

## VII

# Wavelet Bases

One can construct wavelets  $\psi$  such that the dilated and translated family

$$\left\{ \psi_{j,n}(t) = \frac{1}{\sqrt{2^j}} \psi\left(\frac{t - 2^j n}{2^j}\right) \right\}_{(j,n) \in \mathbb{Z}^2}$$

is an orthonormal basis of  $\mathbf{L}^2(\mathbb{R})$ . Behind this simple statement lie very different point of views which open a fruitful exchange between harmonic analysis and discrete signal processing.

Orthogonal wavelets dilated by  $2^j$  carry signal variations at the resolution  $2^{-j}$ . The construction of these bases can thus be related to multiresolution signal approximations. Following this link leads us to an unexpected equivalence between wavelet bases and conjugate mirror filters used in discrete multirate filter banks. These filter banks implement a fast orthogonal wavelet transform that requires only  $O(N)$  operations for signals of size  $N$ . The design of conjugate mirror filters also gives new classes of wavelet orthogonal bases including regular wavelets of compact support. In several dimensions, wavelet bases of  $\mathbf{L}^2(\mathbb{R}^d)$  are constructed with separable products of functions of one variable. Wavelet bases are also adapted to bounded domains and surfaces with lifting algorithms.

## 7.1 Orthogonal Wavelet Bases

Our search for orthogonal wavelets begins with multiresolution approximations. For  $f \in \mathbf{L}^2(\mathbb{R})$ , the partial sum of wavelet coefficients  $\sum_{n=-\infty}^{+\infty} \langle f, \psi_{j,n} \rangle \psi_{j,n}$  can indeed be interpreted as the difference between two approximations of  $f$  at the resolutions  $2^{-j+1}$  and  $2^{-j}$ . Multiresolution approximations compute the approximation of signals at various resolutions with orthogonal projections on different spaces  $\{\mathbf{V}_j\}_{j \in \mathbb{Z}}$ . Section 7.1.3 proves that multiresolution approximations are entirely characterized by a particular discrete filter that governs the loss of information across resolutions. These discrete filters provide a simple procedure for designing and synthesizing orthogonal wavelet bases.

### 7.1.1 Multiresolution Approximations

Adapting the signal resolution allows one to process only the relevant details for a particular task. In computer vision, Burt and Adelson [125] introduced a multiresolution pyramid that can be used to process a low-resolution image first and then selectively increase the resolution when necessary. This section formalizes multiresolution approximations, which set the ground for the construction of orthogonal wavelets.

The approximation of a function  $f$  at a resolution  $2^{-j}$  is specified by a discrete grid of samples that provides local averages of  $f$  over neighborhoods of size proportional to  $2^j$ . A multiresolution approximation is thus composed of embedded grids of approximation. More formally, the approximation of a function at a resolution  $2^{-j}$  is defined as an orthogonal projection on a space

$\mathbf{V}_j \subset \mathbf{L}^2(\mathbb{R})$ . The space  $\mathbf{V}_j$  regroups all possible approximations at the resolution  $2^{-j}$ . The orthogonal projection of  $f$  is the function  $f_j \in \mathbf{V}_j$  that minimizes  $\|f - f_j\|$ . The following definition introduced by Mallat [361] and Meyer [43] specifies the mathematical properties of multiresolution spaces. To avoid confusion, let us emphasize that a scale parameter  $2^j$  is the inverse of the resolution  $2^{-j}$ .

**Definition 7.1** (Multiresolutions). *A sequence  $\{\mathbf{V}_j\}_{j \in \mathbb{Z}}$  of closed subspaces of  $\mathbf{L}^2(\mathbb{R})$  is a multiresolution approximation if the following 6 properties are satisfied:*

$$\forall (j, k) \in \mathbb{Z}^2, \quad f(t) \in \mathbf{V}_j \Leftrightarrow f(t - 2^j k) \in \mathbf{V}_j, \quad (7.1)$$

$$\forall j \in \mathbb{Z}, \quad \mathbf{V}_{j+1} \subset \mathbf{V}_j, \quad (7.2)$$

$$\forall j \in \mathbb{Z}, \quad f(t) \in \mathbf{V}_j \Leftrightarrow f\left(\frac{t}{2}\right) \in \mathbf{V}_{j+1}, \quad (7.3)$$

$$\lim_{j \rightarrow +\infty} \mathbf{V}_j = \bigcap_{j=-\infty}^{+\infty} \mathbf{V}_j = \{0\}, \quad (7.4)$$

$$\lim_{j \rightarrow -\infty} \mathbf{V}_j = \text{Closure} \left( \bigcup_{j=-\infty}^{+\infty} \mathbf{V}_j \right) = \mathbf{L}^2(\mathbb{R}). \quad (7.5)$$

There exists  $\theta$  such that  $\{\theta(t - n)\}_{n \in \mathbb{Z}}$  is a Riesz basis of  $\mathbf{V}_0$ .

Let us give an intuitive explanation of these mathematical properties. Property (7.1) means that  $\mathbf{V}_j$  is invariant by any translation proportional to the scale  $2^j$ . As we shall see later, this space can be assimilated to a uniform grid with intervals  $2^j$ , which characterizes the signal approximation at the resolution  $2^{-j}$ . The inclusion (7.2) is a causality property which proves that an approximation at a resolution  $2^{-j}$  contains all the necessary information to compute an approximation at a coarser resolution  $2^{-j-1}$ . Dilating functions in  $\mathbf{V}_j$  by 2 enlarges the details by 2 and (7.3) guarantees that it defines an approximation at a coarser resolution  $2^{-j-1}$ . When the resolution  $2^{-j}$  goes to 0 (7.4) implies that we lose all the details of  $f$  and

$$\lim_{j \rightarrow +\infty} \|P_{\mathbf{V}_j} f\| = 0. \quad (7.6)$$

On the other hand, when the resolution  $2^{-j}$  goes  $+\infty$ , property (7.5) imposes that the signal approximation converges to the original signal:

$$\lim_{j \rightarrow -\infty} \|f - P_{\mathbf{V}_j} f\| = 0. \quad (7.7)$$

When the resolution  $2^{-j}$  increases, the decay rate of the approximation error  $\|f - P_{\mathbf{V}_j} f\|$  depends on the regularity of  $f$ . Section 9.1.3 relates this error to the uniform Lipschitz regularity of  $f$ .

The existence of a Riesz basis  $\{\theta(t - n)\}_{n \in \mathbb{Z}}$  of  $\mathbf{V}_0$  provides a discretization theorem as explained in Section 3.1.3. The function  $\theta$  can be interpreted as a unit resolution cell; Section 5.1.1 gives the definition of a Riesz basis. It is a family of linearly independent functions such that there exist  $B \geq A > 0$  which satisfy

$$\forall f \in \mathbf{V}_0, \quad A \|f\|^2 \leq \sum_{n=-\infty}^{+\infty} |\langle f(t), \theta(t - n) \rangle|^2 \leq B \|f\|^2. \quad (7.8)$$

This energy equivalence guarantees that signal expansions over  $\{\theta(t - n)\}_{n \in \mathbb{Z}}$  are numerically stable. One verify that the family  $\{2^{-j/2} \theta(2^{-j} t - n)\}_{n \in \mathbb{Z}}$  is a Riesz basis of  $\mathbf{V}_j$  with the same Riesz bounds  $A$  and  $B$  at all scales  $2^j$ . Theorem 3.4 proves that  $\{\theta(t - n)\}_{n \in \mathbb{Z}}$  is a Riesz basis if and only if

$$\forall \omega \in [-\pi, \pi], \quad A \leq \sum_{k=-\infty}^{+\infty} |\hat{\theta}(\omega + 2k\pi)|^2 \leq B. \quad (7.9)$$

**Example 7.1. Piecewise constant approximations** A simple multiresolution approximation is composed of piecewise constant functions. The space  $\mathbf{V}_j$  is the set of all  $g \in \mathbf{L}^2(\mathbb{R})$  such that  $g(t)$  is constant for  $t \in [n2^j, (n+1)2^j)$  and  $n \in \mathbb{Z}$ . The approximation at a resolution  $2^{-j}$  of  $f$  is the closest piecewise constant function on intervals of size  $2^j$ . The resolution cell can be chosen to be the box window  $\theta = \mathbf{1}_{[0,1]}$ . Clearly  $\mathbf{V}_j \subset \mathbf{V}_{j-1}$  since functions constant on intervals of size  $2^j$  are also constant on intervals of size  $2^{j-1}$ . The verification of the other multiresolution properties is left to the reader. It is often desirable to construct approximations that are smooth functions, in which case piecewise constant functions are not appropriate.

**Example 7.2. Shannon approximations** Frequency band-limited functions also yield multiresolution approximations. The space  $\mathbf{V}_j$  is defined as the set of functions whose Fourier transform has a support included in  $[-2^{-j}\pi, 2^{-j}\pi]$ . Theorem 3.5 provides an orthonormal basis  $\{\theta(t-n)\}_{n \in \mathbb{Z}}$  of  $\mathbf{V}_0$  defined by

$$\theta(t) = \frac{\sin \pi t}{\pi t}. \quad (7.10)$$

All other properties of multiresolution approximation are easily verified.

The approximation at the resolution  $2^{-j}$  of  $f \in \mathbf{L}^2(\mathbb{R})$  is the function  $P_{\mathbf{V}_j} f \in \mathbf{V}_j$  that minimizes  $\|P_{\mathbf{V}_j} f - f\|$ . It is proved in (3.12) that its Fourier transform is obtained with a frequency filtering:

$$\widehat{P_{\mathbf{V}_j} f}(\omega) = \hat{f}(\omega) \mathbf{1}_{[-2^{-j}\pi, 2^{-j}\pi]}(\omega).$$

This Fourier transform is generally discontinuous at  $\pm 2^{-j}\pi$ , in which case  $|P_{\mathbf{V}_j} f(t)|$  decays like  $|t|^{-1}$ , for large  $|t|$ , even though  $f$  might have a compact support.

**Example 7.3. Spline approximations** Polynomial spline approximations construct smooth approximations with fast asymptotic decay. The space  $\mathbf{V}_j$  of splines of degree  $m \geq 0$  is the set of functions that are  $m-1$  times continuously differentiable and equal to a polynomial of degree  $m$  on any interval  $[n2^j, (n+1)2^j]$ , for  $n \in \mathbb{Z}$ . When  $m=0$ , it is a piecewise constant multiresolution approximation. When  $m=1$ , functions in  $\mathbf{V}_j$  are piecewise linear and continuous.

A Riesz basis of polynomial splines is constructed with box splines. A box spline  $\theta$  of degree  $m$  is computed by convolving the box window  $\mathbf{1}_{[0,1]}$  with itself  $m+1$  times and centering at 0 or  $1/2$ . Its Fourier transform is

$$\hat{\theta}(\omega) = \left( \frac{\sin(\omega/2)}{\omega/2} \right)^{m+1} \exp\left(\frac{-i\varepsilon\omega}{2}\right). \quad (7.11)$$

If  $m$  is even then  $\varepsilon = 1$  and  $\theta$  has a support centered at  $t = 1/2$ . If  $m$  is odd then  $\varepsilon = 0$  and  $\theta(t)$  is symmetric about  $t = 0$ . Figure 7.1 displays a cubic box spline  $m = 3$  and its Fourier transform. For all  $m \geq 0$ , one can prove that  $\{\theta(t-n)\}_{n \in \mathbb{Z}}$  is a Riesz basis of  $\mathbf{V}_0$  by verifying the condition (7.9). This is done with a closed form expression for the series (7.19).

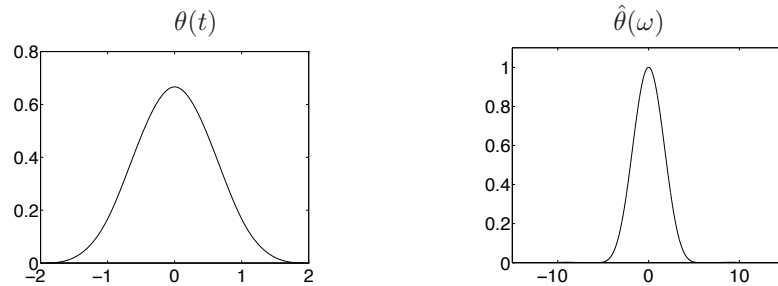


Figure 7.1: Cubic box spline  $\theta$  and its Fourier transform  $\hat{\theta}$ .

### 7.1.2 Scaling Function

The approximation of  $f$  at the resolution  $2^{-j}$  is defined as the orthogonal projection  $P_{\mathbf{V}_j} f$  on  $\mathbf{V}_j$ . To compute this projection, we must find an orthonormal basis of  $\mathbf{V}_j$ . The following theorem

orthogonalizes the Riesz basis  $\{\theta(t - n)\}_{n \in \mathbb{Z}}$  and constructs an orthogonal basis of each space  $\mathbf{V}_j$  by dilating and translating a single function  $\phi$  called a *scaling function*. To avoid confusing the resolution  $2^{-j}$  and the scale  $2^j$ , in the rest of the chapter the notion of resolution is dropped and  $P_{\mathbf{V}_j} f$  is called an approximation at the scale  $2^j$ .

**Theorem 7.1.** *Let  $\{\mathbf{V}_j\}_{j \in \mathbb{Z}}$  be a multiresolution approximation and  $\phi$  be the scaling function whose Fourier transform is*

$$\hat{\phi}(\omega) = \frac{\hat{\theta}(\omega)}{\left(\sum_{k=-\infty}^{+\infty} |\hat{\theta}(\omega + 2k\pi)|^2\right)^{1/2}}. \quad (7.12)$$

Let us denote

$$\phi_{j,n}(t) = \frac{1}{\sqrt{2^j}} \phi\left(\frac{t - n}{2^j}\right).$$

The family  $\{\phi_{j,n}\}_{n \in \mathbb{Z}}$  is an orthonormal basis of  $\mathbf{V}_j$  for all  $j \in \mathbb{Z}$ .

*Proof.* To construct an orthonormal basis, we look for a function  $\phi \in \mathbf{V}_0$ . It can thus be expanded in the basis  $\{\theta(t - n)\}_{n \in \mathbb{Z}}$ :

$$\phi(t) = \sum_{n=-\infty}^{+\infty} a[n] \theta(t - n),$$

which implies that

$$\hat{\phi}(\omega) = \hat{a}(\omega) \hat{\theta}(\omega),$$

where  $\hat{a}$  is a  $2\pi$  periodic Fourier series of finite energy. To compute  $\hat{a}$  we express the orthogonality of  $\{\phi(t - n)\}_{n \in \mathbb{Z}}$  in the Fourier domain. Let  $\bar{\phi}(t) = \phi^*(-t)$ . For any  $(n, p) \in \mathbb{Z}^2$ ,

$$\begin{aligned} \langle \phi(t - n), \phi(t - p) \rangle &= \int_{-\infty}^{+\infty} \phi(t - n) \phi^*(t - p) dt \\ &= \phi \star \bar{\phi}(p - n). \end{aligned} \quad (7.13)$$

Hence  $\{\phi(t - n)\}_{n \in \mathbb{Z}}$  is orthonormal if and only if  $\phi \star \bar{\phi}(n) = \delta[n]$ . Computing the Fourier transform of this equality yields

$$\sum_{k=-\infty}^{+\infty} |\hat{\phi}(\omega + 2k\pi)|^2 = 1. \quad (7.14)$$

Indeed, the Fourier transform of  $\phi \star \bar{\phi}(t)$  is  $|\hat{\phi}(\omega)|^2$ , and we proved in (3.3) that sampling a function periodizes its Fourier transform. The property (7.14) is verified if we choose

$$\hat{a}(\omega) = \left( \sum_{k=-\infty}^{+\infty} |\hat{\theta}(\omega + 2k\pi)|^2 \right)^{-1/2}.$$

We saw in (7.9) that the denominator has a strictly positive lower bound, so  $\hat{a}$  is a  $2\pi$  periodic function of finite energy. ■

**Approximation** The orthogonal projection of  $f$  over  $\mathbf{V}_j$  is obtained with an expansion in the scaling orthogonal basis

$$P_{\mathbf{V}_j} f = \sum_{n=-\infty}^{+\infty} \langle f, \phi_{j,n} \rangle \phi_{j,n}. \quad (7.15)$$

The inner products

$$a_j[n] = \langle f, \phi_{j,n} \rangle \quad (7.16)$$

provide a discrete approximation at the scale  $2^j$ . We can rewrite them as a convolution product:

$$a_j[n] = \int_{-\infty}^{+\infty} f(t) \frac{1}{\sqrt{2^j}} \phi\left(\frac{t - 2^j n}{2^j}\right) dt = f \star \bar{\phi}_j(2^j n), \quad (7.17)$$

with  $\bar{\phi}_j(t) = \sqrt{2^{-j}} \phi(2^{-j}t)$ . The energy of the Fourier transform  $\hat{\phi}$  is typically concentrated in  $[-\pi, \pi]$ , as illustrated by Figure 7.2. As a consequence, the Fourier transform  $\sqrt{2^j} \hat{\phi}^*(2^j \omega)$  of  $\bar{\phi}_j(t)$

is mostly non-negligible in  $[-2^{-j}\pi, 2^{-j}\pi]$ . The discrete approximation  $a_j[n]$  is therefore a low-pass filtering of  $f$  sampled at intervals  $2^j$ . Figure 7.3 gives a discrete multiresolution approximation at scales  $2^{-9} \leq 2^j \leq 2^{-4}$ .

**Example 7.4.** For piecewise constant approximations and Shannon multiresolution approximations we have constructed Riesz bases  $\{\theta(t - n)\}_{n \in \mathbb{Z}}$  which are orthonormal bases, hence  $\phi = \theta$ .

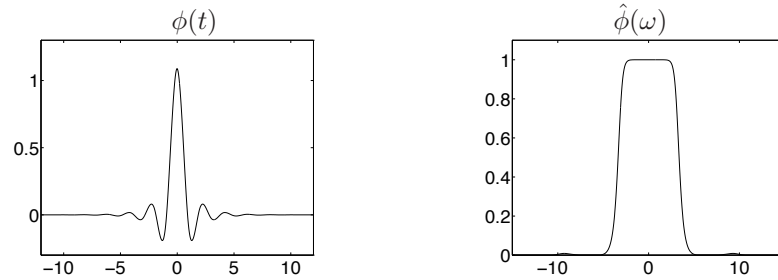


Figure 7.2: Cubic spline scaling function  $\phi$  and its Fourier transform  $\hat{\phi}$  computed with (7.18).

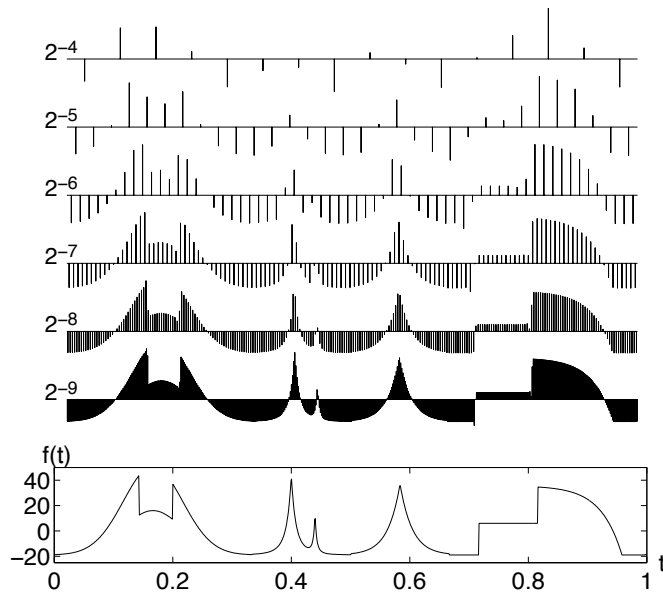


Figure 7.3: Discrete multiresolution approximations  $a_j[n]$  at scales  $2^j$ , computed with cubic splines.

**Example 7.5.** Spline multiresolution approximations admit a Riesz basis constructed with a box spline  $\theta$  of degree  $m$ , whose Fourier transform is given by (7.11). Inserting this expression in (7.12) yields

$$\hat{\phi}(\omega) = \frac{\exp(-i\varepsilon\omega/2)}{\omega^{m+1} \sqrt{S_{2m+2}(\omega)}}, \quad (7.18)$$

with

$$S_n(\omega) = \sum_{k=-\infty}^{+\infty} \frac{1}{(\omega + 2k\pi)^n}, \quad (7.19)$$

and  $\varepsilon = 1$  if  $m$  is even or  $\varepsilon = 0$  if  $m$  is odd. A closed form expression of  $S_{2m+2}(\omega)$  is obtained by computing the derivative of order  $2m$  of the identity

$$S_2(2\omega) = \sum_{k=-\infty}^{+\infty} \frac{1}{(2\omega + 2k\pi)^2} = \frac{1}{4 \sin^2 \omega}.$$

For linear splines  $m = 1$  and

$$S_4(2\omega) = \frac{1 + 2 \cos^2 \omega}{48 \sin^4 \omega}, \quad (7.20)$$

which yields

$$\hat{\phi}(\omega) = \frac{4 \sqrt{3} \sin^2(\omega/2)}{\omega^2 \sqrt{1 + 2 \cos^2(\omega/2)}}. \quad (7.21)$$

The cubic spline scaling function corresponds to  $m = 3$  and  $\hat{\phi}(\omega)$  is calculated with (7.18) by inserting

$$\begin{aligned} S_8(2\omega) &= \frac{5 + 30 \cos^2 \omega + 30 \sin^2 \omega \cos^2 \omega}{105 \cdot 2^8 \sin^8 \omega} \\ &\quad + \frac{70 \cos^4 \omega + 2 \sin^4 \omega \cos^2 \omega + 2/3 \sin^6 \omega}{105 \cdot 2^8 \sin^8 \omega}. \end{aligned} \quad (7.22)$$

This cubic spline scaling function  $\phi$  and its Fourier transform are displayed in Figure 7.2. It has an infinite support but decays exponentially.

### 7.1.3 Conjugate Mirror Filters

A multiresolution approximation is entirely characterized by the scaling function  $\phi$  that generates an orthogonal basis of each space  $\mathbf{V}_j$ . We study the properties of  $\phi$  which guarantee that the spaces  $\mathbf{V}_j$  satisfy all conditions of a multiresolution approximation. It is proved that any scaling function is specified by a discrete filter called a *conjugate mirror filter*.

**Scaling Equation** The multiresolution causality property (7.2) imposes that  $\mathbf{V}_j \subset \mathbf{V}_{j-1}$ . In particular  $2^{-1/2} \phi(t/2) \in \mathbf{V}_1 \subset \mathbf{V}_0$ . Since  $\{\phi(t-n)\}_{n \in \mathbb{Z}}$  is an orthonormal basis of  $\mathbf{V}_0$ , we can decompose

$$\frac{1}{\sqrt{2}} \phi\left(\frac{t}{2}\right) = \sum_{n=-\infty}^{+\infty} h[n] \phi(t-n), \quad (7.23)$$

with

$$h[n] = \left\langle \frac{1}{\sqrt{2}} \phi\left(\frac{t}{2}\right), \phi(t-n) \right\rangle. \quad (7.24)$$

This scaling equation relates a dilation of  $\phi$  by 2 to its integer translations. The sequence  $h[n]$  will be interpreted as a discrete filter.

The Fourier transform of both sides of (7.23) yields

$$\hat{\phi}(2\omega) = \frac{1}{\sqrt{2}} \hat{h}(\omega) \hat{\phi}(\omega) \quad (7.25)$$

for  $\hat{h}(\omega) = \sum_{n=-\infty}^{+\infty} h[n] e^{-in\omega}$ . It is thus tempting to express  $\hat{\phi}(\omega)$  directly as a product of dilations of  $\hat{h}(\omega)$ . For any  $p \geq 0$ , (7.25) implies

$$\hat{\phi}(2^{-p+1}\omega) = \frac{1}{\sqrt{2}} \hat{h}(2^{-p}\omega) \hat{\phi}(2^{-p}\omega). \quad (7.26)$$

By substitution, we obtain

$$\hat{\phi}(\omega) = \left( \prod_{p=1}^P \frac{\hat{h}(2^{-p}\omega)}{\sqrt{2}} \right) \hat{\phi}(2^{-P}\omega). \quad (7.27)$$

If  $\hat{\phi}(\omega)$  is continuous at  $\omega = 0$  then  $\lim_{P \rightarrow +\infty} \hat{\phi}(2^{-P}\omega) = \hat{\phi}(0)$  so

$$\hat{\phi}(\omega) = \prod_{p=1}^{+\infty} \frac{\hat{h}(2^{-p}\omega)}{\sqrt{2}} \hat{\phi}(0). \quad (7.28)$$

The following theorem [361, 43] gives necessary and then sufficient conditions on  $\hat{h}(\omega)$  to guarantee that this infinite product is the Fourier transform of a scaling function.

**Theorem 7.2** (Mallat, Meyer). *Let  $\phi \in \mathbf{L}^2(\mathbb{R})$  be an integrable scaling function. The Fourier series of  $h[n] = \langle 2^{-1/2}\phi(t/2), \phi(t-n) \rangle$  satisfies*

$$\forall \omega \in \mathbb{R} \quad , \quad |\hat{h}(\omega)|^2 + |\hat{h}(\omega + \pi)|^2 = 2, \quad (7.29)$$

and

$$\hat{h}(0) = \sqrt{2}. \quad (7.30)$$

Conversely, if  $\hat{h}(\omega)$  is  $2\pi$  periodic and continuously differentiable in a neighborhood of  $\omega = 0$ , if it satisfies (7.29) and (7.30) and if

$$\inf_{\omega \in [-\pi/2, \pi/2]} |\hat{h}(\omega)| > 0 \quad (7.31)$$

then

$$\hat{\phi}(\omega) = \prod_{p=1}^{+\infty} \frac{\hat{h}(2^{-p}\omega)}{\sqrt{2}} \quad (7.32)$$

is the Fourier transform of a scaling function  $\phi \in \mathbf{L}^2(\mathbb{R})$ .

*Proof.* This theorem is a central result whose proof is long and technical. It is divided in several parts.

- *Proof of the necessary condition (7.29)* The necessary condition is proved to be a consequence of the fact that  $\{\phi(t-n)\}_{n \in \mathbb{Z}}$  is orthonormal. In the Fourier domain, (7.14) gives an equivalent condition:

$$\forall \omega \in \mathbb{R} \quad , \quad \sum_{k=-\infty}^{+\infty} |\hat{\phi}(\omega + 2k\pi)|^2 = 1. \quad (7.33)$$

Inserting  $\hat{\phi}(\omega) = 2^{-1/2}\hat{h}(\omega/2)\hat{\phi}(\omega/2)$  yields

$$\sum_{k=-\infty}^{+\infty} |\hat{h}(\frac{\omega}{2} + k\pi)|^2 |\hat{\phi}(\frac{\omega}{2} + k\pi)|^2 = 2.$$

Since  $\hat{h}(\omega)$  is  $2\pi$  periodic, separating the even and odd integer terms gives

$$|\hat{h}(\frac{\omega}{2})|^2 \sum_{p=-\infty}^{+\infty} \left| \hat{\phi}\left(\frac{\omega}{2} + 2p\pi\right) \right|^2 + \left| \hat{h}\left(\frac{\omega}{2} + \pi\right) \right|^2 \sum_{p=-\infty}^{+\infty} \left| \hat{\phi}\left(\frac{\omega}{2} + \pi + 2p\pi\right) \right|^2 = 2.$$

Inserting (7.33) for  $\omega' = \omega/2$  and  $\omega' = \omega/2 + \pi$  proves that

$$|\hat{h}(\omega')|^2 + |\hat{h}(\omega' + \pi)|^2 = 2.$$

- *Proof of the necessary condition (7.30)* We prove that  $\hat{h}(0) = \sqrt{2}$  by showing that  $\hat{\phi}(0) \neq 0$ . Indeed we know that  $\hat{\phi}(0) = 2^{-1/2}\hat{h}(0)\hat{\phi}(0)$ . More precisely, we verify that  $|\hat{\phi}(0)| = 1$  is a consequence of the completeness property (7.5) of multiresolution approximations.

The orthogonal projection of  $f \in \mathbf{L}^2(\mathbb{R})$  on  $\mathbf{V}_j$  is

$$P_{\mathbf{V}_j} f = \sum_{n=-\infty}^{+\infty} \langle f, \phi_{j,n} \rangle \phi_{j,n}. \quad (7.34)$$

Property (7.5) expressed in the time and Fourier domains with the Plancherel formula implies that

$$\lim_{j \rightarrow -\infty} \|f - P_{\mathbf{V}_j} f\|^2 = \lim_{j \rightarrow -\infty} 2\pi \|\hat{f} - \widehat{P_{\mathbf{V}_j} f}\|^2 = 0. \quad (7.35)$$

To compute the Fourier transform  $\widehat{P_{\mathbf{V}_j} f}(\omega)$ , we denote  $\phi_j(t) = \sqrt{2^{-j}}\phi(2^{-j}t)$ . Inserting the convolution expression (7.17) in (7.34) yields

$$P_{\mathbf{V}_j} f(t) = \sum_{n=-\infty}^{+\infty} f \star \bar{\phi}_j(2^j n) \phi_j(t - 2^j n) = \phi_j \star \sum_{n=-\infty}^{+\infty} f \star \bar{\phi}_j(2^j n) \delta(t - 2^j n).$$

The Fourier transform of  $f \star \bar{\phi}_j(t)$  is  $\sqrt{2^j} \hat{f}(\omega) \hat{\phi}^*(2^j \omega)$ . A uniform sampling has a periodized Fourier transform calculated in (3.3), and hence

$$\widehat{P_{V_j} f}(\omega) = \hat{\phi}(2^j \omega) \sum_{k=-\infty}^{+\infty} \hat{f}\left(\omega - \frac{2k\pi}{2^j}\right) \hat{\phi}^*\left(2^j \left[\omega - \frac{2k\pi}{2^j}\right]\right). \quad (7.36)$$

Let us choose  $\hat{f} = \mathbf{1}_{[-\pi, \pi]}$ . For  $j < 0$  and  $\omega \in [-\pi, \pi]$ , (7.36) gives  $\widehat{P_{V_j} f}(\omega) = |\hat{\phi}(2^j \omega)|^2$ . The mean-square convergence (7.35) implies that

$$\lim_{j \rightarrow -\infty} \int_{-\pi}^{\pi} |1 - |\hat{\phi}(2^j \omega)|^2|^2 d\omega = 0.$$

Since  $\phi$  is integrable,  $\hat{\phi}(\omega)$  is continuous and hence  $\lim_{j \rightarrow -\infty} |\hat{\phi}(2^j \omega)| = |\hat{\phi}(0)| = 1$ .

We now prove that the function  $\phi$  whose Fourier transform is given by (7.32) is a scaling function. This is divided in two intermediate results.

• *Proof that  $\{\phi(t - n)\}_{n \in \mathbb{Z}}$  is orthonormal.* Observe first that the infinite product (7.32) converges and that  $|\hat{\phi}(\omega)| \leq 1$  because (7.29) implies that  $|\hat{h}(\omega)| \leq \sqrt{2}$ . The Parseval formula gives

$$\langle \phi(t), \phi(t - n) \rangle = \int_{-\infty}^{+\infty} \phi(t) \phi^*(t - n) dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |\hat{\phi}(\omega)|^2 e^{in\omega} d\omega.$$

Verifying that  $\{\phi(t - n)\}_{n \in \mathbb{Z}}$  is orthonormal is thus equivalent to showing that

$$\int_{-\infty}^{+\infty} |\hat{\phi}(\omega)|^2 e^{in\omega} d\omega = 2\pi \delta[n].$$

This result is obtained by considering the functions

$$\hat{\phi}_k(\omega) = \prod_{p=1}^k \frac{\hat{h}(2^{-p}\omega)}{\sqrt{2}} \mathbf{1}_{[-2^k\pi, 2^k\pi]}(\omega).$$

and computing the limit, as  $k$  increases to  $+\infty$ , of the integrals

$$I_k[n] = \int_{-\infty}^{+\infty} |\hat{\phi}_k(\omega)|^2 e^{in\omega} d\omega = \int_{-2^k\pi}^{2^k\pi} \prod_{p=1}^k \frac{|\hat{h}(2^{-p}\omega)|^2}{2} e^{in\omega} d\omega.$$

First, let us show that  $I_k[n] = 2\pi \delta[n]$  for all  $k \geq 1$ . To do this, we divide  $I_k[n]$  into two integrals:

$$I_k[n] = \int_{-2^k\pi}^0 \prod_{p=1}^k \frac{|\hat{h}(2^{-p}\omega)|^2}{2} e^{in\omega} d\omega + \int_0^{2^k\pi} \prod_{p=1}^k \frac{|\hat{h}(2^{-p}\omega)|^2}{2} e^{in\omega} d\omega.$$

Let us make the change of variable  $\omega' = \omega + 2^k\pi$  in the first integral. Since  $\hat{h}(\omega)$  is  $2\pi$  periodic, when  $p < k$  then  $|\hat{h}(2^{-p}[\omega' - 2^k\pi])|^2 = |\hat{h}(2^{-p}\omega')|^2$ . When  $k = p$  the hypothesis (7.29) implies that

$$|\hat{h}(2^{-k}[\omega' - 2^k\pi])|^2 + |\hat{h}(2^{-k}\omega')|^2 = 2.$$

For  $k > 1$ , the two integrals of  $I_k[n]$  become

$$I_k[n] = \int_0^{2^k\pi} \prod_{p=1}^{k-1} \frac{|\hat{h}(2^{-p}\omega)|^2}{2} e^{in\omega} d\omega. \quad (7.37)$$

Since  $\prod_{p=1}^{k-1} |\hat{h}(2^{-p}\omega)|^2 e^{in\omega}$  is  $2^k\pi$  periodic we obtain  $I_k[n] = I_{k-1}[n]$ , and by induction  $I_k[n] = I_1[n]$ . Writing (7.37) for  $k = 1$  gives

$$I_1[n] = \int_0^{2\pi} e^{in\omega} d\omega = 2\pi \delta[n],$$

which verifies that  $I_k[n] = 2\pi \delta[n]$ , for all  $k \geq 1$ .

We shall now prove that  $\hat{\phi} \in \mathbf{L}^2(\mathbb{R})$ . For all  $\omega \in \mathbb{R}$

$$\lim_{k \rightarrow \infty} |\hat{\phi}_k(\omega)|^2 = \prod_{p=1}^{\infty} \frac{|\hat{h}(2^{-p}\omega)|^2}{2} = |\hat{\phi}(\omega)|^2.$$



The Fatou Lemma A.1 on positive functions proves that

$$\int_{-\infty}^{+\infty} |\hat{\phi}(\omega)|^2 d\omega \leq \lim_{k \rightarrow \infty} \int_{-\infty}^{+\infty} |\hat{\phi}_k(\omega)|^2 d\omega = 2\pi, \quad (7.38)$$

because  $I_k[0] = 2\pi$  for all  $k \geq 1$ . Since

$$|\hat{\phi}(\omega)|^2 e^{in\omega} = \lim_{k \rightarrow \infty} |\hat{\phi}_k(\omega)|^2 e^{in\omega},$$

we finally verify that

$$\int_{-\infty}^{+\infty} |\hat{\phi}(\omega)|^2 e^{in\omega} d\omega = \lim_{k \rightarrow \infty} \int_{-\infty}^{+\infty} |\hat{\phi}_k(\omega)|^2 e^{in\omega} d\omega = 2\pi \delta[n] \quad (7.39)$$

by applying the dominated convergence Theorem A.1. This requires verifying the upper-bound condition (A.1). This is done in our case by proving the existence of a constant  $C$  such that

$$\left| |\hat{\phi}_k(\omega)|^2 e^{in\omega} \right| = |\hat{\phi}_k(\omega)|^2 \leq C |\hat{\phi}(\omega)|^2. \quad (7.40)$$

Indeed, we showed in (7.38) that  $|\hat{\phi}(\omega)|^2$  is an integrable function.

The existence of  $C > 0$  satisfying (7.40) is trivial for  $|\omega| > 2^k \pi$  since  $\hat{\phi}_k(\omega) = 0$ . For  $|\omega| \leq 2^k \pi$  since  $\hat{\phi}(\omega) = 2^{-1/2} \hat{h}(\omega/2) \hat{\phi}(\omega/2)$ , it follows that

$$|\hat{\phi}(\omega)|^2 = |\hat{\phi}_k(\omega)|^2 |\hat{\phi}(2^{-k}\omega)|^2.$$

To prove (7.40) for  $|\omega| \leq 2^k \pi$ , it is therefore sufficient to show that  $|\hat{\phi}(\omega)|^2 \geq 1/C$  for  $\omega \in [-\pi, \pi]$ .

Let us first study the neighborhood of  $\omega = 0$ . Since  $\hat{h}(\omega)$  is continuously differentiable in this neighborhood and since  $|\hat{h}(\omega)|^2 \leq 2 = |\hat{h}(0)|^2$ , the functions  $|\hat{h}(\omega)|^2$  and  $\log_e |\hat{h}(\omega)|^2$  have derivatives that vanish at  $\omega = 0$ . It follows that there exists  $\varepsilon > 0$  such that

$$\forall |\omega| \leq \varepsilon, \quad 0 \geq \log_e \left( \frac{|\hat{h}(\omega)|^2}{2} \right) \geq -|\omega|.$$

Hence, for  $|\omega| \leq \varepsilon$

$$|\hat{\phi}(\omega)|^2 = \exp \left[ \sum_{p=1}^{+\infty} \log_e \left( \frac{|\hat{h}(2^{-p}\omega)|^2}{2} \right) \right] \geq e^{-|\omega|} \geq e^{-\varepsilon}. \quad (7.41)$$

Now let us analyze the domain  $|\omega| > \varepsilon$ . To do this we take an integer  $l$  such that  $2^{-l}\pi < \varepsilon$ . Condition (7.31) proves that  $K = \inf_{\omega \in [-\pi/2, \pi/2]} |\hat{h}(\omega)| > 0$  so if  $|\omega| \leq \pi$

$$|\hat{\phi}(\omega)|^2 = \prod_{p=1}^l \frac{|\hat{h}(2^{-p}\omega)|^2}{2} \left| \hat{\phi}(2^{-l}\omega) \right|^2 \geq \frac{K^{2l}}{2^l} e^{-\varepsilon} = \frac{1}{C}.$$

This last result finishes the proof of inequality (7.40). Applying the dominated convergence Theorem A.1 proves (7.39) and hence that  $\{\phi(t-n)\}_{n \in \mathbb{Z}}$  is orthonormal. A simple change of variable shows that  $\{\phi_{j,n}\}_{j \in \mathbb{Z}}$  is orthonormal for all  $j \in \mathbb{Z}$ .

• *Proof that  $\{\mathbf{V}_j\}_{j \in \mathbb{Z}}$  is a multiresolution.* To verify that  $\phi$  is a scaling function, we must show that the spaces  $\mathbf{V}_j$  generated by  $\{\phi_{j,n}\}_{j \in \mathbb{Z}}$  define a multiresolution approximation. The multiresolution properties (7.1) and (7.3) are clearly true. The causality  $\mathbf{V}_{j+1} \subset \mathbf{V}_j$  is verified by showing that for any  $p \in \mathbb{Z}$ ,

$$\phi_{j+1,p} = \sum_{n=-\infty}^{+\infty} h[n-2p] \phi_{j,n}.$$

This equality is proved later in (7.107). Since all vectors of a basis of  $\mathbf{V}_{j+1}$  can be decomposed in a basis of  $\mathbf{V}_j$  it follows that  $\mathbf{V}_{j+1} \subset \mathbf{V}_j$ .

To prove the multiresolution property (7.4) we must show that any  $f \in \mathbf{L}^2(\mathbb{R})$  satisfies

$$\lim_{j \rightarrow +\infty} \|P_{\mathbf{V}_j} f\| = 0. \quad (7.42)$$

Since  $\{\phi_{j,n}\}_{n \in \mathbb{Z}}$  is an orthonormal basis of  $\mathbf{V}_j$

$$\|P_{\mathbf{V}_j} f\|^2 = \sum_{n=-\infty}^{+\infty} |\langle f, \phi_{j,n} \rangle|^2.$$

Suppose first that  $f$  is bounded by  $A$  and has a compact support included in  $[2^J, 2^J]$ . The constants  $A$  and  $J$  may be arbitrarily large. It follows that

$$\begin{aligned} \sum_{n=-\infty}^{+\infty} |\langle f, \phi_{j,n} \rangle|^2 &\leq 2^{-j} \left[ \sum_{n=-\infty}^{+\infty} \int_{-2^J}^{2^J} |f(t)| |\phi(2^{-j}t - n)| dt \right]^2 \\ &\leq 2^{-j} A^2 \left[ \sum_{n=-\infty}^{+\infty} \int_{-2^J}^{2^J} |\phi(2^{-j}t - n)| dt \right]^2 \end{aligned}$$

Applying the Cauchy-Schwarz inequality to  $1 \times |\phi(2^{-j}t - n)|$  yields

$$\begin{aligned} \sum_{n=-\infty}^{+\infty} |\langle f, \phi_{j,n} \rangle|^2 &\leq A^2 2^{J+1} \sum_{n=-\infty}^{+\infty} \int_{-2^J}^{2^J} |\phi(2^{-j}t - n)|^2 2^{-j} dt \\ &\leq A^2 2^{J+1} \int_{S_j} |\phi(t)|^2 dt = A^2 2^{J+1} \int_{-\infty}^{+\infty} |\phi(t)|^2 \mathbf{1}_{S_j}(t) dt, \end{aligned}$$

with  $S_j = \cup_{n \in \mathbb{Z}} [n - 2^{J-j}, n + 2^{J-j}]$  for  $j > J$ . For  $t \notin \mathbb{Z}$  we obviously have  $\mathbf{1}_{S_j}(t) \rightarrow 0$  for  $j \rightarrow +\infty$ . The dominated convergence Theorem A.1 applied to  $|\phi(t)|^2 \mathbf{1}_{S_j}(t)$  proves that the integral converges to 0 and hence

$$\lim_{j \rightarrow +\infty} \sum_{n=-\infty}^{+\infty} |\langle f, \phi_{j,n} \rangle|^2 = 0.$$

Property (7.42) is extended to any  $f \in \mathbf{L}^2(\mathbb{R})$  by using the density in  $\mathbf{L}^2(\mathbb{R})$  of bounded function with a compact support, and Theorem A.5.

To prove the last multiresolution property (7.5) we must show that for any  $f \in \mathbf{L}^2(\mathbb{R})$ ,

$$\lim_{j \rightarrow -\infty} \|f - P_{\mathbf{V}_j} f\|^2 = \lim_{j \rightarrow -\infty} (\|f\|^2 - \|P_{\mathbf{V}_j} f\|^2) = 0. \quad (7.43)$$

We consider functions  $f$  whose Fourier transform  $\hat{f}$  has a compact support included in  $[-2^J\pi, 2^J\pi]$  for  $J$  large enough. We proved in (7.36) that the Fourier transform of  $P_{\mathbf{V}_j} f$  is

$$\widehat{P_{\mathbf{V}_j} f}(\omega) = \hat{\phi}(2^j\omega) \sum_{k=-\infty}^{+\infty} \hat{f}(\omega - 2^{-j}2k\pi) \hat{\phi}^*(2^j[\omega - 2^{-j}2k\pi]).$$

If  $j < -J$ , then the supports of  $\hat{f}(\omega - 2^{-j}2k\pi)$  are disjoint for different  $k$  so

$$\begin{aligned} \|P_{\mathbf{V}_j} f\|^2 &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} |\hat{f}(\omega)|^2 |\hat{\phi}(2^j\omega)|^4 d\omega \\ &\quad + \frac{1}{2\pi} \int_{-\infty}^{+\infty} \sum_{\substack{k=-\infty \\ k \neq 0}}^{+\infty} |\hat{f}(\omega - 2^{-j}2k\pi)|^2 |\hat{\phi}(2^j\omega)|^2 |\hat{\phi}(2^j[\omega - 2^{-j}2k\pi])|^2 d\omega. \end{aligned} \quad (7.44)$$

We have already observed that  $|\phi(\omega)| \leq 1$  and (7.41) proves that for  $\omega$  sufficiently small  $|\phi(\omega)| \geq e^{-|\omega|}$  so

$$\lim_{\omega \rightarrow 0} |\hat{\phi}(\omega)| = 1.$$

Since  $|\hat{f}(\omega)|^2 |\hat{\phi}(2^j\omega)|^4 \leq |\hat{f}(\omega)|^2$  and  $\lim_{j \rightarrow -\infty} |\hat{\phi}(2^j\omega)|^4 |\hat{f}(\omega)|^2 = |\hat{f}(\omega)|^2$  one can apply the dominated convergence Theorem A.1, to prove that

$$\lim_{j \rightarrow -\infty} \int_{-\infty}^{+\infty} |\hat{f}(\omega)|^2 |\hat{\phi}(2^j\omega)|^4 d\omega = \int_{-\infty}^{+\infty} |\hat{f}(\omega)|^2 d\omega = \|f\|^2. \quad (7.45)$$

The operator  $P_{\mathbf{V}_j}$  is an orthogonal projector, so  $\|P_{\mathbf{V}_j} f\| \leq \|f\|$ . With (7.44) and (7.45), this implies that  $\lim_{j \rightarrow -\infty} (\|f\|^2 - \|P_{\mathbf{V}_j} f\|^2) = 0$ , and hence verifies (7.43). This property is extended to any  $f \in \mathbf{L}^2(\mathbb{R})$  by using the density in  $\mathbf{L}^2(\mathbb{R})$  of functions whose Fourier transforms have a compact support and the result of Theorem A.5.  $\blacksquare$

Discrete filters whose transfer functions satisfy (7.29) are called *conjugate mirror filters*. As we shall see in Section 7.3, they play an important role in discrete signal processing; they make it possible to decompose discrete signals in separate frequency bands with filter banks. One difficulty

of the proof is showing that the infinite cascade of convolutions that is represented in the Fourier domain by the product (7.32) does converge to a decent function in  $\mathbf{L}^2(\mathbb{R})$ . The sufficient condition (7.31) is not necessary to construct a scaling function, but it is always satisfied in practical designs of conjugate mirror filters. It cannot just be removed as shown by the example  $\hat{h}(\omega) = \cos(3\omega/2)$ , which satisfies all other conditions. In this case, a simple calculation shows that  $\phi = 1/3 \mathbf{1}_{[-3/2, 3/2]}$ . Clearly  $\{\phi(t - n)\}_{n \in \mathbb{Z}}$  is not orthogonal so  $\phi$  is not a scaling function. The condition (7.31) may however be replaced by a weaker but more technical necessary and sufficient condition proved by Cohen [15, 166].

**Example 7.6.** For a Shannon multiresolution approximation,  $\hat{\phi} = \mathbf{1}_{[-\pi, \pi]}$ . We thus derive from (7.32) that

$$\forall \omega \in [-\pi, \pi] \quad , \quad \hat{h}(\omega) = \sqrt{2} \mathbf{1}_{[-\pi/2, \pi/2]}(\omega).$$

**Example 7.7.** For piecewise constant approximations,  $\phi = \mathbf{1}_{[0, 1]}$ . Since  $h[n] = \langle 2^{-1/2} \phi(t/2), \phi(t - n) \rangle$  it follows that

$$h[n] = \begin{cases} 2^{-1/2} & \text{if } n = 0, 1 \\ 0 & \text{otherwise} \end{cases} \quad (7.46)$$

**Example 7.8.** Polynomial splines of degree  $m$  correspond to a conjugate mirror filter  $\hat{h}(\omega)$  that is calculated from  $\hat{\phi}(\omega)$  with (7.25):

$$\hat{h}(\omega) = \sqrt{2} \frac{\hat{\phi}(2\omega)}{\hat{\phi}(\omega)}. \quad (7.47)$$

Inserting (7.18) yields

$$\hat{h}(\omega) = \exp\left(\frac{-i\varepsilon\omega}{2}\right) \sqrt{\frac{S_{2m+2}(\omega)}{2^{2m+1} S_{2m+2}(2\omega)}}, \quad (7.48)$$

where  $\varepsilon = 0$  if  $m$  is odd and  $\varepsilon = 1$  if  $m$  is even. For linear splines  $m = 1$  so (7.20) implies that

$$\hat{h}(\omega) = \sqrt{2} \left[ \frac{1 + 2 \cos^2(\omega/2)}{1 + 2 \cos^2 \omega} \right]^{1/2} \cos^2\left(\frac{\omega}{2}\right). \quad (7.49)$$

For cubic splines, the conjugate mirror filter is calculated by inserting (7.22) in (7.48). Figure 7.4 gives the graph of  $|\hat{h}(\omega)|^2$ . The impulse responses  $h[n]$  of these filters have an infinite support but an exponential decay. For  $m$  odd,  $h[n]$  is symmetric about  $n = 0$ . Table 7.1 gives the coefficients  $h[n]$  above  $10^{-4}$  for  $m = 1, 3$ .

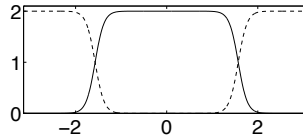


Figure 7.4: The solid line gives  $|\hat{h}(\omega)|^2$  on  $[-\pi, \pi]$ , for a cubic spline multiresolution. The dotted line corresponds to  $|\hat{\phi}(\omega)|^2$ .

#### 7.1.4 In Which Orthogonal Wavelets Finally Arrive

Orthonormal wavelets carry the details necessary to increase the resolution of a signal approximation. The approximations of  $f$  at the scales  $2^j$  and  $2^{j-1}$  are respectively equal to their orthogonal projections on  $\mathbf{V}_j$  and  $\mathbf{V}_{j-1}$ . We know that  $\mathbf{V}_j$  is included in  $\mathbf{V}_{j-1}$ . Let  $\mathbf{W}_j$  be the orthogonal complement of  $\mathbf{V}_j$  in  $\mathbf{V}_{j-1}$ :

$$\mathbf{V}_{j-1} = \mathbf{V}_j \oplus \mathbf{W}_j. \quad (7.50)$$

	n	$h[n]$		n	$h[n]$
$m = 1$	0	0.817645956	$m = 3$	5, -5	0.042068328
	1, -1	0.397296430		6, -6	-0.017176331
	2, -2	-0.069101020		7, -7	-0.017982291
	3, -3	-0.051945337		8, -8	0.008685294
	4, -4	0.016974805		9, -9	0.008201477
	5, -5	0.009990599		10, -10	-0.004353840
	6, -6	-0.003883261		11, -11	-0.003882426
	7, -7	-0.002201945		12, -12	0.002186714
	8, -8	0.000923371		13, -13	0.001882120
	9, -9	0.000511636		14, -14	-0.001103748
	10, -10	-0.000224296		15, -15	-0.000927187
$m = 3$	11, -11	-0.000122686		16, -16	0.000559952
	0	0.766130398		17, -17	0.000462093
	1, -1	0.433923147		18, -18	-0.000285414
	2, -2	-0.050201753		19, -19	-0.000232304
	3, -3	-0.110036987		20, -20	0.000146098
	4, -4	0.032080869			

Table 7.1: Conjugate mirror filters  $h[n]$  corresponding to linear splines  $m = 1$  and cubic splines  $m = 3$ . The coefficients below  $10^{-4}$  are not given.

The orthogonal projection of  $f$  on  $\mathbf{V}_{j-1}$  can be decomposed as the sum of orthogonal projections on  $\mathbf{V}_j$  and  $\mathbf{W}_j$ :

$$P_{\mathbf{V}_{j-1}}f = P_{\mathbf{V}_j}f + P_{\mathbf{W}_j}f. \quad (7.51)$$

The complement  $P_{\mathbf{W}_j}f$  provides the “details” of  $f$  that appear at the scale  $2^{j-1}$  but which disappear at the coarser scale  $2^j$ . The following theorem [43, 361] proves that one can construct an orthonormal basis of  $\mathbf{W}_j$  by scaling and translating a wavelet  $\psi$ .

**Theorem 7.3** (Mallat, Meyer). *Let  $\phi$  be a scaling function and  $h$  the corresponding conjugate mirror filter. Let  $\psi$  be the function whose Fourier transform is*

$$\hat{\psi}(\omega) = \frac{1}{\sqrt{2}} \hat{g}\left(\frac{\omega}{2}\right) \hat{\phi}\left(\frac{\omega}{2}\right), \quad (7.52)$$

with

$$\hat{g}(\omega) = e^{-i\omega} \hat{h}^*(\omega + \pi). \quad (7.53)$$

Let us denote

$$\psi_{j,n}(t) = \frac{1}{\sqrt{2^j}} \psi\left(\frac{t - 2^j n}{2^j}\right).$$

For any scale  $2^j$ ,  $\{\psi_{j,n}\}_{n \in \mathbb{Z}}$  is an orthonormal basis of  $\mathbf{W}_j$ . For all scales,  $\{\psi_{j,n}\}_{(j,n) \in \mathbb{Z}^2}$  is an orthonormal basis of  $\mathbf{L}^2(\mathbb{R})$ .

*Proof.* Let us prove first that  $\hat{\psi}$  can be written as the product (7.52). Necessarily  $\psi(t/2) \in \mathbf{W}_1 \subset \mathbf{V}_0$ . It can thus be decomposed in  $\{\phi(t - n)\}_{n \in \mathbb{Z}}$  which is an orthogonal basis of  $\mathbf{V}_0$ :

$$\frac{1}{\sqrt{2}} \psi\left(\frac{t}{2}\right) = \sum_{n=-\infty}^{+\infty} g[n] \phi(t - n), \quad (7.54)$$

with

$$g[n] = \frac{1}{\sqrt{2}} \left\langle \psi\left(\frac{t}{2}\right), \phi(t - n) \right\rangle. \quad (7.55)$$

The Fourier transform of (7.54) yields

$$\hat{\psi}(2\omega) = \frac{1}{\sqrt{2}} \hat{g}(\omega) \hat{\phi}(\omega). \quad (7.56)$$

The following lemma gives necessary and sufficient conditions on  $\hat{g}$  for designing an orthogonal wavelet.

**Lemma 7.1.** *The family  $\{\psi_{j,n}\}_{n \in \mathbb{Z}}$  is an orthonormal basis of  $\mathbf{W}_j$  if and only if*

$$|\hat{g}(\omega)|^2 + |\hat{g}(\omega + \pi)|^2 = 2 \quad (7.57)$$

and

$$\hat{g}(\omega) \hat{h}^*(\omega) + \hat{g}(\omega + \pi) \hat{h}^*(\omega + \pi) = 0. \quad (7.58)$$

The lemma is proved for  $j = 0$  from which it is easily extended to  $j \neq 0$  with an appropriate scaling. As in (7.14) one can verify that  $\{\psi(t - n)\}_{n \in \mathbb{Z}}$  is orthonormal if and only if

$$\forall \omega \in \mathbb{R}, \quad I(\omega) = \sum_{k=-\infty}^{+\infty} |\hat{\psi}(\omega + 2k\pi)|^2 = 1. \quad (7.59)$$

Since  $\hat{\psi}(\omega) = 2^{-1/2} \hat{g}(\omega/2) \hat{\phi}(\omega/2)$  and  $\hat{g}(\omega)$  is  $2\pi$  periodic,

$$\begin{aligned} I(\omega) &= \sum_{k=-\infty}^{+\infty} |\hat{g}\left(\frac{\omega}{2} + k\pi\right)|^2 |\hat{\phi}\left(\frac{\omega}{2} + k\pi\right)|^2 \\ &= |\hat{g}\left(\frac{\omega}{2}\right)|^2 \sum_{p=-\infty}^{+\infty} |\hat{\phi}\left(\frac{\omega}{2} + 2p\pi\right)|^2 + |\hat{g}\left(\frac{\omega}{2} + \pi\right)|^2 \sum_{p=-\infty}^{+\infty} |\hat{\phi}\left(\frac{\omega}{2} + \pi + 2p\pi\right)|^2. \end{aligned}$$

We know that  $\sum_{p=-\infty}^{+\infty} |\hat{\phi}(\omega + 2p\pi)|^2 = 1$  so (7.59) is equivalent to (7.57).

The space  $\mathbf{W}_0$  is orthogonal to  $\mathbf{V}_0$  if and only if  $\{\phi(t - n)\}_{n \in \mathbb{Z}}$  and  $\{\psi(t - n)\}_{n \in \mathbb{Z}}$  are orthogonal families of vectors. This means that for any  $n \in \mathbb{Z}$

$$\langle \psi(t), \phi(t - n) \rangle = \psi \star \bar{\phi}(n) = 0.$$

The Fourier transform of  $\psi \star \bar{\phi}(t)$  is  $\hat{\psi}(\omega) \hat{\phi}^*(\omega)$ . The sampled sequence  $\psi \star \bar{\phi}(n)$  is zero if its Fourier series computed with (3.3) satisfies

$$\forall \omega \in \mathbb{R}, \quad \sum_{k=-\infty}^{+\infty} \hat{\psi}(\omega + 2k\pi) \hat{\phi}^*(\omega + 2k\pi) = 0. \quad (7.60)$$

By inserting  $\hat{\psi}(\omega) = 2^{-1/2} \hat{g}(\omega/2) \hat{\phi}(\omega/2)$  and  $\hat{\phi}(\omega) = 2^{-1/2} \hat{h}(\omega/2) \hat{\phi}(\omega/2)$  in this equation, since  $\sum_{k=-\infty}^{+\infty} |\hat{\phi}(\omega + 2k\pi)|^2 = 1$  we prove as before that (7.60) is equivalent to (7.58).

We must finally verify that  $\mathbf{V}_{-1} = \mathbf{V}_0 \oplus \mathbf{W}_0$ . Knowing that  $\{\sqrt{2}\phi(2t - n)\}_{n \in \mathbb{Z}}$  is an orthogonal basis of  $\mathbf{V}_{-1}$ , it is equivalent to show that for any  $a[n] \in \ell^2(\mathbb{Z})$  there exist  $b[n] \in \ell^2(\mathbb{Z})$  and  $c[n] \in \ell^2(\mathbb{Z})$  such that

$$\sum_{n=-\infty}^{+\infty} a[n] \sqrt{2} \phi(2[t - 2^{-1}n]) = \sum_{n=-\infty}^{+\infty} b[n] \phi(t - n) + \sum_{n=-\infty}^{+\infty} c[n] \psi(t - n). \quad (7.61)$$

This is done by relating  $\hat{b}(\omega)$  and  $\hat{c}(\omega)$  to  $\hat{a}(\omega)$ . The Fourier transform of (7.61) yields

$$\frac{1}{\sqrt{2}} \hat{a}\left(\frac{\omega}{2}\right) \hat{\phi}\left(\frac{\omega}{2}\right) = \hat{b}(\omega) \hat{\phi}(\omega) + \hat{c}(\omega) \hat{\psi}(\omega).$$

Inserting  $\hat{\psi}(\omega) = 2^{-1/2} \hat{g}(\omega/2) \hat{\phi}(\omega/2)$  and  $\hat{\phi}(\omega) = 2^{-1/2} \hat{h}(\omega/2) \hat{\phi}(\omega/2)$  in this equation shows that it is necessarily satisfied if

$$\hat{a}\left(\frac{\omega}{2}\right) = \hat{b}(\omega) \hat{h}\left(\frac{\omega}{2}\right) + \hat{c}(\omega) \hat{g}\left(\frac{\omega}{2}\right). \quad (7.62)$$

Let us define

$$\hat{b}(2\omega) = \frac{1}{2} [\hat{a}(\omega) \hat{h}^*(\omega) + \hat{a}(\omega + \pi) \hat{h}^*(\omega + \pi)]$$

and

$$\hat{c}(2\omega) = \frac{1}{2} [\hat{a}(\omega) \hat{g}^*(\omega) + \hat{a}(\omega + \pi) \hat{g}^*(\omega + \pi)].$$

When calculating the right-hand side of (7.62) we verify that it is equal to the left-hand side by inserting (7.57), (7.58) and using

$$|\hat{h}(\omega)|^2 + |\hat{h}(\omega + \pi)|^2 = 2. \quad (7.63)$$

Since  $\hat{b}(\omega)$  and  $\hat{c}(\omega)$  are  $2\pi$  periodic they are the Fourier series of two sequences  $b[n]$  and  $c[n]$  that satisfy (7.61). This finishes the proof of the lemma.

The formula (7.53)

$$\hat{g}(\omega) = e^{-i\omega} \hat{h}^*(\omega + \pi)$$

satisfies (7.57) and (7.58) because of (7.63). We thus derive from Lemma 7.1 that  $\{\psi_{j,n}\}_{(j,n) \in \mathbb{Z}^2}$  is an orthogonal basis of  $\mathbf{W}_j$ .

We complete the proof of the theorem by verifying that  $\{\psi_{j,n}\}_{(j,n) \in \mathbb{Z}^2}$  is an orthogonal basis of  $\mathbf{L}^2(\mathbb{R})$ . Observe first that the detail spaces  $\{\mathbf{W}_j\}_{j \in \mathbb{Z}}$  are orthogonal. Indeed  $\mathbf{W}_j$  is orthogonal to  $\mathbf{V}_j$  and  $\mathbf{W}_l \subset \mathbf{V}_{l-1} \subset \mathbf{V}_j$  for  $j < l$ . Hence  $\mathbf{W}_j$  and  $\mathbf{W}_l$  are orthogonal. We can also decompose

$$\mathbf{L}^2(\mathbb{R}) = \bigoplus_{j=-\infty}^{+\infty} \mathbf{W}_j. \quad (7.64)$$

Indeed  $\mathbf{V}_{j-1} = \mathbf{W}_j \oplus \mathbf{V}_j$  and we verify by substitution that for any  $L > J$

$$\mathbf{V}_L = \bigoplus_{j=L-1}^J \mathbf{W}_j \oplus \mathbf{V}_J. \quad (7.65)$$

Since  $\{\mathbf{V}_j\}_{j \in \mathbb{Z}}$  is a multiresolution approximation,  $\mathbf{V}_L$  and  $\mathbf{V}_J$  tend respectively to  $\mathbf{L}^2(\mathbb{R})$  and  $\{0\}$  when  $L$  and  $J$  go respectively to  $-\infty$  and  $+\infty$ , which implies (7.64). A union of orthonormal bases of all  $\mathbf{W}_j$  is therefore an orthonormal basis of  $\mathbf{L}^2(\mathbb{R})$ . ■ ■

The proof of the theorem shows that  $\hat{g}$  is the Fourier series of

$$g[n] = \left\langle \frac{1}{\sqrt{2}} \psi\left(\frac{t}{2}\right), \phi(t-n) \right\rangle, \quad (7.66)$$

which are the decomposition coefficients of

$$\frac{1}{\sqrt{2}} \psi\left(\frac{t}{2}\right) = \sum_{n=-\infty}^{+\infty} g[n] \phi(t-n). \quad (7.67)$$

Calculating the inverse Fourier transform of (7.53) yields

$$g[n] = (-1)^{1-n} h[1-n]. \quad (7.68)$$

This mirror filter plays an important role in the fast wavelet transform algorithm.

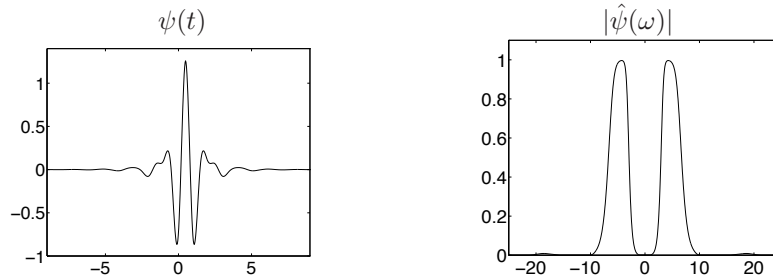


Figure 7.5: Battle-Lemarié cubic spline wavelet  $\psi$  and its Fourier transform modulus.

**Example 7.9.** Figure 7.5 displays the cubic spline wavelet  $\psi$  and its Fourier transform  $\hat{\psi}$  calculated by inserting in (7.52) the expressions (7.18) and (7.48) of  $\hat{\phi}(\omega)$  and  $\hat{h}(\omega)$ . The properties of this Battle-Lemarié spline wavelet are further studied in Section 7.2.2. Like most orthogonal wavelets, the energy of  $\hat{\psi}$  is essentially concentrated in  $[-2\pi, -\pi] \cup [\pi, 2\pi]$ . For any  $\psi$  that generates an orthogonal basis of  $\mathbf{L}^2(\mathbb{R})$ , one can verify that

$$\forall \omega \in \mathbb{R} - \{0\}, \quad \sum_{j=-\infty}^{+\infty} |\hat{\psi}(2^j \omega)|^2 = 1.$$

This is illustrated in Figure 7.6.

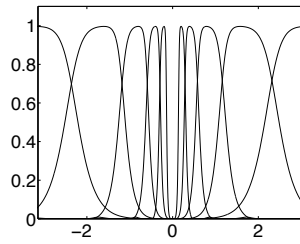


Figure 7.6: Graphs of  $|\hat{\psi}(2^j \omega)|^2$  for the cubic spline Battle-Lemarié wavelet, with  $1 \leq j \leq 5$  and  $\omega \in [-\pi, \pi]$ .

The orthogonal projection of a signal  $f$  in a “detail” space  $\mathbf{W}_j$  is obtained with a partial expansion in its wavelet basis

$$P_{\mathbf{W}_j} f = \sum_{n=-\infty}^{+\infty} \langle f, \psi_{j,n} \rangle \psi_{j,n}.$$

A signal expansion in a wavelet orthogonal basis can thus be viewed as an aggregation of details at all scales  $2^j$  that go from 0 to  $+\infty$

$$f = \sum_{j=-\infty}^{+\infty} P_{\mathbf{W}_j} f = \sum_{j=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} \langle f, \psi_{j,n} \rangle \psi_{j,n}.$$

Figure 7.7 gives the coefficients of a signal decomposed in the cubic spline wavelet orthogonal basis. The calculations are performed with the fast wavelet transform algorithm of Section 7.3. The up or down Diracs give the amplitudes of positive or negative wavelet coefficients, at a distance  $2^j n$  at each scale  $2^j$ . Coefficients are nearly zero at fine scales where the signal is locally regular.

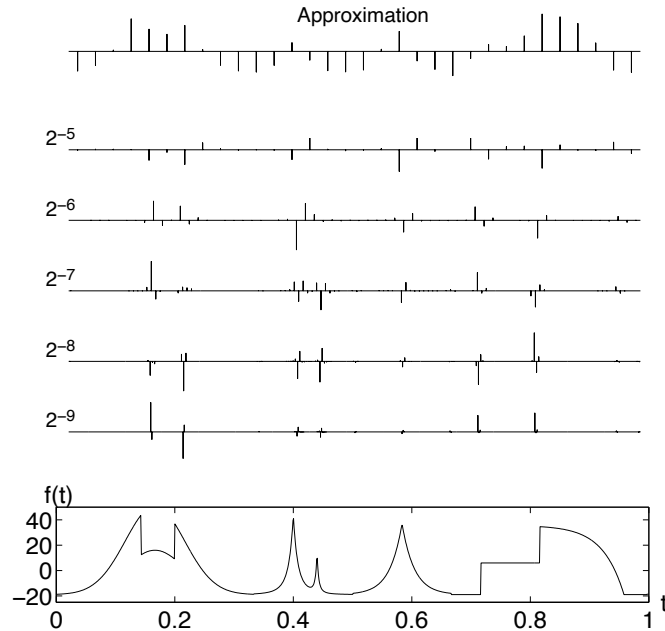


Figure 7.7: Wavelet coefficients  $d_j[n] = \langle f, \psi_{j,n} \rangle$  calculated at scales  $2^j$  with the cubic spline wavelet. Each up or down Dirac gives the amplitude of a positive or negative wavelet coefficient. At the top is the remaining coarse signal approximation  $a_J[n] = \langle f, \phi_{J,n} \rangle$  for  $J = -5$ .

**Wavelet Design** Theorem 7.3 constructs a wavelet orthonormal basis from any conjugate mirror filter  $\hat{h}(\omega)$ . This gives a simple procedure for designing and building wavelet orthogonal bases.

Conversely, we may wonder whether all wavelet orthonormal bases are associated to a multiresolution approximation and a conjugate mirror filter. If we impose that  $\psi$  has a compact support then Lemarié [51] proved that  $\psi$  necessarily corresponds to a multiresolution approximation. It is however possible to construct pathological wavelets that decay like  $|t|^{-1}$  at infinity, and which cannot be derived from any multiresolution approximation. Section 7.2 describes important classes of wavelet bases and explains how to design  $\hat{h}$  to specify the support, the number of vanishing moments and the regularity of  $\psi$ .

## 7.2 Classes of Wavelet Bases

### 7.2.1 Choosing a Wavelet

Most applications of wavelet bases exploit their ability to efficiently approximate particular classes of functions with few non-zero wavelet coefficients. This is true not only for data compression but also for noise removal and fast calculations. The design of  $\psi$  must therefore be optimized to produce a maximum number of wavelet coefficients  $\langle f, \psi_{j,n} \rangle$  that are close to zero. A function  $f$  has few non-negligible wavelet coefficients if most of the fine-scale (high-resolution) wavelet coefficients are small. This depends mostly on the regularity of  $f$ , the number of vanishing moments of  $\psi$  and the size of its support. To construct an appropriate wavelet from a conjugate mirror filter  $h[n]$ , we relate these properties to conditions on  $\hat{h}(\omega)$ .

**Vanishing Moments** Let us recall that  $\psi$  has  $p$  vanishing moments if

$$\int_{-\infty}^{+\infty} t^k \psi(t) dt = 0 \quad \text{for } 0 \leq k < p. \quad (7.69)$$

This means that  $\psi$  is orthogonal to any polynomial of degree  $p-1$ . Section 6.1.3 proves that if  $f$  is regular and  $\psi$  has enough vanishing moments then the wavelet coefficients  $|\langle f, \psi_{j,n} \rangle|$  are small at fine scales  $2^j$ . Indeed, if  $f$  is locally  $\mathbf{C}^k$ , then over a small interval it is well approximated by a Taylor polynomial of degree  $k$ . If  $k < p$ , then wavelets are orthogonal to this Taylor polynomial and thus produce small amplitude coefficients at fine scales. The following theorem relates the number of vanishing moments of  $\psi$  to the vanishing derivatives of  $\hat{\psi}(\omega)$  at  $\omega = 0$  and to the number of zeroes of  $\hat{h}(\omega)$  at  $\omega = \pi$ . It also proves that polynomials of degree  $p-1$  are then reproduced by the scaling functions.

**Theorem 7.4** (Vanishing moments). *Let  $\psi$  and  $\phi$  be a wavelet and a scaling function that generate an orthogonal basis. Suppose that  $|\psi(t)| = O((1+t^2)^{-p/2-1})$  and  $|\phi(t)| = O((1+t^2)^{-p/2-1})$ . The four following statements are equivalent:*

- (i) *The wavelet  $\psi$  has  $p$  vanishing moments.*
- (ii)  *$\hat{\psi}(\omega)$  and its first  $p-1$  derivatives are zero at  $\omega = 0$ .*
- (iii)  *$\hat{h}(\omega)$  and its first  $p-1$  derivatives are zero at  $\omega = \pi$ .*
- (iv) *For any  $0 \leq k < p$ ,*

$$q_k(t) = \sum_{n=-\infty}^{+\infty} n^k \phi(t-n) \quad \text{is a polynomial of degree } k. \quad (7.70)$$

*Proof.* The decay of  $|\phi(t)|$  and  $|\psi(t)|$  implies that  $\hat{\psi}(\omega)$  and  $\hat{\phi}(\omega)$  are  $p$  times continuously differentiable. The  $k^{\text{th}}$  order derivative  $\hat{\psi}^{(k)}(\omega)$  is the Fourier transform of  $(-it)^k \psi(t)$ . Hence

$$\hat{\psi}^{(k)}(0) = \int_{-\infty}^{+\infty} (-it)^k \psi(t) dt.$$

We derive that (i) is equivalent to (ii).

Theorem 7.3 proves that

$$\sqrt{2} \hat{\psi}(2\omega) = e^{-i\omega} \hat{h}^*(\omega + \pi) \hat{\phi}(\omega).$$

Since  $\hat{\phi}(0) \neq 0$ , by differentiating this expression we prove that (ii) is equivalent to (iii).



Let us now prove that (iv) implies (i). Since  $\psi$  is orthogonal to  $\{\phi(t - n)\}_{n \in \mathbb{Z}}$ , it is thus also orthogonal to the polynomials  $q_k$  for  $0 \leq k < p$ . This family of polynomials is a basis of the space of polynomials of degree at most  $p - 1$ . Hence  $\psi$  is orthogonal to any polynomial of degree  $p - 1$  and in particular to  $t^k$  for  $0 \leq k < p$ . This means that  $\psi$  has  $p$  vanishing moments.

To verify that (i) implies (iv) we suppose that  $\psi$  has  $p$  vanishing moments, and for  $k < p$  we evaluate  $q_k(t)$  defined in (7.70). This is done by computing its Fourier transform:

$$\hat{q}_k(\omega) = \hat{\phi}(\omega) \sum_{n=-\infty}^{+\infty} n^k \exp(-in\omega) = (i)^k \hat{\phi}(\omega) \frac{d^k}{d\omega^k} \sum_{n=-\infty}^{+\infty} \exp(-in\omega).$$

Let  $\delta^{(k)}$  be the distribution that is the  $k^{\text{th}}$  order derivative of a Dirac, defined in Appendix A.7. The Poisson formula (2.4) proves that

$$\hat{q}_k(\omega) = (i)^k \frac{1}{2\pi} \hat{\phi}(\omega) \sum_{l=-\infty}^{+\infty} \delta^{(k)}(\omega - 2l\pi). \quad (7.71)$$

With several integrations by parts, we verify the distribution equality

$$\hat{\phi}(\omega) \delta^{(k)}(\omega - 2l\pi) = \hat{\phi}(2l\pi) \delta^{(k)}(\omega - 2l\pi) + \sum_{m=0}^{k-1} a_{m,l}^k \delta^{(m)}(\omega - 2l\pi), \quad (7.72)$$

where  $a_{m,l}^k$  is a linear combination of the derivatives  $\{\hat{\phi}^{(m)}(2l\pi)\}_{0 \leq m \leq k}$ .

For  $l \neq 0$ , let us prove that  $a_{m,l}^k = 0$  by showing that  $\hat{\phi}^{(m)}(2l\pi) = 0$  if  $0 \leq m < p$ . For any  $P > 0$ , (7.27) implies

$$\hat{\phi}(\omega) = \hat{\phi}(2^{-P}\omega) \prod_{p=1}^P \frac{\hat{h}(2^{-p}\omega)}{\sqrt{2}}. \quad (7.73)$$

Since  $\psi$  has  $p$  vanishing moments, we showed in (iii) that  $\hat{h}(\omega)$  has a zero of order  $p$  at  $\omega = \pm\pi$ . But  $\hat{h}(\omega)$  is also  $2\pi$  periodic, so (7.73) implies that  $\hat{\phi}(\omega) = O(|\omega - 2l\pi|^p)$  in the neighborhood of  $\omega = 2l\pi$ , for any  $l \neq 0$ . Hence  $\hat{\phi}^{(m)}(2l\pi) = 0$  if  $m < p$ .

Since  $a_{m,l}^k = 0$  and  $\hat{\phi}(2l\pi) = 0$  when  $l \neq 0$ , it follows from (7.72) that

$$\hat{\phi}(\omega) \delta^{(k)}(\omega - 2l\pi) = 0 \quad \text{for } l \neq 0.$$

The only term that remains in the summation (7.71) is  $l = 0$  and inserting (7.72) yields

$$\hat{q}_k(\omega) = (i)^k \frac{1}{2\pi} \left( \hat{\phi}(0) \delta^{(k)}(\omega) + \sum_{m=0}^{k-1} a_{m,0}^k \delta^{(m)}(\omega) \right).$$

The inverse Fourier transform of  $\delta^{(m)}(\omega)$  is  $(2\pi)^{-1}(-it)^m$  and Theorem 7.2 proves that  $\hat{\phi}(0) \neq 0$ . Hence the inverse Fourier transform  $q_k$  of  $\hat{q}_k$  is a polynomial of degree  $k$ . ■ ■

The hypothesis (iv) is called the Fix-Strang condition [445]. The polynomials  $\{q_k\}_{0 \leq k < p}$  define a basis of the space of polynomials of degree  $p - 1$ . The Fix-Strang condition thus proves that  $\psi$  has  $p$  vanishing moments if and only if any polynomial of degree  $p - 1$  can be written as a linear expansion of  $\{\phi(t - n)\}_{n \in \mathbb{Z}}$ . The decomposition coefficients of the polynomials  $q_k$  do not have a finite energy because polynomials do not have a finite energy.

**Size of Support** If  $f$  has an isolated singularity at  $t_0$  and if  $t_0$  is inside the support of  $\psi_{j,n}(t) = 2^{-j/2} \psi(2^{-j}t - n)$ , then  $\langle f, \psi_{j,n} \rangle$  may have a large amplitude. If  $\psi$  has a compact support of size  $K$ , at each scale  $2^j$  there are  $K$  wavelets  $\psi_{j,n}$  whose support includes  $t_0$ . To minimize the number of high amplitude coefficients we must reduce the support size of  $\psi$ . The following theorem relates the support size of  $h$  to the support of  $\phi$  and  $\psi$ .

**Theorem 7.5** (Compact support). *The scaling function  $\phi$  has a compact support if and only if  $h$  has a compact support and their support are equal. If the support of  $h$  and  $\phi$  is  $[N_1, N_2]$  then the support of  $\psi$  is  $[(N_1 - N_2 + 1)/2, (N_2 - N_1 + 1)/2]$ .*

*Proof.* If  $\phi$  has a compact support, since

$$h[n] = \frac{1}{\sqrt{2}} \left\langle \phi\left(\frac{t}{2}\right), \phi(t-n) \right\rangle,$$

we derive that  $h$  also has a compact support. Conversely, the scaling function satisfies

$$\frac{1}{\sqrt{2}} \phi\left(\frac{t}{2}\right) = \sum_{n=-\infty}^{+\infty} h[n] \phi(t-n). \quad (7.74)$$

If  $h$  has a compact support then one can prove [193] that  $\phi$  has a compact support. The proof is not reproduced here.

To relate the support of  $\phi$  and  $h$ , we suppose that  $h[n]$  is non-zero for  $N_1 \leq n \leq N_2$  and that  $\phi$  has a compact support  $[K_1, K_2]$ . The support of  $\phi(t/2)$  is  $[2K_1, 2K_2]$ . The sum at the right of (7.74) is a function whose support is  $[N_1 + K_1, N_2 + K_2]$ . The equality proves that the support of  $\phi$  is  $[K_1, K_2] = [N_1, N_2]$ .

Let us recall from (7.68) and (7.67) that

$$\frac{1}{\sqrt{2}} \psi\left(\frac{t}{2}\right) = \sum_{n=-\infty}^{+\infty} g[n] \phi(t-n) = \sum_{n=-\infty}^{+\infty} (-1)^{1-n} h[1-n] \phi(t-n).$$

If the supports of  $\phi$  and  $h$  are equal to  $[N_1, N_2]$ , the sum in the right-hand side has a support equal to  $[N_1 - N_2 + 1, N_2 - N_1 + 1]$ . Hence  $\psi$  has a support equal to  $[(N_1 - N_2 + 1)/2, (N_2 - N_1 + 1)/2]$ . ■ ■

If  $h$  has a finite impulse response in  $[N_1, N_2]$ , Theorem 7.5 proves that  $\psi$  has a support of size  $N_2 - N_1$  centered at  $1/2$ . To minimize the size of the support, we must synthesize conjugate mirror filters with as few non-zero coefficients as possible.

**Support Versus Moments** The support size of a function and the number of vanishing moments are a priori independent. However, we shall see in Theorem 7.7 that the constraints imposed on orthogonal wavelets imply that if  $\psi$  has  $p$  vanishing moments then its support is at least of size  $2p - 1$ . Daubechies wavelets are optimal in the sense that they have a minimum size support for a given number of vanishing moments. When choosing a particular wavelet, we thus face a trade-off between the number of vanishing moments and the support size. If  $f$  has few isolated singularities and is very regular between singularities, we must choose a wavelet with many vanishing moments to produce a large number of small wavelet coefficients  $\langle f, \psi_{j,n} \rangle$ . If the density of singularities increases, it might be better to decrease the size of its support at the cost of reducing the number of vanishing moments. Indeed, wavelets that overlap the singularities create high amplitude coefficients.

The multiwavelet construction of Geronimo, Hardin and Massupust [270] offers more design flexibility by introducing several scaling functions and wavelets. Exercise 7.16 gives an example. Better trade-off can be obtained between the multiwavelets supports and their vanishing moments [446]. However, multiwavelet decompositions are implemented with a slightly more complicated filter bank algorithm than a standard orthogonal wavelet transform.

**Regularity** The regularity of  $\psi$  has mostly a cosmetic influence on the error introduced by thresholding or quantizing the wavelet coefficients. When reconstructing a signal from its wavelet coefficients

$$f = \sum_{j=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} \langle f, \psi_{j,n} \rangle \psi_{j,n},$$

an error  $\varepsilon$  added to a coefficient  $\langle f, \psi_{j,n} \rangle$  will add the wavelet component  $\varepsilon \psi_{j,n}$  to the reconstructed signal. If  $\psi$  is smooth, then  $\varepsilon \psi_{j,n}$  is a smooth error. For image coding applications, a smooth error is often less visible than an irregular error, even though they have the same energy. Better quality images are obtained with wavelets that are continuously differentiable than with the discontinuous Haar wavelet. The following theorem due to Tchamitchian [454] relates the uniform Lipschitz regularity of  $\phi$  and  $\psi$  to the number of zeroes of  $\hat{h}(\omega)$  at  $\omega = \pi$ .

**Theorem 7.6** (Tchamitchian). *Let  $\hat{h}(\omega)$  be a conjugate mirror filter with  $p$  zeroes at  $\pi$  and which satisfies the sufficient conditions of Theorem 7.2. Let us perform the factorization*

$$\hat{h}(\omega) = \sqrt{2} \left( \frac{1 + e^{i\omega}}{2} \right)^p \hat{l}(\omega).$$

*If  $\sup_{\omega \in \mathbb{R}} |\hat{l}(\omega)| = B$  then  $\psi$  and  $\phi$  are uniformly Lipschitz  $\alpha$  for*

$$\alpha < \alpha_0 = p - \log_2 B - 1. \quad (7.75)$$

*Proof.* This result is proved by showing that there exist  $C_1 > 0$  and  $C_2 > 0$  such that for all  $\omega \in \mathbb{R}$

$$|\hat{\phi}(\omega)| \leq C_1 (1 + |\omega|)^{-p + \log_2 B} \quad (7.76)$$

$$|\hat{\psi}(\omega)| \leq C_2 (1 + |\omega|)^{-p + \log_2 B}. \quad (7.77)$$

The Lipschitz regularity of  $\phi$  and  $\psi$  is then derived from Theorem 6.1, which shows that if  $\int_{-\infty}^{+\infty} (1 + |\omega|^\alpha) |\hat{f}(\omega)| d\omega < +\infty$ , then  $f$  is uniformly Lipschitz  $\alpha$ .

We proved in (7.32) that  $\hat{\phi}(\omega) = \prod_{j=1}^{+\infty} 2^{-1/2} \hat{h}(2^{-j}\omega)$ . One can verify that

$$\prod_{j=1}^{+\infty} \frac{1 + \exp(i2^{-j}\omega)}{2} = \frac{1 - \exp(i\omega)}{i\omega},$$

hence

$$|\hat{\phi}(\omega)| = \frac{|1 - \exp(i\omega)|^p}{|\omega|^p} \prod_{j=1}^{+\infty} |\hat{l}(2^{-j}\omega)|. \quad (7.78)$$

Let us now compute an upper bound for  $\prod_{j=1}^{+\infty} |\hat{l}(2^{-j}\omega)|$ . At  $\omega = 0$  we have  $\hat{h}(0) = \sqrt{2}$  so  $\hat{l}(0) = 1$ . Since  $\hat{h}(\omega)$  is continuously differentiable at  $\omega = 0$ ,  $\hat{l}(\omega)$  is also continuously differentiable at  $\omega = 0$ . We thus derive that there exists  $\varepsilon > 0$  such that if  $|\omega| < \varepsilon$  then  $|\hat{l}(\omega)| \leq 1 + K|\omega|$ . Consequently

$$\sup_{|\omega| \leq \varepsilon} \prod_{j=1}^{+\infty} |\hat{l}(2^{-j}\omega)| \leq \sup_{|\omega| \leq \varepsilon} \prod_{j=1}^{+\infty} (1 + K|2^{-j}\omega|) \leq e^{K\varepsilon}. \quad (7.79)$$

If  $|\omega| > \varepsilon$ , there exists  $J \geq 1$  such that  $2^{J-1}\varepsilon \leq |\omega| \leq 2^J\varepsilon$  and we decompose

$$\prod_{j=1}^{+\infty} \hat{l}(2^{-j}\omega) = \prod_{j=1}^J \hat{l}(2^{-j}\omega) \prod_{j=1}^{+\infty} \hat{l}(2^{-j-J}\omega). \quad (7.80)$$

Since  $\sup_{\omega \in \mathbb{R}} |\hat{l}(\omega)| = B$ , inserting (7.79) yields for  $|\omega| > \varepsilon$

$$\prod_{j=1}^{+\infty} \hat{l}(2^{-j}\omega) \leq B^J e^{K\varepsilon} = e^{K\varepsilon} 2^{J \log_2 B}. \quad (7.81)$$

Since  $2^J \leq \varepsilon^{-1} 2|\omega|$ , this proves that

$$\forall \omega \in \mathbb{R}, \quad \prod_{j=1}^{+\infty} \hat{l}(2^{-j}\omega) \leq e^{K\varepsilon} \left( 1 + \frac{|2\omega|^{\log_2 B}}{\varepsilon^{\log_2 B}} \right).$$

Equation (7.76) is derived from (7.78) and this last inequality. Since  $|\hat{\psi}(2\omega)| = 2^{-1/2} |\hat{h}(\omega + \pi)| |\hat{\phi}(\omega)|$ , (7.77) is obtained from (7.76). ■

This theorem proves that if  $B < 2^{p-1}$  then  $\alpha_0 > 0$ . It means that  $\phi$  and  $\psi$  are uniformly continuous. For any  $m > 0$ , if  $B < 2^{p-1-m}$  then  $\alpha_0 > m$  so  $\psi$  and  $\phi$  are  $m$  times continuously differentiable. Theorem 7.4 shows that the number  $p$  of zeros of  $\hat{h}(\omega)$  at  $\pi$  is equal to the number of vanishing moments of  $\psi$ . A priori, we are not guaranteed that increasing  $p$  will improve the wavelet regularity, since  $B$  might increase as well. However, for important families of conjugate mirror filters such as splines or Daubechies filters,  $B$  increases more slowly than  $p$ , which implies that wavelet regularity increases with the number of vanishing moments. Let us emphasize that the number of vanishing moments and the regularity of orthogonal wavelets are related but it is the number of vanishing moments and not the regularity that affects the amplitude of the wavelet coefficients at fine scales.

## 7.2.2 Shannon, Meyer and Battle-Lemarié Wavelets

We study important classes of wavelets whose Fourier transforms are derived from the general formula proved in Theorem 7.3,

$$\hat{\psi}(\omega) = \frac{1}{\sqrt{2}} \hat{g}\left(\frac{\omega}{2}\right) \hat{\phi}\left(\frac{\omega}{2}\right) = \frac{1}{\sqrt{2}} \exp\left(\frac{-i\omega}{2}\right) \hat{h}^*\left(\frac{\omega}{2} + \pi\right) \hat{\phi}\left(\frac{\omega}{2}\right). \quad (7.82)$$

**Shannon Wavelet** The Shannon wavelet is constructed from the Shannon multiresolution approximation, which approximates functions by their restriction to low frequency intervals. It corresponds to  $\hat{\phi} = \mathbf{1}_{[-\pi, \pi]}$  and  $\hat{h}(\omega) = \sqrt{2} \mathbf{1}_{[-\pi/2, \pi/2]}(\omega)$  for  $\omega \in [-\pi, \pi]$ . We derive from (7.82) that

$$\hat{\psi}(\omega) = \begin{cases} \exp(-i\omega/2) & \text{if } \omega \in [-2\pi, -\pi] \cup [\pi, 2\pi] \\ 0 & \text{otherwise} \end{cases} \quad (7.83)$$

and hence

$$\psi(t) = \frac{\sin 2\pi(t - 1/2)}{2\pi(t - 1/2)} - \frac{\sin \pi(t - 1/2)}{\pi(t - 1/2)}.$$

This wavelet is  $\mathbf{C}^\infty$  but has a slow asymptotic time decay. Since  $\hat{\psi}(\omega)$  is zero in the neighborhood of  $\omega = 0$ , all its derivatives are zero at  $\omega = 0$ . Theorem 7.4 thus implies that  $\psi$  has an infinite number of vanishing moments.

Since  $\hat{\psi}(\omega)$  has a compact support we know that  $\psi(t)$  is  $\mathbf{C}^\infty$ . However  $|\psi(t)|$  decays only like  $|t|^{-1}$  at infinity because  $\hat{\psi}(\omega)$  is discontinuous at  $\pm\pi$  and  $\pm 2\pi$ .

**Meyer Wavelets** A Meyer wavelet [374] is a frequency band-limited function whose Fourier transform is smooth, unlike the Fourier transform of the Shannon wavelet. This smoothness provides a much faster asymptotic decay in time. These wavelets are constructed with conjugate mirror filters  $\hat{h}(\omega)$  that are  $\mathbf{C}^n$  and satisfy

$$\hat{h}(\omega) = \begin{cases} \sqrt{2} & \text{if } \omega \in [-\pi/3, \pi/3] \\ 0 & \text{if } \omega \in [-\pi, -2\pi/3] \cup [2\pi/3, \pi] \end{cases}. \quad (7.84)$$

The only degree of freedom is the behavior of  $\hat{h}(\omega)$  in the transition bands  $[-2\pi/3, -\pi/3] \cup [\pi/3, 2\pi/3]$ . It must satisfy the quadrature condition

$$|\hat{h}(\omega)|^2 + |\hat{h}(\omega + \pi)|^2 = 2, \quad (7.85)$$

and to obtain  $\mathbf{C}^n$  junctions at  $|\omega| = \pi/3$  and  $|\omega| = 2\pi/3$ , the  $n$  first derivatives must vanish at these abscissa. One can construct such functions that are  $\mathbf{C}^\infty$ .

The scaling function  $\hat{\phi}(\omega) = \prod_{p=1}^{+\infty} 2^{-1/2} \hat{h}(2^{-p}\omega)$  has a compact support and one can verify that

$$\hat{\phi}(\omega) = \begin{cases} 2^{-1/2} \hat{h}(\omega/2) & \text{if } |\omega| \leq 4\pi/3 \\ 0 & \text{if } |\omega| > 4\pi/3 \end{cases}. \quad (7.86)$$

The resulting wavelet (7.82) is

$$\hat{\psi}(\omega) = \begin{cases} 0 & \text{if } |\omega| \leq 2\pi/3 \\ 2^{-1/2} \hat{g}(\omega/2) & \text{if } 2\pi/3 \leq |\omega| \leq 4\pi/3 \\ 2^{-1/2} \exp(-i\omega/2) \hat{h}(\omega/4) & \text{if } 4\pi/3 \leq |\omega| \leq 8\pi/3 \\ 0 & \text{if } |\omega| > 8\pi/3 \end{cases}. \quad (7.87)$$

The functions  $\phi$  and  $\psi$  are  $\mathbf{C}^\infty$  because their Fourier transforms have a compact support. Since  $\hat{\psi}(\omega) = 0$  in the neighborhood of  $\omega = 0$ , all its derivatives are zero at  $\omega = 0$ , which proves that  $\psi$  has an infinite number of vanishing moments.

If  $\hat{h}$  is  $\mathbf{C}^n$  then  $\hat{\psi}$  and  $\hat{\phi}$  are also  $\mathbf{C}^n$ . The discontinuities of the  $(n+1)^{th}$  derivative of  $\hat{h}$  are generally at the junction of the transition band  $|\omega| = \pi/3, 2\pi/3$ , in which case one can show that there exists  $A$  such that

$$|\phi(t)| \leq A(1 + |t|)^{-n-1} \quad \text{and} \quad |\psi(t)| \leq A(1 + |t|)^{-n-1}.$$

Although the asymptotic decay of  $\psi$  is fast when  $n$  is large, its effective numerical decay may be relatively slow, which is reflected by the fact that  $A$  is quite large. As a consequence, a Meyer wavelet transform is generally implemented in the Fourier domain. Section 8.4.2 relates these wavelet bases to lapped orthogonal transforms applied in the Fourier domain. One can prove [18] that there exists no orthogonal wavelet that is  $C^\infty$  and has an exponential decay.

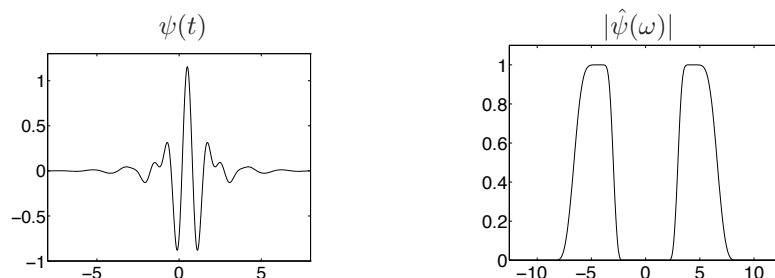


Figure 7.8: Meyer wavelet  $\psi$  and its Fourier transform modulus computed with (7.89).

**Example 7.10.** To satisfy the quadrature condition (7.85), one can verify that  $\hat{h}$  in (7.84) may be defined on the transition bands by

$$\hat{h}(\omega) = \sqrt{2} \cos \left[ \frac{\pi}{2} \beta \left( \frac{3|\omega|}{\pi} - 1 \right) \right] \quad \text{for } |\omega| \in [\pi/3, 2\pi/3],$$

where  $\beta(x)$  is a function that goes from 0 to 1 on the interval  $[0, 1]$  and satisfies

$$\forall x \in [0, 1], \quad \beta(x) + \beta(1 - x) = 1. \quad (7.88)$$

An example due to Daubechies [18] is

$$\beta(x) = x^4 (35 - 84x + 70x^2 - 20x^3). \quad (7.89)$$

The resulting  $\hat{h}(\omega)$  has  $n = 3$  vanishing derivatives at  $|\omega| = \pi/3, 2\pi/3$ . Figure 7.8 displays the corresponding wavelet  $\psi$ .

**Haar Wavelet** The Haar basis is obtained with a multiresolution of piecewise constant functions. The scaling function is  $\phi = \mathbf{1}_{[0,1]}$ . The filter  $h[n]$  given in (7.46) has two non-zero coefficients equal to  $2^{-1/2}$  at  $n = 0$  and  $n = 1$ . Hence

$$\frac{1}{\sqrt{2}} \psi \left( \frac{t}{2} \right) = \sum_{n=-\infty}^{+\infty} (-1)^{1-n} h[1-n] \phi(t-n) = \frac{1}{\sqrt{2}} (\phi(t-1) - \phi(t)),$$

so

$$\psi(t) = \begin{cases} -1 & \text{if } 0 \leq t < 1/2 \\ 1 & \text{if } 1/2 \leq t < 1 \\ 0 & \text{otherwise} \end{cases} \quad (7.90)$$

The Haar wavelet has the shortest support among all orthogonal wavelets. It is not well adapted to approximating smooth functions because it has only one vanishing moment.

**Battle-Lemarié Wavelets** Polynomial spline wavelets introduced by Battle [97] and Lemarié [344] are computed from spline multiresolution approximations. The expressions of  $\hat{\phi}(\omega)$  and  $\hat{h}(\omega)$  are given respectively by (7.18) and (7.48). For splines of degree  $m$ ,  $\hat{h}(\omega)$  and its first  $m$  derivatives are zero at  $\omega = \pi$ . Theorem 7.4 derives that  $\psi$  has  $m + 1$  vanishing moments. It follows from (7.82) that

$$\hat{\psi}(\omega) = \frac{\exp(-i\omega/2)}{\omega^{m+1}} \sqrt{\frac{S_{2m+2}(\omega/2 + \pi)}{S_{2m+2}(\omega) S_{2m+2}(\omega/2)}}.$$

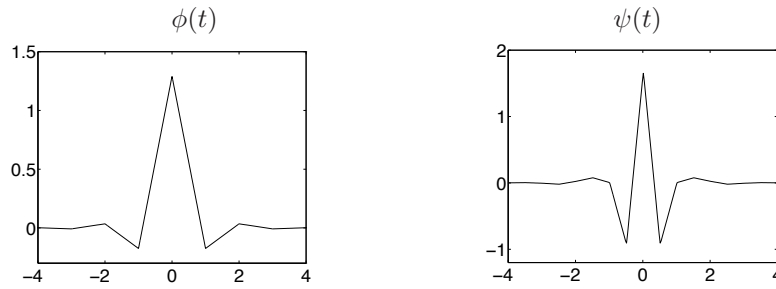


Figure 7.9: Linear spline Battle-Lemarié scaling function  $\phi$  and wavelet  $\psi$ .

This wavelet  $\psi$  has an exponential decay. Since it is a polynomial spline of degree  $m$ , it is  $m - 1$  times continuously differentiable. Polynomial spline wavelets are less regular than Meyer wavelets but have faster time asymptotic decay. For  $m$  odd,  $\psi$  is symmetric about  $1/2$ . For  $m$  even it is antisymmetric about  $1/2$ . Figure 7.5 gives the graph of the cubic spline wavelet  $\psi$  corresponding to  $m = 3$ . For  $m = 1$ , Figure 7.9 displays linear splines  $\phi$  and  $\psi$ . The properties of these wavelets are further studied in [104, 14, 163].

### 7.2.3 Daubechies Compactly Supported Wavelets

Daubechies wavelets have a support of minimum size for any given number  $p$  of vanishing moments. Theorem 7.5 proves that wavelets of compact support are computed with finite impulse response conjugate mirror filters  $h$ . We consider real causal filters  $h[n]$ , which implies that  $\hat{h}$  is a trigonometric polynomial:

$$\hat{h}(\omega) = \sum_{n=0}^{N-1} h[n] e^{-in\omega}.$$

To ensure that  $\psi$  has  $p$  vanishing moments, Theorem 7.4 shows that  $\hat{h}$  must have a zero of order  $p$  at  $\omega = \pi$ . To construct a trigonometric polynomial of minimal size, we factor  $(1 + e^{-i\omega})^p$ , which is a minimum size polynomial having  $p$  zeros at  $\omega = \pi$ :

$$\hat{h}(\omega) = \sqrt{2} \left( \frac{1 + e^{-i\omega}}{2} \right)^p R(e^{-i\omega}). \quad (7.91)$$

The difficulty is to design a polynomial  $R(e^{-i\omega})$  of minimum degree  $m$  such that  $\hat{h}$  satisfies

$$|\hat{h}(\omega)|^2 + |\hat{h}(\omega + \pi)|^2 = 2. \quad (7.92)$$

As a result,  $h$  has  $N = m + p + 1$  non-zero coefficients. The following theorem by Daubechies [193] proves that the minimum degree of  $R$  is  $m = p - 1$ .

**Theorem 7.7** (Daubechies). *A real conjugate mirror filter  $h$ , such that  $\hat{h}(\omega)$  has  $p$  zeroes at  $\omega = \pi$ , has at least  $2p$  non-zero coefficients. Daubechies filters have  $2p$  non-zero coefficients.*

*Proof.* The proof is constructive and computes the Daubechies filters. Since  $h[n]$  is real,  $|\hat{h}(\omega)|^2$  is an even function and can thus be written as a polynomial in  $\cos \omega$ . Hence  $|R(e^{-i\omega})|^2$  defined in (7.91) is a polynomial in  $\cos \omega$  that we can also write as a polynomial  $P(\sin^2(\omega/2))$

$$|\hat{h}(\omega)|^2 = 2 \left( \cos \frac{\omega}{2} \right)^{2p} P \left( \sin^2 \frac{\omega}{2} \right). \quad (7.93)$$

The quadrature condition (7.92) is equivalent to

$$(1 - y)^p P(y) + y^p P(1 - y) = 1, \quad (7.94)$$

for any  $y = \sin^2(\omega/2) \in [0, 1]$ . To minimize the number of non-zero terms of the finite Fourier series  $\hat{h}(\omega)$ , we must find the solution  $P(y) \geq 0$  of minimum degree, which is obtained with the Bezout theorem on polynomials.

**Theorem 7.8** (Bezout). *Let  $Q_1(y)$  and  $Q_2(y)$  be two polynomials of degrees  $n_1$  and  $n_2$  with no common zeroes. There exist two unique polynomials  $P_1(y)$  and  $P_2(y)$  of degrees  $n_2 - 1$  and  $n_1 - 1$  such that*

$$P_1(y) Q_1(y) + P_2(y) Q_2(y) = 1. \quad (7.95)$$

The proof of this classical result is in [18]. Since  $Q_1(y) = (1 - y)^p$  and  $Q_2(y) = y^p$  are two polynomials of degree  $p$  with no common zeros, the Bezout theorem proves that there exist two unique polynomials  $P_1(y)$  and  $P_2(y)$  such that

$$(1 - y)^p P_1(y) + y^p P_2(y) = 1.$$

The reader can verify that  $P_2(y) = P_1(1 - y) = P(1 - y)$  with

$$P(y) = \sum_{k=0}^{p-1} \binom{p-1+k}{k} y^k. \quad (7.96)$$

Clearly  $P(y) \geq 0$  for  $y \in [0, 1]$ . Hence  $P(y)$  is the polynomial of minimum degree satisfying (7.94) with  $P(y) \geq 0$ .

**Minimum Phase Factorization** Now we need to construct a minimum degree polynomial

$$R(e^{-i\omega}) = \sum_{k=0}^m r_k e^{-ik\omega} = r_0 \prod_{k=0}^m (1 - a_k e^{-i\omega})$$

such that  $|R(e^{-i\omega})|^2 = P(\sin^2(\omega/2))$ . Since its coefficients are real,  $R^*(e^{-i\omega}) = R(e^{i\omega})$  and hence

$$|R(e^{-i\omega})|^2 = R(e^{-i\omega}) R(e^{i\omega}) = P\left(\frac{2 - e^{i\omega} - e^{-i\omega}}{4}\right) = Q(e^{-i\omega}). \quad (7.97)$$

This factorization is solved by extending it to the whole complex plane with the variable  $z = e^{-i\omega}$ :

$$R(z) R(z^{-1}) = r_0^2 \prod_{k=0}^m (1 - a_k z) (1 - a_k z^{-1}) = Q(z) = P\left(\frac{2 - z - z^{-1}}{4}\right). \quad (7.98)$$

Let us compute the roots of  $Q(z)$ . Since  $Q(z)$  has real coefficients if  $c_k$  is a root, then  $c_k^*$  is also a root and since it is a function of  $z + z^{-1}$  if  $c_k$  is a root then  $1/c_k$  and hence  $1/c_k^*$  are also roots. To design  $R(z)$  that satisfies (7.98), we choose each root  $a_k$  of  $R(z)$  among a pair  $(c_k, 1/c_k)$  and include  $a_k^*$  as a root to obtain real coefficients. This procedure yields a polynomial of minimum degree  $m = p - 1$ , with  $r_0^2 = Q(0) = P(1/2) = 2^{p-1}$ . The resulting filter  $h$  of minimum size has  $N = p + m + 1 = 2p$  non-zero coefficients.

Among all possible factorizations, the minimum phase solution  $R(e^{i\omega})$  is obtained by choosing  $a_k$  among  $(c_k, 1/c_k)$  to be inside the unit circle  $|a_k| \leq 1$  [50]. The resulting causal filter  $h$  has an energy maximally concentrated at small abscissa  $n \geq 0$ . It is a Daubechies filter of order  $p$ . ■ ■

The constructive proof of this theorem synthesizes causal conjugate mirror filters of size  $2p$ . Table 7.2 gives the coefficients of these Daubechies filters for  $2 \leq p \leq 10$ . The following theorem derives that Daubechies wavelets calculated with these conjugate mirror filters have a support of minimum size.

**Theorem 7.9** (Daubechies). *If  $\psi$  is a wavelet with  $p$  vanishing moments that generates an orthonormal basis of  $\mathbf{L}^2(\mathbb{R})$ , then it has a support of size larger than or equal to  $2p - 1$ . A Daubechies wavelet has a minimum size support equal to  $[-p + 1, p]$ . The support of the corresponding scaling function  $\phi$  is  $[0, 2p - 1]$ .*

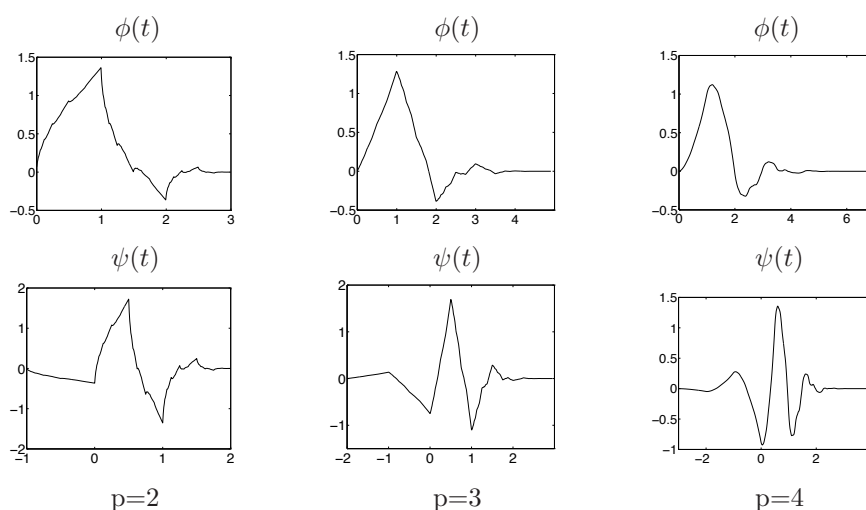
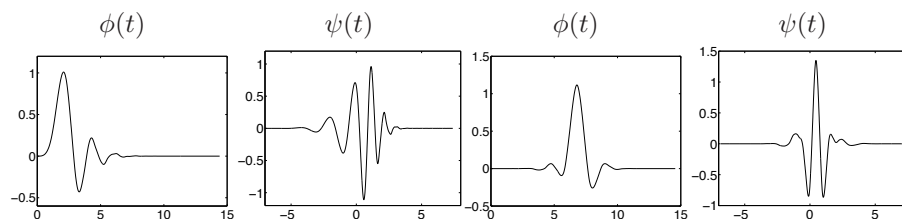
This theorem is a direct consequence of Theorem 7.7. The support of the wavelet, and that of the scaling function, are calculated with Theorem 7.5. When  $p = 1$  we get the Haar wavelet. Figure 7.10 displays the graphs of  $\phi$  and  $\psi$  for  $p = 2, 3, 4$ .

The regularity of  $\phi$  and  $\psi$  is the same since  $\psi(t)$  is a finite linear combination of the  $\phi(2t - n)$ . This regularity is however difficult to estimate precisely. Let  $B = \sup_{\omega \in \mathbb{R}} |R(e^{-i\omega})|$  where  $R(e^{-i\omega})$  is the trigonometric polynomial defined in (7.91). Theorem 7.6 proves that  $\psi$  is at least uniformly

	n	$h_p[n]$		n	$h_p[n]$
p = 2	0	.482962913145	p = 8	0	.054415842243
	1	.836516303738		1	.312871590914
	2	.224143868042		2	.675630736297
	3	-.129409522551		3	.585354683654
p = 3	0	.332670552950		4	-.015829105256
	1	.806891509311		5	-.284015542962
	2	.459877502118		6	.000472484574
	3	.135011020010		7	.128747426620
	4	-.085441273882		8	-.017369301002
p = 4	5	.035226291882		9	-.04408825393
	0	.230377813309		10	.013981027917
	1	.714846570553		11	.008746094047
	2	.630880767930		12	-.004870352993
	3	-.027983769417		13	-.000391740373
	4	-.187034811719		14	.000675449406
	5	.030841381836	p = 9	15	-.000117476784
p = 5	6	.032883011667		0	.038077947364
	7	-.010597401785		1	.243834674613
	0	.160102397974		2	.604823123690
	1	.603829269797		3	.657288078051
	2	.724308528438		4	.133197385825
	3	.138428145901		5	-.293273783279
	4	.242294887066		6	-.096840783223
	5	-.032244869585		7	.148540749338
	6	.077571493840		8	.030725681479
p = 6	7	-.006241490213		9	-.067632829061
	8	-.012580751999		10	.000250947115
	9	.003335725285		11	.022361662124
	0	.111540743350		12	-.004723204758
	1	.494623890398		13	-.004281503682
	2	.751133908021		14	.001847646883
	3	.315250351709		15	.000230385764
	4	-.226264693965		16	-.000251963189
	5	-.129766867567		17	.000039347320
	6	.097501605587	p = 10	0	.026670057901
	7	.027522865530		1	.188176800078
	8	-.031582039317		2	.527201188932
	9	.000553842201		3	.688459039454
	10	.004777257511		4	.281172343661
	11	-.001077301085		5	-.249846424327
p = 7	0	.077852054085		6	-.195946274377
	1	.396539319482		7	.127369340336
	2	.729132090846		8	.093057364604
	3	.469782287405		9	-.071394147166
	4	.143906003929		10	-.029457536822
	5	-.224036184994		11	.033212674059
	6	.071309219267		12	.003606553567
	7	.080612609151		13	-.010733175483
	8	-.038029936935		14	.001395351747
	9	-.016574541631		15	.001992405295
	10	.012550998556		16	-.000685856695
	11	.000429577973		17	-.000116466855
	12	-.001801640704		18	.000093588670
	13	.000353713800		19	-.000013264203

Table 7.2: Daubechies filters for wavelets with  $p$  vanishing moments.



Figure 7.10: Daubechies scaling function  $\phi$  and wavelet  $\psi$  with  $p$  vanishing moments.Figure 7.11: Daubechies (first two) and Symmlets (last two) scaling functions and wavelets with  $p = 8$  vanishing moments.

Lipschitz  $\alpha$  for  $\alpha < p - \log_2 B - 1$ . For Daubechies wavelets,  $B$  increases more slowly than  $p$  and Figure 7.10 shows indeed that the regularity of these wavelets increases with  $p$ . Daubechies and Lagarias [197] have established a more precise technique that computes the exact Lipschitz regularity of  $\psi$ . For  $p = 2$  the wavelet  $\psi$  is only Lipschitz 0.55 but for  $p = 3$  it is Lipschitz 1.08 which means that it is already continuously differentiable. For  $p$  large,  $\phi$  and  $\psi$  are uniformly Lipschitz  $\alpha$ , for  $\alpha$  of the order of  $0.2p$  [167].

**Symmlets** Daubechies wavelets are very asymmetric because they are constructed by selecting the minimum phase square root of  $Q(e^{-i\omega})$  in (7.97). One can show [50] that filters corresponding to a minimum phase square root have their energy optimally concentrated near the starting point of their support. They are thus highly non-symmetric, which yields very asymmetric wavelets.

To obtain a symmetric or antisymmetric wavelet, the filter  $h$  must be symmetric or antisymmetric with respect to the center of its support, which means that  $\hat{h}(\omega)$  has a linear complex phase. Daubechies proved [193] that the Haar filter is the only real compactly supported conjugate mirror filter that has a linear phase. The *Symmlet* filters of Daubechies are obtained by optimizing the choice of the square root  $R(e^{-i\omega})$  of  $Q(e^{-i\omega})$  to obtain an almost linear phase. The resulting wavelets still have a minimum support  $[-p+1, p]$  with  $p$  vanishing moments but they are more symmetric, as illustrated by Figure 7.11 for  $p = 8$ . The coefficients of the Symmlet filters are in WAVELAB. Complex conjugate mirror filters with a compact support and a linear phase can be constructed [351], but they produce complex wavelet coefficients whose real and imaginary parts are redundant when the signal is real.

**Coiflets** For an application in numerical analysis, Coifman asked Daubechies [193] to construct a family of wavelets  $\psi$  that have  $p$  vanishing moments and a minimum size support, but whose

scaling functions also satisfy

$$\int_{-\infty}^{+\infty} \phi(t) dt = 1 \quad \text{and} \quad \int_{-\infty}^{+\infty} t^k \phi(t) dt = 0 \quad \text{for } 1 \leq k < p. \quad (7.99)$$

Such scaling functions are useful in establishing precise quadrature formulas. If  $f$  is  $\mathbf{C}^k$  in the neighborhood of  $2^J n$  with  $k < p$ , then a Taylor expansion of  $f$  up to order  $k$  shows that

$$2^{-J/2} \langle f, \phi_{J,n} \rangle \approx f(2^J n) + O(2^{(k+1)J}). \quad (7.100)$$

At a fine scale  $2^J$ , the scaling coefficients are thus closely approximated by the signal samples. The order of approximation increases with  $p$ . The supplementary condition (7.99) requires increasing the support of  $\psi$ ; the resulting Coiflet has a support of size  $3p-1$  instead of  $2p-1$  for a Daubechies wavelet. The corresponding conjugate mirror filters are tabulated in WAVELAB.

**Audio Filters** The first conjugate mirror filters with finite impulse response were constructed in 1986 by Smith and Barnwell [442] in the context of perfect filter bank reconstruction, explained in Section 7.3.2. These filters satisfy the quadrature condition  $|\hat{h}(\omega)|^2 + |\hat{h}(\omega + \pi)|^2 = 2$ , which is necessary and sufficient for filter bank reconstruction. However,  $\hat{h}(0) \neq \sqrt{2}$  so the infinite product of such filters does not yield a wavelet basis of  $\mathbf{L}^2(\mathbb{R})$ . Instead of imposing any vanishing moments, Smith and Barnwell [442], and later Vaidyanathan and Hoang [470], designed their filters to reduce the size of the transition band, where  $|\hat{h}(\omega)|$  decays from nearly  $\sqrt{2}$  to nearly 0 in the neighborhood of  $\pm\pi/2$ . This constraint is important in optimizing the transform code of audio signals, explained in Section 10.3.3. However, many cascades of these filters exhibit wild behavior. The Vaidyanathan-Hoang filters are tabulated in WAVELAB. Many other classes of conjugate mirror filters with finite impulse response have been constructed [67, 77]. Recursive conjugate mirror filters may also be designed [299] to minimize the size of the transition band for a given number of zeroes at  $\omega = \pi$ . These filters have a fast but non-causal recursive implementation for signals of finite size.

## 7.3 Wavelets and Filter Banks

Decomposition coefficients in a wavelet orthogonal basis are computed with a fast algorithm that cascades discrete convolutions with  $h$  and  $g$ , and subsamples the output. Section 7.3.1 derives this result from the embedded structure of multiresolution approximations. A direct filter bank analysis is performed in Section 7.3.2, which gives more general perfect reconstruction conditions on the filters. Section 7.3.3 shows that perfect reconstruction filter banks decompose signals in a basis of  $\ell^2(\mathbb{Z})$ . This basis is orthogonal for conjugate mirror filters.

### 7.3.1 Fast Orthogonal Wavelet Transform

We describe a fast filter bank algorithm that computes the orthogonal wavelet coefficients of a signal measured at a finite resolution. A fast wavelet transform decomposes successively each approximation  $P_{\mathbf{V}_j} f$  into a coarser approximation  $P_{\mathbf{V}_{j+1}} f$  plus the wavelet coefficients carried by  $P_{\mathbf{W}_{j+1}} f$ . In the other direction, the reconstruction from wavelet coefficients recovers each  $P_{\mathbf{V}_j} f$  from  $P_{\mathbf{V}_{j+1}} f$  and  $P_{\mathbf{W}_{j+1}} f$ .

Since  $\{\phi_{j,n}\}_{n \in \mathbb{Z}}$  and  $\{\psi_{j,n}\}_{n \in \mathbb{Z}}$  are orthonormal bases of  $\mathbf{V}_j$  and  $\mathbf{W}_j$  the projection in these spaces is characterized by

$$a_j[n] = \langle f, \phi_{j,n} \rangle \quad \text{and} \quad d_j[n] = \langle f, \psi_{j,n} \rangle.$$

The following theorem [359, 360] shows that these coefficients are calculated with a cascade of discrete convolutions and subsamplings. We denote  $\bar{x}[n] = x[-n]$  and

$$\tilde{x}[n] = \begin{cases} x[p] & \text{if } n = 2p \\ 0 & \text{if } n = 2p + 1 \end{cases}. \quad (7.101)$$

**Theorem 7.10** (Mallat). *At the decomposition*

$$a_{j+1}[p] = \sum_{n=-\infty}^{+\infty} h[n-2p] a_j[n] = a_j \star \bar{h}[2p], \quad (7.102)$$

$$d_{j+1}[p] = \sum_{n=-\infty}^{+\infty} g[n-2p] a_j[n] = a_j \star \bar{g}[2p]. \quad (7.103)$$

*At the reconstruction,*

$$\begin{aligned} a_j[p] &= \sum_{n=-\infty}^{+\infty} h[p-2n] a_{j+1}[n] + \sum_{n=-\infty}^{+\infty} g[p-2n] d_{j+1}[n] \\ &= \check{a}_{j+1} \star h[p] + \check{d}_{j+1} \star g[p]. \end{aligned} \quad (7.104)$$

*Proof. Proof of (7.102)* Any  $\phi_{j+1,p} \in \mathbf{V}_{j+1} \subset \mathbf{V}_j$  can be decomposed in the orthonormal basis  $\{\phi_{j,n}\}_{n \in \mathbb{Z}}$  of  $\mathbf{V}_j$ :

$$\phi_{j+1,p} = \sum_{n=-\infty}^{+\infty} \langle \phi_{j+1,p}, \phi_{j,n} \rangle \phi_{j,n}. \quad (7.105)$$

With the change of variable  $t' = 2^{-j}t - 2p$  we obtain

$$\begin{aligned} \langle \phi_{j+1,p}, \phi_{j,n} \rangle &= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2^{j+1}}} \phi\left(\frac{t-2^{j+1}p}{2^{j+1}}\right) \frac{1}{\sqrt{2^j}} \phi^*\left(\frac{t-2^jn}{2^j}\right) dt \\ &= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2}} \phi\left(\frac{t}{2}\right) \phi^*(t-n+2p) dt \\ &= \left\langle \frac{1}{\sqrt{2}} \phi\left(\frac{t}{2}\right), \phi(t-n+2p) \right\rangle = h[n-2p]. \end{aligned} \quad (7.106)$$

Hence (7.105) implies that

$$\phi_{j+1,p} = \sum_{n=-\infty}^{+\infty} h[n-2p] \phi_{j,n}. \quad (7.107)$$

Computing the inner product of  $f$  with the vectors on each side of this equality yields (7.102).

*Proof of (7.103)* Since  $\psi_{j+1,p} \in \mathbf{W}_{j+1} \subset \mathbf{V}_j$ , it can be decomposed as

$$\psi_{j+1,p} = \sum_{n=-\infty}^{+\infty} \langle \psi_{j+1,p}, \phi_{j,n} \rangle \phi_{j,n}.$$

As in (7.106), the change of variable  $t' = 2^{-j}t - 2p$  proves that

$$\langle \psi_{j+1,p}, \phi_{j,n} \rangle = \left\langle \frac{1}{\sqrt{2}} \psi\left(\frac{t}{2}\right), \phi(t-n+2p) \right\rangle = g[n-2p] \quad (7.108)$$

and hence

$$\psi_{j+1,p} = \sum_{n=-\infty}^{+\infty} g[n-2p] \phi_{j,n}. \quad (7.109)$$

Taking the inner product with  $f$  on each side gives (7.103).

*Proof of (7.104)* Since  $\mathbf{W}_{j+1}$  is the orthogonal complement of  $\mathbf{V}_{j+1}$  in  $\mathbf{V}_j$  the union of the two bases  $\{\psi_{j+1,n}\}_{n \in \mathbb{Z}}$  and  $\{\phi_{j+1,n}\}_{n \in \mathbb{Z}}$  is an orthonormal basis of  $\mathbf{V}_j$ . Hence any  $\phi_{j,p}$  can be decomposed in this basis:

$$\begin{aligned} \phi_{j,p} &= \sum_{n=-\infty}^{+\infty} \langle \phi_{j,p}, \phi_{j+1,n} \rangle \phi_{j+1,n} \\ &\quad + \sum_{n=-\infty}^{+\infty} \langle \phi_{j,p}, \psi_{j+1,n} \rangle \psi_{j+1,n}. \end{aligned}$$

Inserting (7.106) and (7.108) yields

$$\phi_{j,p} = \sum_{n=-\infty}^{+\infty} h[p-2n] \phi_{j+1,n} + \sum_{n=-\infty}^{+\infty} g[p-2n] \psi_{j+1,n}.$$

Taking the inner product with  $f$  on both sides of this equality gives (7.104). ■ ■

Theorem 7.10 proves that  $a_{j+1}$  and  $d_{j+1}$  are computed by taking every other sample of the convolution of  $a_j$  with  $\bar{h}$  and  $\bar{g}$  respectively, as illustrated by Figure 7.12. The filter  $\bar{h}$  removes the higher frequencies of the inner product sequence  $a_j$  whereas  $\bar{g}$  is a high-pass filter which collects the remaining highest frequencies. The reconstruction (7.104) is an interpolation that inserts zeroes to expand  $a_{j+1}$  and  $d_{j+1}$  and filters these signals, as shown in Figure 7.12.

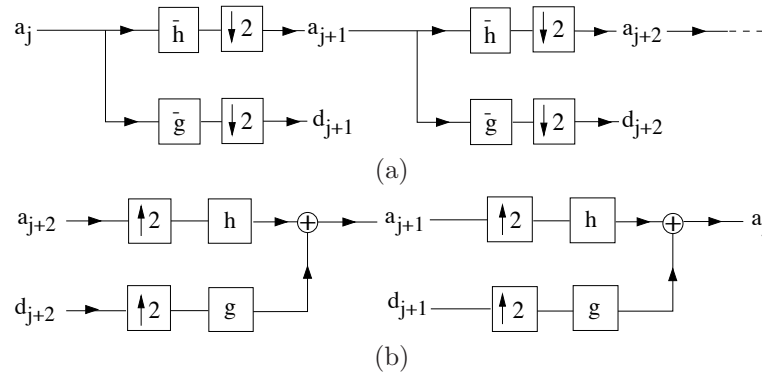


Figure 7.12: (a): A fast wavelet transform is computed with a cascade of filterings with  $\bar{h}$  and  $\bar{g}$  followed by a factor 2 subsampling. (b): A fast inverse wavelet transform reconstructs progressively each  $a_j$  by inserting zeroes between samples of  $a_{j+1}$  and  $d_{j+1}$ , filtering and adding the output.

An *orthogonal wavelet representation* of  $a_L = \langle f, \phi_{L,n} \rangle$  is composed of wavelet coefficients of  $f$  at scales  $2^L < 2^j \leq 2^J$  plus the remaining approximation at the largest scale  $2^J$ :

$$[\{d_j\}_{L < j \leq J}, a_J]. \quad (7.110)$$

It is computed from  $a_L$  by iterating (7.102) and (7.103) for  $L \leq j < J$ . Figure 7.7 gives a numerical example computed with the cubic spline filter of Table 7.1. The original signal  $a_L$  is recovered from this wavelet representation by iterating the reconstruction (7.104) for  $J > j \geq L$ .

**Initialization** Most often the discrete input signal  $b[n]$  is obtained by a finite resolution device that averages and samples an analog input signal. For example, a CCD camera filters the light intensity by the optics and each photo-receptor averages the input light over its support. A pixel value thus measures average light intensity. If the sampling distance is  $N^{-1}$ , to define and compute the wavelet coefficients, we need to associate to  $b[n]$  a function  $f(t) \in \mathbf{V}_L$  approximated at the scale  $2^L = N^{-1}$ , and compute  $a_L[n] = \langle f, \phi_{L,n} \rangle$ . Exercise 7.6 explains how to compute  $a_L[n] = \langle f, \phi_{L,n} \rangle$  so that  $b[n] = f(N^{-1}n)$ .

A simpler and faster approach considers

$$f(t) = \sum_{n=-\infty}^{+\infty} b[n] \phi\left(\frac{t - 2^L n}{2^L}\right) \in \mathbf{V}_L.$$

Since  $\{\phi_{L,n}(t) = 2^{-L/2} \phi(2^{-L}t - n)\}_{n \in \mathbb{Z}}$  is orthonormal and  $2^L = N^{-1}$ ,

$$b[n] = N^{1/2} \langle f, \phi_{L,n} \rangle = N^{1/2} a_L[n].$$

But  $\hat{\phi}(0) = \int_{-\infty}^{\infty} \phi(t) dt = 1$ , so

$$N^{1/2} a_L[n] = \int_{-\infty}^{+\infty} f(t) \frac{1}{N^{-1}} \phi\left(\frac{t - N^{-1}n}{N^{-1}}\right) dt$$

is a weighted average of  $f$  in the neighborhood of  $N^{-1}n$  over a domain proportional to  $N^{-1}$ . Hence if  $f$  is regular,

$$b[n] = N^{1/2} a_L[n] \approx f(N^{-1}n) . \quad (7.111)$$

If  $\psi$  is a Coifflet and  $f(t)$  is regular in the neighborhood of  $N^{-1}n$ , then (7.100) shows that  $N^{-1/2} a_L[n]$  is a high order approximation of  $f(N^{-1}n)$ .

**Finite Signals** Let us consider a signal  $f$  whose support is in  $[0, 1]$  and which is approximated with a uniform sampling at intervals  $N^{-1}$ . The resulting approximation  $a_L$  has  $N = 2^{-L}$  samples. This is the case in Figure 7.7 with  $N = 1024$ . Computing the convolutions with  $\bar{h}$  and  $\bar{g}$  at abscissa close to 0 or close to  $N$  requires knowing the values of  $a_L[n]$  beyond the boundaries  $n = 0$  and  $n = N - 1$ . These boundary problems may be solved with one of the three approaches described in Section 7.5.

Section 7.5.1 explains the simplest algorithm, which periodizes  $a_L$ . The convolutions in Theorem 7.10 are replaced by circular convolutions. This is equivalent to decomposing  $f$  in a periodic wavelet basis of  $\mathbf{L}^2[0, 1]$ . This algorithm has the disadvantage of creating large wavelet coefficients at the borders.

If  $\psi$  is symmetric or antisymmetric, we can use a folding procedure described in Section 7.5.2, which creates smaller wavelet coefficients at the border. It decomposes  $f$  in a folded wavelet basis of  $\mathbf{L}^2[0, 1]$ . However, we mentioned in Section 7.2.3 that Haar is the only symmetric wavelet with a compact support. Higher order spline wavelets have a symmetry but  $h$  must be truncated in numerical calculations.

The most efficient boundary treatment is described in Section 7.5.3, but the implementation is more complicated. Boundary wavelets which keep their vanishing moments are designed to avoid creating large amplitude coefficients when  $f$  is regular. The fast algorithm is implemented with special boundary filters, and requires the same number of calculations as the two other methods.

**Complexity** Suppose that  $h$  and  $g$  have  $K$  non-zero coefficients. Let  $a_L$  be a signal of size  $N = 2^{-L}$ . With appropriate boundary calculations, each  $a_j$  and  $d_j$  has  $2^{-j}$  samples. Equations (7.102) and (7.103) compute  $a_{j+1}$  and  $d_{j+1}$  from  $a_j$  with  $2^{-j}K$  additions and multiplications. The wavelet representation (7.110) is therefore calculated with at most  $2KN$  additions and multiplications. The reconstruction (7.104) of  $a_j$  from  $a_{j+1}$  and  $d_{j+1}$  is also obtained with  $2^{-j}K$  additions and multiplications. The original signal  $a_L$  is thus also recovered from the wavelet representation with at most  $2KN$  additions and multiplications.

**Wavelet Graphs** The graphs of  $\phi$  and  $\psi$  are computed numerically with the inverse wavelet transform. If  $f = \phi$  then  $a_0[n] = \delta[n]$  and  $d_j[n] = 0$  for all  $L < j \leq 0$ . The inverse wavelet transform computes  $a_L$  and (7.111) shows that

$$N^{1/2} a_L[n] \approx \phi(N^{-1}n) .$$

If  $\phi$  is regular and  $N$  is large enough, we recover a precise approximation of the graph of  $\phi$  from  $a_L$ .

Similarly, if  $f = \psi$  then  $a_0[n] = 0$ ,  $d_0[n] = \delta[n]$  and  $d_j[n] = 0$  for  $L < j < 0$ . Then  $a_L[n]$  is calculated with the inverse wavelet transform and  $N^{1/2} a_L[n] \approx \psi(N^{-1}n)$ . The Daubechies wavelets and scaling functions in Figure 7.10 are calculated with this procedure.

### 7.3.2 Perfect Reconstruction Filter Banks

The fast discrete wavelet transform decomposes signals into low-pass and high-pass components subsampled by 2; the inverse transform performs the reconstruction. The study of such classical multirate filter banks became a major signal processing topic in 1976, when Croisier, Esteban and Galand [188] discovered that it is possible to perform such decompositions and reconstructions with *quadrature mirror filters* (Exercise 7.7). However, besides the simple Haar filter, a quadrature mirror filter can not have a finite impulse response. In 1984, Smith and Barnwell [443] and Mintzer [375] found necessary and sufficient conditions for obtaining perfect reconstruction orthogonal filters

with a finite impulse response, that they called *conjugate mirror filters*. The theory was completed by the biorthogonal equations of Vetterli [472, 471] and the general paraunitary matrix theory of Vaidyanathan [469]. We follow this digital signal processing approach which gives a simple understanding of conjugate mirror filter conditions. More complete presentations of filter banks properties can be found in [1, 2, 62, 66, 67].

**Filter Bank** A two-channel multirate filter bank convolves a signal  $a_0$  with a low-pass filter  $\bar{h}[n] = h[-n]$  and a high-pass filter  $\bar{g}[n] = g[-n]$  and subsamples by 2 the output:

$$a_1[n] = a_0 \star \bar{h}[2n] \quad \text{and} \quad d_1[n] = a_0 \star \bar{g}[2n]. \quad (7.112)$$

A reconstructed signal  $\tilde{a}_0$  is obtained by filtering the zero expanded signals with a dual low-pass filter  $\tilde{h}$  and a dual high-pass filter  $\tilde{g}$ , as shown in Figure 7.13. With the zero insertion notation (7.101) it yields

$$\tilde{a}_0[n] = \tilde{a}_1 \star \tilde{h}[n] + \tilde{d}_1 \star \tilde{g}[n]. \quad (7.113)$$

We study necessary and sufficient conditions on  $h$ ,  $g$ ,  $\tilde{h}$  and  $\tilde{g}$  to guarantee a perfect reconstruction  $\tilde{a}_0 = a_0$ .

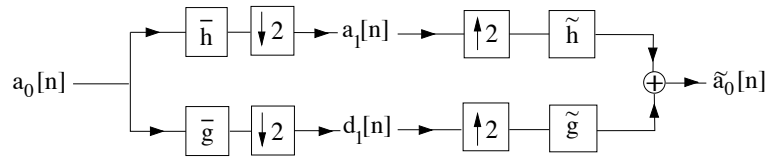


Figure 7.13: The input signal is filtered by a low-pass and a high-pass filter and subsampled. The reconstruction is performed by inserting zeroes and filtering with dual filters  $\tilde{h}$  and  $\tilde{g}$ .

**Subsampling and Zero Interpolation** Subsamplings and expansions with zero insertions have simple expressions in the Fourier domain. Since  $\hat{x}(\omega) = \sum_{n=-\infty}^{+\infty} x[n] e^{-in\omega}$  the Fourier series of the subsampled signal  $y[n] = x[2n]$  can be written

$$\hat{y}(2\omega) = \sum_{n=-\infty}^{+\infty} x[2n] e^{-i2n\omega} = \frac{1}{2} \left( \hat{x}(\omega) + \hat{x}(\omega + \pi) \right). \quad (7.114)$$

The component  $\hat{x}(\omega + \pi)$  creates a frequency folding. This *aliasing* must be canceled at the reconstruction.

The insertion of zeros defines

$$y[n] = \tilde{x}[n] = \begin{cases} x[p] & \text{if } n = 2p \\ 0 & \text{if } n = 2p + 1 \end{cases},$$

whose Fourier transform is

$$\hat{y}(\omega) = \sum_{n=-\infty}^{+\infty} x[n] e^{-i2n\omega} = \hat{x}(2\omega). \quad (7.115)$$

The following theorem gives Vetterli's [471] biorthogonal conditions, which guarantee that  $\tilde{a}_0 = a_0$ .

**Theorem 7.11** (Vetterli). *The filter bank performs an exact reconstruction for any input signal if and only if*

$$\hat{h}^*(\omega + \pi) \hat{h}(\omega) + \hat{g}^*(\omega + \pi) \hat{g}(\omega) = 0, \quad (7.116)$$

and

$$\hat{h}^*(\omega) \hat{h}(\omega) + \hat{g}^*(\omega) \hat{g}(\omega) = 2. \quad (7.117)$$

*Proof.* We first relate the Fourier transform of  $a_1$  and  $d_1$  to the Fourier transform of  $a_0$ . Since  $h$  and  $g$  are real, the transfer functions of  $\hat{h}$  and  $\hat{g}$  are respectively  $\hat{h}(-\omega) = \hat{h}^*(\omega)$  and  $\hat{g}(-\omega) = \hat{g}^*(\omega)$ . By using (7.114), we derive from the definition (7.112) of  $a_1$  and  $d_1$  that

$$\hat{a}_1(2\omega) = \frac{1}{2} \left( \hat{a}_0(\omega) \hat{h}^*(\omega) + \hat{a}_0(\omega + \pi) \hat{h}^*(\omega + \pi) \right), \quad (7.118)$$

$$\hat{d}_1(2\omega) = \frac{1}{2} \left( \hat{a}_0(\omega) \hat{g}^*(\omega) + \hat{a}_0(\omega + \pi) \hat{g}^*(\omega + \pi) \right). \quad (7.119)$$

The expression (7.113) of  $\tilde{a}_0$  and the zero insertion property (7.115) also imply

$$\hat{\tilde{a}}_0(\omega) = \hat{a}_1(2\omega) \hat{h}(\omega) + \hat{d}_1(2\omega) \hat{g}(\omega). \quad (7.120)$$

Hence

$$\begin{aligned} \hat{\tilde{a}}_0(\omega) &= \frac{1}{2} \left( \hat{h}^*(\omega) \hat{h}(\omega) + \hat{g}^*(\omega) \hat{g}(\omega) \right) \hat{a}_0(\omega) + \\ &\quad \frac{1}{2} \left( \hat{h}^*(\omega + \pi) \hat{h}(\omega) + \hat{g}^*(\omega + \pi) \hat{g}(\omega) \right) \hat{a}_0(\omega + \pi). \end{aligned}$$

To obtain  $a_0 = \tilde{a}_0$  for all  $a_0$ , the filters must cancel the aliasing term  $\hat{a}_0(\omega + \pi)$  and guarantee a unit gain for  $\hat{a}_0(\omega)$ , which proves equations (7.116) and (7.117). ■ ■

Theorem 7.11 proves that the reconstruction filters  $\tilde{h}$  and  $\tilde{g}$  are entirely specified by the decomposition filters  $h$  and  $g$ . In matrix form, it can be rewritten

$$\begin{pmatrix} \hat{h}(\omega) & \hat{g}(\omega) \\ \hat{h}(\omega + \pi) & \hat{g}(\omega + \pi) \end{pmatrix} \times \begin{pmatrix} \hat{\tilde{h}}^*(\omega) \\ \hat{\tilde{g}}^*(\omega) \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}. \quad (7.121)$$

The inversion of this  $2 \times 2$  matrix yields

$$\begin{pmatrix} \hat{\tilde{h}}^*(\omega) \\ \hat{\tilde{g}}^*(\omega) \end{pmatrix} = \frac{2}{\Delta(\omega)} \begin{pmatrix} \hat{g}(\omega + \pi) \\ -\hat{h}(\omega + \pi) \end{pmatrix} \quad (7.122)$$

where  $\Delta(\omega)$  is the determinant

$$\Delta(\omega) = \hat{h}(\omega) \hat{g}(\omega + \pi) - \hat{h}(\omega + \pi) \hat{g}(\omega). \quad (7.123)$$

The reconstruction filters are stable only if the determinant does not vanish for all  $\omega \in [-\pi, \pi]$ . Vaidyanathan [469] has extended this result to multirate filter banks with an arbitrary number  $M$  of channels by showing that the resulting matrices of filters satisfy paraunitary properties [66].

**Finite Impulse Response** When all filters have a finite impulse response, the determinant  $\Delta(\omega)$  can be evaluated. This yields simpler relations between the decomposition and reconstruction filters.

**Theorem 7.12.** *Perfect reconstruction filters satisfy*

$$\hat{h}^*(\omega) \hat{h}(\omega) + \hat{h}^*(\omega + \pi) \hat{h}(\omega + \pi) = 2. \quad (7.124)$$

*For finite impulse response filters, there exist  $a \in \mathbb{R}$  and  $l \in \mathbb{Z}$  such that*

$$\hat{g}(\omega) = a e^{-i(2l+1)\omega} \hat{h}^*(\omega + \pi) \quad \text{and} \quad \hat{\tilde{g}}(\omega) = a^{-1} e^{-i(2l+1)\omega} \hat{h}^*(\omega + \pi). \quad (7.125)$$

*Proof.* Equation (7.122) proves that

$$\hat{\tilde{h}}^*(\omega) = \frac{2}{\Delta(\omega)} \hat{g}(\omega + \pi) \quad \text{and} \quad \hat{\tilde{g}}^*(\omega) = \frac{-2}{\Delta(\omega)} \hat{h}(\omega + \pi). \quad (7.126)$$

Hence

$$\hat{g}(\omega) \hat{\tilde{g}}^*(\omega) = -\frac{\Delta(\omega + \pi)}{\Delta(\omega)} \hat{h}^*(\omega + \pi) \hat{h}(\omega + \pi). \quad (7.127)$$

The definition (7.123) implies that  $\Delta(\omega + \pi) = -\Delta(\omega)$ . Inserting (7.127) in (7.117) yields (7.124).

The Fourier transform of finite impulse response filters is a finite series in  $\exp(\pm i n \omega)$ . The determinant  $\Delta(\omega)$  defined by (7.123) is therefore a finite series. Moreover (7.126) proves that  $\Delta^{-1}(\omega)$  must also be a finite series. A finite series in  $\exp(\pm i n \omega)$  whose inverse is also a finite series must have a single term. Since  $\Delta(\omega) = -\Delta(\omega + \pi)$  the exponent  $n$  must be odd. This proves that there exist  $l \in \mathbb{Z}$  and  $a \in \mathbb{R}$  such that

$$\Delta(\omega) = -2a \exp[i(2l+1)\omega]. \quad (7.128)$$

Inserting this expression in (7.126) yields (7.125). ■ ■

The factor  $a$  is a gain which is inverse for the decomposition and reconstruction filters and  $l$  is a reverse shift. We generally set  $a = 1$  and  $l = 0$ . In the time domain (7.125) can then be rewritten

$$g[n] = (-1)^{1-n} \tilde{h}[1-n] \quad \text{and} \quad \tilde{g}[n] = (-1)^{1-n} h[1-n]. \quad (7.129)$$

The two pairs of filters  $(h, g)$  and  $(\tilde{h}, \tilde{g})$  play a symmetric role and can be inverted.

**Conjugate Mirror Filters** If we impose that the decomposition filter  $h$  is equal to the reconstruction filter  $\tilde{h}$ , then (7.124) is the condition of Smith and Barnwell [443] and Mintzer [375] that defines conjugate mirror filters:

$$|\hat{h}(\omega)|^2 + |\hat{h}(\omega + \pi)|^2 = 2. \quad (7.130)$$

It is identical to the filter condition (7.29) that is required in order to synthesize orthogonal wavelets. The next section proves that it is also equivalent to discrete orthogonality properties.

### 7.3.3 Biorthogonal Bases of $\ell^2(\mathbb{Z})$

The decomposition of a discrete signal in a multirate filter bank is interpreted as an expansion in a basis of  $\ell^2(\mathbb{Z})$ . Observe first that the low-pass and high-pass signals of a filter bank computed with (7.112) can be rewritten as inner products in  $\ell^2(\mathbb{Z})$ :

$$a_1[l] = \sum_{n=-\infty}^{+\infty} a_0[n] h[n-2l] = \langle a_0[n], h[n-2l] \rangle, \quad (7.131)$$

$$d_1[l] = \sum_{n=-\infty}^{+\infty} a_0[n] g[n-2l] = \langle a_0[n], g[n-2l] \rangle. \quad (7.132)$$

The signal recovered by the reconstructing filters is

$$a_0[n] = \sum_{l=-\infty}^{+\infty} a_1[l] \tilde{h}[n-2l] + \sum_{l=-\infty}^{+\infty} d_1[l] \tilde{g}[n-2l]. \quad (7.133)$$

Inserting (7.131) and (7.132) yields

$$a_0[n] = \sum_{l=-\infty}^{+\infty} \langle f[k], h[k-2l] \rangle \tilde{h}[n-2l] + \sum_{l=-\infty}^{+\infty} \langle f[k], g[k-2l] \rangle \tilde{g}[n-2l]. \quad (7.134)$$

We recognize the decomposition of  $a_0$  over dual families of vectors  $\{\tilde{h}[n-2l], \tilde{g}[n-2l]\}_{l \in \mathbb{Z}}$  and  $\{h[n-2l], g[n-2l]\}_{l \in \mathbb{Z}}$ . The following theorem proves that these two families are biorthogonal.

**Theorem 7.13.** *If  $h, g, \tilde{h}$  and  $\tilde{g}$  are perfect reconstruction filters whose Fourier transform is bounded then  $\{\tilde{h}[n-2l], \tilde{g}[n-2l]\}_{l \in \mathbb{Z}}$  and  $\{h[n-2l], g[n-2l]\}_{l \in \mathbb{Z}}$  are biorthogonal Riesz bases of  $\ell^2(\mathbb{Z})$ .*

*Proof.* To prove that these families are biorthogonal we must show that for all  $n \in \mathbb{Z}$

$$\langle \tilde{h}[n], h[n-2l] \rangle = \delta[l] \quad (7.135)$$

$$\langle \tilde{g}[n], g[n-2l] \rangle = \delta[l] \quad (7.136)$$



and

$$\langle \tilde{h}[n], g[n - 2l] \rangle = \langle \tilde{g}[n], h[n - 2l] \rangle = 0. \quad (7.137)$$

For perfect reconstruction filters, (7.124) proves that

$$\frac{1}{2} \left( \hat{h}^*(\omega) \hat{h}(\omega) + \hat{h}^*(\omega + \pi) \hat{h}(\omega + \pi) \right) = 1.$$

In the time domain, this equation becomes

$$\bar{h} \star \tilde{h}[2l] = \sum_{k=-\infty}^{+\infty} \tilde{h}[k] \bar{h}[n - 2l] = \delta[l], \quad (7.138)$$

which verifies (7.135). The same proof as for (7.124) shows that

$$\frac{1}{2} \left( \hat{g}^*(\omega) \hat{g}(\omega) + \hat{g}^*(\omega + \pi) \hat{g}(\omega + \pi) \right) = 1.$$

In the time domain, this equation yields (7.136). It also follows from (7.122) that

$$\frac{1}{2} \left( \hat{g}^*(\omega) \hat{h}(\omega) + \hat{g}^*(\omega + \pi) \hat{h}(\omega + \pi) \right) = 0,$$

and

$$\frac{1}{2} \left( \hat{h}^*(\omega) \hat{g}(\omega) + \hat{h}^*(\omega + \pi) \hat{g}(\omega + \pi) \right) = 0.$$

The inverse Fourier transforms of these two equations yield (7.137).

To finish the proof, one must show the existence of Riesz bounds. The reader can verify that this is a consequence of the fact that the Fourier transform of each filter is bounded. ■ ■

**Orthogonal Bases** A Riesz basis is orthonormal if the dual basis is the same as the original basis. For filter banks, this means that  $h = \tilde{h}$  and  $g = \tilde{g}$ . The filter  $h$  is then a conjugate mirror filter

$$|\hat{h}(\omega)|^2 + |\hat{h}(\omega + \pi)|^2 = 2. \quad (7.139)$$

The resulting family  $\{h[n - 2l], g[n - 2l]\}_{l \in \mathbb{Z}}$  is an orthogonal basis of  $\ell^2(\mathbb{Z})$ .

**Discrete Wavelet Bases** The construction of conjugate mirror filters is simpler than the construction of orthogonal wavelet bases of  $\mathbf{L}^2(\mathbb{R})$ . Why then should we bother with continuous time models of wavelets, since in any case all computations are discrete and rely on conjugate mirror filters? The reason is that conjugate mirror filters are most often used in filter banks that cascade several levels of filterings and subsamplings. It is thus necessary to understand the behavior of such a cascade [406]. In a wavelet filter bank tree, the output of the low-pass filter  $\bar{h}$  is sub-decomposed whereas the output of the high-pass filter  $\bar{g}$  is not; this is illustrated in Figure 7.12. Suppose that the sampling distance of the original discrete signal is  $N^{-1}$ . We denote  $a_L[n]$  this discrete signal, with  $2^L = N^{-1}$ . At the depth  $j - L \geq 0$  of this filter bank tree, the low-pass signal  $a_j$  and high-pass signal  $d_j$  can be written

$$a_j[l] = a_L \star \bar{\phi}_j[2^{j-L}l] = \langle a_L[n], \phi_j[n - 2^{j-L}l] \rangle$$

and

$$d_j[l] = a_L \star \bar{\psi}_j[2^{j-L}l] = \langle a_L[n], \psi_j[n - 2^{j-L}l] \rangle.$$

The Fourier transforms of these equivalent filters are

$$\hat{\phi}_j(\omega) = \prod_{p=0}^{j-L-1} \hat{h}(2^p \omega) \quad \text{and} \quad \hat{\psi}_j(\omega) = \hat{g}(2^{j-L-1} \omega) \prod_{p=0}^{j-L-2} \hat{h}(2^p \omega). \quad (7.140)$$

A filter bank tree of depth  $J - L \geq 0$ , decomposes  $a_L$  over the family of vectors

$$\left[ \left\{ \phi_J[n - 2^{J-L}l] \right\}_{l \in \mathbb{Z}}, \left\{ \psi_J[n - 2^{j-L}l] \right\}_{L < j \leq J, l \in \mathbb{Z}} \right]. \quad (7.141)$$

For conjugate mirror filters, one can verify that this family is an orthonormal basis of  $\ell^2(\mathbb{Z})$ . These discrete vectors are close to a uniform sampling of the continuous time scaling functions  $\phi_j(t) = 2^{-j/2}\phi(2^{-j}t)$  and wavelets  $\psi_j(t) = 2^{-j/2}\phi(2^{-j}t)$ . When the number  $L - j$  of successive convolutions increases, one can verify that  $\phi_j[n]$  and  $\psi_j[n]$  converge respectively to  $N^{-1/2}\phi_j(N^{-1}n)$  and  $N^{-1/2}\psi_j(N^{-1}n)$ . The factor  $N^{-1/2}$  normalizes the  $\ell^2(\mathbb{Z})$  norm of these sampled functions. If  $L - j = 4$  then  $\phi_j[n]$  and  $\psi_j[n]$  are already very close to these limit values. The impulse responses  $\phi_j[n]$  and  $\psi_j[n]$  of the filter bank are thus much closer to continuous time scaling functions and wavelets than they are to the original conjugate mirror filters  $h$  and  $g$ . This explains why wavelets provide appropriate models for understanding the applications of these filter banks. Chapter 8 relates more general filter banks to wavelet packet bases.

If the decomposition and reconstruction filters of the filter bank are different, the resulting basis (7.141) is non-orthogonal. The stability of this discrete wavelet basis does not degrade when the depth  $J - L$  of the filter bank increases. The next section shows that the corresponding continuous time wavelet  $\psi(t)$  generates a Riesz basis of  $\mathbf{L}^2(\mathbb{R})$ .

## 7.4 Biorthogonal Wavelet Bases

The stability and completeness properties of biorthogonal wavelet bases are described for perfect reconstruction filters  $h$  and  $\tilde{h}$  having a finite impulse response. The design of linear phase wavelets with compact support is explained in Section 7.4.2.

### 7.4.1 Construction of Biorthogonal Wavelet Bases

An infinite cascade of perfect reconstruction filters  $(h, g)$  and  $(\tilde{h}, \tilde{g})$  yields two scaling functions and wavelets whose Fourier transforms satisfy

$$\hat{\phi}(2\omega) = \frac{1}{\sqrt{2}} \hat{h}(\omega) \hat{\phi}(\omega) \quad , \quad \hat{\tilde{\phi}}(2\omega) = \frac{1}{\sqrt{2}} \hat{\tilde{h}}(\omega) \hat{\tilde{\phi}}(\omega) \quad , \quad (7.142)$$

$$\hat{\psi}(2\omega) = \frac{1}{\sqrt{2}} \hat{g}(\omega) \hat{\phi}(\omega) \quad , \quad \hat{\tilde{\psi}}(2\omega) = \frac{1}{\sqrt{2}} \hat{\tilde{g}}(\omega) \hat{\tilde{\phi}}(\omega) \quad . \quad (7.143)$$

In the time domain, these relations become

$$\phi(t) = \sqrt{2} \sum_{n=-\infty}^{+\infty} h[n] \phi(2t - n) \quad , \quad \tilde{\phi}(t) = \sqrt{2} \sum_{n=-\infty}^{+\infty} \tilde{h}[n] \tilde{\phi}(2t - n) \quad (7.144)$$

$$\psi(t) = \sqrt{2} \sum_{n=-\infty}^{+\infty} g[n] \phi(2t - n) \quad , \quad \tilde{\psi}(t) = \sqrt{2} \sum_{n=-\infty}^{+\infty} \tilde{g}[n] \tilde{\phi}(2t - n) \quad . \quad (7.145)$$

The perfect reconstruction conditions are given by Theorem 7.12. If we normalize the gain and shift to  $a = 1$  and  $l = 0$ , the filters must satisfy

$$\hat{h}^*(\omega) \hat{\tilde{h}}(\omega) + \hat{h}^*(\omega + \pi) \hat{\tilde{h}}(\omega + \pi) = 2, \quad (7.146)$$

and

$$\hat{g}(\omega) = e^{-i\omega} \hat{h}^*(\omega + \pi) \quad , \quad \hat{\tilde{g}}(\omega) = e^{-i\omega} \hat{\tilde{h}}^*(\omega + \pi). \quad (7.147)$$

Wavelets should have a zero average, which means that  $\hat{\psi}(0) = \hat{\tilde{\psi}}(0) = 0$ . This is obtained by setting  $\hat{g}(0) = \hat{\tilde{g}}(0) = 0$  and hence  $\hat{h}(\pi) = \hat{\tilde{h}}(\pi) = 0$ . The perfect reconstruction condition (7.146) implies that  $\hat{h}^*(0) \hat{\tilde{h}}(0) = 2$ . Since both filters are defined up to multiplicative constants respectively equal to  $\lambda$  and  $\lambda^{-1}$ , we adjust  $\lambda$  so that  $\hat{h}(0) = \hat{\tilde{h}}(0) = \sqrt{2}$ .

In the following, we also suppose that  $h$  and  $\tilde{h}$  are finite impulse response filters. One can then prove [18] that

$$\hat{\phi}(\omega) = \prod_{p=1}^{+\infty} \frac{\hat{h}(2^{-p}\omega)}{\sqrt{2}} \quad \text{and} \quad \hat{\tilde{\phi}}(\omega) = \prod_{p=1}^{+\infty} \frac{\hat{\tilde{h}}(2^{-p}\omega)}{\sqrt{2}} \quad (7.148)$$

are the Fourier transforms of distributions of compact support. However, these distributions may exhibit wild behavior and have infinite energy. Some further conditions must be imposed to guarantee that  $\hat{\phi}$  and  $\hat{\tilde{\phi}}$  are the Fourier transforms of finite energy functions. The following theorem gives sufficient conditions on the perfect reconstruction filters for synthesizing biorthogonal wavelet bases of  $\mathbf{L}^2(\mathbb{R})$ .

**Theorem 7.14** (Cohen, Daubechies, Feauveau). *Suppose that there exist strictly positive trigonometric polynomials  $P(e^{i\omega})$  and  $\tilde{P}(e^{i\omega})$  such that*

$$\left| \hat{h}\left(\frac{\omega}{2}\right) \right|^2 P(e^{i\omega/2}) + \left| \hat{h}\left(\frac{\omega}{2} + \pi\right) \right|^2 P(e^{i(\omega/2+\pi)}) = 2P(e^{i\omega}), \quad (7.149)$$

$$\left| \hat{\tilde{h}}\left(\frac{\omega}{2}\right) \right|^2 \tilde{P}(e^{i\omega/2}) + \left| \hat{\tilde{h}}\left(\frac{\omega}{2} + \pi\right) \right|^2 \tilde{P}(e^{i(\omega/2+\pi)}) = 2\tilde{P}(e^{i\omega}) \quad (7.150)$$

and that  $P$  and  $\tilde{P}$  are unique (up to normalization). Suppose that

$$\inf_{\omega \in [-\pi/2, \pi/2]} |\hat{h}(\omega)| > 0, \quad \inf_{\omega \in [-\pi/2, \pi/2]} |\hat{\tilde{h}}(\omega)| > 0. \quad (7.151)$$

- Then the functions  $\hat{\phi}$  and  $\hat{\tilde{\phi}}$  defined in (7.148) belong to  $\mathbf{L}^2(\mathbb{R})$ , and  $\phi, \tilde{\phi}$  satisfy biorthogonal relations

$$\langle \phi(t), \tilde{\phi}(t-n) \rangle = \delta[n]. \quad (7.152)$$

- The two wavelet families  $\{\psi_{j,n}\}_{(j,n) \in \mathbb{Z}^2}$  and  $\{\tilde{\psi}_{j,n}\}_{(j,n) \in \mathbb{Z}^2}$  are biorthogonal Riesz bases of  $\mathbf{L}^2(\mathbb{R})$ .

The proof of this theorem is in [171] and [18]. The hypothesis (7.151) is also imposed by Theorem 7.2, which constructs orthogonal bases of scaling functions. The conditions (7.149) and (7.150) do not appear in the construction of wavelet orthogonal bases because they are always satisfied with  $P(e^{i\omega}) = \tilde{P}(e^{i\omega}) = 1$  and one can prove that constants are the only invariant trigonometric polynomials [340].

Biorthogonality means that for any  $(j, j', n, n') \in \mathbb{Z}^4$ ,

$$\langle \psi_{j,n}, \tilde{\psi}_{j',n'} \rangle = \delta[n-n'] \delta[j-j']. \quad (7.153)$$

Any  $f \in \mathbf{L}^2(\mathbb{R})$  has two possible decompositions in these bases:

$$f = \sum_{n,j=-\infty}^{+\infty} \langle f, \psi_{j,n} \rangle \tilde{\psi}_{j,n} = \sum_{n,j=-\infty}^{+\infty} \langle f, \tilde{\psi}_{j,n} \rangle \psi_{j,n}. \quad (7.154)$$

The Riesz stability implies that there exist  $A > 0$  and  $B > 0$  such that

$$A \|f\|^2 \leq \sum_{n,j=-\infty}^{+\infty} |\langle f, \psi_{j,n} \rangle|^2 \leq B \|f\|^2, \quad (7.155)$$

$$\frac{1}{B} \|f\|^2 \leq \sum_{n,j=-\infty}^{+\infty} |\langle f, \tilde{\psi}_{j,n} \rangle|^2 \leq \frac{1}{A} \|f\|^2. \quad (7.156)$$

**Multiresolutions** Biorthogonal wavelet bases are related to multiresolution approximations. The family  $\{\phi(t-n)\}_{n \in \mathbb{Z}}$  is a Riesz basis of the space  $\mathbf{V}_0$  it generates, whereas  $\{\tilde{\phi}(t-n)\}_{n \in \mathbb{Z}}$  is a Riesz basis of another space  $\tilde{\mathbf{V}}_0$ . Let  $\mathbf{V}_j$  and  $\tilde{\mathbf{V}}_j$  be the spaces defined by

$$\begin{aligned} f(t) \in \mathbf{V}_j &\Leftrightarrow f(2^j t) \in \mathbf{V}_0, \\ f(t) \in \tilde{\mathbf{V}}_j &\Leftrightarrow f(2^j t) \in \tilde{\mathbf{V}}_0. \end{aligned}$$

One can verify that  $\{\mathbf{V}_j\}_{j \in \mathbb{Z}}$  and  $\{\tilde{\mathbf{V}}_j\}_{j \in \mathbb{Z}}$  are two multiresolution approximations of  $\mathbf{L}^2(\mathbb{R})$ . For any  $j \in \mathbb{Z}$ ,  $\{\phi_{j,n}\}_{n \in \mathbb{Z}}$  and  $\{\tilde{\phi}_{j,n}\}_{n \in \mathbb{Z}}$  are Riesz bases of  $\mathbf{V}_j$  and  $\tilde{\mathbf{V}}_j$ . The dilated wavelets  $\{\psi_{j,n}\}_{n \in \mathbb{Z}}$  and  $\{\tilde{\psi}_{j,n}\}_{n \in \mathbb{Z}}$  are bases of two detail spaces  $\mathbf{W}_j$  and  $\tilde{\mathbf{W}}_j$  such that

$$\mathbf{V}_j \oplus \mathbf{W}_j = \mathbf{V}_{j-1} \quad \text{and} \quad \tilde{\mathbf{V}}_j \oplus \tilde{\mathbf{W}}_j = \tilde{\mathbf{V}}_{j-1}.$$

The biorthogonality of the decomposition and reconstruction wavelets implies that  $\mathbf{W}_j$  is not orthogonal to  $\mathbf{V}_j$  but is to  $\tilde{\mathbf{V}}_j$  whereas  $\tilde{\mathbf{W}}_j$  is not orthogonal to  $\tilde{\mathbf{V}}_j$  but is to  $\mathbf{V}_j$ .

**Fast Biorthogonal Wavelet Transform** The perfect reconstruction filter bank studied in Section 7.3.2 implements a fast biorthogonal wavelet transform. For any discrete signal input  $b[n]$  sampled at intervals  $N^{-1} = 2^L$ , there exists  $f \in \mathbf{V}_L$  such that  $a_L[n] = \langle f, \phi_{L,n} \rangle = N^{-1/2} b[n]$ . The wavelet coefficients are computed by successive convolutions with  $\bar{h}$  and  $\bar{g}$ . Let  $a_j[n] = \langle f, \phi_{j,n} \rangle$  and  $d_j[n] = \langle f, \psi_{j,n} \rangle$ . As in Theorem 7.10, one can prove that

$$a_{j+1}[n] = a_j \star \bar{h}[2n] \quad , \quad d_{j+1}[n] = a_j \star \bar{g}[2n] \quad . \quad (7.157)$$

The reconstruction is performed with the dual filters  $\tilde{h}$  and  $\tilde{g}$ :

$$a_j[n] = \check{a}_{j+1} \star \tilde{h}[n] + \check{d}_{j+1} \star \tilde{g}[n]. \quad (7.158)$$

If  $a_L$  includes  $N$  non-zero samples, the biorthogonal wavelet representation  $\{[d_j]_{L < j \leq J}, a_J\}$  is calculated with  $O(N)$  operations, by iterating (7.157) for  $L \leq j < J$ . The reconstruction of  $a_L$  by applying (7.158) for  $J > j \geq L$  requires the same number of operations.

## 7.4.2 Biorthogonal Wavelet Design

The support size, the number of vanishing moments, the regularity and the symmetry of biorthogonal wavelets is controlled with an appropriate design of  $h$  and  $\tilde{h}$ .

**Support** If the perfect reconstruction filters  $h$  and  $\tilde{h}$  have a finite impulse response then the corresponding scaling functions and wavelets also have a compact support. As in Section 7.2.1, one can show that if  $h[n]$  and  $\tilde{h}[n]$  are non-zero respectively for  $N_1 \leq n \leq N_2$  and  $\tilde{N}_1 \leq n \leq \tilde{N}_2$ , then  $\phi$  and  $\tilde{\phi}$  have a support respectively equal to  $[N_1, N_2]$  and  $[\tilde{N}_1, \tilde{N}_2]$ . Since

$$g[n] = (-1)^{1-n} h[1-n] \quad \text{and} \quad \tilde{g}[n] = (-1)^{1-n} \tilde{h}[1-n],$$

the supports of  $\psi$  and  $\tilde{\psi}$  defined in (7.145) are respectively

$$\left[ \frac{N_1 - \tilde{N}_2 + 1}{2}, \frac{N_2 - \tilde{N}_1 + 1}{2} \right] \quad \text{and} \quad \left[ \frac{\tilde{N}_1 - N_2 + 1}{2}, \frac{\tilde{N}_2 - N_1 + 1}{2} \right]. \quad (7.159)$$

Both wavelets thus have a support of the same size and equal to

$$l = \frac{N_2 - N_1 + \tilde{N}_2 - \tilde{N}_1}{2}. \quad (7.160)$$

**Vanishing Moments** The number of vanishing moments of  $\psi$  and  $\tilde{\psi}$  depends on the number of zeroes at  $\omega = \pi$  of  $\hat{h}(\omega)$  and  $\hat{\tilde{h}}(\omega)$ . Theorem 7.4 proves that  $\psi$  has  $\tilde{p}$  vanishing moments if the derivatives of its Fourier transform satisfy  $\hat{\psi}^{(k)}(0) = 0$  for  $k \leq \tilde{p}$ . Since  $\hat{\phi}(0) = 1$ , (7.4.1) implies that it is equivalent to impose that  $\hat{g}(\omega)$  has a zero of order  $\tilde{p}$  at  $\omega = 0$ . Since  $\hat{g}(\omega) = e^{-i\omega} \hat{\tilde{h}}^*(\omega + \pi)$ , this means that  $\hat{\tilde{h}}(\omega)$  has a zero of order  $\tilde{p}$  at  $\omega = \pi$ . Similarly the number of vanishing moments of  $\tilde{\psi}$  is equal to the number  $p$  of zeroes of  $\hat{h}(\omega)$  at  $\pi$ .

**Regularity** Although the regularity of a function is a priori independent of the number of vanishing moments, the smoothness of biorthogonal wavelets is related to their vanishing moments. The regularity of  $\phi$  and  $\psi$  is the same because (7.145) shows that  $\psi$  is a finite linear expansion of  $\phi$  translated. Tchamitchian's Theorem 7.6 gives a sufficient condition for estimating this regularity. If  $\hat{h}(\omega)$  has a zero of order  $p$  at  $\pi$ , we can perform the factorization

$$\hat{h}(\omega) = \left( \frac{1 + e^{-i\omega}}{2} \right)^p \hat{l}(\omega). \quad (7.161)$$

Let  $B = \sup_{\omega \in [-\pi, \pi]} |\hat{l}(\omega)|$ . Theorem 7.6 proves that  $\phi$  is uniformly Lipschitz  $\alpha$  for

$$\alpha < \alpha_0 = p - \log_2 B - 1.$$

Generally,  $\log_2 B$  increases more slowly than  $p$ . This implies that the regularity of  $\phi$  and  $\psi$  increases with  $p$ , which is equal to the number of vanishing moments of  $\tilde{\psi}$ . Similarly, one can show that the regularity of  $\tilde{\psi}$  and  $\tilde{\phi}$  increases with  $\tilde{p}$ , which is the number of vanishing moments of  $\psi$ . If  $\hat{h}$  and  $\tilde{h}$  have different numbers of zeroes at  $\pi$ , the properties of  $\psi$  and  $\tilde{\psi}$  can therefore be very different.

**Ordering of Wavelets** Since  $\psi$  and  $\tilde{\psi}$  might not have the same regularity and number of vanishing moments, the two reconstruction formulas

$$f = \sum_{n,j=-\infty}^{+\infty} \langle f, \psi_{j,n} \rangle \tilde{\psi}_{j,n}, \quad (7.162)$$

$$f = \sum_{n,j=-\infty}^{+\infty} \langle f, \tilde{\psi}_{j,n} \rangle \psi_{j,n} \quad (7.163)$$

are not equivalent. The decomposition (7.162) is obtained with the filters  $(h, g)$  at the decomposition and  $(\tilde{h}, \tilde{g})$  at the reconstruction. The inverse formula (7.163) corresponds to  $(\tilde{h}, \tilde{g})$  at the decomposition and  $(h, g)$  at the reconstruction.

To produce small wavelet coefficients in regular regions we must compute the inner products using the wavelet with the maximum number of vanishing moments. The reconstruction is then performed with the other wavelet, which is generally the smoothest one. If errors are added to the wavelet coefficients, for example with a quantization, a smooth wavelet at the reconstruction introduces a smooth error. The number of vanishing moments of  $\psi$  is equal to the number  $\tilde{p}$  of zeroes at  $\pi$  of  $\hat{h}$ . Increasing  $\tilde{p}$  also increases the regularity of  $\tilde{\psi}$ . It is thus better to use  $h$  at the decomposition and  $\tilde{h}$  at the reconstruction if  $\hat{h}$  has fewer zeroes at  $\pi$  than  $\hat{\tilde{h}}$ .

**Symmetry** It is possible to construct smooth biorthogonal wavelets of compact support which are either symmetric or antisymmetric. This is impossible for orthogonal wavelets, besides the particular case of the Haar basis. Symmetric or antisymmetric wavelets are synthesized with perfect reconstruction filters having a linear phase. If  $h$  and  $\tilde{h}$  have an odd number of non-zero samples and are symmetric about  $n = 0$ , the reader can verify that  $\phi$  and  $\tilde{\phi}$  are symmetric about  $t = 0$  while  $\psi$  and  $\tilde{\psi}$  are symmetric with respect to a shifted center. If  $h$  and  $\tilde{h}$  have an even number of non-zero samples and are symmetric about  $n = 1/2$ , then  $\phi(t)$  and  $\tilde{\phi}(t)$  are symmetric about  $t = 1/2$ , while  $\psi$  and  $\tilde{\psi}$  are antisymmetric with respect to a shifted center. When the wavelets are symmetric or antisymmetric, wavelet bases over finite intervals are constructed with the folding procedure of Section 7.5.2.

### 7.4.3 Compactly Supported Biorthogonal Wavelets

We study the design of biorthogonal wavelets with a minimum size support for a specified number of vanishing moments. Symmetric or antisymmetric compactly supported spline biorthogonal wavelet bases are constructed with a technique introduced in [171].

**Theorem 7.15** (Cohen, Daubechies, Feauveau). *Biorthogonal wavelets  $\psi$  and  $\tilde{\psi}$  with respectively  $\tilde{p}$  and  $p$  vanishing moments have a support of size at least  $p + \tilde{p} - 1$ . CDF biorthogonal wavelets have a minimum support of size  $p + \tilde{p} - 1$ .*

*Proof.* The proof follows the same approach as the proof of Daubechies's Theorem 7.7. One can verify that  $p$  and  $\tilde{p}$  must necessarily have the same parity. We concentrate on filters  $h[n]$  and  $\tilde{h}[n]$  that have a symmetry with respect to  $n = 0$  or  $n = 1/2$ . The general case proceeds similarly. We can then factor

$$\hat{h}(\omega) = \sqrt{2} \exp\left(\frac{-i\varepsilon\omega}{2}\right) \left(\cos \frac{\omega}{2}\right)^p L(\cos \omega), \quad (7.164)$$

$$\hat{\tilde{h}}(\omega) = \sqrt{2} \exp\left(\frac{-i\varepsilon\omega}{2}\right) \left(\cos \frac{\omega}{2}\right)^{\tilde{p}} \tilde{L}(\cos \omega), \quad (7.165)$$

with  $\varepsilon = 0$  for  $p$  and  $\tilde{p}$  even and  $\varepsilon = 1$  for odd values. Let  $q = (p + \tilde{p})/2$ . The perfect reconstruction condition

$$\hat{h}^*(\omega) \hat{\tilde{h}}(\omega) + \hat{h}^*(\omega + \pi) \hat{\tilde{h}}(\omega + \pi) = 2$$

n	$p, \tilde{p}$	$h[n]$	$\tilde{h}[n]$
0	$p = 2, \tilde{p} = 4$	0.70710678118655	0.99436891104358
1, -1		0.35355339059327	0.41984465132951
2, -2			-0.17677669529664
3, -3			-0.06629126073624
4, -4			0.03314563036812
0, 1	$p = 3, \tilde{p} = 7$	0.53033008588991	0.95164212189718
-1, 2		0.17677669529664	-0.02649924094535
-2, 3			-0.30115912592284
-3, 4			0.03133297870736
-4, 5			0.07466398507402
-5, 6			-0.01683176542131
-6, 7			-0.00906325830378
-7, 8			0.00302108610126

Table 7.3: Perfect reconstruction filters  $h$  and  $\tilde{h}$  for compactly supported spline wavelets, with  $\hat{h}$  and  $\hat{\tilde{h}}$  having respectively  $\tilde{p}$  and  $p$  zeros at  $\omega = \pi$ .

is imposed by writing

$$L(\cos \omega) \tilde{L}(\cos \omega) = P\left(\sin^2 \frac{\omega}{2}\right), \quad (7.166)$$

where the polynomial  $P(y)$  must satisfy for all  $y \in [0, 1]$

$$(1 - y)^q P(y) + y^q P(1 - y) = 1. \quad (7.167)$$

We saw in (7.96) that the polynomial of minimum degree satisfying this equation is

$$P(y) = \sum_{k=0}^{q-1} \binom{q-1+k}{k} y^k. \quad (7.168)$$

The spectral factorization (7.166) is solved with a root attribution similar to (7.98). The resulting minimum support of  $\psi$  and  $\tilde{\psi}$  specified by (7.160) is then  $p + \tilde{p} - 1$ . ■ ■

**Spline Biorthogonal Wavelets** Let us choose

$$\hat{h}(\omega) = \sqrt{2} \exp\left(\frac{-i\varepsilon\omega}{2}\right) \left(\cos \frac{\omega}{2}\right)^p \quad (7.169)$$

with  $\varepsilon = 0$  for  $p$  even and  $\varepsilon = 1$  for  $p$  odd. The scaling function computed with (7.148) is then a box spline of degree  $p - 1$

$$\hat{\phi}(\omega) = \exp\left(\frac{-i\varepsilon\omega}{2}\right) \left(\frac{\sin(\omega/2)}{\omega/2}\right)^p.$$

Since  $\psi$  is a linear combination of box splines  $\phi(2t - n)$ , it is a compactly supported polynomial spline of same degree.

The number of vanishing moments  $\tilde{p}$  of  $\psi$  is a free parameter, which must have the same parity as  $p$ . Let  $q = (p + \tilde{p})/2$ . The biorthogonal filter  $\tilde{h}$  of minimum length is obtained by observing that  $L(\cos \omega) = 1$  in (7.164). The factorization (7.166) and (7.168) thus imply that

$$\hat{\tilde{h}}(\omega) = \sqrt{2} \exp\left(\frac{-i\varepsilon\omega}{2}\right) \left(\cos \frac{\omega}{2}\right)^{\tilde{p}} \sum_{k=0}^{q-1} \binom{q-1+k}{k} \left(\sin \frac{\omega}{2}\right)^{2k}. \quad (7.170)$$

These filters satisfy the conditions of Theorem 7.14 and thus generate biorthogonal wavelet bases. Table 7.3 gives the filter coefficients for  $(p = 2, \tilde{p} = 4)$  and  $(p = 3, \tilde{p} = 7)$ . The resulting dual wavelet and scaling functions are shown in Figure 7.13.

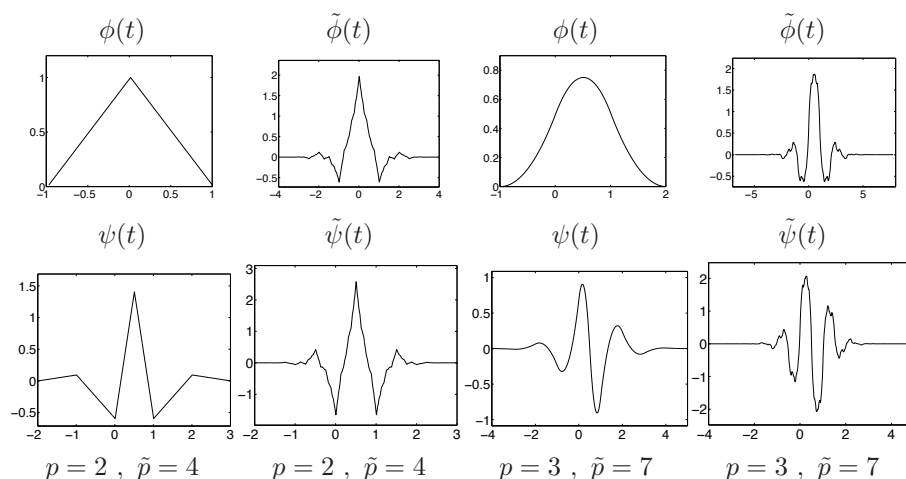


Figure 7.14: Spline biorthogonal wavelets and scaling functions of compact support corresponding to the filters of Table 7.3.

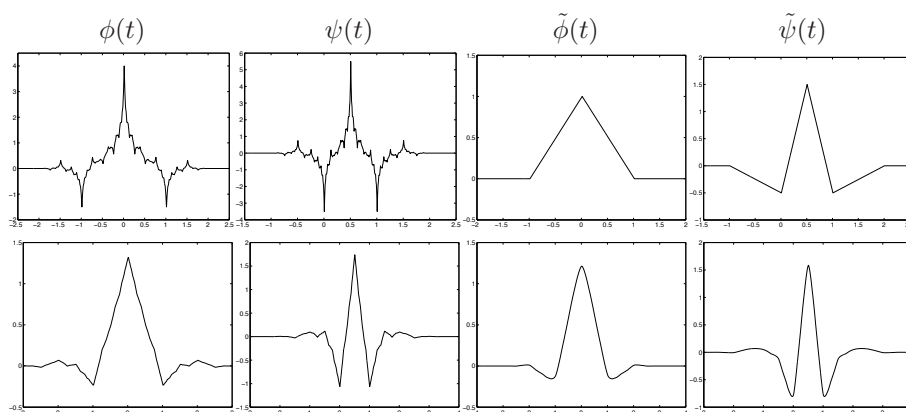


Figure 7.15: Biorthogonal wavelets and scaling functions calculated with the filters of Table 7.4, with  $p = 2$  and  $\tilde{p} = 2$  (top row) and with  $p = 4$  and  $\tilde{p} = 4$  (bottom row).

**Closer Filter Length** Biorthogonal filters  $h$  and  $\tilde{h}$  of more similar length are obtained by factoring the polynomial  $P(\sin^2 \frac{\omega}{2})$  in (7.166) with two polynomial  $L(\cos \omega)$  and  $\tilde{L}(\cos \omega)$  of similar degree. There is a limited number of possible factorizations. For  $q = (p + \tilde{p})/2 < 4$ , the only solution is  $L(\cos \omega) = 1$ . For  $q = 4$  there is one non-trivial factorization and for  $q = 5$  there are two. Table 7.4 gives the resulting coefficients of the filters  $h$  and  $\tilde{h}$  of most similar length, computed by Cohen, Daubechies and Feauveau [171]. These filters also satisfy the conditions of Theorem 7.14 and therefore define biorthogonal wavelet bases. Figure 7.15 gives the scaling functions and wavelets for  $p = \tilde{p} = 2$  and  $p = \tilde{p} = 4$ , which correspond respectively to filter sizes 5/3 and 9/7. For  $p = \tilde{p} = 4$ ,  $\phi, \psi$  are similar to  $\tilde{\phi}, \tilde{\psi}$ , which indicates that this basis is nearly orthogonal. This particular set of filters is often used in image compression and recommended for JPEG-2000. The quasi-orthogonality guarantees a good numerical stability and the symmetry allows one to use the folding procedure of Section 7.5.2 at the boundaries. There are also enough vanishing moments to create small wavelet coefficients in regular image domains. Section 7.8.5 describes their lifting implementation which is simple and efficient. The filters 5/3 are also recommended for lossless compression with JPEG-2000, because they use integer operations with a lifting algorithm. The design of other compactly supported biorthogonal filters is discussed extensively in [171, 473].

$p, \tilde{p}$	$n$	$h[n]$	$\tilde{h}[n]$
$p = 2$ $\tilde{p} = 2$	0	1.06066017177982	0.70710678118655
	-1, 1	0.35355339059327	0.35355339059327
	-2, 2	-0.17677669529664	0
$p = 4$ $\tilde{p} = 4$	0	0.85269867900889	0.78848561640637
	-1, 1	0.37740285561283	0.41809227322204
	-2, 2	-0.11062440441844	-0.04068941760920
	-3, 3	-0.02384946501956	-0.06453888262876
	-4, 4	0.03782845554969	0
$p = 5$ $\tilde{p} = 5$	0	0.89950610974865	0.73666018142821
	-1, 1	0.47680326579848	0.34560528195603
	-2, 2	-0.09350469740094	-0.05446378846824
	-3, 3	-0.13670658466433	0.00794810863724
	-4, 4	-0.00269496688011	0.03968708834741
	-5, 5	0.01345670945912	0
$p = 5$ $\tilde{p} = 5$	0	0.54113273169141	1.32702528570780
	-1, 1	0.34335173921766	0.47198693379091
	-2, 2	0.06115645341349	-0.36378609009851
	-3, 3	0.00027989343090	-0.11843354319764
	-4, 4	0.02183057133337	0.05382683783789
	-5, 5	0.00992177208685	0

Table 7.4: Perfect reconstruction filters of most similar length.

## 7.5 Wavelet Bases on an Interval

To decompose signals  $f$  defined over an interval  $[0, 1]$ , it is necessary to construct wavelet bases of  $\mathbf{L}^2[0, 1]$ . Such bases are synthesized by modifying the wavelets  $\psi_{j,n}(t) = 2^{-j/2}\psi(2^{-j}t - n)$  of a basis  $\{\psi_{j,n}\}_{(j,n) \in \mathbb{Z}^2}$  of  $\mathbf{L}^2(\mathbb{R})$ . The *inside* wavelets  $\psi_{j,n}$  whose support are included in  $[0, 1]$  are not modified. The *boundary* wavelets  $\psi_{j,n}$  whose supports overlap  $t = 0$  or  $t = 1$  are transformed into functions having a support in  $[0, 1]$ , which are designed in order to provide the necessary complement to generate a basis of  $\mathbf{L}^2[0, 1]$ . If  $\psi$  has a compact support then there is a constant number of boundary wavelets at each scale.

The main difficulty is to construct boundary wavelets that keep their vanishing moments. The next three sections describe different approaches to constructing boundary wavelets. Periodic wavelets have no vanishing moments at the boundary, whereas folded wavelets have one vanishing moment. The custom-designed boundary wavelets of Section 7.5.3 have as many vanishing moments as the inside wavelets but are more complicated to construct. Scaling functions  $\phi_{j,n}$  are also restricted to  $[0, 1]$  by modifying the scaling functions  $\phi_{j,n}(t) = 2^{-j/2}\phi(2^{-j}t - n)$  associated to the wavelets  $\psi_{j,n}$ . The resulting wavelet basis of  $\mathbf{L}^2[0, 1]$  is composed of  $2^{-J}$  scaling functions at a coarse scale  $2^J < 1$ , plus  $2^{-j}$  wavelets at each scale  $2^j \leq 2^J$ :

$$\left[ \{\phi_{J,n}^{\text{int}}\}_{0 \leq n < 2^{-J}} , \{\psi_{j,n}^{\text{int}}\}_{-\infty < j \leq J, 0 \leq n < 2^{-j}} \right] . \quad (7.171)$$

On any interval  $[a, b]$ , a wavelet orthonormal basis of  $\mathbf{L}^2[a, b]$  is constructed with a dilation by  $b - a$  and a translation by  $a$  of the wavelets in (7.171).

**Discrete Basis of  $\mathbb{C}^N$**  The decomposition of a signal in a wavelet basis over an interval is computed by modifying the fast wavelet transform algorithm of Section 7.3.1. A discrete signal  $b[n]$  of  $N$  samples is associated to the approximation of a signal  $f \in \mathbf{L}^2[0, 1]$  at a scale  $N^{-1} = 2^L$  with (7.111):

$$N^{-1/2} b[n] = a_L[n] = \langle f, \phi_{L,n}^{\text{int}} \rangle \quad \text{for } 0 \leq n < 2^{-L} .$$

Its wavelet coefficients can be calculated at scales  $1 \geq 2^j > 2^L$ . We set

$$a_j[n] = \langle f, \phi_{j,n}^{\text{int}} \rangle \quad \text{and} \quad d_j[n] = \langle f, \psi_{j,n}^{\text{int}} \rangle \quad \text{for } 0 \leq n < 2^{-j} . \quad (7.172)$$



The wavelets and scaling functions with support inside  $[0, 1]$  are identical to the wavelets and scaling functions of a basis of  $\mathbf{L}^2(\mathbb{R})$ . The corresponding coefficients  $a_j[n]$  and  $d_j[n]$  can thus be calculated with the decomposition and reconstruction equations given by Theorem 7.10. These convolution formulas must however be modified near the boundary where the wavelets and scaling functions are modified. Boundary calculations depend on the specific design of the boundary wavelets, as explained in the next three sections. The resulting filter bank algorithm still computes the  $N$  coefficients of the wavelet representation  $[a_J, \{d_j\}_{L < j \leq J}]$  of  $a_L$  with  $O(N)$  operations.

Wavelet coefficients can also be written as discrete inner products of  $a_L$  with discrete wavelets:

$$a_j[n] = \langle a_L[m], \phi_{j,n}^{\text{int}}[m] \rangle \quad \text{and} \quad d_j[n] = \langle a_L[m], \psi_{j,n}^{\text{int}}[m] \rangle. \quad (7.173)$$

As in Section 7.3.3, we verify that

$$[\{\phi_{j,n}^{\text{int}}[m]\}_{0 \leq n < 2^{-j}}, \{\psi_{j,n}^{\text{int}}[m]\}_{L < j \leq J, 0 \leq n < 2^{-j}}]$$

is an orthonormal basis of  $\mathbb{C}^N$ .

### 7.5.1 Periodic Wavelets

A wavelet basis  $\{\psi_{j,n}\}_{(j,n) \in \mathbb{Z}^2}$  of  $\mathbf{L}^2(\mathbb{R})$  is transformed into a wavelet basis of  $\mathbf{L}^2[0, 1]$  by periodizing each  $\psi_{j,n}$ . The periodization of  $f \in \mathbf{L}^2(\mathbb{R})$  over  $[0, 1]$  is defined by

$$f^{\text{pér}}(t) = \sum_{k=-\infty}^{+\infty} f(t+k). \quad (7.174)$$

The resulting periodic wavelets are

$$\psi_{j,n}^{\text{pér}}(t) = \frac{1}{\sqrt{2^j}} \sum_{k=-\infty}^{+\infty} \psi\left(\frac{t - 2^j n + k}{2^j}\right).$$

For  $j \leq 0$ , there are  $2^{-j}$  different  $\psi_{j,n}^{\text{pér}}$  indexed by  $0 \leq n < 2^{-j}$ . If the support of  $\psi_{j,n}$  is included in  $[0, 1]$  then  $\psi_{j,n}^{\text{pér}}(t) = \psi_{j,n}(t)$  for  $t \in [0, 1]$ . The restriction to  $[0, 1]$  of this periodization thus modifies only the boundary wavelets whose supports overlap  $t = 0$  or  $t = 1$ . As indicated in Figure 7.16, such wavelets are transformed into boundary wavelets which have two disjoint components near  $t = 0$  and  $t = 1$ . Taken separately, the components near  $t = 0$  and  $t = 1$  of these boundary wavelets have no vanishing moments, and thus create large signal coefficients, as we shall see later. The following theorem proves that periodic wavelets together with periodized scaling functions  $\phi_{j,n}^{\text{pér}}$  generate an orthogonal basis of  $\mathbf{L}^2[0, 1]$ .



Figure 7.16: The restriction to  $[0, 1]$  of a periodic wavelet  $\psi_{j,n}^{\text{pér}}$  has two disjoint components near  $t = 0$  and  $t = 1$ .

**Theorem 7.16.** For any  $J \leq 0$

$$\left[ \{\psi_{j,n}^{\text{pér}}\}_{-\infty < j \leq J, 0 \leq n < 2^{-j}}, \{\phi_{j,n}^{\text{pér}}\}_{0 \leq n < 2^{-j}} \right] \quad (7.175)$$

is an orthogonal basis of  $\mathbf{L}^2[0, 1]$ .

*Proof.* The orthogonality of this family is proved with the following lemma.

**Lemma 7.2.** Let  $\alpha(t), \beta(t) \in \mathbf{L}^2(\mathbb{R})$ . If  $\langle \alpha(t), \beta(t+k) \rangle = 0$  for all  $k \in \mathbb{Z}$  then

$$\int_0^1 \alpha^{\text{pér}}(t) \beta^{\text{pér}}(t) dt = 0. \quad (7.176)$$

To verify (7.176) we insert the definition (7.174) of periodized functions:

$$\begin{aligned} \int_0^1 \alpha^{\text{pér}}(t) \beta^{\text{pér}}(t) dt &= \int_{-\infty}^{+\infty} \alpha(t) \beta^{\text{pér}}(t) dt \\ &= \sum_{k=-\infty}^{+\infty} \int_{-\infty}^{+\infty} \alpha(t) \beta(t+k) dt = 0. \end{aligned}$$

Since  $\{\psi_{j,n}\}_{-\infty < j \leq J, n \in \mathbb{Z}}$ ,  $\{\phi_{J,n}\}_{n \in \mathbb{Z}}$  is orthogonal in  $\mathbf{L}^2(\mathbb{R})$ , we can verify that any two different wavelets or scaling functions  $\alpha^{\text{pér}}$  and  $\beta^{\text{pér}}$  in (7.175) have necessarily a non-periodized version that satisfies  $\langle \alpha(t), \beta(t+k) \rangle = 0$  for all  $k \in \mathbb{Z}$ . Lemma 7.2 thus proves that (7.175) is orthogonal in  $\mathbf{L}^2[0, 1]$ .

To prove that this family generates  $\mathbf{L}^2[0, 1]$ , we extend  $f \in \mathbf{L}^2[0, 1]$  with zeros outside  $[0, 1]$  and decompose it in the wavelet basis of  $\mathbf{L}^2(\mathbb{R})$ :

$$f = \sum_{j=-\infty}^J \sum_{n=-\infty}^{+\infty} \langle f, \psi_{j,n} \rangle \psi_{j,n} + \sum_{n=-\infty}^{+\infty} \langle f, \phi_{J,n} \rangle \phi_{J,n}. \quad (7.177)$$

This zero extension is periodized with the sum (7.174), which defines  $f^{\text{pér}}(t) = f(t)$  for  $t \in [0, 1]$ . Periodizing (7.177) proves that  $f$  can be decomposed over the periodized wavelet family (7.175) in  $\mathbf{L}^2[0, 1]$ . ■

Theorem 7.16 shows that periodizing a wavelet orthogonal basis of  $\mathbf{L}^2(\mathbb{R})$  defines a wavelet orthogonal basis of  $\mathbf{L}^2[0, 1]$ . If  $J = 0$  then there is a single scaling function, and one can verify that  $\phi_{0,0}(t) = 1$ . The resulting scaling coefficient  $\langle f, \phi_{0,0} \rangle$  is the average of  $f$  over  $[0, 1]$ .

Periodic wavelet bases have the disadvantage of creating high amplitude wavelet coefficients in the neighborhood of  $t = 0$  and  $t = 1$ , because the boundary wavelets have separate components with no vanishing moments. If  $f(0) \neq f(1)$ , the wavelet coefficients behave as if the signal were discontinuous at the boundaries. This can also be verified by extending  $f \in \mathbf{L}^2[0, 1]$  into an infinite 1 periodic signal  $f^{\text{pér}}$  and by showing that

$$\int_0^1 f(t) \psi_{j,n}^{\text{pér}}(t) dt = \int_{-\infty}^{+\infty} f^{\text{pér}}(t) \psi_{j,n}(t) dt. \quad (7.178)$$

If  $f(0) \neq f(1)$  then  $f^{\text{pér}}(t)$  is discontinuous at  $t = 0$  and  $t = 1$ , which creates high amplitude wavelet coefficients when  $\psi_{j,n}$  overlaps the interval boundaries.

**Periodic Discrete Transform** For  $f \in \mathbf{L}^2[0, 1]$  let us consider

$$a_j[n] = \langle f, \phi_{j,n}^{\text{pér}} \rangle \quad \text{and} \quad d_j[n] = \langle f, \psi_{j,n}^{\text{pér}} \rangle.$$

We verify as in (7.178) that these inner products are equal to the coefficients of a periodic signal decomposed in a non-periodic wavelet basis:

$$a_j[n] = \langle f^{\text{pér}}, \phi_{j,n} \rangle \quad \text{and} \quad d_j[n] = \langle f^{\text{pér}}, \psi_{j,n} \rangle.$$

The convolution formulas of Theorem 7.10 thus apply if we take into account the periodicity of  $f^{\text{pér}}$ . This means that  $a_j[n]$  and  $d_j[n]$  are considered as discrete signals of period  $2^{-j}$ , and all convolutions in (7.102-7.104) must therefore be replaced by circular convolutions. Despite the poor behavior of periodic wavelets near the boundaries, they are often used because the numerical implementation is particularly simple.

## 7.5.2 Folded Wavelets

Decomposing  $f \in \mathbf{L}^2[0, 1]$  in a periodic wavelet basis was shown in (7.178) to be equivalent to a decomposition of  $f^{\text{pér}}$  in a regular basis of  $\mathbf{L}^2(\mathbb{R})$ . Let us extend  $f$  with zeros outside  $[0, 1]$ . To avoid creating discontinuities with such a periodization, the signal is folded with respect to  $t = 0$ :

$f_0(t) = f(t) + f(-t)$ . The support of  $f_0$  is  $[-1, 1]$  and it is transformed into a 2 periodic signal, as illustrated in Figure 7.17

$$f^{\text{repl}}(t) = \sum_{k=-\infty}^{+\infty} f_0(t-2k) = \sum_{k=-\infty}^{+\infty} f(t-2k) + \sum_{k=-\infty}^{+\infty} f(2k-t). \quad (7.179)$$

Clearly  $f^{\text{repl}}(t) = f(t)$  if  $t \in [0, 1]$ , and it is symmetric with respect to  $t = 0$  and  $t = 1$ . If  $f$  is continuously differentiable then  $f^{\text{repl}}$  is continuous at  $t = 0$  and  $t = 1$ , but its derivative is discontinuous at  $t = 0$  and  $t = 1$  if  $f'(0) \neq 0$  and  $f'(1) \neq 0$ .

Decomposing  $f^{\text{repl}}$  in a wavelet basis  $\{\psi_{j,n}\}_{(j,n) \in \mathbb{Z}^2}$  is equivalent to decomposing  $f$  on a folded wavelet basis. Let  $\psi_{j,n}^{\text{repl}}$  be the folding of  $\psi_{j,n}$  with the summation (7.179). One can verify that

$$\int_0^1 f(t) \psi_{j,n}^{\text{repl}}(t) dt = \int_{-\infty}^{+\infty} f^{\text{repl}}(t) \psi_{j,n}(t) dt. \quad (7.180)$$

Suppose that  $f$  is regular over  $[0, 1]$ . Then  $f^{\text{repl}}$  is continuous at  $t = 0, 1$  and hence produces smaller boundary wavelet coefficients than  $f^{\text{pér}}$ . However, it is not continuously differentiable at  $t = 0, 1$ , which creates bigger wavelet coefficients at the boundary than inside.

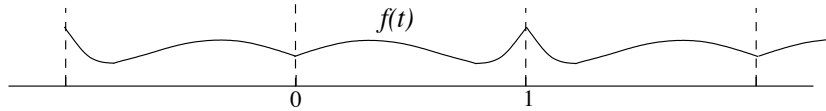


Figure 7.17: The folded signal  $f^{\text{repl}}(t)$  is 2 periodic, symmetric about  $t = 0$  and  $t = 1$ , and equal to  $f(t)$  on  $[0, 1]$ .

To construct a basis of  $\mathbf{L}^2[0, 1]$  with the folded wavelets  $\psi_{j,n}^{\text{repl}}$ , it is sufficient for  $\psi(t)$  to be either symmetric or antisymmetric with respect to  $t = 1/2$ . The Haar wavelet is the only real compactly supported wavelet that is symmetric or antisymmetric and which generates an orthogonal basis of  $\mathbf{L}^2(\mathbb{R})$ . On the other hand, if we loosen up the orthogonality constraint, Section 7.4 proves that there exist biorthogonal bases constructed with compactly supported wavelets that are either symmetric or antisymmetric. Let  $\{\psi_{j,n}\}_{(j,n) \in \mathbb{Z}^2}$  and  $\{\tilde{\psi}_{j,n}\}_{(j,n) \in \mathbb{Z}^2}$  be such biorthogonal wavelet bases. If we fold the wavelets as well as the scaling functions then for  $J \leq 0$

$$\left[ \{\psi_{j,n}^{\text{repl}}\}_{-\infty < j \leq J, 0 \leq n < 2^{-j}}, \{\phi_{J,n}^{\text{repl}}\}_{0 \leq n < 2^{-J}} \right] \quad (7.181)$$

is a Riesz basis of  $\mathbf{L}^2[0, 1]$  [173]. The biorthogonal basis is obtained by folding the dual wavelets  $\tilde{\psi}_{j,n}$  and is given by

$$\left[ \{\tilde{\psi}_{j,n}^{\text{repl}}\}_{-\infty < j \leq J, 0 \leq n < 2^{-j}}, \{\tilde{\phi}_{J,n}^{\text{repl}}\}_{0 \leq n < 2^{-J}} \right]. \quad (7.182)$$

If  $J = 0$  then  $\phi_{0,0}^{\text{repl}} = \tilde{\phi}_{0,0}^{\text{repl}} = 1$ .

Biorthogonal wavelets of compact support are characterized by a pair of finite perfect reconstruction filters  $(h, \tilde{h})$ . The symmetry of these wavelets depends on the symmetry and size of the filters, as explained in Section 7.4.2. A fast folded wavelet transform is implemented with a modified filter bank algorithm, where the treatment of boundaries is slightly more complicated than for periodic wavelets. The symmetric and antisymmetric cases are considered separately.

**Folded Discrete Transform** For  $f \in \mathbf{L}^2[0, 1]$ , we consider

$$a_j[n] = \langle f, \phi_{j,n}^{\text{repl}} \rangle \quad \text{and} \quad d_j[n] = \langle f, \psi_{j,n}^{\text{repl}} \rangle.$$

We verify as in (7.180) that these inner products are equal to the coefficients of a folded signal decomposed in a non-folded wavelet basis:

$$a_j[n] = \langle f^{\text{repl}}, \phi_{j,n} \rangle \quad \text{and} \quad d_j[n] = \langle f^{\text{repl}}, \psi_{j,n} \rangle.$$

The convolution formulas of Theorem 7.10 thus apply if we take into account the symmetry and periodicity of  $f^{\text{repl}}$ . The symmetry properties of  $\phi$  and  $\psi$  imply that  $a_j[n]$  and  $d_j[n]$  also have symmetry and periodicity properties, which must be taken into account in the calculations of (7.102-7.104).

Symmetric biorthogonal wavelets are constructed with perfect reconstruction filters  $h$  and  $\hat{h}$  of odd size that are symmetric about  $n = 0$ . Then  $\phi$  is symmetric about 0, whereas  $\psi$  is symmetric about  $1/2$ . As a result, one can verify that  $a_j[n]$  is  $2^{-j+1}$  periodic and symmetric about  $n = 0$  and  $n = 2^{-j}$ . It is thus characterized by  $2^{-j} + 1$  samples, for  $0 \leq n \leq 2^{-j}$ . The situation is different for  $d_j[n]$  which is  $2^{-j+1}$  periodic but symmetric with respect to  $-1/2$  and  $2^{-j} - 1/2$ . It is characterized by  $2^{-j}$  samples, for  $0 \leq n < 2^{-j}$ .

To initialize this algorithm, the original signal  $a_L[n]$  defined over  $0 \leq n < N - 1$  must be extended by one sample at  $n = N$ , and considered to be symmetric with respect to  $n = 0$  and  $n = N$ . The extension is done by setting  $a_L[N] = a_L[N - 1]$ . For any  $J < L$ , the resulting discrete wavelet representation  $[\{d_j\}_{L < j \leq J}, a_J]$  is characterized by  $N + 1$  coefficients. To avoid adding one more coefficient, one can modify symmetry at the right boundary of  $a_L$  by considering that it is symmetric with respect to  $N - 1/2$  instead of  $N$ . The symmetry of the resulting  $a_j$  and  $d_j$  at the right boundary is modified accordingly by studying the properties of the convolution formula (7.157). As a result, these signals are characterized by  $2^{-j}$  samples and the wavelet representation has  $N$  coefficients. A simpler implementation of this folding technique is given with a lifting in Section 7.8.5. This folding approach is used in most applications because it leads to simpler data structures which keep constant the number of coefficients. However, the discrete coefficients near the right boundary can not be written as inner products of some function  $f(t)$  with dilated boundary wavelets.

Antisymmetric biorthogonal wavelets are obtained with perfect reconstruction filters  $h$  and  $\hat{h}$  of even size that are symmetric about  $n = 1/2$ . In this case  $\phi$  is symmetric about  $1/2$  and  $\psi$  is antisymmetric about  $1/2$ . As a result  $a_j$  and  $d_j$  are  $2^{-j+1}$  periodic and respectively symmetric and antisymmetric about  $-1/2$  and  $2^{-j} - 1/2$ . They are both characterized by  $2^{-j}$  samples, for  $0 \leq n < 2^{-j}$ . The algorithm is initialized by considering that  $a_L[n]$  is symmetric with respect to  $-1/2$  and  $N - 1/2$ . There is no need to add another sample. The resulting discrete wavelet representation  $[\{d_j\}_{L < j \leq J}, a_J]$  is characterized by  $N$  coefficients.

### 7.5.3 Boundary Wavelets

Wavelet coefficients are small in regions where the signal is regular only if the wavelets have enough vanishing moments. The restriction of periodic and folded “boundary” wavelets to the neighborhood of  $t = 0$  and  $t = 1$  have respectively 0 and 1 vanishing moment. These boundary wavelets thus cannot fully take advantage of the signal regularity. They produce large inner products, as if the signal were discontinuous or had a discontinuous derivative. To avoid creating large amplitude wavelet coefficients at the boundaries, one must synthesize boundary wavelets that have as many vanishing moments as the original wavelet  $\psi$ . Initially introduced by Meyer, this approach has been refined by Cohen, Daubechies and Vial [173]. The main results are given without proofs.

**Multiresolution of  $L^2[0, 1]$**  A wavelet basis of  $L^2[0, 1]$  is constructed with a multiresolution approximation  $\{\mathbf{V}_j^{\text{int}}\}_{-\infty < j \leq 0}$ . A wavelet has  $p$  vanishing moments if it is orthogonal to all polynomials of degree  $p - 1$  or smaller. Since wavelets at a scale  $2^j$  are orthogonal to functions in  $\mathbf{V}_j^{\text{int}}$ , to guarantee that they have  $p$  vanishing moments we make sure that polynomials of degree  $p - 1$  are inside  $\mathbf{V}_j^{\text{int}}$ .

We define an approximation space  $\mathbf{V}_j^{\text{int}} \subset L^2[0, 1]$  with a compactly supported Daubechies scaling function  $\phi$ , associated to a wavelet with  $p$  vanishing moments. Theorem 7.7 proves that the support of  $\phi$  has size  $2p - 1$ . We translate  $\phi$  so that its support is  $[-p + 1, p]$ . At a scale  $2^j \leq (2p)^{-1}$ , there are  $2^{-j} - 2p$  scaling functions with a support inside  $[0, 1]$ :

$$\phi_{j,n}^{\text{int}}(t) = \phi_{j,n}(t) = \frac{1}{\sqrt{2^j}} \phi\left(\frac{t - 2^j n}{2^j}\right) \quad \text{for } p \leq n < 2^{-j} - p.$$

To construct an approximation space  $\mathbf{V}_j^{\text{int}}$  of dimension  $2^{-j}$  we add  $p$  scaling functions with a

support on the left boundary near  $t = 0$ :

$$\phi_{j,n}^{\text{int}}(t) = \frac{1}{\sqrt{2^j}} \phi_n^{\text{left}}\left(\frac{t}{2^j}\right) \quad \text{for } 0 \leq n < p ,$$

and  $p$  scaling functions on the right boundary near  $t = 1$ :

$$\phi_{j,n}^{\text{int}}(t) = \frac{1}{\sqrt{2^j}} \phi_{2^{-j}-1-n}^{\text{right}}\left(\frac{t-1}{2^j}\right) \quad \text{for } 2^{-j} - p \leq n < 2^{-j} .$$

The following theorem constructs appropriate boundary scaling functions  $\{\phi_n^{\text{left}}\}_{0 \leq n < p}$  and  $\{\phi_n^{\text{right}}\}_{0 \leq n < p}$ .

**Theorem 7.17** (Cohen, Daubechies, Vial). *One can construct boundary scaling functions  $\phi_n^{\text{left}}$  and  $\phi_n^{\text{right}}$  so that if  $2^{-j} \geq 2p$  then  $\{\phi_{j,n}^{\text{int}}\}_{0 \leq n < 2^{-j}}$  is an orthonormal basis of a space  $\mathbf{V}_j^{\text{int}}$  satisfying*

$$\begin{aligned} \mathbf{V}_j^{\text{int}} &\subset \mathbf{V}_{j-1}^{\text{int}} \\ \lim_{j \rightarrow -\infty} \mathbf{V}_j^{\text{int}} &= \text{Closure} \left( \bigcup_{j=-\infty}^{-\log_2(2p)} \mathbf{V}_j^{\text{int}} \right) = \mathbf{L}^2[0, 1] , \end{aligned}$$

and the restrictions to  $[0, 1]$  of polynomials of degree  $p-1$  are in  $\mathbf{V}_j^{\text{int}}$ .

*Proof.* A sketch of the proof is given. All details can be found in [173]. Since the wavelet  $\psi$  corresponding to  $\phi$  has  $p$  vanishing moments, the Fix-Strang condition (7.70) implies that

$$q_k(t) = \sum_{n=-\infty}^{+\infty} n^k \phi(t-n) \quad (7.183)$$

is a polynomial of degree  $k$ . At any scale  $2^j$ ,  $q_k(2^{-j}t)$  is still a polynomial of degree  $k$ , and for  $0 \leq k < p$  this family defines a basis of polynomials of degree  $p-1$ . To guarantee that polynomials of degree  $p-1$  are in  $\mathbf{V}_j^{\text{int}}$  we impose that the restriction of  $q_k(2^{-j}t)$  to  $[0, 1]$  can be decomposed in the basis of  $\mathbf{V}_j^{\text{int}}$ :

$$\begin{aligned} q_k(2^{-j}t) \mathbf{1}_{[0,1]}(t) &= \sum_{n=0}^{p-1} a[n] \phi_n^{\text{left}}(2^{-j}t) + \sum_{n=p}^{2^{-j}-p-1} n^k \phi(2^{-j}t-n) + \\ &\quad \sum_{n=0}^{p-1} b[n] \phi_n^{\text{right}}(2^{-j}t-2^{-j}) . \end{aligned} \quad (7.184)$$

Since the support of  $\phi$  is  $[-p+1, p]$ , the condition (7.184) together with (7.183) can be separated into two non-overlapping left and right conditions. With a change of variable, we verify that (7.184) is equivalent to

$$\sum_{n=-p+1}^p n^k \phi(t-n) \mathbf{1}_{[0,+\infty)}(t) = \sum_{n=0}^{p-1} a[n] \phi_n^{\text{left}}(t), \quad (7.185)$$

and

$$\sum_{n=-p}^{p-1} n^k \phi(t-n) \mathbf{1}_{(-\infty,0]}(t) = \sum_{n=0}^{p-1} b[n] \phi_n^{\text{right}}(t). \quad (7.186)$$

The embedding property  $\mathbf{V}_j^{\text{int}} \subset \mathbf{V}_{j-1}^{\text{int}}$  is obtained by imposing that the boundary scaling functions satisfy scaling equations. We suppose that  $\phi_n^{\text{left}}$  has a support  $[0, p+n]$  and satisfies a scaling equation of the form

$$2^{-1/2} \phi_n^{\text{left}}(2^{-1}t) = \sum_{l=0}^{p-1} H_{n,l}^{\text{left}} \phi_l^{\text{left}}(t) + \sum_{m=p}^{p+2n} h_{n,m}^{\text{left}} \phi(t-m) , \quad (7.187)$$

whereas  $\phi_n^{\text{right}}$  has a support  $[-p-n, 0]$  and satisfies a similar scaling equation on the right. The constants  $H_{n,l}^{\text{left}}$ ,  $h_{n,m}^{\text{left}}$ ,  $H_{n,l}^{\text{right}}$  and  $h_{n,m}^{\text{right}}$  are adjusted to verify the polynomial reproduction equations (7.185) and (7.186), while producing orthogonal scaling functions. The resulting family  $\{\phi_{j,n}^{\text{int}}\}_{0 \leq n < 2^{-j}}$  is an orthonormal basis of a space  $\mathbf{V}_j^{\text{int}}$ .

The convergence of the spaces  $\mathbf{V}_j^{\text{int}}$  to  $\mathbf{L}^2[0, 1]$  when  $2^j$  goes to 0 is a consequence of the fact that the multiresolution spaces  $\mathbf{V}_j$  generated by the Daubechies scaling function  $\{\phi_{j,n}\}_{n \in \mathbb{Z}}$  converge to  $\mathbf{L}^2(\mathbb{R})$ . ■

The proof constructs the scaling functions through scaling equations specified by discrete filters. At the boundaries, the filter coefficients are adjusted to construct orthogonal scaling functions with a support in  $[0, 1]$ , and to guarantee that polynomials of degree  $p - 1$  are reproduced by these scaling functions. Table 7.5 gives the filter coefficients for  $p = 2$ .

**Wavelet Basis of  $L^2[0, 1]$**  Let  $\mathbf{W}_j^{\text{int}}$  be the orthogonal complement of  $\mathbf{V}_j^{\text{int}}$  in  $\mathbf{V}_{j-1}^{\text{int}}$ . The support of the Daubechies wavelet  $\psi$  with  $p$  vanishing moments is  $[-p + 1, p]$ . Since  $\psi_{j,n}$  is orthogonal to any  $\phi_{j,l}$ , we verify that an orthogonal basis of  $\mathbf{W}_j^{\text{int}}$  can be constructed with the  $2^{-j} - 2p$  inside wavelets with support in  $[0, 1]$ :

$$\psi_{j,n}^{\text{int}}(t) = \psi_{j,n}(t) = \frac{1}{\sqrt{2^j}} \psi\left(\frac{t - 2^j n}{2^j}\right) \quad \text{for } p \leq n < 2^{-j} - p,$$

to which are added  $2p$  left and right boundary wavelets

$$\psi_{j,n}^{\text{int}}(t) = \frac{1}{\sqrt{2^j}} \psi_n^{\text{left}}\left(\frac{t}{2^j}\right) \quad \text{for } 0 \leq n < p,$$

$$\psi_{j,n}^{\text{int}}(t) = \frac{1}{\sqrt{2^j}} \psi_{2^{-j}-1-n}^{\text{right}}\left(\frac{t-1}{2^j}\right) \quad \text{for } 2^{-j} - p \leq n < 2^{-j}.$$

Since  $\mathbf{W}_j^{\text{int}} \subset \mathbf{V}_{j-1}^{\text{int}}$ , the left and right boundary wavelets at any scale  $2^j$  can be expanded into scaling functions at the scale  $2^{j-1}$ . For  $j = 1$  we impose that the left boundary wavelets satisfy equations of the form

$$\frac{1}{\sqrt{2}} \psi_n^{\text{left}}\left(\frac{t}{2}\right) = \sum_{l=0}^{p-1} G_{n,l}^{\text{left}} \phi_l^{\text{left}}(t) + \sum_{m=p}^{p+2n} g_{n,m}^{\text{left}} \phi(t-m). \quad (7.188)$$

The right boundary wavelets satisfy similar equations. The coefficients  $G_{n,l}^{\text{left}}$ ,  $g_{n,m}^{\text{left}}$ ,  $G_{n,l}^{\text{right}}$ ,  $g_{n,m}^{\text{right}}$  are computed so that  $\{\psi_{j,n}^{\text{int}}\}_{0 \leq n < 2^{-j}}$  is an orthonormal basis of  $\mathbf{W}_j^{\text{int}}$ . Table 7.5 gives the values of these coefficients for  $p = 2$ .

For any  $2^J \leq (2p)^{-1}$  the multiresolution properties prove that

$$L^2[0, 1] = \mathbf{V}_J^{\text{int}} \oplus_{j=-\infty}^J \mathbf{W}_j^{\text{int}},$$

which implies that

$$[\{\phi_{J,n}^{\text{int}}\}_{0 \leq n < 2^{-J}}, \{\psi_{j,n}^{\text{int}}\}_{-\infty < j \leq J, 0 \leq n < 2^{-j}}] \quad (7.189)$$

is an orthonormal wavelet basis of  $L^2[0, 1]$ . The boundary wavelets, like the inside wavelets, have  $p$  vanishing moments because polynomials of degree  $p - 1$  are included in the space  $\mathbf{V}_J^{\text{int}}$ . Figure 7.18 displays the  $2p = 4$  boundary scaling functions and wavelets.

$k$	$l$	$H_{k,l}^{\text{left}}$	$G_{k,l}^{\text{left}}$	$k$	$m$	$h_{k,m}^{\text{left}}$	$g_{k,m}^{\text{left}}$
0	0	0.6033325119	-0.7965436169	0	2	-0.398312997	-0.2587922483
0	1	0.690895531	0.5463927140	1	2	0.8500881025	0.227428117
1	0	0.03751746045	0.01003722456	1	3	0.2238203570	-0.8366028212
1	1	0.4573276599	0.1223510431	1	4	-0.1292227434	0.4830129218
$k$	$l$	$H_{k,l}^{\text{right}}$	$G_{k,l}^{\text{right}}$	$k$	$m$	$h_{k,m}^{\text{right}}$	$g_{k,m}^{\text{right}}$
-2	-2	0.1901514184	-0.3639069596	-2	-5	0.4431490496	0.235575950
-2	-1	-0.1942334074	0.3717189665	-2	-4	0.7675566693	0.4010695194
-1	-2	0.434896998	0.8014229620	-2	-3	0.3749553316	-0.7175799994
-2	-1	0.8705087534	-0.2575129195	-1	-3	0.2303890438	-0.5398225007
		$h[-1]$	$h[0]$			$h[1]$	$h[2]$
		0.482962913145	0.836516303738			0.224143868042	-0.129409522551

Table 7.5: Left and right border coefficients for a Daubechies wavelet with  $p = 2$  vanishing moments. The inside filter coefficients are at the bottom of the table. A table of coefficients for  $p \geq 2$  vanishing moments can be retrieved over the Internet at the FTP site <ftp://math.princeton.edu/pub/user/ingrid/interval-tables>.

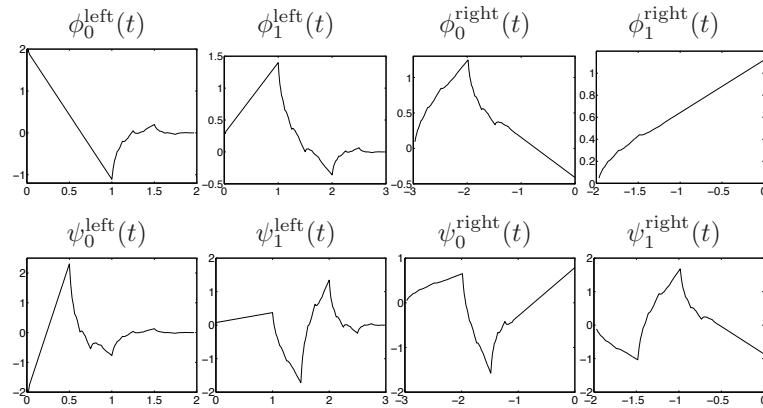


Figure 7.18: Boundary scaling functions and wavelets with  $p = 2$  vanishing moments.

**Fast Discrete Algorithm** For any  $f \in \mathbf{L}^2[0, 1]$  we denote

$$a_j[n] = \langle f, \phi_{j,n}^{\text{int}} \rangle \quad \text{and} \quad d_j[n] = \langle f, \psi_{j,n}^{\text{int}} \rangle \quad \text{for } 0 \leq n \leq 2^{-j}.$$

Wavelet coefficients are computed with a cascade of convolutions identical to Theorem 7.10 as long as filters do not overlap the signal boundaries. A Daubechies filter  $h$  is considered here to have a support located at  $[-p + 1, p]$ . At the boundary, the usual Daubechies filters are replaced by the boundary filters that relate the boundary wavelets and scaling functions to the finer-scale scaling functions in (7.187) and (7.188).

**Theorem 7.18** (Cohen, Daubechies, Vial).

If  $0 \leq k < p$

$$\begin{aligned} a_j[k] &= \sum_{l=0}^{p-1} H_{k,l}^{\text{left}} a_{j-1}[l] + \sum_{m=p}^{p+2k} h_{k,m}^{\text{left}} a_{j-1}[m], \\ d_j[k] &= \sum_{l=0}^{p-1} G_{k,l}^{\text{left}} a_{j-1}[l] + \sum_{m=p}^{p+2k} g_{k,m}^{\text{left}} a_{j-1}[m]. \end{aligned}$$

If  $p \leq k < 2^{-j} - p$

$$\begin{aligned} a_j[k] &= \sum_{l=-\infty}^{+\infty} h[l-2k] a_{j-1}[l], \\ d_j[k] &= \sum_{l=-\infty}^{+\infty} g[l-2k] a_{j-1}[l]. \end{aligned}$$

If  $-p \leq k < 0$

$$\begin{aligned} a_j[2^{-j} + k] &= \sum_{l=-p}^{-1} H_{k,l}^{\text{right}} a_{j-1}[2^{-j+1} + l] + \sum_{m=-p+2k+1}^{-p-1} h_{k,m}^{\text{right}} a_{j-1}[2^{-j+1} + m], \\ d_j[2^{-j} + k] &= \sum_{l=-p}^{-1} G_{k,l}^{\text{right}} a_{j-1}[2^{-j+1} + l] + \sum_{m=-p+2k+1}^{-p-1} g_{k,m}^{\text{right}} a_{j-1}[2^{-j+1} + m]. \end{aligned}$$

This cascade algorithm decomposes  $a_L$  into a discrete wavelet transform  $[a_J, \{d_j\}_{L < j \leq J}]$  with  $O(N)$  operations. The maximum scale must satisfy  $2^J \leq (2p)^{-1}$ , because the number of boundary coefficients remains equal to  $2p$  at all scales. The implementation is more complicated than the folding and periodic algorithms described in Sections 7.5.1 and 7.5.2, but does not require more computations. The signal  $a_L$  is reconstructed from its wavelet coefficients, by inverting the decomposition formula in Theorem 7.18.

**Theorem 7.19** (Cohen, Daubechies, Vial).

If  $0 \leq l \leq p-1$

$$a_{j-1}[l] = \sum_{k=0}^{p-1} H_{k,l}^{\text{left}} a_j[k] + \sum_{k=0}^{p-1} G_{k,l}^{\text{left}} d_j[k].$$

If  $p \leq l \leq 3p-2$

$$\begin{aligned} a_{j-1}[l] &= \sum_{k=(l-p)/2}^{p-1} h_{k,l}^{\text{left}} a_j[k] + \sum_{k=-\infty}^{+\infty} h[l-2k] a_j[k] + \\ &\quad \sum_{k=(l-p)/2}^{p-1} g_{k,l}^{\text{left}} d_j[k] + \sum_{k=-\infty}^{+\infty} g[l-2k] d_j[k]. \end{aligned}$$

If  $3p-1 \leq l \leq 2^{-j+1} - 3p$

$$a_{j-1}[l] = \sum_{k=-\infty}^{+\infty} h[l-2k] a_j[k] + \sum_{k=-\infty}^{+\infty} g[l-2k] d_j[k].$$

If  $-p-1 \geq l \geq -3p+1$

$$\begin{aligned} a_{j-1}[2^{-j+1} + l] &= \sum_{k=-p}^{(l+p-1)/2} h_{k,l}^{\text{right}} a_j[2^{-j} + k] + \sum_{k=-\infty}^{+\infty} h[l-2k] a_j[2^{-j} + k] + \\ &\quad \sum_{k=-p}^{(l+p-1)/2} g_{k,l}^{\text{right}} d_j[2^{-j} + k] + \sum_{k=-\infty}^{+\infty} g[l-2k] d_j[2^{-j} + k]. \end{aligned}$$

If  $-1 \geq l \geq -p$

$$a_{j-1}[2^{-j+1} + l] = \sum_{k=-p}^{-1} H_{k,l}^{\text{right}} a_j[2^{-j} + k] + \sum_{k=-p}^{-1} G_{k,l}^{\text{right}} d_j[2^{-j} + k].$$

The original signal  $a_L$  is reconstructed from the orthogonal wavelet representation  $[a_J, \{d_j\}_{L < j \leq J}]$  by iterating these equations for  $L < j \leq J$ . This reconstruction is performed with  $O(N)$  operations.



## 7.6 Multiscale Interpolations

Multiresolution approximations are closely connected to the generalized interpolations and sampling theorems studied in Section 3.1.3. The next section constructs general classes of interpolation functions from orthogonal scaling functions and derives new sampling theorems. Interpolation bases have the advantage of easily computing the decomposition coefficients from the sample values of the signal. Section 7.6.2 constructs interpolation wavelet bases.

### 7.6.1 Interpolation and Sampling Theorems

Section 3.1.3 explains that a sampling scheme approximates a signal by its orthogonal projection onto a space  $\mathbf{U}_s$  and samples this projection at intervals  $s$ . The space  $\mathbf{U}_s$  is constructed so that any function in  $\mathbf{U}_s$  can be recovered by interpolating a uniform sampling at intervals  $s$ . We relate the construction of interpolation functions to orthogonal scaling functions and compute the orthogonal projector on  $\mathbf{U}_s$ .

We call *interpolation function* any  $\phi$  such that  $\{\phi(t-n)\}_{n \in \mathbb{Z}}$  is a Riesz basis of the space  $\mathbf{U}_1$  it generates, and which satisfies

$$\phi(n) = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n \neq 0 \end{cases}. \quad (7.190)$$

Any  $f \in \mathbf{U}_1$  is recovered by interpolating its samples  $f(n)$ :

$$f(t) = \sum_{n=-\infty}^{+\infty} f(n) \phi(t-n). \quad (7.191)$$

Indeed, we know that  $f$  is a linear combination of the basis vector  $\{\phi(t-n)\}_{n \in \mathbb{Z}}$  and the interpolation property (7.190) yields (7.191). The Whittaker sampling Theorem 3.2 is based on the interpolation function

$$\phi(t) = \frac{\sin \pi t}{\pi t}.$$

In this case, the space  $\mathbf{U}_1$  is the set of functions whose Fourier transforms are included in  $[-\pi, \pi]$ .

Scaling an interpolation function yields a new interpolation for a different sampling interval. Let us define  $\phi_s(t) = \phi(t/s)$  and

$$\mathbf{U}_s = \{f \in \mathbf{L}^2(\mathbb{R}) \text{ with } f(st) \in \mathbf{U}_1\}.$$

One can verify that any  $f \in \mathbf{U}_s$  can be written

$$f(t) = \sum_{n=-\infty}^{+\infty} f(ns) \phi_s(t - ns). \quad (7.192)$$

**Scaling Autocorrelation** We denote by  $\phi_o$  an orthogonal scaling function, defined by the fact that  $\{\phi_o(t-n)\}_{n \in \mathbb{Z}}$  is an orthonormal basis of a space  $\mathbf{V}_0$  of a multiresolution approximation. Theorem 7.2 proves that this scaling function is characterized by a conjugate mirror filter  $h_o$ . The following theorem defines an interpolation function from the autocorrelation of  $\phi_o$  [422].

**Theorem 7.20.** *Let  $\bar{\phi}_o(t) = \phi_o(-t)$  and  $\bar{h}_o[n] = h_o[-n]$ . If  $|\hat{\phi}_o(\omega)| = O((1 + |\omega|)^{-1})$  then*

$$\phi(t) = \int_{-\infty}^{+\infty} \phi_o(u) \phi_o(u-t) du = \phi_o \star \bar{\phi}_o(t) \quad (7.193)$$

*is an interpolation function. Moreover*

$$\phi\left(\frac{t}{2}\right) = \sum_{n=-\infty}^{+\infty} h[n] \phi(t-n) \quad (7.194)$$

*with*

$$h[n] = \sum_{m=-\infty}^{+\infty} h_o[m] h_o[m-n] = h_o \star \bar{h}_o[n]. \quad (7.195)$$

*Proof.* Observe first that

$$\phi(n) = \langle \phi_o(t), \phi_o(t-n) \rangle = \delta[n],$$

which prove the interpolation property (7.190). To prove that  $\{\phi(t-n)\}_{n \in \mathbb{Z}}$  is a Riesz basis of the space  $\mathbf{U}_1$  it generates, we verify the condition (7.9). The autocorrelation  $\phi(t) = \phi_o \star \bar{\phi}_o(t)$  has a Fourier transform  $\hat{\phi}(\omega) = |\hat{\phi}_o(\omega)|^2$ . Condition (7.9) thus means that there exist  $B \geq A > 0$  such that

$$\forall \omega \in [-\pi, \pi] \quad , \quad A \leq \sum_{k=-\infty}^{+\infty} |\hat{\phi}_o(\omega - 2k\pi)|^4 \leq B. \quad (7.196)$$

We proved in (7.14) that the orthogonality of a family  $\{\phi_o(t-n)\}_{n \in \mathbb{Z}}$  is equivalent to

$$\forall \omega \in [-\pi, \pi] \quad , \quad \sum_{k=-\infty}^{+\infty} |\hat{\phi}_o(\omega + 2k\pi)|^2 = 1. \quad (7.197)$$

The right inequality of (7.196) is therefore valid for  $A = 1$ . Let us prove the left inequality. Since  $|\hat{\phi}_o(\omega)| = O((1+|\omega|)^{-1})$ , one can verify that there exists  $K > 0$  such that for all  $\omega \in [-\pi, \pi]$ ,  $\sum_{|k| > K} |\hat{\phi}_o(\omega + 2k\pi)|^2 < 1/2$ , so (7.197) implies that  $\sum_{k=-K}^K |\hat{\phi}_o(\omega + 2k\pi)|^2 \geq 1/2$ . It follows that

$$\sum_{k=-K}^K |\hat{\phi}_o(\omega + 2k\pi)|^4 \geq \frac{1}{4(2K+1)},$$

which proves (7.196) for  $A^{-1} = 4(2K+1)$ .

Since  $\phi_o$  is a scaling function, (7.23) proves that there exists a conjugate mirror filter  $h_o$  such that

$$\frac{1}{\sqrt{2}} \phi_o\left(\frac{t}{2}\right) = \sum_{n=-\infty}^{+\infty} h_o[n] \phi_o(t-n).$$

Computing  $\phi(t) = \phi_o \star \bar{\phi}_o(t)$  yields (7.194) with  $h[n] = h_o \star \bar{h}_o[n]$ . ■

Theorem 7.20 proves that the autocorrelation of an orthogonal scaling function  $\phi_o$  is an interpolation function  $\phi$  that also satisfies a scaling equation. One can design  $\phi$  to approximate regular signals efficiently by their orthogonal projection in  $\mathbf{U}_s$ . Definition 6.1 measures the regularity of  $f$  with a Lipschitz exponent, which depends on the difference between  $f$  and its Taylor polynomial expansion. The following theorem gives a condition for recovering polynomials by interpolating their samples with  $\phi$ . It derives an upper bound for the error when approximating  $f$  by its orthogonal projection in  $\mathbf{U}_s$ .

**Theorem 7.21** (Fix, Strang). *Any polynomial  $q(t)$  of degree smaller or equal to  $p-1$  is decomposed into*

$$q(t) = \sum_{n=-\infty}^{+\infty} q(n) \phi(t-n) \quad (7.198)$$

*if and only if  $\hat{h}(\omega)$  has a zero of order  $p$  at  $\omega = \pi$ .*

*Suppose that this property is satisfied. If  $f$  has a compact support and is uniformly Lipschitz  $\alpha \leq p$  then there exists  $C > 0$  such that*

$$\forall s > 0 \quad , \quad \|f - P_{\mathbf{U}_s} f\| \leq C s^\alpha. \quad (7.199)$$

*Proof.* The main steps of the proof are given, without technical detail. Let us set  $s = 2^j$ . One can verify that the spaces  $\{\mathbf{V}_j = \mathbf{U}_{2^j}\}_{j \in \mathbb{Z}}$  define a multiresolution approximation of  $\mathbf{L}^2(\mathbb{R})$ . The Riesz basis of  $\mathbf{V}_0$  required by Definition 7.1 is obtained with  $\theta = \phi$ . This basis is orthogonalized by Theorem 7.1 to obtain an orthogonal basis of scaling functions. Theorem 7.3 derives a wavelet orthonormal basis  $\{\psi_{j,n}\}_{(j,n) \in \mathbb{Z}^2}$  of  $\mathbf{L}^2(\mathbb{R})$ .

Using Theorem 7.4, one can verify that  $\psi$  has  $p$  vanishing moments if and only if  $\hat{h}(\omega)$  has  $p$  zeros at  $\pi$ . Although  $\phi$  is not the orthogonal scaling function, the Fix-Strang condition (7.70) remains valid. It is thus also equivalent that for  $k < p$

$$q_k(t) = \sum_{n=-\infty}^{+\infty} n^k \phi(t-n)$$

is a polynomial of degree  $k$ . The interpolation property (7.191) implies that  $q_k(n) = n^k$  for all  $n \in \mathbb{Z}$  so  $q_k(t) = t^k$ . Since  $\{t^k\}_{0 \leq k < p}$  is a basis for polynomials of degree  $p-1$ , any polynomial  $q(t)$  of degree  $p-1$  can be decomposed over  $\{\phi(t-n)\}_{n \in \mathbb{Z}}$  if and only if  $\hat{h}(\omega)$  has  $p$  zeros at  $\pi$ .

We indicate how to prove (7.199) for  $s = 2^j$ . The truncated family of wavelets  $\{\psi_{l,n}\}_{l \leq j, n \in \mathbb{Z}}$  is an orthogonal basis of the orthogonal complement of  $\mathbf{U}_{2^j} = \mathbf{V}_j$  in  $\mathbf{L}^2(\mathbb{R})$ . Hence

$$\|f - P_{\mathbf{U}_{2^j}} f\|^2 = \sum_{l=-\infty}^j \sum_{n=-\infty}^{+\infty} |\langle f, \psi_{l,n} \rangle|^2.$$

If  $f$  is uniformly Lipschitz  $\alpha$ , since  $\psi$  has  $p$  vanishing moments, Theorem 6.3 proves that there exists  $A > 0$  such that

$$|Wf(2^l n, 2^l)| = |\langle f, \psi_{l,n} \rangle| \leq A 2^{(\alpha+1/2)l}.$$

To simplify the argument we suppose that  $\psi$  has a compact support, although this is not required. Since  $f$  also has a compact support, one can verify that the number of non-zero  $\langle f, \psi_{l,n} \rangle$  is bounded by  $K 2^{-l}$  for some  $K > 0$ . Hence

$$\|f - P_{\mathbf{U}_{2^j}} f\|^2 \leq \sum_{l=-\infty}^j K 2^{-l} A^2 2^{(2\alpha+1)l} \leq \frac{K A^2}{1 - 2^{-\alpha}} 2^{2\alpha j},$$

which proves (7.199) for  $s = 2^j$ . ■

As long as  $\alpha \leq p$ , the larger the Lipschitz exponent  $\alpha$  the faster the error  $\|f - P_{\mathbf{U}_s} f\|$  decays to zero when the sampling interval  $s$  decreases. If a signal  $f$  is  $\mathbf{C}^k$  with a compact support then it is uniformly Lipschitz  $k$ , so Theorem 7.21 proves that  $\|f - P_{\mathbf{U}_s} f\| \leq C s^k$ .

**Example 7.11.** A cubic spline interpolation function is obtained from the linear spline scaling function  $\phi_o$ . The Fourier transform expression (7.5) yields

$$\hat{\phi}(\omega) = |\hat{\phi}_o(\omega)|^2 = \frac{48 \sin^4(\omega/2)}{\omega^4 (1 + 2 \cos^2(\omega/2))}. \quad (7.200)$$

Figure 7.19(a) gives the graph of  $\phi$ , which has an infinite support but exponential decay. With Theorem 7.21 one can verify that this interpolation function recovers polynomials of degree 3 from a uniform sampling. The performance of spline interpolation functions for generalized sampling theorems is studied in [161, 468].

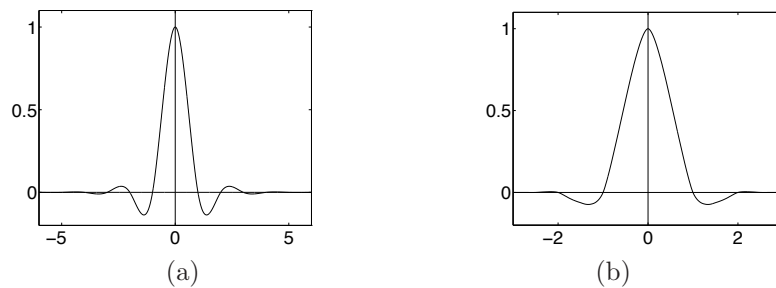


Figure 7.19: (a): Cubic spline interpolation function. (b): Deslaurier-Dubuc interpolation function of degree 3.

**Example 7.12.** Deslaurier-Dubuc [205] interpolation functions of degree  $2p-1$  are compactly supported interpolation functions of minimal size that decompose polynomials of degree  $2p-1$ . One can verify that such an interpolation function is the autocorrelation of a scaling function  $\phi_o$ . To reproduce polynomials of degree  $2p-1$ , Theorem 7.21 proves that  $\hat{h}(\omega)$  must have a zero of order  $2p$  at  $\pi$ . Since  $h[n] = h_o \star \bar{h}_o[n]$  it follows that  $\hat{h}(\omega) = |\hat{h}_o(\omega)|^2$ , and hence  $\hat{h}_o(\omega)$  has a zero of order  $p$  at  $\pi$ . Daubechies's Theorem 7.7 designs minimum size conjugate mirror filters  $h_o$  which satisfy this condition. Daubechies filters  $h_o$  have  $2p$  non-zero coefficients and the resulting scaling function  $\phi_o$

has a support of size  $2p - 1$ . The autocorrelation  $\phi$  is the Deslaurier-Dubuc interpolation function, whose support is  $[-2p + 1, 2p - 1]$ .

For  $p = 1$ ,  $\phi_o = \mathbf{1}_{[0,1]}$  and  $\phi$  is the piecewise linear tent function whose support is  $[-1, 1]$ . For  $p = 2$ , the Deslaurier-Dubuc interpolation function  $\phi$  is the autocorrelation of the Daubechies 2 scaling function, shown in Figure 7.10. The graph of this interpolation function is in Figure 7.19(b). Polynomials of degree  $2p - 1 = 3$  are interpolated by this function.

The scaling equation (7.194) implies that any autocorrelation filter verifies  $h[2n] = 0$  for  $n \neq 0$ . For any  $p \geq 0$ , the non-zero values of the resulting filter are calculated from the coefficients of the polynomial (7.168) that is factored to synthesize Daubechies filters. The support of  $h$  is  $[-2p + 1, 2p - 1]$  and

$$h[2n + 1] = (-1)^{p-n} \frac{\prod_{k=0}^{2p-1} (k - p + 1/2)}{(n + 1/2)(p - n - 1)!(p + n)!} \quad \text{for } -p \leq n < p. \quad (7.201)$$

**Dual Basis** If  $f \notin \mathbf{U}_s$  then it is approximated by its orthogonal projection  $P_{\mathbf{U}_s} f$  on  $\mathbf{U}_s$  before the samples at intervals  $s$  are recorded. This orthogonal projection is computed with a biorthogonal basis  $\{\tilde{\phi}_s(t - ns)\}_{n \in \mathbb{Z}}$  [80]. Theorem 3.4 proves that  $\tilde{\phi}_s(t) = s^{-1} \tilde{\phi}(s^{-1}t)$  where the Fourier transform of  $\tilde{\phi}$  is

$$\hat{\tilde{\phi}}(\omega) = \frac{\hat{\phi}^*(\omega)}{\sum_{k=-\infty}^{+\infty} |\hat{\phi}(\omega + 2k\pi)|^2}. \quad (7.202)$$

Figure 7.20 gives the graph of the cubic spline  $\tilde{\phi}$  associated to the cubic spline interpolation function. The orthogonal projection of  $f$  over  $\mathbf{U}_s$  is computed by decomposing  $f$  in the biorthogonal bases

$$P_{\mathbf{U}_s} f(t) = \sum_{n=-\infty}^{+\infty} \langle f(u), \tilde{\phi}_s(u - ns) \rangle \phi_s(t - ns). \quad (7.203)$$

Let  $\bar{\tilde{\phi}}_s(t) = \tilde{\phi}_s(-t)$ . The interpolation property (7.190) implies that

$$P_{\mathbf{U}_s} f(ns) = \langle f(u), \tilde{\phi}_s(u - ns) \rangle = f \star \bar{\tilde{\phi}}_s(ns). \quad (7.204)$$

This discretization of  $f$  through a projection onto  $\mathbf{U}_s$  is therefore obtained by a filtering with  $\bar{\tilde{\phi}}_s$  followed by a uniform sampling at intervals  $s$ . The best linear approximation of  $f$  is recovered with the interpolation formula (7.203).

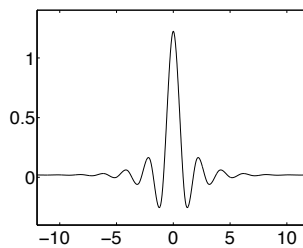


Figure 7.20: The dual cubic spline  $\tilde{\phi}(t)$  associated to the spline interpolation function  $\phi(t)$  shown in Figure 7.19(a).

## 7.6.2 Interpolation Wavelet Basis

An interpolation function  $\phi$  can recover a signal  $f$  from a uniform sampling  $\{f(ns)\}_{n \in \mathbb{Z}}$  if  $f$  belongs to an appropriate subspace  $\mathbf{U}_s$  of  $\mathbf{L}^2(\mathbb{R})$ . Donoho [212] has extended this approach by constructing interpolation wavelet bases of the whole space of uniformly continuous signals, with the sup norm. The decomposition coefficients are calculated from sample values instead of inner product integrals.

**Subdivision Scheme** Let  $\phi$  be an interpolation function, which is the autocorrelation of an orthogonal scaling function  $\phi_o$ . Let  $\phi_{j,n}(t) = \phi(2^{-j}t - n)$ . The constant  $2^{-j/2}$  that normalizes the energy of  $\phi_{j,n}$  is not added because we shall use a sup norm  $\|f\|_\infty = \sup_{t \in \mathbb{R}} |f(t)|$  instead of the  $\mathbf{L}^2(\mathbb{R})$  norm, and

$$\|\phi_{j,n}\|_\infty = \|\phi\|_\infty = |\phi(0)| = 1.$$

We define the interpolation space  $\mathbf{V}_j$  of functions

$$g = \sum_{n=-\infty}^{+\infty} a[n] \phi_{j,n},$$

where  $a[n]$  has at most a polynomial growth in  $n$ . Since  $\phi$  is an interpolation function,  $a[n] = g(2^j n)$ . This space  $\mathbf{V}_j$  is not included in  $\mathbf{L}^2(\mathbb{R})$  since  $a[n]$  may not have a finite energy. The scaling equation (7.194) implies that  $\mathbf{V}_{j+1} \subset \mathbf{V}_j$  for any  $j \in \mathbb{Z}$ . If the autocorrelation filter  $h$  has a Fourier transform  $\hat{h}(\omega)$  which has a zero of order  $p$  at  $\omega = \pi$ , then Theorem 7.21 proves that polynomials of degree smaller than  $p - 1$  are included in  $\mathbf{V}_j$ .

For  $f \notin \mathbf{V}_j$ , we define a simple projector on  $\mathbf{V}_j$  that interpolates the dyadic samples  $f(2^j n)$ :

$$P_{\mathbf{V}_j} f(t) = \sum_{n=-\infty}^{+\infty} f(2^j n) \phi_j(t - 2^j n). \quad (7.205)$$

This projector has no orthogonality property but satisfies  $P_{\mathbf{V}_j} f(2^j n) = f(2^j n)$ . Let  $\mathbf{C}_0$  be the space of functions that are uniformly continuous over  $\mathbb{R}$ . The following theorem proves that any  $f \in \mathbf{C}_0$  can be approximated with an arbitrary precision by  $P_{\mathbf{V}_j} f$  when  $2^j$  goes to zero.

**Theorem 7.22** (Donoho). *Suppose that  $\phi$  has an exponential decay. If  $f \in \mathbf{C}_0$  then*

$$\lim_{j \rightarrow -\infty} \|f - P_{\mathbf{V}_j} f\|_\infty = \lim_{j \rightarrow -\infty} \sup_{t \in \mathbb{R}} |f(t) - P_{\mathbf{V}_j} f(t)| = 0. \quad (7.206)$$

*Proof.* Let  $\omega(\delta, f)$  denote the modulus of continuity

$$\omega(\delta, f) = \sup_{|h| \leq \delta} \sup_{t \in \mathbb{R}} |f(t+h) - f(t)|. \quad (7.207)$$

By definition,  $f \in \mathbf{C}_0$  if  $\lim_{\delta \rightarrow 0} \omega(\delta, f) = 0$ .

Any  $t \in \mathbb{R}$  can be written  $t = 2^j(n+h)$  with  $n \in \mathbb{Z}$  and  $|h| \leq 1$ . Since  $P_{\mathbf{V}_j} f(2^j n) = f(2^j n)$ ,

$$\begin{aligned} |f(2^j(n+h)) - P_{\mathbf{V}_j} f(2^j(n+h))| &\leq |f(2^j(n+h)) - f(2^j n)| \\ &\quad + |P_{\mathbf{V}_j} f(2^j(n+h)) - P_{\mathbf{V}_j} f(2^j n)| \\ &\leq \omega(2^j, f) + \omega(2^j, P_{\mathbf{V}_j} f). \end{aligned}$$

The next lemma proves that  $\omega(2^j, P_{\mathbf{V}_j} f) \leq C_\phi \omega(2^j, f)$  where  $C_\phi$  is a constant independent of  $j$  and  $f$ . Taking a sup over  $t = 2^j(n+h)$  implies the final result:

$$\sup_{t \in \mathbb{R}} |f(t) - P_{\mathbf{V}_j} f(t)| \leq (1 + C_\phi) \omega(2^j, f) \rightarrow 0 \quad \text{when } j \rightarrow -\infty.$$

**Lemma 7.3.** *There exists  $C_\phi > 0$  such that for all  $j \in \mathbb{Z}$  and  $f \in \mathbf{C}_0$*

$$\omega(2^j, P_{\mathbf{V}_j} f) \leq C_\phi \omega(2^j, f). \quad (7.208)$$

Let us set  $j = 0$ . For  $|h| \leq 1$ , a summation by parts gives

$$P_{\mathbf{V}_0} f(t+h) - P_{\mathbf{V}_0} f(t) = \sum_{n=-\infty}^{+\infty} (f(n+1) - f(n)) \theta_h(t-n)$$

where

$$\theta_h(t) = \sum_{k=1}^{+\infty} (\phi(t+h-k) - \phi(t-k)).$$

Hence

$$|P_{\mathbf{V}_0}f(t+h) - P_{\mathbf{V}_0}f(t)| \leq \sup_{n \in \mathbb{Z}} |f(n+1) - f(n)| \sum_{n=-\infty}^{+\infty} |\theta_h(t-n)|. \quad (7.209)$$

Since  $\phi$  has an exponential decay, there exists a constant  $C_\phi$  such that if  $|h| \leq 1$  and  $t \in \mathbb{R}$  then  $\sum_{n=-\infty}^{+\infty} |\theta_h(t-n)| \leq C_\phi$ . Taking a sup over  $t$  in (7.209) proves that

$$\omega(1, P_{\mathbf{V}_0}f) \leq C_\phi \sup_{n \in \mathbb{Z}} |f(n+1) - f(n)| \leq C_\phi \omega(1, f).$$

Scaling this result by  $2^j$  yields (7.208). ■

**Interpolation Wavelets** The projection  $P_{\mathbf{V}_j}f(t)$  interpolates the values  $f(2^j n)$ . When reducing the scale by 2, we obtain a finer interpolation  $P_{\mathbf{V}_{j-1}}f(t)$  which also goes through the intermediate samples  $f(2^j(n+1/2))$ . This refinement can be obtained by adding “details” that compensate for the difference between  $P_{\mathbf{V}_j}f(2^j(n+1/2))$  and  $f(2^j(n+1/2))$ . To do this, we create a “detail” space  $\mathbf{W}_j$  that provides the values  $f(t)$  at intermediate dyadic points  $t = 2^j(n+1/2)$ . This space is constructed from interpolation functions centered at these locations, namely  $\phi_{j-1,2n+1}$ . We call *interpolation wavelets*

$$\psi_{j,n} = \phi_{j-1,2n+1}.$$

Observe that  $\psi_{j,n}(t) = \psi(2^{-j}t - n)$  with

$$\psi(t) = \phi(2t - 1).$$

The function  $\psi$  is not truly a wavelet since it has no vanishing moment. However, we shall see that it plays the same role as a wavelet in this decomposition. We define  $\mathbf{W}_j$  to be the space of all sums  $\sum_{n=-\infty}^{+\infty} a[n] \psi_{j,n}$ . The following theorem proves that it is a (non-orthogonal) complement of  $\mathbf{V}_j$  in  $\mathbf{V}_{j-1}$ .

**Theorem 7.23.** *For any  $j \in \mathbb{Z}$*

$$\mathbf{V}_{j-1} = \mathbf{V}_j \oplus \mathbf{W}_j.$$

*If  $f \in \mathbf{V}_{j-1}$  then*

$$f = \sum_{n=-\infty}^{+\infty} f(2^j n) \phi_{j,n} + \sum_{n=-\infty}^{+\infty} d_j[n] \psi_{j,n}$$

*with*

$$d_j[n] = f\left(2^j\left(n + \frac{1}{2}\right)\right) - P_{\mathbf{V}_j}f\left(2^j\left(n + \frac{1}{2}\right)\right). \quad (7.210)$$

*Proof.* Any  $f \in \mathbf{V}_{j-1}$  can be written

$$f = \sum_{n=-\infty}^{+\infty} f(2^{j-1}n) \phi_{j-1,n}.$$

The function  $f - P_{\mathbf{V}_j}f$  belongs to  $\mathbf{V}_{j-1}$  and vanishes at  $\{2^j n\}_{n \in \mathbb{Z}}$ . It can thus be decomposed over the intermediate interpolation functions  $\phi_{j-1,2n+1} = \psi_{j,n}$ :

$$f(t) - P_{\mathbf{V}_j}f(t) = \sum_{n=-\infty}^{+\infty} d_j[n] \psi_{j,n}(t) \in \mathbf{W}_j.$$

This proves that  $\mathbf{V}_{j-1} \subset \mathbf{V}_j \oplus \mathbf{W}_j$ . By construction we know that  $\mathbf{W}_j \subset \mathbf{V}_{j-1}$  so  $\mathbf{V}_{j-1} = \mathbf{V}_j \oplus \mathbf{W}_j$ . Setting  $t = 2^{j-1}(2n+1)$  in this formula also verifies (7.210). ■

Theorem 7.23 refines an interpolation from a coarse grid  $2^j n$  to a finer grid  $2^{j-1}n$  by adding “details” whose coefficients  $d_j[n]$  are the interpolation errors  $f(2^j(n+1/2)) - P_{\mathbf{V}_j}f(2^j(n+1/2))$ . The following theorem defines a interpolation wavelet basis of  $\mathbf{C}_0$  in the sense of uniform convergence.

**Theorem 7.24.** *If  $f \in \mathbf{C}_0$  then*

$$\lim_{\substack{m \rightarrow +\infty \\ l \rightarrow -\infty}} \|f - \sum_{n=-m}^m f(2^J n) \phi_{J,n} - \sum_{j=l}^J \sum_{n=-m}^m d_j[n] \psi_{j,n}\|_\infty = 0. \quad (7.211)$$

The formula (7.211) decomposes  $f$  into a coarse interpolation at intervals  $2^J$  plus layers of details that give the interpolation errors on successively finer dyadic grids. The proof is done by choosing  $f$  to be a continuous function with a compact support, in which case (7.211) is derived from Theorem 7.23 and (7.206). The density of such functions in  $\mathbf{C}_0$  (for the sup norm) allows us to extend this result to any  $f$  in  $\mathbf{C}_0$ . We shall write

$$f = \sum_{n=-\infty}^{+\infty} f(2^J n) \phi_{J,n} + \sum_{j=-\infty}^J \sum_{n=-\infty}^{+\infty} d_j[n] \psi_{j,n},$$

which means that  $[\{\phi_{J,n}\}_{n \in \mathbb{Z}}, \{\psi_{j,n}\}_{n \in \mathbb{Z}, j \leq J}]$  is a basis of  $\mathbf{C}_0$ . In  $\mathbf{L}^2(\mathbb{R})$ , “biorthogonal” scaling functions and wavelets are formally defined by

$$\begin{aligned} f(2^J n) &= \langle f, \tilde{\phi}_{J,n} \rangle = \int_{-\infty}^{+\infty} f(t) \tilde{\phi}_{J,n}(t) dt, \\ d_j[n] &= \langle f, \tilde{\psi}_{j,n} \rangle = \int_{-\infty}^{+\infty} f(t) \tilde{\psi}_{j,n}(t) dt. \end{aligned} \quad (7.212)$$

Clearly  $\tilde{\phi}_{J,n}(t) = \delta(t - 2^J n)$ . Similarly, (7.210) and (7.205) implies that  $\tilde{\psi}_{j,n}$  is a finite sum of Diracs. These dual scaling functions and wavelets do not have a finite energy, which illustrates the fact that  $[\{\phi_{J,n}\}_{n \in \mathbb{Z}}, \{\psi_{j,n}\}_{n \in \mathbb{Z}, j \leq J}]$  is not a Riesz basis of  $\mathbf{L}^2(\mathbb{R})$ .

If  $\hat{h}(\omega)$  has  $p$  zeros at  $\pi$  then one can verify that  $\tilde{\psi}_{j,n}$  has  $p$  vanishing moments. With similar derivations as in the proof of (6.20) in Theorem 6.4, one can show that if  $f$  is uniformly Lipschitz  $\alpha \leq p$  then there exists  $A > 0$  such that

$$|\langle f, \tilde{\psi}_{j,n} \rangle| = |d_j[n]| \leq A 2^{\alpha j}.$$

A regular signal yields small amplitude wavelet coefficients at fine scales. We can thus neglect these coefficients and still reconstruct a precise approximation of  $f$ .

**Fast Calculations** The interpolating wavelet transform of  $f$  is calculated at scale  $1 \geq 2^j > N^{-1} = 2^L$  from its sample values  $\{f(N^{-1}n)\}_{n \in \mathbb{Z}}$ . At each scale  $2^j$ , the values of  $f$  in between samples  $\{2^j n\}_{n \in \mathbb{Z}}$  are calculated with the interpolation (7.205):

$$\begin{aligned} P_{V_j} f(2^j(n + 1/2)) &= \sum_{k=-\infty}^{+\infty} f(2^j k) \phi(n - k + 1/2) \\ &= \sum_{k=-\infty}^{+\infty} f(2^j k) h_i[n - k], \end{aligned} \quad (7.213)$$

where the interpolation filter  $h_i$  is a subsampling of the autocorrelation filter  $h$  in (7.195):

$$h_i[n] = \phi(n + 1/2) = h[2n + 1]. \quad (7.214)$$

The wavelet coefficients are computed with (7.210):

$$d_j[n] = f(2^j(n + 1/2)) - P_{V_j} f(2^j(n + 1/2)).$$

The reconstruction of  $f(N^{-1}n)$  from the wavelet coefficients is performed recursively by recovering the samples  $f(2^{j-1}n)$  from the coarser sampling  $f(2^j n)$  with the interpolation (7.213) to which is added  $d_j[n]$ . If  $h_i[n]$  is a finite filter of size  $K$  and if  $f$  has a support in  $[0, 1]$  then the decomposition and reconstruction algorithms require  $KN$  multiplications and additions.

A Deslauriers-Dubuc interpolation function  $\phi$  has the shortest support while including polynomials of degree  $2p - 1$  in the spaces  $\mathbf{V}_j$ . The corresponding interpolation filter  $h_i[n]$  defined by (7.214) has  $2p$  non-zero coefficients for  $-p \leq n < p$ , which are calculated in (7.201). If  $p = 2$  then  $h_i[1] = h_i[-2] = -1/16$  and  $h_i[0] = h_i[-1] = 9/16$ . Suppose that  $q(t)$  is a polynomial of degree smaller or equal to  $2p - 1$ . Since  $q = P_{\mathbf{V}_j} q$ , (7.213) implies a Lagrange interpolation formula

$$q\left(2^j(n+1/2)\right) = \sum_{k=-\infty}^{+\infty} q(2^j k) h_i[n-k].$$

The Lagrange filter  $h_i$  of size  $2p$  is the shortest filter that recovers intermediate values of polynomials of degree  $2p - 1$  from a uniform sampling.

To restrict the wavelet interpolation bases to a finite interval  $[0, 1]$  while reproducing polynomials of degree  $2p - 1$ , the filter  $h_i$  is modified at the boundaries. Suppose that  $f(N^{-1}n)$  is defined for  $0 \leq n < N$ . When computing the interpolation

$$P_{\mathbf{V}_j} f\left(2^j(n+1/2)\right) = \sum_{k=-\infty}^{+\infty} f(2^j k) h_i[n-k],$$

if  $n$  is too close to 0 or to  $2^{-j} - 1$  then  $h_i$  must be modified to ensure that the support of  $h_i[n-k]$  remains inside  $[0, 2^{-j} - 1]$ . The interpolation  $P_{\mathbf{V}_j} f(2^j(n+1/2))$  is then calculated from the closest  $2p$  samples  $f(2^j k)$  for  $2^j k \in [0, 1]$ . The new interpolation coefficients are computed in order to recover exactly all polynomials of degree  $2p - 1$  [450]. For  $p = 2$ , the problem occurs only at  $n = 0$  and the appropriate boundary coefficients are

$$h_i[0] = \frac{5}{16}, \quad h_i[-1] = \frac{15}{16}, \quad h_i[-2] = \frac{-5}{16}, \quad h_i[-3] = \frac{1}{16}.$$

The symmetric boundary filter  $h_i[-n]$  is used on the other side at  $n = 2^{-j} - 1$ .

## 7.7 Separable Wavelet Bases

To any wavelet orthonormal basis  $\{\psi_{j,n}\}_{(j,n) \in \mathbb{Z}^2}$  of  $\mathbf{L}^2(\mathbb{R})$ , one can associate a separable wavelet orthonormal basis of  $\mathbf{L}^2(\mathbb{R}^2)$ :

$$\left\{ \psi_{j_1, n_1}(x_1) \psi_{j_2, n_2}(x_2) \right\}_{(j_1, j_2, n_1, n_2) \in \mathbb{Z}^4}. \quad (7.215)$$

The functions  $\psi_{j_1, n_1}(x_1) \psi_{j_2, n_2}(x_2)$  mix information at two different scales  $2^{j_1}$  and  $2^{j_2}$  along  $x_1$  and  $x_2$ , which we often want to avoid. Separable multiresolutions lead to another construction of separable wavelet bases whose elements are products of functions dilated at the same scale. These multiresolution approximations also have important applications in computer vision, where they are used to process images at different levels of details. Lower resolution images are indeed represented by fewer pixels and might still carry enough information to perform a recognition task.

Signal decompositions in separable wavelet bases are computed with a separable extension of the filter bank algorithm described in Section 7.7.3. Section 7.7.4 constructs separable wavelet bases in any dimension, and explains the corresponding fast wavelet transform algorithm. Non-separable wavelet bases can also be constructed [83, 333] but they are used less often in image processing. Section 7.8.3 gives examples of non-separable quincunx biorthogonal wavelet bases, which have a single quasi-isotropic wavelet at each scale.

### 7.7.1 Separable Multiresolutions

As in one dimension, the notion of resolution is formalized with orthogonal projections in spaces of various sizes. The approximation of an image  $f(x_1, x_2)$  at the resolution  $2^{-j}$  is defined as the orthogonal projection of  $f$  on a space  $\mathbf{V}_j^2$  that is included in  $\mathbf{L}^2(\mathbb{R}^2)$ . The space  $\mathbf{V}_j^2$  is the set of all approximations at the resolution  $2^{-j}$ . When the resolution decreases, the size of  $\mathbf{V}_j^2$



decreases as well. The formal definition of a multiresolution approximation  $\{\mathbf{V}_j^2\}_{j \in \mathbb{Z}}$  of  $\mathbf{L}^2(\mathbb{R}^2)$  is a straightforward extension of Definition 7.1 that specifies multiresolutions of  $\mathbf{L}^2(\mathbb{R})$ . The same causality, completeness and scaling properties must be satisfied.

We consider the particular case of separable multiresolutions. Let  $\{\mathbf{V}_j\}_{j \in \mathbb{Z}}$  be a multiresolution of  $\mathbf{L}^2(\mathbb{R})$ . A separable two-dimensional multiresolution is composed of the tensor product spaces

$$\mathbf{V}_j^2 = \mathbf{V}_j \otimes \mathbf{V}_j. \quad (7.216)$$

The space  $\mathbf{V}_j^2$  is the set of finite energy functions  $f(x_1, x_2)$  that are linear expansions of separable functions:

$$f(x_1, x_2) = \sum_{m=-\infty}^{+\infty} a[m] f_m(x_1) g_m(x_2) \quad \text{with } f_m \in \mathbf{V}_j, \quad g_m \in \mathbf{V}_j.$$

Section A.5 reviews the properties of tensor products. If  $\{\mathbf{V}_j\}_{j \in \mathbb{Z}}$  is a multiresolution approximation of  $\mathbf{L}^2(\mathbb{R})$  then  $\{\mathbf{V}_j^2\}_{j \in \mathbb{Z}}$  is a multiresolution approximation of  $\mathbf{L}^2(\mathbb{R}^2)$ .

Theorem 7.1 demonstrates the existence of a scaling function  $\phi$  such that  $\{\phi_{j,m}\}_{m \in \mathbb{Z}}$  is an orthonormal basis of  $\mathbf{V}_j$ . Since  $\mathbf{V}_j^2 = \mathbf{V}_j \otimes \mathbf{V}_j$ , Theorem A.6 proves that for  $x = (x_1, x_2)$  and  $n = (n_1, n_2)$

$$\left\{ \phi_{j,n}^2(x) = \phi_{j,n_1}(x_1) \phi_{j,n_2}(x_2) = \frac{1}{2^j} \phi\left(\frac{x_1 - 2^j n_1}{2^j}\right) \phi\left(\frac{x_2 - 2^j n_2}{2^j}\right) \right\}_{n \in \mathbb{Z}^2}$$

is an orthonormal basis of  $\mathbf{V}_j^2$ . It is obtained by scaling by  $2^j$  the two-dimensional separable scaling function  $\phi^2(x) = \phi(x_1) \phi(x_2)$  and translating it on a two-dimensional square grid with intervals  $2^j$ .

**Example 7.13. Piecewise constant approximation** Let  $\mathbf{V}_j$  be the approximation space of functions that are constant on  $[2^j m, 2^j(m+1)]$  for any  $m \in \mathbb{Z}$ . The tensor product defines a two-dimensional piecewise constant approximation. The space  $\mathbf{V}_j^2$  is the set of functions that are constant on any square  $[2^j n_1, 2^j(n_1+1)] \times [2^j n_2, 2^j(n_2+1)]$ , for  $(n_1, n_2) \in \mathbb{Z}^2$ . The two dimensional scaling function is

$$\phi^2(x) = \phi(x_1) \phi(x_2) = \begin{cases} 1 & \text{if } 0 \leq x_1 \leq 1 \text{ and } 0 \leq x_2 \leq 1 \\ 0 & \text{otherwise} \end{cases}.$$

**Example 7.14. Shannon approximation** Let  $\mathbf{V}_j$  be the space of functions whose Fourier transforms have a support included in  $[-2^{-j}\pi, 2^{-j}\pi]$ . The space  $\mathbf{V}_j^2$  is the set of functions whose two-dimensional Fourier transforms have a support included in the low-frequency square  $[-2^{-j}\pi, 2^{-j}\pi] \times [-2^{-j}\pi, 2^{-j}\pi]$ . The two-dimensional scaling function is a perfect two-dimensional low-pass filter whose Fourier transform is

$$\hat{\phi}(\omega_1) \hat{\phi}(\omega_2) = \begin{cases} 1 & \text{if } |\omega_1| \leq 2^{-j}\pi \text{ and } |\omega_2| \leq 2^{-j}\pi \\ 0 & \text{otherwise} \end{cases}.$$

**Example 7.15. Spline approximation** Let  $\mathbf{V}_j$  be the space of polynomial spline functions of degree  $p$  that are  $\mathbf{C}^{p-1}$ , with nodes located at  $2^{-j}m$  for  $m \in \mathbb{Z}$ . The space  $\mathbf{V}_j^2$  is composed of two-dimensional polynomial spline functions that are  $p-1$  times continuously differentiable. The restriction of  $f(x_1, x_2) \in \mathbf{V}_j^2$  to any square  $[2^j n_1, 2^j(n_1+1)] \times [2^j n_2, 2^j(n_2+1)]$  is a separable product  $q_1(x_1)q_2(x_2)$  of two polynomials of degree at most  $p$ .

**Multiresolution Vision** An image of 512 by 512 pixels often includes too much information for real time vision processing. Multiresolution algorithms process less image data by selecting the relevant details that are necessary to perform a particular recognition task [57]. The human visual system uses a similar strategy. The distribution of photoreceptors on the retina is not uniform. The visual acuity is greatest at the center of the retina where the density of receptors is maximum. When moving apart from the center, the resolution decreases proportionally to the distance from the retina center [427].

The high resolution visual center is called the *fovea*. It is responsible for high acuity tasks such as reading or recognition. A retina with a uniform resolution equal to the highest fovea resolution



Figure 7.21: Multiresolution approximations  $a_j[n_1, n_2]$  of an image at scales  $2^j$ , for  $-5 \geq j \geq -8$ .

would require about 10,000 times more photoreceptors. Such a uniform resolution retina would increase considerably the size of the optic nerve that transmits the retina information to the visual cortex and the size of the visual cortex that processes this data.

Active vision strategies [81] compensate the non-uniformity of visual resolution with eye saccades, which move successively the fovea over regions of a scene with a high information content. These saccades are partly guided by the lower resolution information gathered at the periphery of the retina. This multiresolution sensor has the advantage of providing high resolution information at selected locations, and a large field of view, with relatively little data.

Multiresolution algorithms implement in software [124] the search for important high resolution data. A uniform high resolution image is measured by a camera but only a small part of this information is processed. Figure 7.21 displays a pyramid of progressively lower resolution images calculated with a filter bank presented in Section 7.7.3. Coarse to fine algorithms analyze first the lower resolution image and selectively increase the resolution in regions where more details are needed. Such algorithms have been developed for object recognition, and stereo calculations [283].

## 7.7.2 Two-Dimensional Wavelet Bases

A separable wavelet orthonormal basis of  $\mathbf{L}^2(\mathbb{R}^2)$  is constructed with separable products of a scaling function  $\phi$  and a wavelet  $\psi$ . The scaling function  $\phi$  is associated to a one-dimensional multiresolution approximation  $\{\mathbf{V}_j\}_{j \in \mathbb{Z}}$ . Let  $\{\mathbf{V}_j^2\}_{j \in \mathbb{Z}}$  be the separable two-dimensional multiresolution defined by  $\mathbf{V}_j^2 = \mathbf{V}_j \otimes \mathbf{V}_j$ . Let  $\mathbf{W}_j^2$  be the detail space equal to the orthogonal complement of the lower resolution approximation space  $\mathbf{V}_j^2$  in  $\mathbf{V}_{j-1}^2$ :

$$\mathbf{V}_{j-1}^2 = \mathbf{V}_j^2 \oplus \mathbf{W}_j^2. \quad (7.217)$$

To construct a wavelet orthonormal basis of  $\mathbf{L}^2(\mathbb{R}^2)$ , the following theorem builds a wavelet basis of each detail space  $\mathbf{W}_j^2$ .

**Theorem 7.25.** *Let  $\phi$  be a scaling function and  $\psi$  be the corresponding wavelet generating a wavelet orthonormal basis of  $\mathbf{L}^2(\mathbb{R})$ . We define three wavelets:*

$$\psi^1(x) = \phi(x_1)\psi(x_2) \quad , \quad \psi^2(x) = \psi(x_1)\phi(x_2) \quad , \quad \psi^3(x) = \psi(x_1)\psi(x_2), \quad (7.218)$$

and denote for  $1 \leq k \leq 3$

$$\psi_{j,n}^k(x) = \frac{1}{2^j} \psi^k\left(\frac{x_1 - 2^j n_1}{2^j}, \frac{x_2 - 2^j n_2}{2^j}\right).$$

The wavelet family

$$\{\psi_{j,n}^1, \psi_{j,n}^2, \psi_{j,n}^3\}_{n \in \mathbb{Z}^2} \quad (7.219)$$

is an orthonormal basis of  $\mathbf{W}_j^2$  and

$$\{\psi_{j,n}^1, \psi_{j,n}^2, \psi_{j,n}^3\}_{(j,n) \in \mathbb{Z}^3} \quad (7.220)$$

is an orthonormal basis of  $\mathbf{L}^2(\mathbb{R}^2)$ .

*Proof.* Equation (7.217) is rewritten

$$\mathbf{V}_{j-1} \otimes \mathbf{V}_{j-1} = (\mathbf{V}_j \otimes \mathbf{V}_j) \oplus \mathbf{W}_j^2. \quad (7.221)$$

The one-dimensional multiresolution space  $\mathbf{V}_{j-1}$  can also be decomposed into  $\mathbf{V}_{j-1} = \mathbf{V}_j \oplus \mathbf{W}_j$ . By inserting this in (7.221), the distributivity of  $\oplus$  with respect to  $\otimes$  proves that

$$\mathbf{W}_j^2 = (\mathbf{V}_j \otimes \mathbf{W}_j) \oplus (\mathbf{W}_j \otimes \mathbf{V}_j) \oplus (\mathbf{W}_j \otimes \mathbf{W}_j). \quad (7.222)$$

Since  $\{\phi_{j,m}\}_{m \in \mathbb{Z}}$  and  $\{\psi_{j,m}\}_{m \in \mathbb{Z}}$  are orthonormal bases of  $\mathbf{V}_j$  and  $\mathbf{W}_j$ , we derive that

$$\{\phi_{j,n_1}(x_1) \psi_{j,n_2}(x_2), \psi_{j,n_1}(x_1) \phi_{j,n_2}(x_2), \psi_{j,n_1}(x_1) \psi_{j,n_2}(x_2)\}_{(n_1, n_2) \in \mathbb{Z}^2}$$

is an orthonormal basis of  $\mathbf{W}_j^2$ . As in the one-dimensional case, the overall space  $\mathbf{L}^2(\mathbb{R}^2)$  can be decomposed as an orthogonal sum of the detail spaces at all resolutions:

$$\mathbf{L}^2(\mathbb{R}^2) = \oplus_{j=-\infty}^{+\infty} \mathbf{W}_j^2. \quad (7.223)$$

Hence

$$\{\phi_{j,n_1}(x_1) \psi_{j,n_2}(x_2), \psi_{j,n_1}(x_1) \phi_{j,n_2}(x_2), \psi_{j,n_1}(x_1) \psi_{j,n_2}(x_2)\}_{(j, n_1, n_2) \in \mathbb{Z}^3}$$

is an orthonormal basis of  $\mathbf{L}^2(\mathbb{R}^2)$ . ■

The three wavelets extract image details at different scales and in different directions. Over positive frequencies,  $\hat{\phi}$  and  $\hat{\psi}$  have an energy mainly concentrated respectively on  $[0, \pi]$  and  $[\pi, 2\pi]$ . The separable wavelet expressions (7.218) imply that

$$\hat{\psi}^1(\omega_1, \omega_2) = \hat{\phi}(\omega_1) \hat{\psi}(\omega_2), \quad \hat{\psi}^2(\omega_1, \omega_2) = \hat{\psi}(\omega_1) \hat{\phi}(\omega_2)$$

and  $\hat{\psi}^3(\omega_1, \omega_2) = \hat{\psi}(\omega_1) \hat{\psi}(\omega_2)$ . Hence  $|\hat{\psi}^1(\omega_1, \omega_2)|$  is large at low horizontal frequencies  $\omega_1$  and high vertical frequencies  $\omega_2$ , whereas  $|\hat{\psi}^2(\omega_1, \omega_2)|$  is large at high horizontal frequencies and low vertical frequencies, and  $|\hat{\psi}^3(\omega_1, \omega_2)|$  is large at high horizontal and vertical frequencies. Figure 7.22 displays the Fourier transform of separable wavelets and scaling functions calculated from a one-dimensional Daubechies 4 wavelet.

Suppose that  $\psi(t)$  has  $p$  vanishing moments and is thus orthogonal to one-dimensional polynomials of degree  $p-1$ . The wavelet  $\psi^1$  has  $p$  one-dimensional directional vanishing moments along  $x_2$  in the sense that it is orthogonal to any function  $f(x_1, x_2)$  that is a polynomial of degree  $p-1$  along  $x_2$  for  $x_1$  fixed. It is a horizontal directional wavelet that yields large coefficients for horizontal edges, as explained in Section 5.5.1. Similarly,  $\psi^2$  has  $p-1$  directional vanishing moments along  $x_1$  and is a vertical directional wavelet. This is illustrated by the decomposition of a square in Figure 7.24. The wavelet  $\psi^3$  has directional vanishing moments along both  $x_1$  and  $x_2$  and is therefore not a directional wavelet. It produces large coefficients at corners or junctions. The three wavelets  $\psi^k$  for  $k = 1, 2, 3$  are orthogonal to two-dimensional polynomials of degree  $p-1$ .

**Example 7.16.** For a Shannon multiresolution approximation, the resulting two-dimensional wavelet basis paves the two-dimensional Fourier plane  $(\omega_1, \omega_2)$  with dilated rectangles. The Fourier transforms  $\hat{\phi}$  and  $\hat{\psi}$  are the indicator functions respectively of  $[-\pi, \pi]$  and  $[-2\pi, -\pi] \cup [\pi, 2\pi]$ . The separable space  $\mathbf{V}_j^2$  contains functions whose two-dimensional Fourier transforms have a support included in the low-frequency square  $[-2^{-j}\pi, 2^{-j}\pi] \times [-2^{-j}\pi, 2^{-j}\pi]$ . This corresponds to the support of  $\hat{\phi}_{j,n}^2$  indicated in Figure 7.23. The detail space  $\mathbf{W}_j^2$  is the orthogonal complement of  $\mathbf{V}_j^2$  in  $\mathbf{V}_{j-1}^2$  and thus includes functions whose Fourier transforms have a support in the frequency annulus between the two squares  $[-2^{-j}\pi, 2^{-j}\pi] \times [-2^{-j}\pi, 2^{-j}\pi]$  and  $[-2^{-j+1}\pi, 2^{-j+1}\pi] \times [-2^{-j+1}\pi, 2^{-j+1}\pi]$ . As shown in Figure 7.23, this annulus is decomposed in three separable frequency regions, which are the Fourier supports of  $\hat{\psi}_{j,n}^k$  for  $1 \leq k \leq 3$ . Dilating these supports at all scales  $2^j$  yields an exact cover of the frequency plane  $(\omega_1, \omega_2)$ .

For general separable wavelet bases, Figure 7.23 gives only an indication of the domains where the energy of the different wavelets is concentrated. When the wavelets are constructed with a one-dimensional wavelet of compact support, the resulting Fourier transforms have side lobes that appear in Figure 7.22.

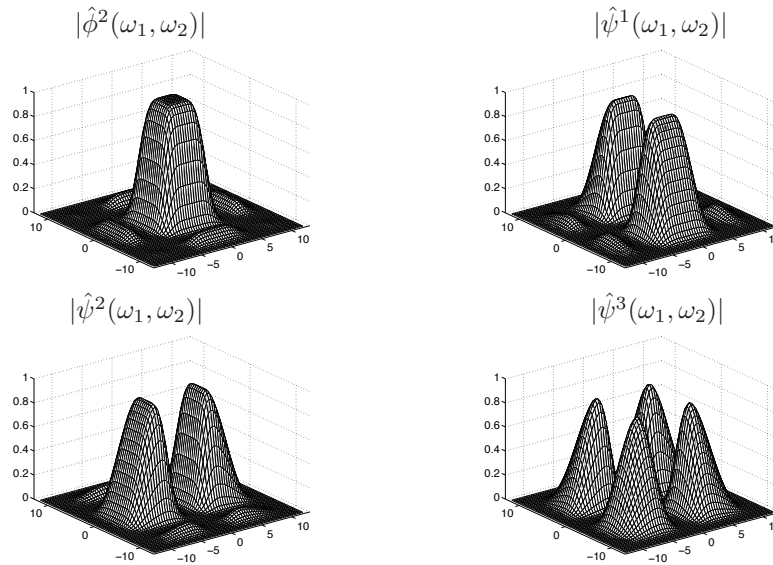


Figure 7.22: Fourier transforms of a separable scaling function and of 3 separable wavelets calculated from a one-dimensional Daubechies 4 wavelet.

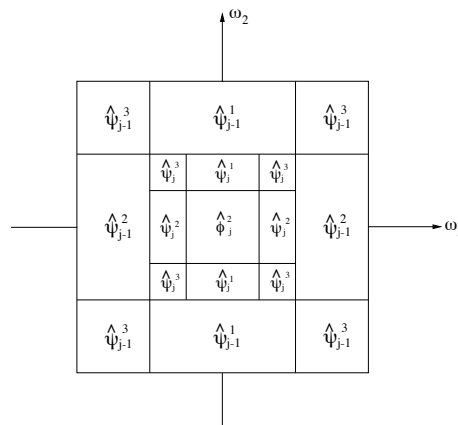


Figure 7.23: These dyadic rectangles indicate the regions where the energy of  $\hat{\psi}_{j,n}^k$  is mostly concentrated, for  $1 \leq k \leq 3$ . Image approximations at the scale  $2^j$  are restricted to the lower frequency square.

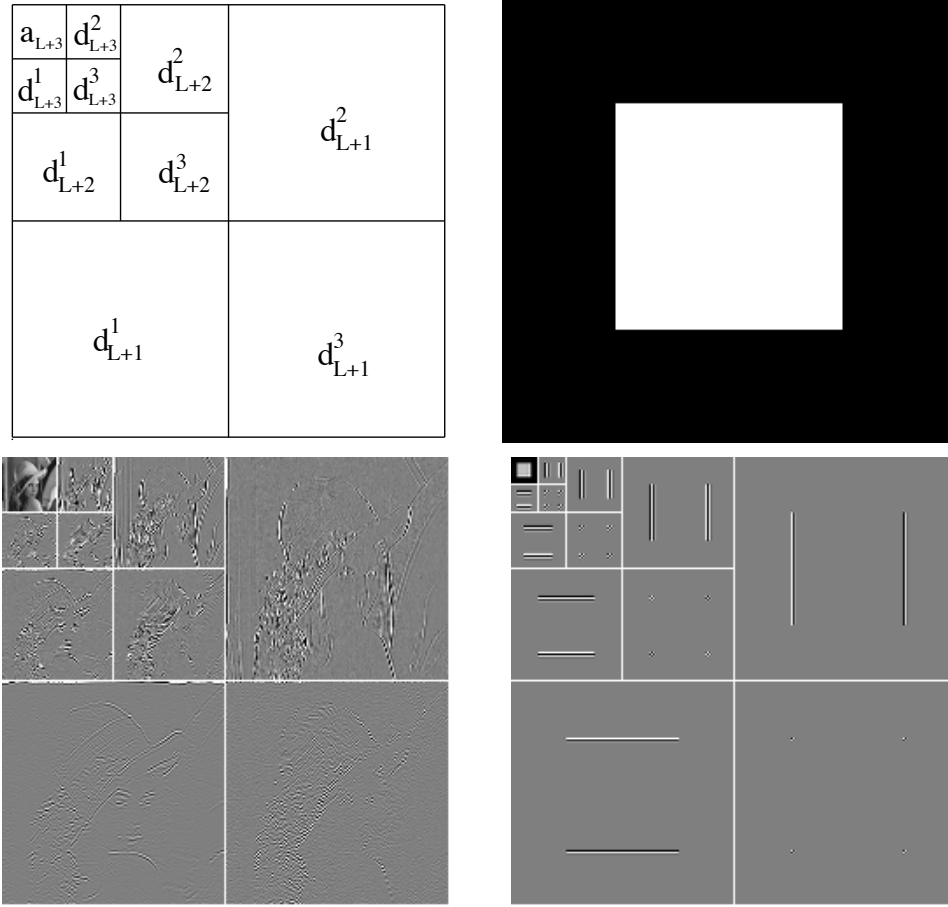


Figure 7.24: Separable wavelet transforms of Lena and of a white square in a black background, decomposed respectively on 3 and 4 octaves. Black, grey and white pixels correspond respectively to positive, zero and negative wavelet coefficients. The disposition of wavelet image coefficients  $d_j^k[n, m] = \langle f, \psi_{j,n}^k \rangle$  is illustrated on the top left.

**Example 7.17.** Figure 7.24 gives two examples of wavelet transforms computed using separable Daubechies wavelets with  $p = 4$  vanishing moments. They are calculated with the filter bank algorithm of Section 7.7.3. Coefficients of large amplitude in  $d_j^1$ ,  $d_j^2$  and  $d_j^3$  correspond respectively to vertical high frequencies (horizontal edges), horizontal high frequencies (vertical edges), and high frequencies in both directions (corners). Regions where the image intensity varies smoothly yield nearly zero coefficients, shown in grey. The large number of nearly zero coefficients makes it particularly attractive for compact image coding.

**Separable Biorthogonal Bases** One-dimensional biorthogonal wavelet bases are extended to separable biorthogonal bases of  $\mathbf{L}^2(\mathbb{R}^2)$  with the same approach as in Theorem 7.25. Let  $\phi, \psi$  and  $\tilde{\phi}, \tilde{\psi}$  be two dual pairs of scaling functions and wavelets that generate biorthogonal wavelet bases of  $\mathbf{L}^2(\mathbb{R})$ . The dual wavelets of  $\psi^1, \psi^2$  and  $\psi^3$  defined by (7.218) are

$$\tilde{\psi}^1(x) = \tilde{\phi}(x_1) \tilde{\psi}(x_2), \quad \tilde{\psi}^2(x) = \tilde{\psi}(x_1) \tilde{\phi}(x_2), \quad \tilde{\psi}^3(x) = \tilde{\psi}(x_1) \tilde{\psi}(x_2). \quad (7.224)$$

One can verify that

$$\{\psi_{j,n}^1, \psi_{j,n}^2, \psi_{j,n}^3\}_{(j,n) \in \mathbb{Z}^3} \quad (7.225)$$

and

$$\{\tilde{\psi}_{j,n}^1, \tilde{\psi}_{j,n}^2, \tilde{\psi}_{j,n}^3\}_{(j,n) \in \mathbb{Z}^3} \quad (7.226)$$

are biorthogonal Riesz bases of  $\mathbf{L}^2(\mathbb{R}^2)$ .

### 7.7.3 Fast Two-Dimensional Wavelet Transform

The fast wavelet transform algorithm presented in Section 7.3.1 is extended in two dimensions. At all scales  $2^j$  and for any  $n = (n_1, n_2)$ , we denote

$$a_j[n] = \langle f, \phi_{j,n}^2 \rangle \quad \text{and} \quad d_j^k[n] = \langle f, \psi_{j,n}^k \rangle \quad \text{for } 1 \leq k \leq 3 .$$

For any pair of one-dimensional filters  $y[m]$  and  $z[m]$  we write the product filter  $yz[n] = y[n_1]z[n_2]$ , and  $\bar{y}[m] = y[-m]$ . Let  $h[m]$  and  $g[m]$  be the conjugate mirror filters associated to the wavelet  $\psi$ .

The wavelet coefficients at the scale  $2^{j+1}$  are calculated from  $a_j$  with two-dimensional separable convolutions and subsamplings. The decomposition formula are obtained by applying the one-dimensional convolution formula (7.103) and (7.102) of Theorem 7.10 to the separable two-dimensional wavelets and scaling functions for  $n = (n_1, n_2)$ :

$$a_{j+1}[n] = a_j \star \bar{h}\bar{h}[2n] , \quad (7.227)$$

$$d_{j+1}^1[n] = a_j \star \bar{h}\bar{g}[2n] , \quad (7.228)$$

$$d_{j+1}^2[n] = a_j \star \bar{g}\bar{h}[2n] , \quad (7.229)$$

$$d_{j+1}^3[n] = a_j \star \bar{g}\bar{g}[2n] . \quad (7.230)$$

We showed in (3.70) that a separable two-dimensional convolution can be factored into one-dimensional convolutions along the rows and columns of the image. With the factorization illustrated in Figure 7.25(a), these four convolutions equations are computed with only six groups of one-dimensional convolutions. The rows of  $a_j$  are first convolved with  $\bar{h}$  and  $\bar{g}$  and subsampled by 2. The columns of these two output images are then convolved respectively with  $\bar{h}$  and  $\bar{g}$  and subsampled, which gives the four subsampled images  $a_{j+1}$ ,  $d_{j+1}^1$ ,  $d_{j+1}^2$  and  $d_{j+1}^3$ .

We denote by  $\check{y}[n] = \check{y}[n_1, n_2]$  the image twice the size of  $y[n]$ , obtained by inserting a row of zeros and a column of zeros between pairs of consecutive rows and columns. The approximation  $a_j$  is recovered from the coarser scale approximation  $a_{j+1}$  and the wavelet coefficients  $d_{j+1}^k$  with two-dimensional separable convolutions derived from the one-dimensional reconstruction formula (7.104)

$$a_j[n] = \check{a}_{j+1} \star hh[n] + \check{d}_{j+1}^1 \star hg[n] + \check{d}_{j+1}^2 \star gh[n] + \check{d}_{j+1}^3 \star gg[n] . \quad (7.231)$$

These four separable convolutions can also be factored into six groups of one-dimensional convolutions along rows and columns, illustrated in Figure 7.25(b).

Let  $b[n]$  be an input image whose pixels have a distance  $2^L$ . We associate to  $b[n]$  a function  $f(x) \in \mathbf{V}_L^2$  approximated at the scale  $2^L$ . Its coefficients  $a_L[n] = \langle f, \phi_{L,n}^2 \rangle$  are defined like in (7.111) by

$$b[n] = 2^{-L} a_L[n] \approx f(2^L n) . \quad (7.232)$$

The wavelet image representation of  $a_L$  is computed by iterating (7.227-7.230) for  $L \leq j < J$ :

$$[a_J , \{d_j^1, d_j^2, d_j^3\}_{L < j \leq J}] . \quad (7.233)$$

The image  $a_L$  is recovered from this wavelet representation by computing (7.231) for  $J > j \geq L$ .

**Finite Image and Complexity** When  $a_L$  is a finite image of  $N = N_1 N_2$  pixels, we face boundary problems when computing the convolutions (7.227-7.231). Since the decomposition algorithm is separable along rows and columns, we use one of the three one-dimensional boundary techniques described in Section 7.5. The resulting values are decomposition coefficients in a wavelet basis of  $\mathbf{L}^2[0, 1]^2$ . Depending on the boundary treatment, this wavelet basis is a periodic basis, a folded basis or a boundary adapted basis.

For square images with  $N_1 = N_2$ , the resulting images  $a_j$  and  $d_j^k$  have  $2^{-2j}$  samples. The images of the wavelet representation (7.233) thus include a total of  $N$  samples. If  $h$  and  $g$  have size  $K$ , the reader can verify that  $2K2^{-2(j-1)}$  multiplications and additions are needed to compute the four convolutions (7.227-7.230) with the factorization of Figure 7.25(a). The wavelet representation (7.233) is thus calculated with fewer than  $8/3 KN$  operations. The reconstruction of  $a_L$  by factoring the reconstruction equation (7.231) requires the same number of operations.

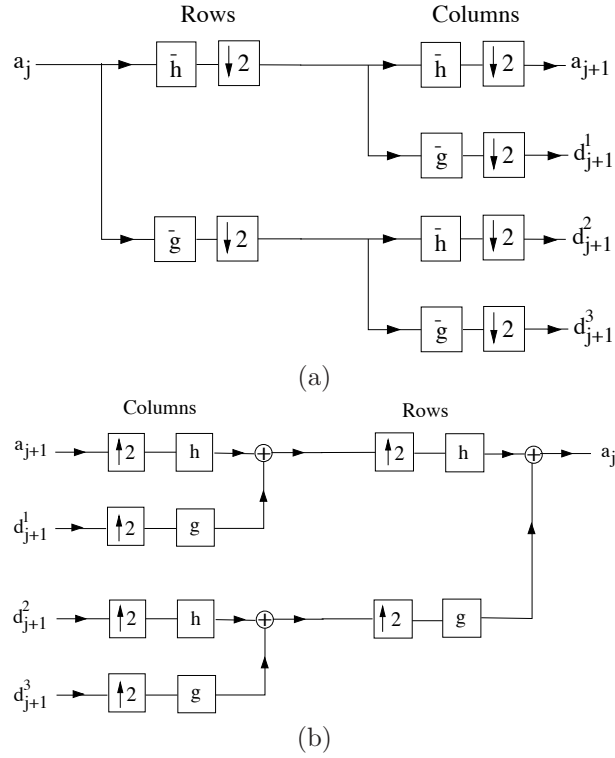


Figure 7.25: (a): Decomposition of  $a_j$  with 6 groups of one-dimensional convolutions and subsamplings along the image rows and columns. (b): Reconstruction of  $a_j$  by inserting zeros between the rows and columns of  $a_{j+1}$  and  $d_{j+1}^k$ , and filtering the output.

**Fast Biorthogonal Wavelet Transform** The decomposition of an image in a biorthogonal wavelet basis is performed with the same fast wavelet transform algorithm. Let  $(\tilde{h}, \tilde{g})$  be the perfect reconstruction filters associated to  $(h, g)$ . The inverse wavelet transform is computed by replacing the filters  $(h, g)$  that appear in (7.231) by  $(\tilde{h}, \tilde{g})$ .

#### 7.7.4 Wavelet Bases in Higher Dimensions

Separable wavelet orthonormal bases of  $\mathbf{L}^2(\mathbb{R}^p)$  are constructed for any  $p \geq 2$ , with a procedure similar to the two-dimensional extension. Let  $\phi$  be a scaling function and  $\psi$  a wavelet that yields an orthogonal basis of  $\mathbf{L}^2(\mathbb{R})$ . We denote  $\theta^0 = \phi$  and  $\theta^1 = \psi$ . To any integer  $0 \leq \varepsilon < 2^p$  written in binary form  $\varepsilon = \varepsilon_1 \dots, \varepsilon_p$  we associate the  $p$ -dimensional functions defined in  $x = (x_1, \dots, x_p)$  by

$$\psi^\varepsilon(x) = \theta^{\varepsilon_1}(x_1) \dots \theta^{\varepsilon_p}(x_p),$$

For  $\varepsilon = 0$ , we obtain a  $p$ -dimensional scaling function

$$\psi^0(x) = \phi(x_1) \dots \phi(x_p).$$

Non-zero indexes  $\varepsilon$  correspond to  $2^p - 1$  wavelets. At any scale  $2^j$  and for  $n = (n_1, \dots, n_p)$  we denote

$$\psi_{j,n}^\varepsilon(x) = 2^{-pj/2} \psi^\varepsilon\left(\frac{x_1 - 2^j n_1}{2^j}, \dots, \frac{x_p - 2^j n_p}{2^j}\right).$$

**Theorem 7.26.** *The family obtained by dilating and translating the  $2^p - 1$  wavelets for  $\varepsilon \neq 0$*

$$\left\{ \psi_{j,n}^\varepsilon \right\}_{1 \leq \varepsilon < 2^p, (j,n) \in \mathbb{Z}^{p+1}} \quad (7.234)$$

*is an orthonormal basis of  $\mathbf{L}^2(\mathbb{R}^p)$ .*



The proof is done by induction on  $p$ . It follows the same steps as the proof of Theorem 7.25 which associates to a wavelet basis of  $\mathbf{L}^2(\mathbb{R})$  a separable wavelet basis of  $\mathbf{L}^2(\mathbb{R}^2)$ . For  $p = 2$ , we verify that the basis (7.234) includes 3 elementary wavelets. For  $p = 3$ , there are 7 different wavelets.

**Fast Wavelet Transform** Let  $b[n]$  be an input  $p$ -dimensional discrete signal sampled at intervals  $2^L$ . We associate to  $b[n]$  an approximation  $f$  at the scale  $2^L$  whose scaling coefficients  $a_L[n] = \langle f, \psi_{L,n}^0 \rangle$  satisfy

$$b[n] = 2^{-Lp/2} a_L[n] \approx f(2^L n) .$$

The wavelet coefficients of  $f$  at scales  $2^j > 2^L$  are computed with separable convolutions and subsamplings along the  $p$  signal dimensions. We denote

$$a_j[n] = \langle f, \psi_{j,n}^0 \rangle \quad \text{and} \quad d_j^\varepsilon[n] = \langle f, \psi_{j,n}^\varepsilon \rangle \quad \text{for } 0 < \varepsilon < 2^p .$$

The fast wavelet transform is computed with filters that are separable products of the one-dimensional filters  $h$  and  $g$ . The separable  $p$ -dimensional low-pass filter is

$$h^0[n] = h[n_1] \dots h[n_p] .$$

Let us denote  $u^0[m] = h[m]$  and  $u^1[m] = g[m]$ . To any integer  $\varepsilon = \varepsilon_1 \dots \varepsilon_p$  written in a binary form, we associate a separable  $p$ -dimensional band-pass filter

$$g^\varepsilon[n] = u^{\varepsilon_1}[n_1] \dots u^{\varepsilon_p}[n_p] .$$

Let  $\bar{g}^\varepsilon[n] = g^\varepsilon[-n]$ . One can verify that

$$a_{j+1}[n] = a_j \star \bar{h}^0[2n] , \tag{7.235}$$

$$d_{j+1}^\varepsilon[n] = a_j \star \bar{g}^\varepsilon[2n] . \tag{7.236}$$

We denote by  $\check{y}[n]$  the signal obtained by adding a zero between any two samples of  $y[n]$  that are adjacent in the  $p$ -dimensional lattice  $n = (n_1, \dots, n_p)$ . It doubles the size of  $y[n]$  along each direction. If  $y[n]$  has  $M^p$  samples, then  $\check{y}[n]$  has  $(2M)^p$  samples. The reconstruction is performed with

$$a_j[n] = \check{a}_{j+1} \star h^0[n] + \sum_{\varepsilon=1}^{2^p-1} \check{d}_{j+1}^\varepsilon \star g^\varepsilon[n] . \tag{7.237}$$

The  $2^p$  separable convolutions needed to compute  $a_j$  and  $\{d_j^\varepsilon\}_{1 \leq \varepsilon \leq 2^p}$  as well as the reconstruction (7.237) can be factored in  $2^{p+1} - 2$  groups of one-dimensional convolutions along the rows of  $p$ -dimensional signals. This is a generalization of the two-dimensional case, illustrated in Figures 7.25. The wavelet representation of  $a_L$  is

$$[\{d_j^\varepsilon\}_{1 \leq \varepsilon < 2^p, L < j \leq J}, a_J] . \tag{7.238}$$

It is computed by iterating (7.235) and (7.236) for  $L \leq j < J$ . The reconstruction of  $a_L$  is performed with the partial reconstruction (7.237) for  $J > j \geq L$ .

If  $a_L$  is a finite signal of size  $N_1 \dots N_p$ , the one-dimensional convolutions are modified with one of the three boundary techniques described in Section 7.5. The resulting algorithm computes decomposition coefficients in a separable wavelet basis of  $\mathbf{L}^2[0, 1]^p$ . If  $N_1 = \dots = N_p$ , the signals  $a_j$  and  $d_j^\varepsilon$  have  $2^{-pj}$  samples. Like  $a_L$ , the wavelet representation (7.238) is composed of  $N$  samples. If the filter  $h$  has  $K$  non-zero samples then the separable factorization of (7.235) and (7.236) requires  $pK2^{-p(j-1)}$  multiplications and additions. The wavelet representation (7.238) is thus computed with fewer than  $p(1 - 2^{-p})^{-1}KN$  multiplications and additions. The reconstruction is performed with the same number of operations.