EXACT TAIL ASYMPTOTICS OF A QUEUE WITH LRD INPUT TRAFFIC

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In this work we compute the exact tail asymptotics of the stationary workload W, associated to a discrete-time single server queue, with constant release rate, infinite buffer capacity, and with $M/G/\infty$ input traffic exhibiting long-range dependence. We choose a regularly varying distribution with parameter $\alpha > 1$ for the general distribution G. We show that the exact asymptotics of the workload is a specific regularly varying function under some assumptions on the parameters.

1. Introduction

The rapid increase in the number and complexity of available communications services, such as the Web traffic [10] and the WAN traffic [15] for example, have made traditional traffic models based on exponential assumptions about inter-arrival times and resource holding requirements obsolete. The statistical profile of network traffic exhibits empirical correlations that decay to zero as a power rather than exponentially as those seen in traditional telephony.

For this reason, there has been broad interest in the literature in the study of service systems exhibiting some forms of long-range dependence of the input process in which the correlation functions are not summable.

Some models propose fractional Brownian motion and its discrete-time analog, namely fractional Gaussian noise, as input traffic processes with long-range dependence property. For instance, in [9, 11, 13] the logarithm asymptotics of the workload for a single server queue is derived. All these studies arrive at the same result: the stationary workload associated to a single service queue is asymptotically Weibullian.

In this paper we take instead the busy server process of the $M/G/\infty$ queue as the input traffic process for a discrete-time single service system with constant release rate.

The fundamental observation about $M/G/\infty$ input traffic was made in [3]: if the distribution function F of the service times in the $M/G/\infty$ queue has infinite variance, then the busy server process has nonsummable correlations, and thus provides an input process with long-range dependence for the queue of our interest.

In order to estimate the performances of the service system of interest, we will estimate the *asymptotic tail*, $\mathsf{P}(W > x)$, as $x \to \infty$, where W is the stationary workload under the Loynes's stability conditions [1, 7, 18]. This study was made by many authors. In [8, 14] an upper and a lower bound for the logarithmic tail asymptotics was derived. A more precise result under reasonable general conditions was obtained for the Weibull, Lognormal and Pareto case. In the special case F Pareto with tail parameter $\alpha > 1$ we have

$$\log \mathsf{P}(W > x) \sim -(\alpha - 1) \log x, \tag{1}$$

where $a(x) \sim b(x)$ means $a(x)/b(x) \to 1,\ x \to \infty$. The main result of this work is an improvement of the previous result. Under the same assumptions on the parameters of the model but assuming only that F is a distribution function with regularly varying tails with parameter $\alpha > 1$, rather than simply Pareto, we get

$$\mathsf{P}(W > x) \sim L(x) x^{-(\alpha-1)}, \tag{2}$$

where $L(x)$ is a specific slowly varying function.

The result (2) is in accordance with the exact asymptotics tail for a single server queue generated by the $M/G/\infty$ fluid input process [16], that is, the analogous queue in continuous time and under the same assumptions on the general distribution G.

The techniques used to obtain the result (2) are different from the large deviation arguments of [8, 14, 16].

Our main tool is the analysis performed by Baccelli and Foss [2] of Veraverbeke's theorem [5, 17] on the asymptotic tail of the supremum of a random walk with i.i.d. subexponential increments with negative mean.

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This analysis consists in identifying *typical events* responsible for the fact that the random walk has crossed the level x, up to higher-order probabilities as $x \to \infty$.

In order to use this analysis, we first derive an upper bound $W \leqslant U + V$, where U and V are independent and V is the stationary workload of a single server queue with independent service times. This upper bound allows us to use Theorem 8 of [2], which shows that the probability $\mathsf{P}(W > x)$ is mainly due to one very big service time associated to a customer in the $M/G/\infty$ queue, whereas the others remain close to their mean.

2. Model and main result

We consider a discrete time single server queue with infinite buffer capacity and constant release rate of a cells/slot with FIFO service discipline. The input process is represented by the sequence $\{\sigma_k, k = 1, 2, \ldots\}$, where σ_{k+1} is the number of new cells arriving at the start of time slot $[k, k+1)$, $k = 1, 2, \ldots$. Let W_k denote the number of cells remaining in the buffer by the end of slot $[k-1, k)$ with $k \geqslant 1$, and suppose Y is the number of cells in the buffer at time slot $[0, 1)$, so that Lindley's recursion is

$$\begin{cases} W_0 & = Y, \\ W_{k+1} & = [W_k + \sigma_{k+1} - a]^+, \ k = 0, 1, \ldots. \end{cases} \tag{3}$$

Let $W_n^{[Y]}$ be the solution of (3). If $\{\sigma_k, \ k = 1, 2, \ldots\}$ is a stationary and ergodic process, under the Loynes's stability assumption [1,7,18] $\mathsf{E}[\sigma_1] < a$, the system is stable in the sense that $W_n^{[Y]} \xrightarrow{\text{Law}} W_\infty$, as $n \to \infty$ for some finite random variable W_∞ and any initial condition Y.

2.1. Stochastic Assumptions

The sequence $\{\sigma_k, \ k = 1, 2, \ldots\}$ is generated by the $M/G/\infty$ input process of Cox [3] in the following way. Suppose that during time slot $[k-1, k)$, $k = 1, 2, \ldots$, b_k new customers arrive in the system. For $i = 1, \ldots, b_k$, $X_{k,i}$ is the service time of the ith customer. Customer i is presented to its own server and begins service by the start of slot $[k, k+1)$, $k \geqslant 1$. Moreover b_0 is the number of customers in the system at time 0 and for $i = 1, \ldots, b_0$, $X_{0,i}$ is the residual service time of the ith customer present in the queue at time 0. Let σ_k denote the number of busy servers, or equivalently, the number of customers still present in the system at the beginning of time slot $[k, k+1)$. The busy server process $\{\sigma_k, \ k = 0, 1, 2, \ldots\}$ is what we refer to as the $M/G/\infty$ input process.

We have that

$$\sigma_k = \sum_{i=1}^{b_0} 1_{\{X_{0,i} > k\}} + \sum_{j=1}^{k} \sum_{i=1}^{b_j} 1_{\{X_{j,i} > k-j\}}, \quad k \geqslant 0. \tag{4}$$

Now the random variables

$$b_0, \ \{b_k, \ k = 1, 2, \ldots\}, \ \{X_{0,i}, \ i = 1, 2, \ldots, b_0\}, \ \{X_{k,i}, \ k = 1, 2, \ldots, i = 1, 2, \ldots, b_k\},$$

are assumed to satisfy the following assumptions:

(A1) all the sequences of random variables are mutually independent;

(A2) $\{b_k, \ k = 1, 2, \ldots\}$ are i.i.d. Poisson random variables with parameter $\lambda > 0$ and b_0 is a Poisson random variable independent of everything else with parameter $\gamma > 0$;

(A3) $\{X_{k,i}, \ k = 1, 2, \ldots, \ i = 1, 2, \ldots, b_k\}$ are i.i.d. random variables valued in $\mathbb{N}$ with the same distribution function of, say X, with $\mathsf{E}[X] < \infty$.

Moreover, $\{X_{0,i}, i = 1, 2, \ldots, b_0\}$ are assumed to be i.i.d random variables with the same distribution of the random variable X_0.

Moreover, we will assume that

(A4) b_0 has a Poisson distribution with mean $\gamma = \lambda \mathsf{E}[X]$;

(A5) $X_{0,i}$ has a distribution $\mathsf{P}(X_0 \leqslant k) = \dfrac{1}{\mathsf{E}[X]} \displaystyle\sum_{n=1}^{k} \mathsf{P}(X \geqslant n)$, for $k = 1, 2, \ldots$.

Proposition 1 ([3], [12]). *Under the assumptions* $(A1), (A2), (A3), (A4),$ *and* $(A5)$ *the process* $\{\sigma_k, k = 0, 1, 2, \ldots\}$ *is stationary, ergodic, and reversible such that*

$$\gamma = \mathsf{E}[\sigma_k] = \lambda \mathsf{E}[X], \quad \forall k \geqslant 0. \tag{5}$$

Its covariance function is given by

$$\Gamma(h) = \operatorname{cov}(\sigma_k, \sigma_{k+h}) = \lambda \mathsf{E}[X] \mathsf{P}(X_0 > h), \quad \forall k, h = 0, 1, \ldots. \tag{6}$$

Moreover,

$$\sum_{h=0}^{\infty} \Gamma(h) = \lambda \mathsf{E}[X] \mathsf{E}[X_0] = \frac{\lambda}{2} \mathsf{E}[X(X+1)]. \tag{7}$$

In the rest of the paper, we will make the following assumption on $\overline{F}$, the tail distribution of X:

(A6) $\quad \overline{F}(x) = \mathsf{P}(X > x) \sim x^{-\alpha} L(x), \quad$ with $\alpha > 1$ and L a slowly varying function.

We define by $\overline{F^s}$ the integrated tail distribution as follows:

$$\overline{F^s}(x) = \min\left\{1, \int_x^{\infty} \overline{F}(u) du\right\}. \tag{8}$$

Thanks to the assumption on F, by Karamata's Theorem [4] the tail of F^s is such that

$$\overline{F^s}(x) \sim x^{-(\alpha-1)} L^{'}(x), \quad \text{with } L^{'} \text{ a slowly varying function.} \tag{9}$$

In this case we have the following important characteristics of the input process:

Proposition 2 [3, 12]. *Under the hypotheses* (A1), (A2), (A3), (A4), (A5), *and* (A6) *the stationary process* $\{\sigma_k, \ k = 0, 1, 2, \ldots\}$ *is long-range dependent, that is*

$$\sum_{h=0}^{\infty} \Gamma(h) = \infty,$$

if and only if $1 < \alpha < 2$. *Moreover, the process* $\{\sigma_k, \ k = 0, 1, 2, \ldots\}$ *is asymptotically* (*second order*) *self-similar with Hurst parameter* $H = \frac{3-\alpha}{2}$.

Now, thanks to (5), $\mathsf{E}[\sigma_1] = \gamma$, and then under the Loynes's stability condition $\gamma < a$ we have

$$W_\infty \stackrel{\text{Law}}{=} \sup_{k \geqslant 1} \left[\sum_{k=1}^{\infty} (\sigma_k - a)\right]^+, \tag{10}$$

where the formula on the right is the solution of (3) when $W_0 = 0$, obtained thanks to the reversibility property of the process $\{\sigma_k, \ k = 0, 1, 2, \ldots\}$.

2.2. Exact asymptotics for the stationary workload

Theorem 1 (Main Result). *Under the assumption* (A1), (A2), (A3), (A4), (A5), (A6) *and*

$$\star \quad \gamma < a, \tag{11}$$

$$\star \quad 1 + \gamma > a, \tag{12}$$

we have

$$\mathsf{P}(W_\infty > x) \sim \frac{\lambda}{a - \gamma} (1 + \gamma - a)^{\alpha - 1} \overline{F^s}(x), \ as \quad x \to \infty. \tag{13}$$

We observe that the assumption $\gamma < a$ is the Loynes's stability condition. Moreover, since γ is the mean number of customer in the $M/G/\infty$ queue, the assumption $1 + \gamma > a$ says that the service system becomes unstable when one customer stays indefinitely in the queue.

3. Proof of the main result

We briefly outline the argument of the proof. The main idea is to apply the main result of [2] to our framework. We first derive an upper bound $W \leqslant U + V$, where U and V are independent and V is the stationary workload of a $GI/GI/1/\infty$ queue. This upper bound allows us to use Theorem 8 of [2], which shows that the probability $\mathsf{P}(W > x)$ is mainly due to one very large $X_{j,i}$ whereas the others remain close to their mean. Denote

$$W = \sup_{k\geqslant 1}\left[\sum_{k=1}^{\infty}(\sigma_k - a)\right]^+. \tag{14}$$

3.1. An upper bound and the single big event theorem

We first derive an upper bound for the random variable W. We have

$$\sum_{k=1}^{n}\sum_{i=1}^{b_0} 1_{\{X_{0,i}>k\}} = \sum_{i=1}^{b_0}\sum_{k=1}^{n} 1_{\{X_{0,i}>k\}} = \sum_{i=1}^{b_0}[X_{0,i}\wedge n], \quad \forall n \geqslant 1,$$

and

$$\sum_{k=1}^{n}\sum_{j=1}^{k}\sum_{i=1}^{b_j} 1_{\{X_{j,i}>k-j\}} = \sum_{j=1}^{n}\sum_{i=1}^{b_j}\sum_{k=1}^{n} 1_{\{X_{j,i}>k-j\}} = \sum_{j=1}^{n}\sum_{i=1}^{b_j}[X_{j,i}\wedge(n-j)], \quad \forall n \geqslant 1.$$

Hence

$$W = \sup_{n\geqslant 1}\left[\sum_{i=1}^{b_0}(X_{0,i}\wedge n) + \sum_{j=1}^{n}\sum_{i=1}^{b_j}(X_{j,i}\wedge(n-j)) - na\right]^+ = \sup_{n\geqslant 1}(W_n) \leqslant \tag{15}$$

$$\leqslant \sum_{i=1}^{b_0} X_{0,i} + \sup_{n\geqslant 1}\left[\sum_{j=1}^{n}\Big(Y_j - a\Big)\right]^+ = U + V = Z, \tag{16}$$

where

$$U = \sum_{i=1}^{b_0} X_{0,i}, \quad Y_j = \sum_{i=1}^{b_j} X_{j,i}, \quad V = \sup_{n\geqslant 1}\Big[\sum_{j=1}^{n}(Y_j - a)\Big]^+. \tag{17}$$

Note that V is finite and corresponds to the stationary waiting time of a $D/GI/1$ system with service times Y_j with $\mathsf{E}[Y_1] = \gamma < a$.

Let N_x be a function such that $N_x \uparrow \infty$ and $N_x\overline{F}(x) = o(\overline{F^s}(x))$ whose existence is ensured by the heavy-tail property [4, 6, 18]. Using Theorem 8 of [2], we derive an asymptoptic equivalence of $\mathsf{P}(W > x)$ as $x \to \infty$, given by the formula (20).

Proposition 3. *For any x, let $\{K_{n,x}, n \geqslant N_x\}$ be a sequence of events such that*

a) for any $n \geqslant N_x$, the events $K_{n,x}$ and Y_n are independent;

b) $\lim_{x\to\infty}\inf_{n\geqslant N_x}\mathsf{P}(K_{n,x}) = 1$,

and define for any sequence η_n tending to 0*, as $n \to \infty$,*

$$A_{n,x} = K_{n,x} \cap \{Y_n > x + n(a - \gamma + \eta_n)\}. \tag{18}$$

Let L_x be an event such that

c) the event L_x and the random variable U are independent;

d) $\lim_{x\to\infty}\mathsf{P}(L_x) = 1$, *and define*

$$B_x = \{U > x\} \cap L_x. \tag{19}$$

Then, as $x \to \infty$,

$$\mathsf{P}(W > x) = \sum_{n \geqslant N_x} \mathsf{P}(W > x, A_{n,x}) + \mathsf{P}(W > x, B_x) + o\Big(\overline{F^s}(x)\Big). \tag{20}$$

Proof. We note that $\mathsf{P}(U > x) \sim \lambda \overline{F^s}(x)$ [4, 6, 18] as $x \to \infty$, and thanks to Veraverbeke's Theorem $\mathsf{P}(V > x) \sim \dfrac{\lambda}{a-\gamma}\overline{F^s}(x)$ as $x \to \infty$. Moreover U and V are independent, and then

$$\mathsf{P}(Z > x) = \mathsf{P}(U + V > x) \sim \lambda \left(1 + \frac{1}{a-\gamma}\right) \overline{F^s}(x), \quad x \to \infty. \tag{21}$$

Let $K_{n,x}$, L_x, $A_{n,x}$ be events similar to the assumptions of this proposition. Denote $A_x = \cup_{n \geqslant N_x} A_{n,x}$ and $C_x =$ $= A_x \cup B_x$. Moreover we define $\tilde{K}_{n,x} = K_{n,x} \cap \{U \leqslant x\}$, and $\tilde{A}_{n,x}$ the corresponding events as in (18); and then $\tilde{A}_x$ and $\tilde{C}_x$. Evidently $\tilde{K}_{n,x}$ and $\tilde{A}_{n,x}$ satisfy again the assumptions a), b) of this proposition. Now the following bound holds:

$$\mathsf{P}(Z > x) \geqslant \mathsf{P}(Z > x, C_x) \geqslant \mathsf{P}(V > x, \tilde{A}_x) + \mathsf{P}(B_x). \tag{22}$$

Now V, $\tilde{K}_{n,x}$, and $\tilde{A}_{n,x}$ satisfy the assumptions of the Corollary 2 of [2]. Therefore,

$$\mathsf{P}(V > x, \tilde{A}_x) + o(\overline{F^s}(x)) = \mathsf{P}(V > x, A_x) + o(\overline{F^s}(x)) = \mathsf{P}(V > x) \sim \frac{\lambda}{a-\gamma}\overline{F^s}(x). \tag{23}$$

By the previous equality and since $\mathsf{P}(B_x) = \mathsf{P}(U > x)\mathsf{P}(L_x) \sim \lambda \overline{F^s}(x)$ as $x \to \infty$, we have from (22)

$$\mathsf{P}(Z > x) = \mathsf{P}(Z > x, C_x) + o(\overline{F^s}(x)). \tag{24}$$

Now we have

$$\mathsf{P}(W > x, C_x) \leqslant \mathsf{P}(W > x) \leqslant \mathsf{P}(W > x, C_x) + \mathsf{P}(Z > x, C_x^c) = \mathsf{P}(W > x, C_x) + o\Big(\overline{F^s}(x)\Big),$$

where the last equality follows from (24). Then $\mathsf{P}(W > x) = \mathsf{P}(W > x, C_x) + o(\overline{F^s}(x))$. Now we can construct disjoint events $K^{'}_{n,x}$ for any $n \geqslant N_x$ that satisfy the assumptions $a), b)$, using similar arguments of Corollary 2 of [2]. Then

$$\mathsf{P}(W > x, C^{'}_x) = \sum_{n \geqslant N_x} \mathsf{P}(W > x, A^{'}_{n,x}) + \mathsf{P}(W > x, B_x) + o\Big(\overline{F^s}(x)\Big).$$

Moreover, it is not difficult to prove that

$$\sum_{n \geqslant N_x} \mathsf{P}(W > x, A^{'}_{n,x}) - \sum_{n \geqslant N_x} \mathsf{P}(W > x, A_{n,x}) = o\Big(\overline{F^s}(x)\Big)$$

and then the equivalence (20) follows. See [18] for details.

3.2. Asymptotic equivalence for $\mathsf{P}(W > x, B_x)$

Proposition 4. *Under the previous assumptions, we have*

$$\lim_{x \to \infty} \frac{\mathsf{P}(W > x, B_x)}{\overline{F^s}(x)} = \lambda(1 + \gamma - a)^{\alpha - 1}. \tag{25}$$

We will prove this proposition in two steps. First we derive a lower bound and then an upper bound.

3.2.1. Lower bound

We take

$$L_x = \left\{ \frac{\sum_{j=1}^{m}\sum_{i=1}^{b_j}(X_{j,i}\wedge(m-j))}{m} > \gamma - \varepsilon_m,\ \forall m > x \right\}. \tag{26}$$

By definition, L_x satisfies the assumption c) of Proposition 3, and d) thanks to Lemma 1 for some sequence ε_n such that $\varepsilon_n \downarrow 0$ and $n\varepsilon_n \to \infty$ as $n \to \infty$. We observe that for all n we have

$$W \geqslant \sum_{i=1}^{b_0}(X_{0,i}\wedge n) + \sum_{j=1}^{n}\sum_{i=1}^{b_j}(X_{j,i}\wedge(n-j)) - na.$$

Taking $n = U = \sum_{i=1}^{b_0} X_{0,i}$, we have

$$W \geqslant U + \sum_{j=1}^{U}\sum_{i=1}^{b_j}(X_{j,i}\wedge(U-j)) - Ua = U(1-a) + \sum_{j=1}^{U}\sum_{i=1}^{b_j}(X_{j,i}\wedge(U-j)).$$

On the event $B_x = \{U > x, L_x\}$ we have

$$W \geqslant U(1-a) + U(\gamma - \varepsilon_U) = U(1+\gamma-a-\varepsilon_U).$$

We recall that $\varepsilon_n \downarrow 0$ as $n \to \infty$ and that the condition $1+\gamma-a > 0$ is assumed. Let $\delta > 0$ be such that $1+\gamma-a-\delta > 0$. For $n \geqslant n_0$ with n_0 large enough, $\varepsilon_n < \delta$; therefore, for x large enough, if $U > x \geqslant n_0$ we have $\varepsilon_U < \varepsilon_x < \delta$. Moreover, since $\gamma < a$, we have also $1+\gamma-a < 1$. Hence

$$\begin{aligned} \mathsf{P}(W > x, B_x) &\geqslant \mathsf{P}\Big(U(1+\gamma-a-\varepsilon_x) > x, U > x, L_x\Big) = \\ &= \mathsf{P}\Big(U(1+\gamma-a-\varepsilon_x) > x\Big)\mathsf{P}(L_x) = \\ &\geqslant \mathsf{P}\Big(U > \frac{x}{1+\gamma-a-\delta}\Big)\mathsf{P}(L_x). \end{aligned}$$

Now since $\mathsf{P}(U > x) \sim \lambda\overline{F^s}(x)$ as $x \to \infty$ [4, 6, 18], $\overline{F^s}(x) \sim x^{-(\alpha-1)}L(x)$ with L a slowly varying function, and finally sending δ to 0, we conclude that

$$\liminf_{x\to\infty} \frac{\mathsf{P}(W > x, B_x)}{\overline{F^s}(x)} \geqslant \lambda(1+\gamma-a)^{\alpha-1}. \tag{27}$$

3.2.2. Upper bound

Let $\alpha_x = x^{\frac{1}{2}-\varepsilon}$ for some $0 < \varepsilon < \frac{1}{2}$. Next take

$$\begin{aligned} R_x &= \left\{ \sum_{j=1}^{n}\Big(\sum_{i=1}^{b_j}(X_{j,i}\wedge(n-j)) - \gamma\Big) \leqslant n\varepsilon_n,\ \forall n \geqslant \alpha_x \right\}, \\ Q_x &= \left\{ \sum_{j=1}^{n}\Big(\sum_{i=1}^{b_j}(X_{j,i}\wedge(n-j)) - a\Big) \leqslant \alpha_x,\ \forall n < \alpha_x \right\}. \end{aligned}$$

Define

$$L_x = R_x \cap Q_x. \tag{28}$$

Observe that L_x satisfies the assumptions of Proposition 3. In fact, for all n

$$\sum_{j=1}^{n}\Big(\sum_{i=1}^{b_j}(X_{j,i}\wedge(n-j)) - a\Big) \leqslant \sum_{j=1}^{n}(Y_j - a) \leqslant V < \infty \text{ a.s.}$$

Therefore $\mathsf{P}(Q_x) \to 1$, as $x \to \infty$. Moreover $\mathsf{P}(R_x) \to 1$ as $x \to \infty$ thanks to Lemma 1. Now we use Lemma 2, with $N = b_0$, $Z_i = X_{i,0}$, we have $S_N = U$. Moreover, we set $M_0 \triangleq M_N$ and $M_0^1 \triangleq M_N^1$, so that

$$\mathsf{P}(W > x, B_x) = \mathsf{P}(W > x, U > x, L_x) = \mathsf{P}(W > x, M_0 > x, M_0^1 \leqslant \alpha_x, L_x) + o(\overline{F^s}(x)).$$

We have that

$$W = \sup_{n \geqslant 1} W_n \leqslant \max\left\{\sup_{n \leqslant \alpha_x} W_n, \sup_{n > \alpha_x} W_n\right\}.$$

Let $\delta > 0$ such that $\gamma - a + \delta < 0$. Since $\varepsilon_n \downarrow 0$, there exists an n_0 such that for all $n \geqslant n_0$, $\varepsilon_n < \delta$. Hence, if x is large enough, for $n > \alpha_x \geqslant n_0$, $\varepsilon_n < \delta$. Now on the event $\{M_0 > x, M_0^1 \leqslant \alpha_x\}$, for all n we have

$$\sum_{i=1}^{b_0} (X_{0,i} \wedge n) \leqslant M_0 \wedge n + b_0(\alpha_x \wedge n).$$

Again, on the event $\{M_0 > x, M_0^1 \leqslant \alpha_x, L_x\}$, we have

$$\begin{aligned} W &\leqslant \max\left\{\sup_{n \leqslant \alpha_x} [M_0 \wedge n + b_0(\alpha_x \wedge n) + \alpha_x]^+, \sup_{n > \alpha_x} [M_0 \wedge n + b_0(\alpha_x \wedge n) + n(\gamma - a + \delta)]^+\right\} \leqslant \\ &\leqslant \max\left\{[\alpha_x + b_0\alpha_x + \alpha_x], M_0(1 + \gamma - a + \delta) + b_0\alpha_x\right\} \leqslant \\ &\leqslant M_0(1 + \gamma - a + \delta) + (b_0 + 2)\alpha_x, \end{aligned}$$

since for x large enough $x > \alpha_x$ and by assumption $1 + \gamma - a > 0$. Therefore,

$$\mathsf{P}(W > x, B_x) \leqslant \mathsf{P}\Big(M_0 > \frac{x}{1 + \gamma - a + \delta} - \frac{(2 + b_0)\alpha_x}{1 + \gamma - a + \delta}\Big).$$

Let $c_\delta = (1 + \gamma - a + \delta)^{-1}$. Then by definition of M_0

$$\mathsf{P}(W > x, B_x) \leqslant \sum_{n=1}^{\infty} n\mathsf{P}(b_0 = n)\mathsf{P}\Big(X_0 > c_\delta x - c_\delta(n + 2)\alpha_x\Big).$$

For any continuous, positive real function h we can write

$$\mathsf{P}(W > x, B_x) \leqslant \sum_{n=1}^{h(x)} n\mathsf{P}(b_0 = n)\mathsf{P}\Big(X_0 > c_\delta x - c_\delta(n + 2)\alpha_x\Big) + \sum_{n > h(x)} n\mathsf{P}(b_0 = n).$$

Now let $h(x) = \frac{\sqrt{x}}{c_\delta} - 2$, for x large; then $c_\delta(2 + n)\alpha_x \leqslant c_\delta(2 + h(x))\alpha_x = x^{1-\varepsilon}$, for $n \leqslant h(x)$. Moreover

$$\sum_{n > h(x)} n\mathsf{P}(b_0 = n) = \gamma\mathsf{P}(b_0 \geqslant h(x)) \leqslant Ae^{-h(x)},$$

where A is some positive constant. Therefore,

$$\begin{aligned} \mathsf{P}(W > x, B_x) &\leqslant \mathsf{P}\Big(X_0 > c_\delta x - x^{1-\varepsilon}\Big) \sum_{n=1}^{\infty} n\mathsf{P}(b_0 = n) + Ae^{-h(x)} = \\ &= \mathsf{P}\Big(X_0 > c_\delta x - x^{1-\varepsilon}\Big)\lambda\mathsf{E}[X] + Ae^{-h(x)}. \end{aligned}$$

Now divide both members by $\overline{F^s}(x)$. We have $Ae^{-h(x)} = o(\overline{F^s}(x))$. In fact, since $\overline{F^s}(x) \sim L(x)x^{-(\alpha-1)}$ as $x \to \infty$ for some L slowly varying function, we have

$$\log\left(\frac{Ae^{-h(x)}}{\overline{F^s}(x)}\right) = \log(A) - h(x) - \log(L(x)) + (\alpha - 1)\log(x) \to -\infty,$$

recalling the representation theorem for the slowly varying function L [4]. Now we observe that $\mathsf{P}(X_0 > x) \sim \sim (\mathsf{E}[X])^{-1}\overline{F^s}(x)$. Moreover, since $x^{1-\varepsilon} = o(x)$, $\overline{F^s}(c_\delta x - x^{1-\varepsilon}) \sim \overline{F^s}(c_\delta x) \sim c_\delta^{-(\alpha-1)}\overline{F^s}(x)$, as $x \to \infty$. Therefore, sending δ to zero, $c_\delta \to (1+\gamma-a)^{-1}$ and then we obtain

$$\limsup_{x\to\infty} \frac{\mathsf{P}(W > x, B_x)}{\overline{F^s}(x)} \leqslant \lambda(1+\gamma-a)^{\alpha-1}. \tag{29}$$

Therefore by (27) and (29), the proposition 4 is proved.

3.3. Asymptotic equivalence for $\sum_{n\geqslant N_x} \mathsf{P}(W > x, A_{n,x})$

Proposition 5. *Under the previous assumptions*

$$\lim_{x\to\infty} \frac{\sum\limits_{n\geqslant N_x} \mathsf{P}(W > x, A_{n,x})}{\overline{F^s}(x)} = \frac{\lambda(1+\gamma-a)}{a-\gamma}(1+\gamma-a)^{\alpha-1}. \tag{30}$$

We will prove this proposition in two steps. First we derive a lower bound and then an upper bound.

3.3.1. Lower bound

We take $\forall n \geqslant N_x$

$$K_{n,x} = \left\{ \frac{\sum\limits_{j=1}^{n-1}\sum\limits_{i=1}^{b_j}(X_{j,i}\wedge(k-j)) + \sum\limits_{j=n+1}^{k}\sum\limits_{i=1}^{b_j}(X_{j,i}\wedge(k-j))}{k} - \gamma \geqslant -\eta_k,\ \forall k \geqslant n \right\}, \tag{31}$$

with some sequence η_n such that $\eta_n \downarrow 0$ and $n\eta_n \to \infty$ as $n \to \infty$. Hence $K_{n,x}$ satisfies the hypothesis a) of Proposition 3 and b) thanks to lemma 1.

We have for all $k \geqslant n$,

$$W \quad \geqslant \quad Y_n \wedge (k-n) + \sum_{j=1}^{n-1}\sum_{i=1}^{b_j}(X_{j,i}\wedge(k-j)) + \sum_{j=n+1}^{k}\sum_{i=1}^{b_j}(X_{j,i}\wedge(k-j)) - ka.$$

On the event $K_{n,x}$, taking $k = n + Y_n$, we have

$$W \quad \geqslant \quad Y_n + (n+Y_n)(\gamma - a - \eta_n) = Y_n(1+\gamma-a-\eta_n) + n(\gamma-a-\eta_n).$$

Since $1+\gamma-a<1$, we have

$$\sum_{n\geqslant N_x} \mathsf{P}\Big(W > x, A_{n,x}\Big) \geqslant \sum_{n\geqslant N_x} \mathsf{P}\Big(Y_n(1+\gamma-a-\eta_n) \geqslant x + n(a-\gamma+\eta_n)\Big)\mathsf{P}(K_{n,x}).$$

Let $\delta > 0$ such that $1+\gamma-a-\delta > 0$. Since $\eta_n \downarrow 0$ as $n \to \infty$, there exists n_0 such that for all $n \geqslant n_0$, $\eta_n < \delta$. Recalling that $N_x \to \infty$ as $x \to \infty$, if x is large enough, and for all $n \geqslant N_x \geqslant n_0$ we have

$$\sum_{n\geqslant N_x} \mathsf{P}\Big(W > x, A_{n,x}\Big) \geqslant \sum_{n\geqslant N_x} \mathsf{P}\Big(Y_n(1+\gamma-a-\delta) \geqslant x + n(a-\gamma+\delta)\Big)\mathsf{P}(K_{n,x}).$$

Therefore

$$\liminf_{x\to\infty} \frac{\sum\limits_{n\geqslant N_x} \mathsf{P}(W > x, A_{n,x})}{\overline{F^s}(x)} \geqslant \lambda\frac{(1+\gamma-a-\delta)}{a-\gamma+\delta}(1+\gamma-a-\delta)^{\alpha-1}, \tag{32}$$

where the formula on the right follows thanks to assumption b) of Proposition 3,the subexponentiality of Y_1, and since $N_x\overline{F}(x) = o(\overline{F^s}(x))$ [2,18].

Finally, sending δ to zero, we obtain

$$\liminf_{x\to\infty} \frac{\sum\limits_{n\geqslant N_x} \mathsf{P}(W > x, A_{n,x})}{\overline{F^s}(x)} \quad \geqslant \quad \lambda\frac{(1+\gamma-a)}{a-\gamma}(1+\gamma-a)^{\alpha-1}. \tag{33}$$

3.3.2. Upper bound

We choose a function α_x such that $\alpha_x \uparrow \infty$, $\alpha_x = o(x)$ as $x \to \infty$. Therefore, $\overline{F^s}(x-\alpha_x) \sim \overline{F^s}(x)$ as $x \to \infty$. Define $U_k = \left[\sum_{j=1}^k \sum_{i=1}^{b_j} (X_{j,i} \wedge (k-j)) - ka\right]^+$. For any n fixed

$$W \leqslant U + \max\left\{ \sup_{1\leqslant k\leqslant n-1} U_k,\ \sup_{k\geqslant n} \left[\sum_{j=1}^{n-1}\sum_{i=1}^{b_j}(X_{j,i}\wedge(k-j)) + \right.\right.$$

$$\left.\left. + \sum_{i=1}^{b_n}(X_{n,i}\wedge(k-n)) + \sum_{j=n+1}^{k}\sum_{i=1}^{b_j}(X_{j,i}\wedge(k-j)) - ka\right]^+ \right\}.$$

Now we note that $\forall n \in \mathbb{N}, \sup_{1\leqslant k\leqslant n-1} U_k \leqslant V < \infty$, a.s.; then

$$\lim_{x\to\infty} \mathsf{P}\left(\sup_{1\leqslant k\leqslant n-1} U_k < \alpha_x \right) = 1.$$

For all $n \geqslant N_x$ define

$$K'_{n,x} = \left\{ \sum_{j=1}^{n-1}\sum_{i=1}^{b_j}(X_{j,i}\wedge(k-j)) + \sum_{j=n+1}^{k}\sum_{i=1}^{b_j}(X_{j,i}\wedge(k-j)) - k\gamma < k\eta_k,\ \forall k \geqslant n \right\},$$

where η_n is a some sequence such that $\eta_n \downarrow 0$, $n\eta_n \to \infty$ as $n \to \infty$. Now define $\forall n \geqslant N_x$

$$K_{n,x} = K'_{n,x} \cap \left\{ U \leqslant \alpha_x \right\} \cap \left\{ \sup_{1\leqslant k\leqslant n-1} U_k \leqslant \alpha_x \right\}, \tag{34}$$

which is independent of Y_n and satisfies also the assumption b) of Proposition 3.

Thanks to the Lemma 3 with $Y_n = S_n$ we have

$$\sum_{n\geqslant N_x} \mathsf{P}\Big(W > x, K_{n,x}, Y_n > x + n(a-\gamma+\eta_n)\Big) =$$

$$= \sum_{n\geqslant N_x} \mathsf{P}\Big(W > x, K_{n,x}, M_n > x + n(a-\gamma+\eta_n), M_n^1 \leqslant \alpha_x\Big) + o(\overline{F^s}(x)).$$

We observe that on the event $\{M_n > x + n(a-\gamma+\eta_n), M_n^1 \leqslant \alpha_x\}$ we have

$$\sum_{i=1}^{b_n}(X_{n,i}\wedge(k-n)) \leqslant M_n \wedge (k-n) + b_n(\alpha_x \wedge (k-n));$$

moreover, on the event $\{M_n > x + n(a-\gamma+\eta_n), M_n^1 \leqslant \alpha_x, K_{n,x}\}$ we have

$$W \quad\leqslant\quad \alpha_x + \max\left(\alpha_x,\ \sup_{k\geqslant n}\left[M_n\wedge(k-n) + b_n(\alpha_x\wedge(k-n)) + k(\gamma - a + \eta_k)\right]^+ \right).$$

Now let $\delta > 0$ such that $\gamma - a + \delta < 0$. We have $\eta_k < \delta$, for all $k \geqslant n \geqslant N_x$, for x large enough. Hence

$$W \quad\leqslant\quad \alpha_x + \max\left(\alpha_x,\quad M_n(1+\gamma-a+\delta) + n(\gamma-a+\delta) + b_n\alpha_x\right),$$

since $-1 < \gamma - a + \delta < 0$. Therefore

$$\sum_{n\geqslant N_x} \mathsf{P}\Big(W > x, K_{n,x}, Y_n > x + n(a-\gamma+\eta_n)\Big) \leqslant$$

$$\leqslant \quad \sum_{n\geqslant N_x} \mathsf{P}\Big(M_n(1+\gamma-a+\delta) + n(\gamma-a+\delta) + b_n\alpha_x > x\Big) =$$

$$= \quad \sum_{n\geqslant N_x}\sum_{k=0}^{\infty} \mathsf{P}(b_n = k)\mathsf{P}\Big(\max_{i=1}^{k} X_{n,i}(1+\gamma-a+\delta) > x - k\alpha_x + n(a-\gamma-\delta)\Big).$$

With $c_\delta = (1+\gamma-a+\delta)^{-1}$ and $d_\delta = \frac{a-\gamma-\delta}{1+\gamma-a+\delta}$, we have

$$\begin{aligned}
&\leqslant \sum_{n\geqslant N_x}\sum_{k=1}^{\infty} \mathsf{P}(b_1=k)\mathsf{P}\Big(\max_{i=1}^{k} X_i > c_\delta x - c_\delta k\alpha_x + nd_\delta\Big) \leqslant \\
&\leqslant \sum_{n\geqslant N_x}\sum_{k=1}^{\infty} k\mathsf{P}(b_1=k)\mathsf{P}\Big(X > c_\delta x - c_\delta k\alpha_x + nd_\delta\Big) = \\
&= \frac{1}{d_\delta}\sum_{k=1}^{\infty} \mathsf{P}(b_1=k)k\overline{F^s}\left(c_\delta x - c_\delta k\alpha_x\right).
\end{aligned}$$

With similar arguments used for the upper bound of $\mathsf{P}(W>x,B_x)$ we have that

$$\limsup_{x\to\infty} \frac{\sum_{n>N_x}\mathsf{P}(W>x,A_{n,x})}{\overline{F^s}(x)} \leqslant \lambda\frac{1+\gamma-a+\delta}{a-\gamma-\delta}(1+\gamma-a+\delta)^{\alpha-1}.$$

Finally sending δ to zero, we conclude that

$$\limsup_{x\to\infty} \frac{\sum_{n\geqslant N_x}\mathsf{P}(W>x,A_{n,x})}{\overline{F^s}(x)} \quad\leqslant\quad \lambda\frac{(1+\gamma-a)}{a-\gamma}(1+\gamma-a)^{\alpha-1}, \tag{35}$$

which ends the proof of Proposition 5.

4. Appendix

Lemma 1. *Under the hypotheses* (A1),(A2), *and* (A3), *we have*

$$\frac{1}{n}\sum_{j=1}^{n}\sum_{i=1}^{b_j}(X_{j,i}\wedge(n-j)) \to \gamma \quad \text{a.s. as } n\to\infty.$$

Proof. By the strong law of large numbers (SLLN)

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\left(\sum_{i=1}^{b_j}(X_{j,i}\wedge(n-j))-\gamma\right) \leqslant \frac{1}{n}\sum_{j=1}^{n}\left(\sum_{i=1}^{b_j}X_{j,i}-\gamma\right)\leqslant 0, \text{ a.s.}$$

Now $\forall M<n$,

$$\begin{aligned}
&\frac{1}{n}\sum_{j=1}^{n}\left(\sum_{i=1}^{b_j}(X_{j,i}\wedge(n-j))-\gamma\right) \geqslant \\
&\geqslant \frac{1}{n}\sum_{j=1}^{n-M}\left(\sum_{i=1}^{b_j}(X_{j,i}\wedge(n-j))-\lambda\mathsf{E}[X]\right) - \frac{M\lambda\mathsf{E}[X]}{n} \geqslant \\
&\geqslant \frac{1}{n}\sum_{j=1}^{n-M}\left(\sum_{i=1}^{b_j}(X_{j,i}\wedge M)-\lambda\mathsf{E}[X\wedge M]\right) + (n-M)\left(\frac{\lambda\mathsf{E}[X\wedge M]-\lambda\mathsf{E}[X]}{n-M}\right) - \frac{M\lambda\mathsf{E}[X]}{n}.
\end{aligned}$$

Therefore, again by the SLLN, for any M

$$\liminf_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\left(\sum_{i=1}^{b_j}(X_{j,i}\wedge(n-j))-\gamma\right) \quad\geqslant\quad \lambda\mathsf{E}[X\wedge M]-\lambda\mathsf{E}[X], \text{ a.s.}$$

Since as $M\to\infty$, it holds that $\mathsf{E}[X\wedge M]\uparrow\mathsf{E}[X]$,

$$\liminf_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\left(\sum_{i=1}^{b_j}(X_{j,i}\wedge(n-j))-\gamma\right)\geqslant 0, \ a.s.,$$

and the lemma is proved.

Lemma 2. *Let N be a Poisson random variable, and $\{Z_i\}_{i\in\mathbb{N}}$ be a sequence of i.i.d sub-exponential random variables with distribution function G. Let*

$$S_N = \sum_{i=1}^{N} Z_i \quad \textit{and}, \quad M_N = \max_{i=1}^{N} Z_i.$$

Moreover let $i_0 = \arg\max_{i=1}^{N} Z_i$ and $M_N^1 = \max_{i=1, i\neq i_0}^{N} Z_i$. Then for any given event E and for any function $\alpha_x \uparrow \infty$ as $x \to \infty$ such that $\alpha_x = o(x)$, we have

$$\begin{aligned} \mathsf{P}[E,\ S_N > x] &= \mathsf{P}[E,\ M_N > x] + o(\overline{G}(x)) = &(36)\\ &= \mathsf{P}[E,\ M_N > x, M_N^1 \leqslant \alpha_x] + o(\overline{G}(x)). &(37)\end{aligned}$$

Proof. Let $p_n = \mathsf{P}(N = n)$. Since $\alpha_x \leqslant x$ for x large,

$$\frac{\mathsf{P}[M_N > x, M_N^1 \leqslant \alpha_x]}{\overline{G}(x)} = \frac{\sum_{n=1}^{\infty} p_n \sum_{k=1}^{n} \mathsf{P}[Y_k > x, Y_i \leqslant \alpha_x\ \forall i \neq k]}{\overline{G}(x)} = \sum_{n=1}^{\infty} n p_n G(\alpha_x)^{n-1}.$$

By the dominated convergence theorem,

$$\lim_{x\to\infty} \frac{\mathsf{P}[M_N > x, M_N^1 \leqslant \alpha_x]}{\overline{G}(x)} = \sum_{n=1}^{\infty} n p_n = \mathsf{E}[N]. \tag{38}$$

Therefore, since

$$\frac{\lim_{x\to\infty} \mathsf{P}(M_N > x)}{\overline{G}(x)} = \mathsf{E}[N]$$

it follows from (38) that

$$\mathsf{P}(M_N > x, M_N^1 > \alpha_x) = o(\overline{G}(x)). \tag{39}$$

Since $M_N \leqslant S_N$, we have clearly $\mathsf{P}[E,\ S_N > x] \geqslant \mathsf{P}[E,\ M_N > x, M_N^1 \leqslant \alpha_x]$. Moreover, we have

$$\mathsf{P}[E,\ S_N > x] \leqslant \mathsf{P}[E,\ M_N > x, M_N^1 \leqslant \alpha_x] + \mathsf{P}[\ M_N > x, M_N^1 > \alpha_x] + \mathsf{P}[S_N > x,\ M_N \leqslant x].$$

By the subexponentiality [4,6,18], we have $\mathsf{P}[S_N > x] \sim \mathsf{P}[M_N > x] \sim \mathsf{E}[N]\overline{G}(x),\ x \to \infty$. Hence

$$\mathsf{P}[S_N > x,\ M_N \leqslant x] = \mathsf{P}[S_N > x] - \mathsf{P}[M_N > x] = o(\overline{G}(x)). \tag{40}$$

Finally by (39) and (40),

$$\mathsf{P}(E,\ S_N > x) = \mathsf{P}(E,\ M_N > x, M_N^1 \leqslant \alpha_x) + o(\overline{G}(x)).$$

Lemma 3. *Under the conditions* (A1), (A2), *and* (A3) *define*

$$S_n = \sum_{i=1}^{b_n} X_{n,i}, \quad M_n = \max_{i=1}^{b_n} X_{n,i}, \quad i_n = \arg\max_{i=1,\cdots,b_n} X_{n,i}, \quad M_n^1 = \max_{i\neq i_n, i=1}^{b_n} X_{n,i}.$$

Then, $\forall c > 0$, $\eta_n \downarrow 0$, as $n \to \infty$, for all events E_n and for any function $\alpha_x \uparrow \infty$ as $x \to \infty$ with $\alpha_x = o(x)$, we have

$$\sum_{n\geqslant N_x} \mathsf{P}(E_n, S_n > x + n(c+\eta_n)) = \sum_{n\geqslant N_x} \mathsf{P}(E_n, M_n > x + n(c+\eta_n), M_n^1 \leqslant \alpha_x) + o(\overline{F^s}(x)).$$

Proof. We omit in the following proof the sequence η_n, but the same arguments are true, including η_n. For x large enough

$$\begin{aligned}\sum_{n\geqslant 1} \mathsf{P}(M_n > x + nc, M_n^1 \leqslant \alpha_x) &= \sum_{n\geqslant 1}\sum_{k=1}^{\infty} \mathsf{P}(b_n = k) \sum_{i=1}^{k} \mathsf{P}(X_{n,i} > x + nc, X_{n,j} \leqslant \alpha_x\ \forall j \neq i) = \\ &= \sum_{n\geqslant 1} \mathsf{P}(X > x + nc) \left(\sum_{k=1}^{\infty} \mathsf{P}(b_1 = k) k [\mathsf{P}(X \leqslant \alpha_x)]^{k-1}\right).\end{aligned}$$

Therefore by the dominated convergence theorem, we have

$$\lim_{x\to\infty}\frac{\sum_{n\geqslant 1}\mathsf{P}(M_n > x+nc, M_n^1\leqslant \alpha_x)}{\overline{F^s}(x)} \quad = \quad \frac{1}{c}\sum_{k=1}^{\infty}\mathsf{P}(b_1=k)k=\frac{\lambda}{c}. \tag{41}$$

On the other hand,

$$\sum_{n\geqslant 1}\mathsf{P}(M_n > x+nc)=\sum_{n\geqslant 1}\mathsf{P}(M_n > x+nc, M_n^1\leqslant\alpha_x)+\sum_{n\geqslant 1}\mathsf{P}(M_n > x+nc, M_n^1>\alpha_x).$$

By the definition of M_n, we have also

$$\lim_{x\to\infty}\frac{\sum_{n\geqslant 1}\mathsf{P}(M_n > x+nc)}{\overline{F^s}(x)}=\frac{\lambda}{c}. \tag{42}$$

Therefore, by (41) and (42),

$$\sum_{n\geqslant 1}\mathsf{P}(M_n > x+nc, M_n^1>\alpha_x)=o(\overline{F^s}(x)),\ x\to\infty. \tag{43}$$

Moreover, since $S_n\geqslant M_n$,

$$\sum_{n\geqslant 1}\mathsf{P}(E_n, S_n > x+nc) \quad \geqslant \quad \sum_{n\geqslant 1}\mathsf{P}(E_n, M_n > x+nc, M_n^1\leqslant\alpha_x).$$

Again

$$\begin{aligned}\sum_{n\geqslant 1}\mathsf{P}(E_n, S_n > x+nc) \quad &\leqslant \quad \sum_{n\geqslant 1}\mathsf{P}(E_n, M_n > x+nc, M_n^1\leqslant\alpha_x)+\sum_{n\geqslant 1}\mathsf{P}(M_n > x+nc, M_n^1>\alpha_x)+\\ &+\sum_{n\geqslant 1}\mathsf{P}(S_n > x+nc, M_n\leqslant x+nc).\end{aligned}$$

Now by the subexponentiality [2, 18], we have

$$\sum_{n\geqslant 1}\mathsf{P}(S_n > x+nc)\sim\sum_{n\geqslant 1}\mathsf{P}(M_n > x+nc)\sim\lambda c^{-1}\overline{F^s}(x),\ x\to\infty.$$

Hence

$$\sum_{n\geqslant 1}P(S_n > x+nc, M_n\leqslant x+nc)=o(\overline{F^s}(x)). \tag{44}$$

Therefore, by (43) and (44),

$$\sum_{n\geqslant 1}\mathsf{P}(E_n, S_n > x+nc)=\sum_{n\geqslant 1}\mathsf{P}(E_n, M_n > x+nc, M_n^1\leqslant\alpha_x)+o(\overline{F^s}(x)).$$

We conclude, observing that

$$\begin{aligned}&\sum_{n=1}^{N_x}\mathsf{P}(E_n, S_n > x+nc)\leqslant N_x\mathsf{P}(S_1>x)=o(\overline{F^s}(x)),\\ &\sum_{n=1}^{N_x}\mathsf{P}(E_n, M_n > x+nc, M_n^1\leqslant\alpha_x)\leqslant N_x\mathsf{P}(M_1>x)=o(\overline{F^s}(x)).\end{aligned}$$

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