ACN 915: Algorithms for networked information
Homework 2

all homeworks need to be returned on February 6

Exercise 1

Let A be a $n \times d$ matrix with singular values $\sigma_1, \ldots, \sigma_r$ and right singular vectors $v_1, \ldots v_r$. We define the Frobenius norm of A by $\|A\|_F = \sqrt{\sum_{j,k} a_{jk}^2}$.

1. For a_j a row of A, show that $\sum_{i=1}^{r} (a_j^T v_i)^2 = \|a_j\|^2$.
2. Show that $\|A\|_F = \sum_{i=1}^{r} \sigma_i^2$.

For $1 \leq k \leq r$, we define $A_k = \sum_{i=1}^{k} \sigma_i u_i v_i^T$, where u_i are the left singular vectors.

3. Show that A_k has rank k.

Let $V_k = Vect(v_1, \ldots, v_k)$.

4. Show that the rows of A_k are the projections of the rows of A onto V_k.

Consider B with rank $k \leq r$ minimizing $\|A - B\|_F$ and let V be the space spanned by the rows of B.

5. Show that $dim(V) \leq k$ and that each row of B is the projection of the corresponding row of A onto V.
6. Show that $\|A - A_k\|_F \leq \|A - B\|_F$.

Exercise 2

We consider a set V of candidates. The extended Condorcet criterion requires: if X, Y are a partition of the set V of all candidates and for all $x \in X$, $y \in Y$, x beats y in direct comparison (i.e. more voters prefer x to y than vice versa), then all of X should precede all of Y.

Recall that a tournament graph G on V is a complete oriented graph on V.

1. Show that each tournament graph G contains a Hamiltonian path (i.e. a path going through each node exactly once).
2. Show that for any profile of voters, there is a rank aggregation (i.e. a consensus ordering) satisfying the extended Condorcet criterion.