VI

Wavelet Zoom

A wavelet transform can focus on localized signal structures with a zooming procedure that progressively reduces the scale parameter. Singularities and irregular structures often carry essential information in a signal. For example, discontinuities in images may correspond to occlusion contours of objects in a scene. The wavelet transform amplitude across scales is related to the local signal regularity and Lipschitz exponents. Singularities and edges are detected from wavelet transform local maxima at multiple scales. These maxima define a geometric scale-space support, from which signal and image approximations are recovered.

Non-isolated singularities appear in highly irregular signals such as multifractals. The wavelet transform takes advantage of multifractal self-similarities to compute the distribution of their singularities. This singularity spectrum characterizes multifractal properties. Throughout this chapter, wavelets are real functions.

6.1 Lipschitz Regularity

To characterize singular structures, it is necessary to precisely quantify the local regularity of a signal $f(t)$. Lipschitz exponents provide uniform regularity measurements over time intervals, but also at any point v. If f has a singularity at v, which means that it is not differentiable at v, then the Lipschitz exponent at v characterizes this singular behavior.

The next section relates the uniform Lipschitz regularity of f over $\mathbb{R}$ to the asymptotic decay of the amplitude of its Fourier transform. This global regularity measurement is useless in analyzing the signal properties at particular locations. Section 6.1.3 studies zooming procedures that measure local Lipschitz exponents from the decay of the wavelet transform amplitude at fine scales.

6.1.1 Lipschitz Definition and Fourier Analysis

The Taylor formula relates the differentiability of a signal to local polynomial approximations. Suppose that f is m times differentiable in $[v-h, v+h]$. Let p_v be the Taylor polynomial in the neighborhood of v:

$$p_v(t) = \sum_{k=0}^{m-1} \frac{f^{(k)}(v)}{k!} (t-v)^k. \tag{6.1}$$

The Taylor formula proves that the approximation error

$$\varepsilon_v(t) = f(t) - p_v(t)$$

satisfies

$$\forall t \in [v-h, v+h] \quad , \quad |\varepsilon_v(t)| \leqslant \frac{|t-v|^m}{m!} \sup_{u\in[v-h,v+h]} |f^m(u)|. \tag{6.2}$$

The m^{th} order differentiability of f in the neighborhood of v yields an upper bound on the error $\varepsilon_v(t)$ when t tends to v. The Lipschitz regularity refines this upper bound with non-integer exponents. Lipschitz exponents are also called *Hölder* exponents in the mathematical literature.

Definition 6.1 (Lipschitz).

- *A function f is pointwise Lipschitz $\alpha \geqslant 0$ at v, if there exist $K > 0$, and a polynomial p_v of degree $m = \lfloor \alpha \rfloor$ such that*

$$\forall t \in \mathbb{R} \quad , \quad |f(t) - p_v(t)| \leqslant K\,|t - v|^{\alpha}. \tag{6.3}$$

- *A function f is uniformly Lipschitz α over $[a, b]$ if it satisfies (6.3) for all $v \in [a, b]$, with a constant K that is independent of v.*
- *The Lipschitz regularity of f at v or over $[a, b]$ is the sup of the α such that f is Lipschitz α.*

At each v the polynomial $p_v(t)$ is uniquely defined. If f is $m = \lfloor \alpha \rfloor$ times continuously differentiable in a neighborhood of v, then p_v is the Taylor expansion of f at v. Pointwise Lipschitz exponents may vary arbitrarily from abscissa to abscissa. One can construct multifractal functions with non-isolated singularities, where f has a different Lipschitz regularity at each point. In contrast, uniform Lipschitz exponents provide a more global measurement of regularity, which applies to a whole interval. If f is uniformly Lipschitz $\alpha > m$ in the neighborhood of v then one can verify that f is necessarily m times continuously differentiable in this neighborhood.

If $0 \leqslant \alpha < 1$ then $p_v(t) = f(v)$ and the Lipschitz condition (6.3) becomes

$$\forall t \in \mathbb{R} \quad , \quad |f(t) - f(v)| \leqslant K\,|t - v|^{\alpha}.$$

A function that is bounded but discontinuous at v is Lipschitz 0 at v. If the Lipschitz regularity is $\alpha < 1$ at v, then f is not differentiable at v and α characterizes the singularity type.

Fourier Condition The uniform Lipschitz regularity of f over $\mathbb{R}$ is related to the asymptotic decay of its Fourier transform. The following theorem can be interpreted as a generalization of Theorem 2.5.

Theorem 6.1. *A function f is bounded and uniformly Lipschitz α over $\mathbb{R}$ if*

$$\int_{-\infty}^{+\infty} |\hat{f}(\omega)|\,(1 + |\omega|^{\alpha})\,d\omega < +\infty. \tag{6.4}$$

Proof. To prove that f is bounded, we use the inverse Fourier integral (2.8) and (6.4) which shows that

$$|f(t)| \leqslant \int_{-\infty}^{+\infty} |\hat{f}(\omega)|\,d\omega < +\infty.$$

Let us now verify the Lipschitz condition (6.3) when $0 \leqslant \alpha \leqslant 1$. In this case $p_v(t) = f(v)$ and the uniform Lipschitz regularity means that there exists $K > 0$ such that for all $(t, v) \in \mathbb{R}^2$

$$\frac{|f(t) - f(v)|}{|t - v|^{\alpha}} \leqslant K.$$

Since

$$\begin{aligned} f(t) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{f}(\omega)\,\exp(i\omega t)\,d\omega, \\ \frac{|f(t) - f(v)|}{|t - v|^{\alpha}} &\leqslant \frac{1}{2\pi} \int_{-\infty}^{+\infty} |\hat{f}(\omega)|\,\frac{|\exp(i\omega t) - \exp(i\omega v)|}{|t - v|^{\alpha}}\,d\omega. \end{aligned} \tag{6.5}$$

For $|t - v|^{-1} \leqslant |\omega|$,

$$\frac{|\exp(i\omega t) - \exp(i\omega v)|}{|t - v|^{\alpha}} \leqslant \frac{2}{|t - v|^{\alpha}} \leqslant 2\,|\omega|^{\alpha}.$$

For $|t - v|^{-1} \geqslant |\omega|$,

$$\frac{|\exp(i\omega t) - \exp(i\omega v)|}{|t - v|^{\alpha}} \leqslant \frac{|\omega|\,|t - v|}{|t - v|^{\alpha}} \leqslant |\omega|^{\alpha}.$$

Cutting the integral (6.5) in two for $|\omega| < |t-v|^{-1}$ and $|\omega| \geqslant |t-v|^{-1}$ yields

$$\frac{|f(t)-f(v)|}{|t-v|^\alpha} \leqslant \frac{1}{2\pi}\int_{-\infty}^{+\infty} 2\,|\hat f(\omega)|\,|\omega|^\alpha\, d\omega = K.$$

If (6.4) is satisfied, then $K < +\infty$ so f is uniformly Lipschitz α.

Let us extend this result for $m = \lfloor \alpha \rfloor > 0$. We proved in (2.42) that (6.4) implies that f is m times continuously differentiable. One can verify that f is uniformly Lipschitz α over $\mathbb{R}$ if and only if $f^{(m)}$ is uniformly Lipschitz $\alpha - m$ over $\mathbb{R}$. The Fourier transform of $f^{(m)}$ is $(i\omega)^m \hat f(\omega)$. Since $0 \leqslant \alpha - m < 1$, we can use our previous result which proves that $f^{(m)}$ is uniformly Lipschitz $\alpha - m$, and hence that f is uniformly Lipschitz α. ■ ■

The Fourier transform is a powerful tool for measuring the minimum global regularity of functions. However, it is not possible to analyze the regularity of f at a particular point v from the decay of $|\hat f(\omega)|$ at high frequencies ω. In contrast, since wavelets are well localized in time, the wavelet transform gives Lipschitz regularity over intervals *and* at points.

6.1.2 Wavelet Vanishing Moments

To measure the local regularity of a signal, it is not so important to use a wavelet with a narrow frequency support, but vanishing moments are crucial. If the wavelet has n vanishing moments then we show that the wavelet transform can be interpreted as a multiscale differential operator of order n. This yields a first relation between the differentiability of f and its wavelet transform decay at fine scales.

Polynomial Suppression The Lipschitz property (6.3) approximates f with a polynomial p_v in the neighborhood of v:

$$f(t) = p_v(t) + \varepsilon_v(t) \quad \text{with} \quad |\varepsilon_v(t)| \leqslant K\,|t-v|^\alpha\ . \tag{6.6}$$

A wavelet transform estimates the exponent α by ignoring the polynomial p_v. For this purpose, we use a wavelet that has $n > \alpha$ *vanishing moments*:

$$\int_{-\infty}^{+\infty} t^k\,\psi(t)\,dt = 0 \quad \text{for} \quad 0 \leqslant k < n\ .$$

A wavelet with n vanishing moments is orthogonal to polynomials of degree $n-1$. Since $\alpha < n$, the polynomial p_v has degree at most $n-1$. With the change of variable $t' = (t-u)/s$ we verify that

$$Wp_v(u,s) = \int_{-\infty}^{+\infty} p_v(t)\,\frac{1}{\sqrt{s}}\,\psi\left(\frac{t-u}{s}\right)\,dt = 0. \tag{6.7}$$

Since $f = p_v + \varepsilon_v$,

$$Wf(u,s) = W\varepsilon_v(u,s). \tag{6.8}$$

Section 6.1.3 explains how to measure α from $|Wf(u,s)|$ when u is in the neighborhood of v.

Multiscale Differential Operator The following theorem proves that a wavelet with n vanishing moments can be written as the n^{th} order derivative of a function θ; the resulting wavelet transform is a multiscale differential operator. We suppose that ψ has a fast decay which means that for any decay exponent $m \in \mathbb{N}$ there exists C_m such that

$$\forall t \in \mathbb{R}\ ,\ |\psi(t)| \leqslant \frac{C_m}{1+|t|^m}\ . \tag{6.9}$$

Theorem 6.2. *A wavelet ψ with a fast decay has n vanishing moments if and only if there exists θ with a fast decay such that*

$$\psi(t) = (-1)^n\,\frac{d^n\theta(t)}{dt^n}. \tag{6.10}$$

As a consequence

$$Wf(u,s) = s^n \frac{d^n}{du^n}(f \star \bar{\theta}_s)(u) \ , \tag{6.11}$$

with $\bar{\theta}_s(t) = s^{-1/2}\theta(-t/s)$. *Moreover,* ψ *has no more than* n *vanishing moments if and only if* $\int_{-\infty}^{+\infty} \theta(t)\, dt \neq 0$.

Proof. The fast decay of ψ implies that $\hat{\psi}$ is $\mathbf{C}^\infty$. This is proved by setting $f = \hat{\psi}$ in Theorem 2.5. The integral of a function is equal to its Fourier transform evaluated at $\omega = 0$. The derivative property (2.22) implies that for any $k < n$

$$\int_{-\infty}^{+\infty} t^k\, \psi(t)\, dt = (i)^k \hat{\psi}^{(k)}(0) = 0. \tag{6.12}$$

We can therefore make the factorization

$$\hat{\psi}(\omega) = (-i\omega)^n\, \hat{\theta}(\omega), \tag{6.13}$$

and $\hat{\theta}(\omega)$ is bounded. The fast decay of θ is proved with an induction on n. For $n = 1$,

$$\theta(t) = \int_{-\infty}^{t} \psi(u)\, du = \int_{t}^{+\infty} \psi(u)\, du,$$

and the fast decay of θ is derived from (6.9). We then similarly verify that increasing by 1 the order of integration up to n maintains the fast decay of θ.

Conversely, $|\hat{\theta}(\omega)| \leqslant \int_{-\infty}^{+\infty} |\theta(t)|\, dt < +\infty$, because θ has a fast decay. The Fourier transform of (6.10) yields (6.13) which implies that $\hat{\psi}^{(k)}(0) = 0$ for $k < n$. It follows from (6.12) that ψ has n vanishing moments.

To test whether ψ has more than n vanishing moments, we compute with (6.13)

$$\int_{-\infty}^{+\infty} t^n\, \psi(t)\, dt = (i)^n\, \hat{\psi}^{(n)}(0) = (-i)^n\, n!\, \hat{\theta}(0).$$

Clearly, ψ has no more than n vanishing moments if and only if $\hat{\theta}(0) = \int_{-\infty}^{+\infty} \theta(t)\, dt \neq 0$.

The wavelet transform (4.32) can be written

$$Wf(u,s) = f \star \bar{\psi}_s(u) \ \text{ with } \ \bar{\psi}_s(t) = \frac{1}{\sqrt{s}}\, \psi\left(\frac{-t}{s}\right) \ . \tag{6.14}$$

We derive from (6.10) that $\bar{\psi}_s(t) = s^n\, \frac{d^n \bar{\theta}_s(t)}{dt^n}$. Commuting the convolution and differentiation operators yields

$$Wf(u,s) = s^n\, f \star \frac{d^n \bar{\theta}_s}{dt^n}(u) = s^n\, \frac{d^n}{du^n}(f \star \bar{\theta}_s)(u).$$

■ ■

If $K = \int_{-\infty}^{+\infty} \theta(t)\, dt \neq 0$ then the convolution $f \star \bar{\theta}_s(t)$ can be interpreted as a weighted average of f with a kernel dilated by s. So (6.11) proves that $Wf(u,s)$ is an n^{th} order derivative of an averaging of f over a domain proportional to s. Figure 6.1 shows a wavelet transform calculated with $\psi = -\theta'$, where θ is a Gaussian. The resulting $Wf(u,s)$ is the derivative of f averaged in the neighborhood of u with a Gaussian kernel dilated by s.

Since θ has a fast decay, one can verify that

$$\lim_{s \to 0} \frac{1}{\sqrt{s}}\, \bar{\theta}_s = K\, \delta \ ,$$

in the sense of the weak convergence (A.27). This means that for any ϕ that is continuous at u,

$$\lim_{s \to 0} \phi \star \frac{1}{\sqrt{s}}\, \bar{\theta}_s(u) = K\, \phi(u).$$

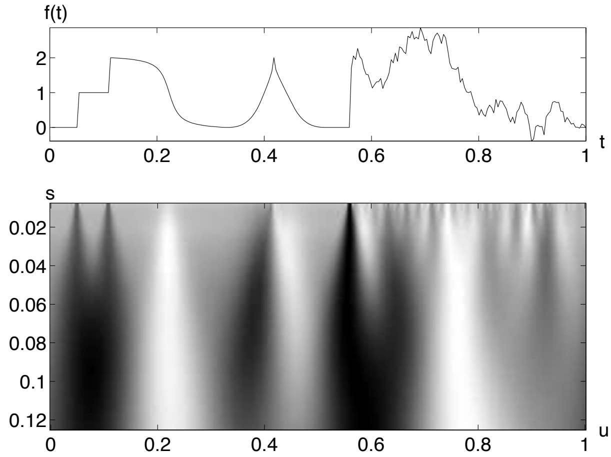

Figure 6.1: Wavelet transform $Wf(u,s)$ calculated with $\psi = -\theta'$ where θ is a Gaussian, for the signal f shown above. The position parameter u and the scale s vary respectively along the horizontal and vertical axes. Black, grey and white points correspond respectively to positive, zero and negative wavelet coefficients. Singularities create large amplitude coefficients in their cone of influence.

If f is n times continuously differentiable in the neighborhood of u then (6.11) implies that

$$\lim_{s\to 0} \frac{Wf(u,s)}{s^{n+1/2}} = \lim_{s\to 0} f^{(n)} \star \frac{1}{\sqrt{s}}\,\bar{\theta}_s(u) = K\, f^{(n)}(u)\ . \tag{6.15}$$

In particular, if f is $\mathbf{C}^n$ with a bounded n^{th} order derivative then $|Wf(u,s)| = O(s^{n+1/2})$. This is a first relation between the decay of $|Wf(u,s)|$ when s decreases and the uniform regularity of f. Finer relations are studied in the next section.

6.1.3 Regularity Measurements with Wavelets

The decay of the wavelet transform amplitude across scales is related to the uniform and pointwise Lipschitz regularity of the signal. Measuring this asymptotic decay is equivalent to zooming into signal structures with a scale that goes to zero. We suppose that the wavelet ψ has n vanishing moments and is $\mathbf{C}^n$ with derivatives that have a fast decay. This means that for any $0 \leqslant k \leqslant n$ and $m \in \mathbb{N}$ there exists C_m such that

$$\forall t \in \mathbb{R}\ ,\ |\psi^{(k)}(t)| \leqslant \frac{C_m}{1+|t|^m}\ . \tag{6.16}$$

The following theorem relates the uniform Lipschitz regularity of f on an interval to the amplitude of its wavelet transform at fine scales.

Theorem 6.3. *If $f \in \mathbf{L^2}(\mathbb{R})$ is uniformly Lipschitz $\alpha \leqslant n$ over $[a,b]$, then there exists $A > 0$ such that*

$$\forall (u,s) \in [a,b] \times \mathbb{R}^+\quad ,\quad |Wf(u,s)| \leqslant A\, s^{\alpha+1/2}. \tag{6.17}$$

Conversely, suppose that f is bounded and that $Wf(u,s)$ satisfies (6.17) for an $\alpha < n$ that is not an integer. Then f is uniformly Lipschitz α on $[a+\varepsilon, b-\varepsilon]$, for any $\varepsilon > 0$.

Proof. This theorem is proved with minor modifications in the proof of Theorem 6.4. Since f is Lipschitz α at any $v \in [a,b]$, Theorem 6.4 shows in (6.20) that

$$\forall (u,s) \in \mathbb{R} \times \mathbb{R}^+\quad ,\quad |Wf(u,s)| \leqslant\ A\, s^{\alpha+1/2} \left(1 + \left|\frac{u-v}{s}\right|^{\alpha}\right)\ .$$

For $u \in [a,b]$, we can choose $v = u$, which implies that $|Wf(u,s)| \leqslant A\, s^{\alpha+1/2}$. We verify from the proof of (6.20) that the constant A does not depend on v because the Lipschitz regularity is uniform over $[a,b]$.

To prove that f is uniformly Lipschitz α over $[a+\varepsilon, b-\varepsilon]$ we must verify that there exists K such that for all $v \in [a+\varepsilon, b-\varepsilon]$ we can find a polynomial p_v of degree $\lfloor\alpha\rfloor$ such that

$$\forall t \in \mathbb{R} \quad , \quad |f(t) - p_v(t)| \leqslant K\,|t-v|^\alpha \ . \tag{6.18}$$

When $t \notin [a+\varepsilon/2, b-\varepsilon/2]$ then $|t-v| \geqslant \varepsilon/2$ and since f is bounded, (6.18) is verified with a constant K that depends on ε. For $t \in [a+\varepsilon/2, b-\varepsilon/2]$, the proof follows the same derivations as the proof of pointwise Lipschitz regularity from (6.21) in Theorem 6.4. The upper bounds (6.26) and (6.27) are replaced by

$$\forall t \in [a+\varepsilon/2, b-\varepsilon/2] \quad , \quad |\Delta_j^{(k)}(t)| \leqslant K\,2^{(\alpha-k)j} \quad \text{for } 0 \leqslant k \leqslant \lfloor\alpha\rfloor + 1 \ . \tag{6.19}$$

This inequality is verified by computing an upper bound integral similar to (6.25) but which is divided in two, for $u \in [a,b]$ and $u \notin [a,b]$. When $u \in [a,b]$, the condition (6.21) is replaced by $|Wf(u,s)| \leqslant A\,s^{\alpha+1/2}$ in (6.25). When $u \notin [a,b]$, we just use the fact that $|Wf(u,s)| \leqslant \|f\|\,\|\psi\|$ and derive (6.19) from the fast decay of $|\psi^{(k)}(t)|$, by observing that $|t-u| \geqslant \varepsilon/2$ for $t \in [a+\varepsilon/2, b-\varepsilon/2]$. The constant K depends on A and ε but not on v. The proof then proceeds like the proof of Theorem 6.4, and since the resulting constant K in (6.29) does not depend on v, the Lipschitz regularity is uniform over $[a-\varepsilon, b+\varepsilon]$. ■ ■

The inequality (6.17) is really a condition on the asymptotic decay of $|Wf(u,s)|$ when s goes to zero. At large scales it does not introduce any constraint since the Cauchy-Schwarz inequality guarantees that the wavelet transform is bounded:

$$|Wf(u,s)| = |\langle f, \psi_{u,s}\rangle| \leqslant \|f\|\,\|\psi\|.$$

When the scale s decreases, $Wf(u,s)$ measures fine scale variations in the neighborhood of u. Theorem 6.3 proves that $|Wf(u,s)|$ decays like $s^{\alpha+1/2}$ over intervals where f is uniformly Lipschitz α.

Observe that the upper bound (6.17) is similar to the sufficient Fourier condition of Theorem 6.1, which supposes that $|\hat{f}(\omega)|$ decays faster than $\omega^{-\alpha}$. The wavelet scale s plays the role of a "localized" inverse frequency ω^{-1}. As opposed to the Fourier transform Theorem 6.1, the wavelet transform gives a Lipschitz regularity condition that is localized over any finite interval and it provides a necessary condition which is nearly sufficient. When $[a,b] = \mathbb{R}$ then (6.17) is a necessary and sufficient condition for f to be uniformly Lipschitz α on $\mathbb{R}$.

If ψ has exactly n vanishing moments then the wavelet transform decay gives no information concerning the Lipschitz regularity of f for $\alpha > n$. If f is uniformly Lipschitz $\alpha > n$ then it is $\mathbf{C}^n$ and (6.15) proves that $\lim_{s\to 0} s^{-n-1/2}\,Wf(u,s) = K\,f^{(n)}(u)$ with $K \neq 0$. This proves that $|Wf(u,s)| \sim s^{n+1/2}$ at fine scales despite the higher regularity of f.

If the Lipschitz exponent α is an integer then (6.17) is not sufficient in order to prove that f is uniformly Lipschitz α. When $[a,b] = \mathbb{R}$, if $\alpha = 1$ and ψ has two vanishing moments, then the class of functions that satisfy (6.17) is called the *Zygmund class* [43]. It is slightly larger than the set of functions that are uniformly Lipschitz 1. For example, $f(t) = t\,\log_e t$ belongs to the Zygmund class although it is not Lipschitz 1 at $t = 0$.

Pointwise Lipschitz Regularity The study of pointwise Lipschitz exponents with the wavelet transform is a delicate and beautiful topic which finds its mathematical roots in the characterization of Sobolev spaces by Littlewood and Paley in the 1930's. Characterizing the regularity of f at a point v can be difficult because f may have very different types of singularities that are aggregated in the neighborhood of v. In 1984, Bony [116] introduced the "two-microlocalization" theory which refines the Littlewood-Paley approach to provide pointwise characterization of singularities, which he used to study the solution of hyperbolic partial differential equations. These technical results became much simpler through the work of Jaffard [311] who proved that the two-microlocalization properties are equivalent to specific decay conditions on the wavelet transform amplitude. The following theorem gives a necessary condition and a sufficient condition on the wavelet transform for estimating the Lipschitz regularity of f at a point v. Remember that the wavelet ψ has n vanishing moments and n derivatives having a fast decay.

Theorem 6.4 (Jaffard)**.** *If $f \in \mathbf{L^2}(\mathbb{R})$ is Lipschitz $\alpha \leqslant n$ at v, then there exists A such that*

$$\forall (u,s) \in \mathbb{R} \times \mathbb{R}^+ \quad , \quad |Wf(u,s)| \leqslant A\, s^{\alpha+1/2} \left(1 + \left|\frac{u-v}{s}\right|^{\alpha}\right) . \tag{6.20}$$

Conversely, if $\alpha < n$ is not an integer and there exist A and $\alpha' < \alpha$ such that

$$\forall (u,s) \in \mathbb{R} \times \mathbb{R}^+ \quad , \quad |Wf(u,s)| \leqslant A\, s^{\alpha+1/2} \left(1 + \left|\frac{u-v}{s}\right|^{\alpha'}\right) \tag{6.21}$$

then f is Lipschitz α at v.

Proof. The necessary condition is relatively simple to prove but the sufficient condition is much more difficult.

• *Proof of (6.20)* Since f is Lipschitz α at v, there exists a polynomial p_v of degree $\lfloor \alpha \rfloor < n$ and K such that $|f(t) - p_v(t)| \leqslant K|t-v|^{\alpha}$. Since ψ has n vanishing moments, we saw in (6.7) that $Wp_v(u,s) = 0$ and hence

$$\begin{aligned} |Wf(u,s)| &= \left| \int_{-\infty}^{+\infty} \Big(f(t) - p_v(t)\Big) \frac{1}{\sqrt{s}}\, \psi\Big(\frac{t-u}{s}\Big)\, dt \right| \\ &\leqslant \int_{-\infty}^{+\infty} K\,|t-v|^{\alpha}\, \frac{1}{\sqrt{s}} \left|\psi\Big(\frac{t-u}{s}\Big)\right| dt. \end{aligned}$$

The change of variable $x = (t-u)/s$ gives

$$|Wf(u,s)| \leqslant \sqrt{s} \int_{-\infty}^{+\infty} K\,|sx + u - v|^{\alpha}\, |\psi(x)|\, dx.$$

Since $|a+b|^{\alpha} \leqslant 2^{\alpha}\,(|a|^{\alpha} + |b|^{\alpha})$,

$$|Wf(u,s)| \leqslant K\, 2^{\alpha} \sqrt{s} \left(s^{\alpha} \int_{-\infty}^{+\infty} |x|^{\alpha}\, |\psi(x)|\, dx + |u-v|^{\alpha} \int_{-\infty}^{+\infty} |\psi(x)|\, dx \right)$$

which proves (6.20).

• *Proof of (6.21)* The wavelet reconstruction formula (4.37) proves that f can be decomposed in a Littlewood-Paley type sum

$$f(t) = \sum_{j=-\infty}^{+\infty} \Delta_j(t) \tag{6.22}$$

with

$$\Delta_j(t) = \frac{1}{C_\psi} \int_{-\infty}^{+\infty} \int_{2^j}^{2^{j+1}} Wf(u,s)\, \frac{1}{\sqrt{s}}\, \psi\left(\frac{t-u}{s}\right) \frac{ds}{s^2}\, du . \tag{6.23}$$

Let $\Delta_j^{(k)}$ be its k^{th} order derivative. To prove that f is Lipschitz α at v we shall approximate f with a polynomial that generalizes the Taylor polynomial

$$p_v(t) = \sum_{k=0}^{\lfloor \alpha \rfloor} \left(\sum_{j=-\infty}^{+\infty} \Delta_j^{(k)}(v) \right) \frac{(t-v)^k}{k!} . \tag{6.24}$$

If f is n times differentiable at v then p_v corresponds to the Taylor polynomial but this is not necessarily true. We shall first prove that $\sum_{j=-\infty}^{+\infty} \Delta_j^{(k)}(v)$ is finite by getting upper bounds on $|\Delta_j^{(k)}(t)|$. These sums may be thought of as a generalization of pointwise derivatives.

To simplify the notation, we denote by K a generic constant which may change value from one line to the next but that does not depend on j and t. The hypothesis (6.21) and the asymptotic decay condition (6.16) imply that

$$\begin{aligned} |\Delta_j(t)| &= \frac{1}{C_\psi} \int_{-\infty}^{+\infty} \int_{2^j}^{2^{j+1}} A\, s^{\alpha} \left(1 + \left|\frac{u-v}{s}\right|^{\alpha'}\right) \frac{C_m}{1 + |(t-u)/s|^m}\, \frac{ds}{s^2}\, du \\ &\leqslant K \int_{-\infty}^{+\infty} 2^{\alpha j} \left(1 + \left|\frac{u-v}{2^j}\right|^{\alpha'}\right) \frac{1}{1 + |(t-u)/2^j|^m}\, \frac{du}{2^j} \end{aligned} \tag{6.25}$$

Since $|u-v|^{\alpha'} \leqslant 2^{\alpha'}(|u-t|^{\alpha'} + |t-v|^{\alpha'})$, the change of variable $u' = 2^{-j}(u-t)$ yields

$$|\Delta_j(t)| \leqslant K\, 2^{\alpha j} \int_{-\infty}^{+\infty} \frac{1+|u'|^{\alpha'} + \left|(v-t)/2^j\right|^{\alpha'}}{1+|u'|^m}\, du'.$$

Choosing $m = \alpha' + 2$ yields

$$|\Delta_j(t)| \leqslant K\, 2^{\alpha j} \left(1 + \left|\frac{v-t}{2^j}\right|^{\alpha'}\right). \tag{6.26}$$

The same derivations applied to the derivatives of $\Delta_j(t)$ yield

$$\forall k \leqslant \lfloor\alpha\rfloor + 1 \quad , \quad |\Delta_j^{(k)}(t)| \leqslant K\, 2^{(\alpha-k)j} \left(1 + \left|\frac{v-t}{2^j}\right|^{\alpha'}\right). \tag{6.27}$$

At $t = v$ it follows that

$$\forall k \leqslant \lfloor\alpha\rfloor \quad , \quad |\Delta_j^{(k)}(v)| \leqslant K\, 2^{(\alpha-k)j}\, . \tag{6.28}$$

This guarantees a fast decay of $|\Delta_j^{(k)}(v)|$ when 2^j goes to zero, because α is not an integer so $\alpha > \lfloor\alpha\rfloor$. At large scales 2^j, since $|Wf(u,s)| \leqslant \|f\|\,\|\psi\|$ with the change of variable $u' = (t-u)/s$ in (6.23) we have

$$|\Delta_j^{(k)}(v)| \leqslant \frac{\|f\|\,\|\psi\|}{C_\psi} \int_{-\infty}^{+\infty} |\psi^{(k)}(u')|\, du' \int_{2^j}^{2^{j+1}} \frac{ds}{s^{3/2+k}}$$

and hence $|\Delta_j^{(k)}(v)| \leqslant K 2^{-(k+1/2)j}$. Together with (6.28) this proves that the polynomial p_v defined in (6.24) has coefficients that are finite.

With the Littlewood-Paley decomposition (6.22) we compute

$$|f(t) - p_v(t)| = \left| \sum_{j=-\infty}^{+\infty} \left(\Delta_j(t) - \sum_{k=0}^{\lfloor\alpha\rfloor} \Delta_j^{(k)}(v)\, \frac{(t-v)^k}{k!} \right) \right| .$$

The sum over scales is divided in two at 2^J such that $2^J \geqslant |t-v| \geqslant 2^{J-1}$. For $j \geqslant J$, we can use the classical Taylor theorem to bound the Taylor expansion of Δ_j:

$$\begin{aligned} I &= \sum_{j=J}^{+\infty} \left| \Delta_j(t) - \sum_{k=0}^{\lfloor\alpha\rfloor} \Delta_j^{(k)}(v) \frac{(t-v)^k}{k!} \right| \\ &\leqslant \sum_{j=J}^{+\infty} \frac{(t-v)^{\lfloor\alpha\rfloor+1}}{(\lfloor\alpha\rfloor+1)!} \sup_{h\in[t,v]} |\Delta_j^{\lfloor\alpha\rfloor+1}(h)|\ . \end{aligned}$$

Inserting (6.27) yields

$$I \leqslant K\, |t-v|^{\lfloor\alpha\rfloor+1} \sum_{j=J}^{+\infty} 2^{-j(\lfloor\alpha\rfloor+1-\alpha)} \left|\frac{v-t}{2^j}\right|^{\alpha'}$$

and since $2^J \geqslant |t-v| \geqslant 2^{J-1}$ we get $I \leqslant K\, |v-t|^{\alpha}$.

Let us now consider the case $j < J$

$$\begin{aligned} II &= \sum_{j=-\infty}^{J-1} \left| \Delta_j(t) - \sum_{k=0}^{\lfloor\alpha\rfloor} \Delta_j^{(k)}(v)\, \frac{(t-v)^k}{k!} \right| \\ &\leqslant K \sum_{j=-\infty}^{J-1} \left(2^{\alpha j} \left(1 + \left|\frac{v-t}{2^j}\right|^{\alpha'}\right) + \sum_{k=0}^{\lfloor\alpha\rfloor} \frac{(t-v)^k}{k!}\, 2^{j(\alpha-k)} \right) \\ &\leqslant K \left(2^{\alpha J} + 2^{(\alpha-\alpha')J}\, |t-v|^{\alpha'} + \sum_{k=0}^{\lfloor\alpha\rfloor} \frac{(t-v)^k}{k!}\, 2^{J(\alpha-k)} \right) \end{aligned}$$

and since $2^J \geqslant |t-v| \geqslant 2^{J-1}$ we get $II \leqslant K\, |v-t|^{\alpha}$. As a result

$$|f(t) - p_v(t)| \leqslant I + II \leqslant K\, |v-t|^{\alpha} \tag{6.29}$$

which proves that f is Lipschitz α at v. ■

Cone of Influence To interpret more easily the necessary condition (6.20) and the sufficient condition (6.21), we shall suppose that ψ has a compact support equal to $[-C, C]$. The *cone of influence* of v in the scale-space plane is the set of points (u, s) such that v is included in the support of $\psi_{u,s}(t) = s^{-1/2}\,\psi((t-u)/s)$. Since the support of $\psi((t-u)/s)$ is equal to $[u - Cs, u + Cs]$, the cone of influence of v is defined by

$$|u - v| \leqslant Cs. \tag{6.30}$$

It is illustrated in Figure 6.2. If u is in the cone of influence of v then $Wf(u,s) = \langle f, \psi_{u,s}\rangle$ depends on the value of f in the neighborhood of v. Since $|u-v|/s \leqslant C$, the conditions (6.20,6.21) can be written

$$|Wf(u,s)| \leqslant A'\, s^{\alpha+1/2}$$

which is identical to the uniform Lipschitz condition (6.17) given by Theorem 6.3. In Figure 6.1, the high amplitude wavelet coefficients are in the cone of influence of each singularity.

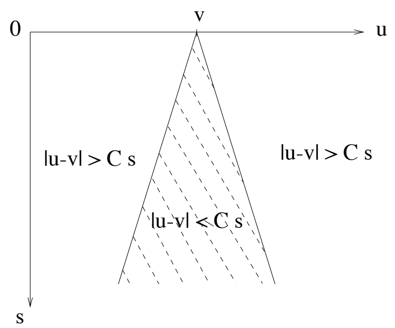

Figure 6.2: The cone of influence of an abscissa v consists of the scale-space points (u, s) for which the support of $\psi_{u,s}$ intersects $t = v$.

Oscillating Singularities It may seem surprising that (6.20) and (6.21) also impose a condition on the wavelet transform outside the cone of influence of v. Indeed, this corresponds to wavelets whose support does not intersect v. For $|u - v| > Cs$ we get

$$|Wf(u,s)| \leqslant A'\, s^{\alpha-\alpha'+1/2}\, |u-v|^{\alpha}. \tag{6.31}$$

We shall see that it is indeed necessary to impose this decay when u tends to v in order to control the oscillations of f that might generate singularities.

Let us consider the generic example of a highly oscillatory function

$$f(t) = \sin\frac{1}{t}\ ,$$

which is discontinuous at $v = 0$ because of the acceleration of its oscillations. Since ψ is a smooth $\mathbf{C}^n$ function, if it is centered close to zero then the rapid oscillations of $\sin t^{-1}$ produce a correlation integral $\langle \sin t^{-1}, \psi_{u,s}\rangle$ that is very small. With an integration by parts, one can verify that if (u,s) is in the cone of influence of $v = 0$, then $|Wf(u,s)| \leqslant A\, s^{2+1/2}$. This looks as if f is Lipschitz 2 at 0. However, Figure 6.3 shows high energy wavelet coefficients outside the cone of influence of $v = 0$, which are responsible for the discontinuity. To guarantee that f is Lipschitz α, the amplitude of such coefficients is controlled by the upper bound (6.31).

To explain why the high frequency oscillations appear outside the cone of influence of v, we use the results of Section 4.4.2 on the estimation of instantaneous frequencies with wavelet ridges. The instantaneous frequency of $\sin t^{-1} = \sin\theta(t)$ is $|\theta'(t)| = t^{-2}$. Let ψ^a be the analytic part of ψ, defined in (4.47). The corresponding complex analytic wavelet transform is $W^a f(u,s) = \langle f, \psi^a_{u,s}\rangle$. It was proved in (4.109) that for a fixed time u, the maximum of $s^{-1/2}|W^a f(u,s)|$ is located at the scale

$$s(u) = \frac{\eta}{\theta'(u)} = \eta\, u^2,$$

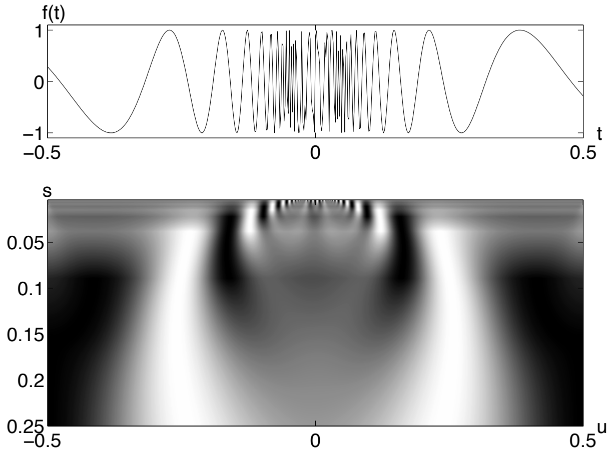

Figure 6.3: Wavelet transform of $f(t) = \sin(a\,t^{-1})$ calculated with $\psi = -\theta'$ where θ is a Gaussian. High amplitude coefficients are along a parabola outside the cone of influence of $t = 0$.

where η is the center frequency of $\hat{\psi}^a(\omega)$. When u varies, the set of points $(u, s(u))$ define a *ridge* that is a parabola located outside the cone of influence of $v = 0$ in the plane (u, s). Since $\psi = \mathrm{Re}[\psi^a]$, the real wavelet transform is

$$Wf(u,s) = \mathrm{Re}[W^a f(u,s)].$$

The high amplitude values of $Wf(u,s)$ are thus located along the same parabola ridge curve in the scale-space plane, which clearly appears in Figure 6.3. Real wavelet coefficients $Wf(u,s)$ change sign along the ridge because of the variations of the complex phase of $W^a f(u,s)$.

The example of $f(t) = \sin t^{-1}$ can be extended to general oscillating singularities [31]. A function f has an oscillating singularity at v if there exist $\alpha \geqslant 0$ and $\beta > 0$ such that for t in a neighborhood of v

$$f(t) \sim |t - v|^\alpha \, g\left(\frac{1}{|t-v|^\beta}\right) ,$$

where $g(t)$ is a $\mathbf{C}^\infty$ oscillating function whose primitives at any order are bounded. The function $g(t) = \sin t^{-1}$ is a typical example. The oscillations have an instantaneous frequency $\theta'(t)$ that increases to infinity faster than $|t|^{-1}$ when t goes to v. High energy wavelet coefficients are located along the ridge $s(u) = \eta/\theta'(u)$, and this curve is necessarily outside the cone of influence $|u - v| \leqslant Cs$.

6.2 Wavelet Transform Modulus Maxima

Theorems 6.3 and 6.4 prove that the local Lipschitz regularity of f at v depends on the decay at fine scales of $|Wf(u,s)|$ in the neighborhood of v. Measuring this decay directly in the time-scale plane (u,s) is not necessary. The decay of $|Wf(u,s)|$ can indeed be controlled from its local maxima values. Section 6.2.1 studies the detection and characterization of singularities from wavelet local maxima. Signal approximations are recovered in Section 6.2.2, from the scale-space support of these local maxima at dyadic scales.

6.2.1 Detection of Singularities

Singularities are detected by finding the abscissa where the wavelet modulus maxima converge at fine scales. A wavelet *modulus maximum* is defined as a point (u_0, s_0) such that $|Wf(u, s_0)|$ is locally maximum at $u = u_0$. This implies that

$$\frac{\partial Wf(u_0, s_0)}{\partial u} = 0.$$