III

Discrete Revolution

Digital signal processing has taken over. First used in the 1950's at the service of analog signal processing to simulate analog transforms, digital algorithms have invaded most traditional fortresses, including phones, music recording, cameras, televisions, and all information processing. Analog computations performed with electronic circuits are faster than digital algorithms implemented with microprocessors, but are less precise and less flexible. Thus analog circuits are often replaced by digital chips once the computational performance of microprocessors is sufficient to operate in real time for a given application.

Whether sound recordings or images, most discrete signals are obtained by sampling an analog signal. An analog to digital conversion is a linear approximation which introduces an error which depends upon the sampling rate. Once more, the Fourier transform is unavoidable because the eigenvectors of discrete time-invariant operators are sinusoidal waves. The Fourier transform is discretized for signals of finite size and implemented with a Fast Fourier Transform algorithm.

3.1 Sampling Analog Signals

The simplest way to discretize an analog signal f is to record its sample values $\{f(ns)\}_{n\in\mathbb{Z}}$ at intervals s. An approximation of $f(t)$ at any $t \in \mathbb{R}$ may be recovered by interpolating these samples. The Shannon-Whittaker sampling theorem gives a sufficient condition on the support of the Fourier transform $\hat{f}$ to recover $f(t)$ exactly. Aliasing and approximation errors are studied when this condition is not satisfied.

Digital acquisition devices often do not satisfy the restrictive hypothesis of the Shannon-Whittaker sampling theorem. General linear analog to discrete conversion is introduced in Section 3.1.3, showing that a stable uniform discretization is a linear approximation. A digital conversion also approximates discrete coefficients with a given precision, to store them with a limited number of bits. This quantization aspect is studied in Chapter 10.

3.1.1 Shannon-Whittaker Sampling Theorem

Sampling is first studied from the more classical Shannon-Whittaker point of view, which tries to recover $f(t)$ from its samples $\{f(ns)\}_{n\in\mathbb{Z}}$. A discrete signal may be represented as a sum of Diracs. We associate to any sample $f(ns)$ a Dirac $f(ns)\delta(t-ns)$ located at $t=ns$. A uniform sampling of f thus corresponds to the weighted Dirac sum

$$f_d(t) = \sum_{n=-\infty}^{+\infty} f(ns)\,\delta(t-ns). \tag{3.1}$$

The Fourier transform of $\delta(t-ns)$ is $\mathrm{e}^{-ins\omega}$ so the Fourier transform of f_d is a Fourier series:

$$\hat{f}_d(\omega) = \sum_{n=-\infty}^{+\infty} f(ns)\,\mathrm{e}^{-ins\omega}. \tag{3.2}$$

To understand how to compute $f(t)$ from the sample values $f(ns)$ and hence f from f_d, we relate their Fourier transforms $\hat{f}$ and $\hat{f}_d$.

Theorem 3.1. *The Fourier transform of the discrete signal obtained by sampling f at intervals s is*

$$\hat{f}_d(\omega) = \frac{1}{s} \sum_{k=-\infty}^{+\infty} \hat{f}\left(\omega - \frac{2k\pi}{s}\right). \tag{3.3}$$

Proof. Since $\delta(t - ns)$ is zero outside $t = ns$,

$$f(ns)\,\delta(t - ns) = f(t)\,\delta(t - ns),$$

so we can rewrite (3.1) as multiplication with a Dirac comb:

$$f_d(t) = f(t) \sum_{n=-\infty}^{+\infty} \delta(t - ns) = f(t)\,c(t). \tag{3.4}$$

Computing the Fourier transform yields

$$\hat{f}_d(\omega) = \frac{1}{2\pi}\,\hat{f} \star \hat{c}(\omega). \tag{3.5}$$

The Poisson formula (2.4) proves that

$$\hat{c}(\omega) = \frac{2\pi}{s} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{s}\right). \tag{3.6}$$

Since $\hat{f} \star \delta(\omega - \xi) = \hat{f}(\omega - \xi)$, inserting (3.6) in (3.5) proves (3.3). ■ ■

Theorem 3.1 proves that sampling f at intervals s is equivalent to making its Fourier transform $2\pi/s$ periodic by summing all its translations $\hat{f}(\omega - 2k\pi/s)$. The resulting sampling theorem was first proved by Whittaker [482] in 1935 in a book on interpolation theory. Shannon rediscovered it in 1949 for applications to communication theory [428].

Theorem 3.2 (Shannon, Whittaker)**.** *If the support of $\hat{f}$ is included in $[-\pi/s, \pi/s]$ then*

$$f(t) = \sum_{n=-\infty}^{+\infty} f(ns)\,\phi_s(t - ns), \tag{3.7}$$

with

$$\phi_s(t) = \frac{\sin(\pi t/s)}{\pi t/s}. \tag{3.8}$$

Proof. If $n \neq 0$, the support of $\hat{f}(\omega - n\pi/s)$ does not intersect the support of $\hat{f}(\omega)$ because $\hat{f}(\omega) = 0$ for $|\omega| > \pi/s$. So (3.3) implies

$$\hat{f}_d(\omega) = \frac{\hat{f}(\omega)}{s} \quad \text{if} \quad |\omega| \leqslant \frac{\pi}{s}. \tag{3.9}$$

The Fourier transform of ϕ_s is $\hat{\phi}_s = s\,\mathbf{1}_{[-\pi/s,\pi/s]}$. Since the support of $\hat{f}$ is in $[-\pi/s, \pi/s]$ it results from (3.9) that $\hat{f}(\omega) = \hat{\phi}_s(\omega)\,\hat{f}_d(\omega)$. The inverse Fourier transform of this equality gives

$$\begin{aligned} f(t) = \phi_s \star f_d(t) &= \phi_s \star \sum_{n=-\infty}^{+\infty} f(ns)\,\delta(t - ns) \\ &= \sum_{n=-\infty}^{+\infty} f(ns)\,\phi_s(t - ns). \end{aligned}$$

■ ■

The sampling theorem imposes that the support of $\hat{f}$ is included in $[-\pi/s, \pi/s]$, which guarantees that f has no brutal variations between consecutive samples, and can thus be recovered with a smooth interpolation. Section 3.1.3 shows that one can impose other smoothness conditions to recover f from its samples. Figure 3.1 illustrates the different steps of a sampling and reconstruction from samples, in both the time and Fourier domains.

3.1.2 Aliasing

The sampling interval s is often imposed by computation or storage constraints and the support of $\hat{f}$ is generally not included in $[-\pi/s, \pi/s]$. In this case the interpolation formula (3.7) does not recover f. We analyze the resulting error and a filtering procedure to reduce it.

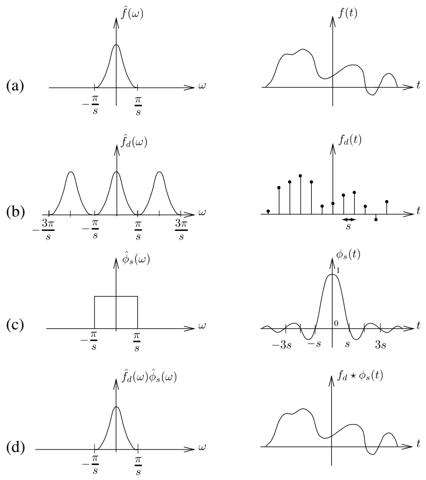

Figure 3.1: (a): Signal f and its Fourier transform $\hat{f}$. (b): A uniform sampling of f makes its Fourier transform periodic. (c): Ideal low-pass filter. (d): The filtering of (b) with (c) recovers f.

Theorem 3.1 proves that

$$\hat{f}_d(\omega) = \frac{1}{s} \sum_{k=-\infty}^{+\infty} \hat{f}\left(\omega - \frac{2k\pi}{s}\right). \tag{3.10}$$

Suppose that the support of $\hat{f}$ goes beyond $[-\pi/s, \pi/s]$. In general the support of $\hat{f}(\omega - 2k\pi/s)$ intersects $[-\pi/s, \pi/s]$ for several $k \neq 0$, as shown in Figure 3.2. This folding of high frequency components over a low frequency interval is called *aliasing*. In the presence of aliasing, the interpolated signal

$$\phi_s \star f_d(t) = \sum_{n=-\infty}^{+\infty} f(ns)\, \phi_s(t - ns)$$

has a Fourier transform

$$\hat{f}_d(\omega)\, \hat{\phi}_s(\omega) = s\, \hat{f}_d(\omega)\, \mathbf{1}_{[-\pi/s,\pi/s]}(\omega) = \mathbf{1}_{[-\pi/s,\pi/s]}(\omega) \sum_{k=-\infty}^{+\infty} \hat{f}\left(\omega - \frac{2k\pi}{s}\right) \tag{3.11}$$

which may be completely different from $\hat{f}(\omega)$ over $[-\pi/s, \pi/s]$. The signal $\phi_s \star f_d$ may not even be a good approximation of f, as shown by Figure 3.2.

Example 3.1. *Let us consider a high frequency oscillation*

$$f(t) = \cos(\omega_0 t) = \frac{\mathrm{e}^{i\omega_0 t} + \mathrm{e}^{-i\omega_0 t}}{2}.$$

Its Fourier transform is

$$\hat{f}(\omega) = \pi\Big(\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\Big).$$

If $2\pi/s > \omega_0 > \pi/s$ then (3.11) yields

$$\begin{aligned}
&\hat{f}_d(\omega)\, \hat{\phi}_s(\omega) \\
&\quad = \pi\, \mathbf{1}_{[-\pi/s,\pi/s]}(\omega) \sum_{k=-\infty}^{+\infty} \left(\delta\Big(\omega - \omega_0 - \frac{2k\pi}{s}\Big) + \delta\Big(\omega + \omega_0 - \frac{2k\pi}{s}\Big)\right) \\
&\quad = \pi\Big(\delta(\omega - \frac{2\pi}{s} + \omega_0) + \delta(\omega + \frac{2\pi}{s} - \omega_0)\Big),
\end{aligned}$$

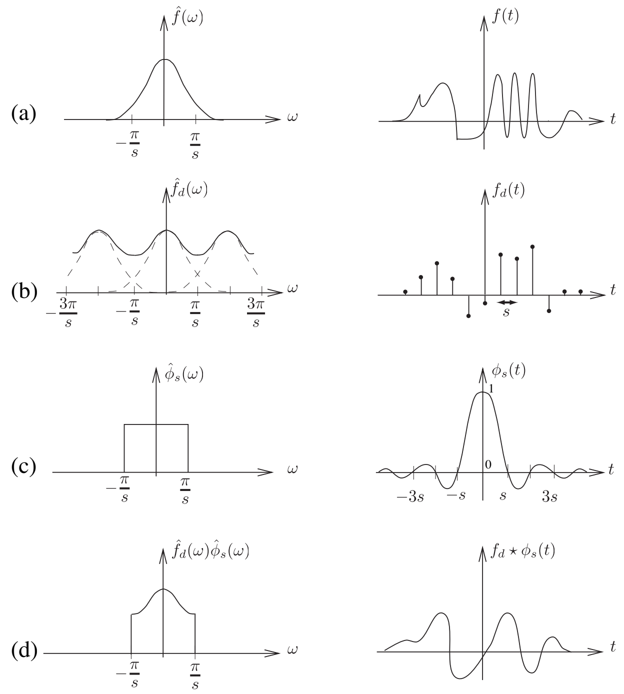

Figure 3.2: (a): Signal f and its Fourier transform $\hat{f}$. (b): Aliasing produced by an overlapping of $\hat{f}(\omega - 2k\pi/s)$ for different k, shown in dashed lines. (c): Ideal low-pass filter. (d): The filtering of (b) with (c) creates a low-frequency signal that is different from f.

so

$$f_d \star \phi_s(t) = \cos\left[\left(\frac{2\pi}{s} - \omega_0\right)t\right].$$

The aliasing reduces the high frequency ω_0 to a lower frequency $2\pi/s - \omega_0 \in [-\pi/s, \pi/s]$. The same frequency folding is observed in a film that samples a fast moving object without enough images per second. A wheel turning rapidly appears as turning much more slowly in the film.

Removal of Aliasing To apply the sampling theorem, f is approximated by the closest signal $\tilde{f}$ whose Fourier transform has a support in $[-\pi/s, \pi/s]$. The Plancherel formula (2.26) proves that

$$\begin{aligned} \|f - \tilde{f}\|^2 &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} |\hat{f}(\omega) - \widehat{\tilde{f}}(\omega)|^2\, d\omega \\ &= \frac{1}{2\pi}\int_{|\omega|>\pi/s} |\hat{f}(\omega)|^2\, d\omega + \frac{1}{2\pi}\int_{|\omega|\leqslant\pi/s} |\hat{f}(\omega) - \widehat{\tilde{f}}(\omega)|^2\, d\omega. \end{aligned}$$

This distance is minimum when the second integral is zero and hence

$$\widehat{\tilde{f}}(\omega) = \hat{f}(\omega)\, \mathbf{1}_{[-\pi/s,\pi/s]}(\omega) = \frac{1}{s}\, \hat{\phi}_s(\omega)\, \hat{f}(\omega). \tag{3.12}$$

It corresponds to $\tilde{f} = \frac{1}{s}\, f \star \phi_s$. The filtering of f by ϕ_s avoids the aliasing by removing any frequency larger than π/s. Since $\widehat{\tilde{f}}$ has a support in $[-\pi/s, \pi/s]$, the sampling theorem proves that $\tilde{f}(t)$ can be recovered from the samples $\tilde{f}(ns)$. An analog to digital converter is therefore composed of a filter that limits the frequency band to $[-\pi/s, \pi/s]$, followed by a uniform sampling at intervals s.

3.1.3 General Sampling and Linear Analog Conversions

The Shannon-Whittaker theorem is a particular example of linear discrete to analog conversion, which does not apply to all digital acquisition devices. This section describes general analog to discrete conversion and reverse discrete to analog conversion, with a general linear filtering and uniform sampling. Analog signals are approximated by linear projections on approximation spaces.

Sampling Theorems We want to recover a stable approximation of $f \in \mathbf{L^2}(\mathbb{R})$ from a filtering and uniform sampling that outputs $\{f \star \bar{\phi}_s(ns)\}_{n\in\mathbb{Z}}$, for some real filter $\bar{\phi}_s(t)$. These samples can be written as inner products in $\mathbf{L^2}(\mathbb{R})$:

$$f \star \phi_s(ns) = \int_{-\infty}^{+\infty} f(t)\, \bar{\phi}_s(ns - t)\, dt = \langle f(t), \phi_s(t - ns)\rangle \tag{3.13}$$

with $\phi_s(t) = \bar{\phi}_s(-t)$. Let $\mathbf{U}_s$ be the approximation space generated by linear combinations of the $\{\phi_s(t-ns)\}_{n\in\mathbb{Z}}$. The approximation $\tilde{f} \in \mathbf{U}_s$ which minimizes the maximum possible error $\|f - \tilde{f}\|$ is the orthognal projection of f on $\mathbf{U}_s$ (Exercice 3.5). The calculation of this orthogonal projection is stable if $\{\phi_s(t - ns)\}_{n\in\mathbb{Z}}$ is a Riesz basis of $\mathbf{U}_s$, as defined in Section 5.1.1. Following Definition 5.1, a Riesz basis is a family of linearly independent functions that yields inner product satisfying an energy equivalence. There exists $B \geqslant A > 0$ such that for any $f \in \mathbf{U}_s$

$$A\,\|f\|^2 \leqslant \sum_{n=-\infty}^{+\infty} |\langle f(t), \phi_s(t - ns)\rangle|^2 \leqslant B\,\|f\|^2. \tag{3.14}$$

The basis is orthogonal if and only if $A = B$. The following generalized sampling theorem computes the orthogonal projection on the approximation space $\mathbf{U}_s$ [468].

Theorem 3.3 (Linear sampling). *Let $\{\phi_s(t-ns)\}_{n\in\mathbb{Z}}$ be a Riesz basis of $\mathbf{U}_s$ and $\bar{\phi}_s(t) = \phi_s(-t)$. There exists a biorthogonal basis $\{\tilde{\phi}_s(t - ns)\}_{n\in\mathbb{Z}}$ of $\mathbf{U}_s$ such that*

$$\forall f \in \mathbf{L^2}(\mathbb{R}) \quad , \quad P_{\mathbf{U}_s} f(t) = \sum_{n=-\infty}^{+\infty} f \star \bar{\phi}_s(ns)\, \tilde{\phi}_s(t - ns)\ . \tag{3.15}$$

Proof. For any Riesz basis, Section 5.1.2 proves that there exists a biorthogonal basis $\{\tilde{\phi}_{s,n}(t)\}_{n\in\mathbb{Z}}$ that satisfies the biorthogonality relations

$$\forall (n,m) \in \mathbb{Z}^2 \quad , \quad \langle \phi_s(t - ns)\, ,\, \tilde{\phi}_{s,m}(t - ms)\rangle = \delta[n - m]\ . \tag{3.16}$$

Since $\langle \phi_s(t - (n-m)s)\, ,\, \tilde{\phi}_{s,0}(t)\rangle = \langle \phi_s(t - ns)\, ,\, \tilde{\phi}_{s,0}(t - ms)\rangle = 0$ and since the dual basis is unique, necessarily $\tilde{\phi}_{s,m}(t) = \tilde{\phi}_{s,0}(t - ms)$. Section 5.1.2 proves in (5.20) that the orthogonal projection in $\mathbf{U}_s$ can be written

$$P_{\mathbf{U}_s} f(t) = \sum_{n=-\infty}^{+\infty} \langle f(t), \phi_s(t - ns)\rangle\, \tilde{\phi}_s(t - ns)$$

which proves (3.15). ■

The orthogonal projection (3.15) can be rewritten as an analog filtering of the discrete signal $f_d(t) = \sum_{n=-\infty}^{+\infty} f \star \bar{\phi}_s(ns)\, \delta(t - ns)$:

$$P_{\mathbf{U}_s} f(t) = f_d \star \tilde{\phi}_s(t)\ . \tag{3.17}$$

If $f \in \mathbf{U}_s$ then $P_{\mathbf{U}_s} f = f$ so it is exactly reconstructed by filtering the uniformly sampled discrete signal $\{f \star \bar{\phi}_s(ns)\}_{n\in\mathbb{Z}}$ with the analog filter $\tilde{\phi}(t)$. If $f \notin \mathbf{U}_s$ then (3.17) recovers the best linear approximation approximation of f in $\mathbf{U}_s$. Section 9.1 shows that the linear approximation error $\|f - P_{\mathbf{U}_s} f\|$ depends essentially on the uniform regularity of f. Given some prior information on f, optimizing the analog discretization filter ϕ_s amounts to optimize the approximation space $\mathbf{U}_s$ in order to minimize this error. The following theorem characterizes filters ϕ_s that generate a Riesz basis and computes the dual filter.

Theorem 3.4. *A filter ϕ_s generates a Riesz basis $\{\phi_s(t-ns)\}_{n\in\mathbb{Z}}$ of a space $\mathbf{U}_s$ if and only if there exists $B \geqslant A > 0$ such that*

$$\forall\omega \in [0, 2\pi/s] \quad , \quad A \leqslant \frac{1}{s}\sum_{k=-\infty}^{+\infty} |\hat{\phi}_s(\omega - \frac{2k\pi}{s})|^2 \leqslant B \ . \tag{3.18}$$

The biorthogonal basis $\{\tilde{\phi}_s(t-ns)\}_{n\in\mathbb{Z}}$ is defined by the dual filter $\tilde{\phi}_s$ which satisfies:

$$\widehat{\tilde{\phi}_s}(\omega) = \frac{s\,\hat{\phi}_s^*(\omega)}{\sum_{k=-\infty}^{+\infty} |\hat{\phi}_s(\omega - 2k\pi/s)|^2} \ . \tag{3.19}$$

Proof. Theorem 5.1 proves that $\{\phi_s(t-ns)\}_{n\in\mathbb{Z}}$ is a Riesz basis of $\mathbf{U}_s$ with Riesz bounds $B \geqslant A > 0$ if and only if

$$\forall a \in \boldsymbol{\ell}^2(\mathbb{Z}) \quad , \quad A\|a\|^2 \leqslant \|\sum_{n\in\mathbb{Z}} a[ns]\,\phi_s(t-ns)\|^2 \leqslant B\|a\|^2 \ , \tag{3.20}$$

with $\|a\|^2 = \sum_{n\in\mathbb{Z}} |a[ns]|^2$.

Let us first write these conditions in the Fourier domain. The Fourier transform of $f(t) = \sum_{n=-\infty}^{+\infty} a[ns]\,\phi_s(t-ns)$ is

$$\hat{f}(\omega) = \sum_{n=-\infty}^{+\infty} a[ns]\,\mathrm{e}^{-ins\omega}\,\hat{\phi}_s(\omega) = \hat{a}(\omega)\,\hat{\phi}_s(\omega) \tag{3.21}$$

where $\hat{a}(\omega)$ is the Fourier series $\hat{a}(\omega) = \sum_{n=-\infty}^{+\infty} a[ns]\,\mathrm{e}^{-ins\omega}$. Let us relate the norm of f and $\hat{a}$. Since $\hat{a}(\omega)$ is $2\pi/s$ periodic, inserting (3.21) in the Plancherel formula (2.26) gives

$$\begin{aligned} \|f\|^2 &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} |\hat{f}(\omega)|^2\,d\omega = \frac{1}{2\pi}\int_0^{2\pi/s} \sum_{k=-\infty}^{+\infty} |\hat{a}(\omega+2k\pi/s)|^2\,|\hat{\phi}_s(\omega+2k\pi/s)|^2\,d\omega \\ &= \frac{1}{2\pi}\int_0^{2\pi/s} |\hat{a}(\omega)|^2 \sum_{k=-\infty}^{+\infty} |\hat{\phi}_s(\omega+2k\pi/s)|^2\,d\omega \ . \end{aligned} \tag{3.22}$$

Section 3.2.2 on Fourier series proves that

$$\|a\|^2 = \sum_{n=-\infty}^{+\infty} |a[ns]|^2 = \frac{s}{2\pi}\int_0^{2\pi/s} |\hat{a}(\omega)|^2\,d\omega \ . \tag{3.23}$$

As a consequence of (3.22) and (3.23), the Riesz bound inequalities (3.20) are equivalent to

$$\forall \hat{a} \in \mathbf{L^2}[0,2\pi/s] \quad , \quad \frac{1}{2\pi}\int_0^{2\pi/s} |\hat{a}(\omega)|^2 \sum_{k=-\infty}^{+\infty} |\hat{\phi}_s(\omega+2k\pi/s)|^2\,d\omega \leqslant \frac{B\,s}{2\pi}\int_0^{2\pi/s} |\hat{a}(\omega)|^2\,d\omega \tag{3.24}$$

and

$$\forall \hat{a} \in \mathbf{L^2}[0,2\pi/s] \quad , \quad \frac{1}{2\pi}\int_0^{2\pi/s} |\hat{a}(\omega)|^2 \sum_{k=-\infty}^{+\infty} |\hat{\phi}_s(\omega+2k\pi/s)|^2\,d\omega \geqslant \frac{A\,s}{2\pi}\int_0^{2\pi/s} |\hat{a}(\omega)|^2\,d\omega \ . \tag{3.25}$$

If $\hat{\phi}_s$ satisfies (3.18) then clearly (3.24) and (3.25) are valid, which proves (3.22).

Conversely, if $\{\phi_s(ns-t)\}_{n\in\mathbb{Z}}$ is a Riesz basis. Suppose that either the upper or the lower bound of (3.18) is not satisfied for ω in a set of non-zero measure. Let $\hat{a}$ be the indicator function of this set. Then either (3.24) or (3.25) are not valid for this $\hat{a}$. This implies that the Riesz bounds (3.20) are not valid for a and hence that it is not a Riesz basis, which contradicts our hypothesis. So (3.18) is indeed valid for almost all $\omega \in [0, 2\pi/s]$.

To compute the biorthogonal basis, we are looking for $\tilde{\phi}_s \in \mathbf{U}_s$ such that $\{\tilde{\phi}_s(t-ns)\}_{n\in\mathbb{Z}}$ satisfies the biorthogonal relations (3.16). Since $\tilde{\phi}_s \in \mathbf{U}_s$ we saw in (3.21) that its Fourier transform can be written $\widehat{\tilde{\phi}_s}(\omega) = \hat{a}(\omega)\hat{\phi}_s(\omega)$ where $\hat{a}(\omega)$ is $2\pi/s$ periodic. Let us define $g(t) = \bar{\phi}_s \star \tilde{\phi}_s(t)$. Its Fourier transform is

$$\hat{g}(\omega) = \hat{\phi}_s^*(\omega)\,\hat{\tilde{\phi}}_s(\omega) = \hat{a}(\omega)|\hat{\phi}_s(\omega)|^2 \ .$$

The biorthogonal relations (3.16) are satisfied if and only if $g(ns) = 0$ if $n \neq 0$ and $g(0) = 1$. It results that $g_d(t) = \sum_{n=-\infty}^{+\infty} g(ns)\,\delta(t - ns) = \delta(t)$. Theorem 3.1 derives in (3.3) that

$$\hat{g}_d(\omega) = \frac{1}{s} \sum_{k=-\infty}^{+\infty} \hat{g}\left(\omega - 2k\pi/s\right) = \frac{\hat{a}(\omega)}{s} \sum_{k=-\infty}^{+\infty} |\hat{\phi}_s(\omega - 2k\pi/s)|^2 = 1.$$

It results that

$$\hat{a}(\omega) = s \left(\sum_{k=-\infty}^{+\infty} |\hat{\phi}_s(\omega - 2k\pi/s)|^2 \right)^{-1}$$

which proves (3.19). ■ ■

This theorem gives a necessary and sufficient condition on the low-pass filter $\bar{\phi}_s(t) = \phi_s(-t)$ to recover a stable signal approximation from a uniform sampling at intervals s. For various sampling intervals s, the low-pass filter can be obtained by dilating a single filter $\phi_s(t) = s^{-1/2}\phi(t/s)$ and hence $\hat{\phi}_s(\omega) = s^{1/2}\hat{\phi}(s\omega)$. The necessary and sufficient Riesz basis condition (3.18) is then satisfied if and only if

$$\forall \omega \in [-\pi, \pi] \quad , \quad A \leqslant \sum_{k=-\infty}^{+\infty} |\hat{\phi}(\omega - 2k\pi)|^2 \leqslant B. \tag{3.26}$$

It results from (3.19) that the dual filter satisfies $\widehat{\tilde{\phi}_s}(\omega) = s^{1/2}\widehat{\tilde{\phi}}(s\omega)$ and hence $\tilde{\phi}_s(t) = s^{-1/2}\tilde{\phi}(t/s)$. When $A = B = 1$ the Riesz basis is an orthonormal basis, which proves the following corollary.

Corollary 3.1. *The family $\{\phi_s(t - ns)\}_{n\in\mathbb{Z}}$ is an orthonormal basis of the space $\mathbf{U}_s$ it generates, with $\phi_s(t) = s^{-1/2}\phi(t/s)$, if and only if*

$$\forall \omega \in [0, 2\pi] \quad , \quad \sum_{k=-\infty}^{+\infty} |\hat{\phi}(\omega - 2k\pi)|^2 = 1 \ , \tag{3.27}$$

and the dual filter is $\tilde{\phi}_s = \phi_s$.

Shannon-Whittaker revisited The Shannon-Whittaker theorem 3.2 is defined with a sine-cardinal perfect low-pass filter ϕ_s, which we renormalize here to have a unit norm. The following theorem proves that it samples functions in an orthonormal basis.

Theorem 3.5. *If $\phi_s(t) = s^{1/2}\sin(\pi s^{-1}t)/(\pi t)$ then $\{\phi_s(t - ns)\}_{n\in\mathbb{Z}}$ is an orthonormal basis of the space $\mathbf{U}_s$ of functions whose Fourier transforms have a support included in $[-\pi/s, \pi/s]$. If $f \in \mathbf{U}_s$ then*

$$f(nT) = s^{-1/2}\, f \star \phi_s(ns) \ . \tag{3.28}$$

Proof. The filter satisfies $\phi_s(t) = s^{-1/2}\phi(t/s)$ with $\phi(t) = \sin(\pi t)/(\pi t)$. The Fourier transform $\hat{\phi}(\omega) = \mathbf{1}_{[-\pi,\pi]}(\omega)$ satisfies the condition (3.27) of Corollary 3.1, which proves that $\{\phi_s(t - ns)\}_{n\in\mathbb{Z}}$ is an orthonormal basis of a space $\mathbf{U}_s$.

Any $f(t) = \sum_{n=-\infty}^{+\infty} a[ns]\,\phi_s(t - ns) \in \mathbf{U}_s$ has a Fourier transform which can be written

$$\hat{f}(\omega) = \sum_{n=-\infty}^{+\infty} a[ns]\,\mathrm{e}^{-ins\omega}\,\hat{\phi}_s(\omega) = \hat{a}(\omega)\,s^{1/2}\,\mathbf{1}_{[-\pi/s,\pi/s]} \ , \tag{3.29}$$

which implies that $f \in \mathbf{U}_s$ if and only if f has a Fourier transform supported in $[-\pi/s, \pi/s]$.

If $f \in \mathbf{U}_s$ then decomposing it the orthonormal basis $\{\phi_s(t - ns)\}_{n\in\mathbb{Z}}$ gives

$$f(t) = P_{\mathbf{U}_s} f(t) = \sum_{n\in\mathbb{Z}} \langle f(u), \phi_s(u - ns)\rangle\, \phi_s(t - ns) \ .$$

Since $\phi_s(ps) = s^{-1/2}\delta[ps]$ and $\phi_s(-t) = \phi_s(t)$, it results that

$$f(ns) = s^{-1/2}\langle f(u), \phi_s(u - ns)\rangle = s^{-1/2}\, f \star \phi_s(ns) \ .$$

■

This theorem proves that in the particular case of the Shannon-Whittaker sampling theorem, if $f \in \mathbf{U}_s$ then the sampled low-pass filtered values $f \star \phi_s(ns)$ are proportional to the signal samples $f(ns)$. This comes from the fact that the sine-cardinal $\phi(t) = \sin(\pi t/s)/(\pi t/s)$ satisfies the interpolation property $\phi(ns) = \delta[ns]$. A generalization of such multiscale interpolations is studied in Section 7.6.

The Shannon-Whittaker sampling approximates signals by restricting their Fourier transform to a low frequency interval. It is particularly effective for smooth signals whose Fourier transform have an energy concentrated at low frequencies. It is also well adapted to sound recordings, which are well approximated by lower frequency harmonics.

For discontinuous signals such as images, a low-frequency restriction produces Gibbs oscillations studied in Section 2.3.3. The image visual quality is degraded by these oscillations, which have a total variation (2.65) that is infinite. A piecewise constant approximation has the advantage of creating no such spurious oscillations.

Block Sampler A block sampler approximates signals with piecewise constant functions. The approximation space $\mathbf{U}_s$ is the set of all functions that are constant on intervals $[ns, (n+1)s]$, for any $n \in \mathbb{Z}$. Let $\phi_s(t) = s^{-1/2}\, \mathbf{1}_{[0,s]}(t)$. The family $\{\phi_s(t - ns]\}_{n\in\mathbb{Z}}$ is an orthonormal basis of $\mathbf{U}_s$ (Exercise 3.1). If $f \notin \mathbf{U}_s$ then its orthogonal projection on $\mathbf{U}_s$ is calculated with a partial decomposition in the block orthonormal basis of $\mathbf{U}_s$

$$P_{\mathbf{U}_s} f(t) = \sum_{n=-\infty}^{+\infty} \langle f(u), \phi_s(u - ns)\rangle\, \phi_s(t - ns), \tag{3.30}$$

and each coefficient is proportional to the signal average on $[ns, (n+1)s]$

$$\langle f(u), \phi_s(u - ns)\rangle = f \star \phi_s(ns) = s^{-1/2} \int_{ns}^{(n+1)s} f(u)\, du \;.$$

This block analog to digital conversion is particularly simple to implement in analog electronic, where the integration is performed by a capacity.

In domains where f is a regular function, a piecewise constant approximation $P_{\mathbf{U}_s} f$ is not very precise and can be significantly improved. More precise approximations are obtained with approximation spaces $\mathbf{U}_s$ of higher order polynomial splines. The resulting approximations can introduce small Gibbs oscillations, but these oscillations have a finite total variation.

Spline Sampling Block samplers are generalized by a spline sampling with a space $\mathbf{U}_s$ of splines functions that are $m-1$ times continuously differentiable and equal to a polynomial of degree m on any interval $[ns, (n+1)s)$, for $n \in \mathbb{Z}$. When $m = 1$, functions in $\mathbf{U}_s$ are piecewise linear and continuous.

A Riesz basis of polynomial splines is constructed with *box splines*. A box spline ϕ of degree m is computed by convolving the box window $\mathbf{1}_{[0,1]}$ with itself $m+1$ times and centering it at 0 or 1/2. Its Fourier transform is

$$\hat{\phi}(\omega) = \left(\frac{\sin(\omega/2)}{\omega/2}\right)^{m+1} \exp\left(\frac{-i\varepsilon\omega}{2}\right). \tag{3.31}$$

If m is even then $\varepsilon = 1$ and ϕ has a support centered at $t = 1/2$. If m is odd then $\varepsilon = 0$ and $\phi(t)$ is symmetric about $t = 0$. One can verify that $\hat{\phi}(\omega)$ satisfies the sampling condition (3.26) using a closed form expression (7.19) of the resulting series. It results that for any $s > 0$, a box splines family $\{\phi_s(t - ns)\}_{n\in\mathbb{Z}}$ defines a Riesz basis of $\mathbf{U}_s$, and thus a stable sampling.

3.2 Discrete Time-Invariant Filters

3.2.1 Impulse Response and Transfer Function

Classical discrete signal processing algorithms are mostly based on time-invariant linear operators [50, 54]. The time-invariance is limited to translations on the sampling grid. To simplify notation,