of Theorem 5.17.

Theorem 5.18. *Let $g[n]$ be an N periodic discrete window whose support restricted to $[-N/2,N/2]$ is included in $[-K/2,K/2-1]$. If M divides N and*

$$\forall 0 \leqslant n < N \ , \quad K \sum_{m=0}^{N/M-1} |g[n-mM]|^2 = A > 0 \tag{5.86}$$

then $\{g_{m,k}[n] = g[n-mM]\, e^{i2\pi kn/K}\}_{0\leqslant k<K, 0\leqslant m<N/M}$ is a tight frame $\mathbb{C}^N$ with a frame bound equal to A.

The proof of this theorem follows the same steps as the proof of Theorem 5.17. It is left in Exercise 5.10. There are N/M translated windows and hence NK/M windowed Fourier coefficients. For a fixed window position indexed by m, the discrete windowed Fourier coefficients are the discrete Fourier coefficients of the windowed signal

$$Sf[m,k] = \langle f, g_{m,k}\rangle = \sum_{n=K/2}^{K/2-1} f[n]\, g[n-mM]\, e^{-i2\pi kn/K} \quad \text{for} \quad 0 \leqslant k < K.$$

They are computed with $O(K\log_2 K)$ operations with an FFT. Over all windows, this requires a total of $O(NK/M \log_2 K)$ operations. We generally choose $1 < K/M \leqslant 2$ so that only consecutive windows overlap. The square root of a Hanning window $g[n] = \sqrt{2/K}\cos(\pi n/K)$ satisfies (5.86) for $M = K/2$ and a redundancy factor $A = 2$. Figure 5.7 shows the log spectrogram $\log|Sf[m,k]|^2$ of the windowed Fourier frame coefficients computed with a square root Hanning window for a musical recording.

Necessary Frame Conditions For general windowed Fourier frames of $\mathbf{L^2}(\mathbb{R}^2)$, Daubechies [18] proved several necessary conditions on g, u_0 and ξ_0 to guarantee that $\{g_{n,k}\}_{(n,k)\in\mathbb{Z}^2}$ is a frame of $\mathbf{L^2}(\mathbb{R})$. We do not reproduce the proofs, but summarize the main results.

Theorem 5.19 (Daubechies)**.** *The windowed Fourier family $\{g_{n,k}\}_{(n,k)\in\mathbb{Z}^2}$ is a frame only if*

$$\frac{2\pi}{u_0\,\xi_0} \geqslant 1. \tag{5.87}$$

The frame bounds A and B necessarily satisfy

$$A \leqslant \frac{2\pi}{u_0\,\xi_0} \leqslant B, \tag{5.88}$$

$$\forall t \in \mathbb{R} \quad , \quad A \leqslant \frac{2\pi}{\xi_0} \sum_{n=-\infty}^{+\infty} |g(t-nu_0)|^2 \leqslant B, \tag{5.89}$$

$$\forall \omega \in \mathbb{R} \quad , \quad A \leqslant \frac{1}{u_0} \sum_{k=-\infty}^{+\infty} |\hat{g}(\omega - k\xi_0)|^2 \leqslant B. \tag{5.90}$$

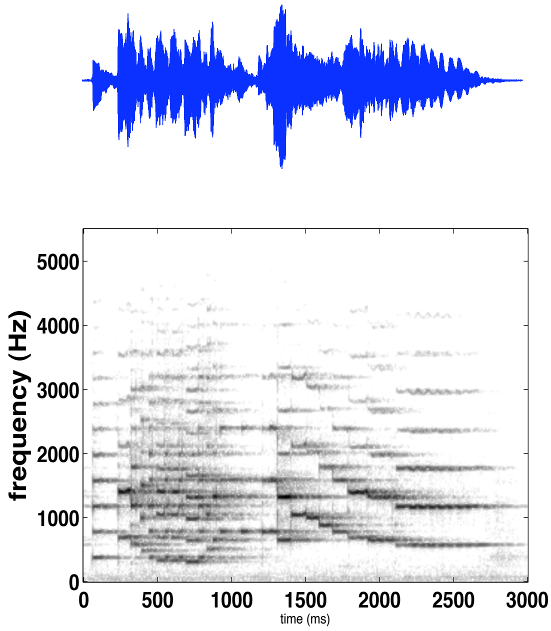

Figure 5.7: Top: Musical recording. Bottom: Log spectrogram $\log |Sf[m,k]|^2$ computed with a square root Hanning window.

The ratio $2\pi/(u_0\xi_0)$ measures the density of windowed Fourier atoms in the time-frequency plane. The first condition (5.87) ensures that this density is greater than 1 because the covering ability of each atom is limited. The inequalities (5.89) and (5.90) are proved in full generality by Chui and Shi [162]. They show that the uniform time translations of g must completely cover the time axis, and the frequency translations of its Fourier transform $\hat{g}$ must similarly cover the frequency axis.

Since all windowed Fourier vectors are normalized, the frame is an orthogonal basis only if $A = B = 1$. The frame bound condition (5.88) shows that this is possible only at the critical sampling density $u_0\xi_0 = 2\pi$. The Balian-Low Theorem [91] proves that g is then either non-smooth or has a slow time decay.

Theorem 5.20 (Balian-Low)**.** *If $\{g_{n,k}\}_{(n,k)\in\mathbb{Z}^2}$ is a windowed Fourier frame with $u_0\xi_0 = 2\pi$, then*

$$\int_{-\infty}^{+\infty} t^2\, |g(t)|^2\, dt = +\infty \quad \text{or} \quad \int_{-\infty}^{+\infty} \omega^2\, |\hat{g}(\omega)|^2\, d\omega = +\infty. \tag{5.91}$$

This theorem proves that we cannot construct an orthogonal windowed Fourier basis with a differentiable window g of compact support. On the other hand, one can verify that the discontinuous rectangular window

$$g = \frac{1}{\sqrt{u_0}}\, \mathbf{1}_{[-u_0/2, u_0/2]}$$

yields an orthogonal windowed Fourier basis for $u_0\xi_0 = 2\pi$. This basis is rarely used because of the bad frequency localization of $\hat{g}$.

Sufficient Conditions The following theorem proved by Daubechies [194] gives sufficient conditions on u_0, ξ_0 and g for constructing a windowed Fourier frame.

Theorem 5.21 (Daubechies)**.** *Let us define*

$$\theta(u) = \sup_{0\leqslant t\leqslant u_0} \sum_{n=-\infty}^{+\infty} |g(t - nu_0)|\, |g(t - nu_0 + u)| \tag{5.92}$$

and

$$\Delta = \sum_{\substack{k=-\infty \\ k\neq 0}}^{+\infty} \left[\theta\left(\frac{2\pi k}{\xi_0}\right) \theta\left(\frac{-2\pi k}{\xi_0}\right) \right]^{1/2}. \tag{5.93}$$

If u_0 and ξ_0 satisfy

$$A_0 = \frac{2\pi}{\xi_0} \left(\int_{0\leqslant t\leqslant u_0} \sum_{n=-\infty}^{+\infty} |g(t-nu_0)|^2 - \Delta \right) > 0 \tag{5.94}$$

and

$$B_0 = \frac{2\pi}{\xi_0} \left(\sup_{0\leqslant t\leqslant u_0} \sum_{n=-\infty}^{+\infty} |g(t-nu_0)|^2 + \Delta \right) < +\infty, \tag{5.95}$$

then $\{g_{n,k}\}_{(n,k)\in\mathbb{Z}^2}$ is a frame. The constants A_0 and B_0 are respectively lower bounds and upper bounds of the frame bounds A and B.

Observe that the only difference between the sufficient conditions (5.94, 5.95) and the necessary condition (5.89) is the addition and subtraction of Δ. If Δ is small compared to $\inf_{0\leqslant t\leqslant u_0} \sum_{n=-\infty}^{+\infty} |g(t-nu_0)|^2$ then A_0 and B_0 are close to the optimal frame bounds A and B.

Dual Frame Theorem 5.5 proves that the dual windowed frame vectors are

$$\tilde{g}_{n,k} = (\Phi^*\Phi)^{-1} g_{n,k}. \tag{5.96}$$

The following theorem shows that this dual frame is also a windowed Fourier frame, which means that its vectors are time and frequency translations of a new window $\tilde{g}$.

Theorem 5.22. *Dual windowed Fourier vectors can be rewritten*

$$\tilde{g}_{n,k}(t) = \tilde{g}(t-nu_0)\, \exp(ik\xi_0 t)$$

where $\tilde{g}$ is the dual window

$$\tilde{g} = (\Phi^*\Phi)^{-1} g. \tag{5.97}$$

Proof. This result is proved by showing first that $\Phi^*\Phi$ commutes with time and frequency translations proportional to u_0 and ξ_0. If $\phi \in \mathbf{L^2}(\mathbb{R})$ and $\phi_{m,l}(t) = \phi(t-mu_0)\, \exp(il\xi_0 t)$ we verify that

$$\Phi^*\Phi\phi_{m,l}(t) = \exp(il\xi_0 t)\, \Phi^*\Phi h(t-mu_0).$$

Indeed

$$\Phi^*\Phi\phi_{m,l} = \sum_{(n,k)\in\mathbb{Z}^2} \langle \phi_{m,l}, g_{n,k}\rangle\, g_{n,k}$$

and a change of variable yields

$$\langle \phi_{m,l}, g_{n,k}\rangle = \langle \phi, g_{n-m,k-l}\rangle.$$

Consequently

$$\begin{aligned} \Phi^*\Phi\phi_{m,l}(t) &= \sum_{(n,k)\in\mathbb{Z}^2} \langle \phi, g_{n-m,k-l}\rangle\, \exp(il\xi_0 t)\, g_{n-m,k-l}(t-mu_0) \\ &= \exp(il\xi_0 t)\, \Phi^*\Phi\phi(t-mu_0). \end{aligned}$$

Since $\Phi^*\Phi$ commutes with these translations and frequency modulations we verify that $(\Phi^*\Phi)^{-1}$ necessarily commutes with the same group operations. Hence

$$\tilde{g}_{n,k}(t) = (\Phi^*\Phi)^{-1} g_{n,k} = \exp(ik\xi_0)\, (\Phi^*\Phi)^{-1} g_{0,0}(t-nu_0) = \exp(ik\xi_0)\, \tilde{g}(t-nu_0).$$

■ ■