Foster criteria and Maximally stable distributed scheduling

Laurent Massoulié

MSR-Inria Joint Centre

September 29, 2015

Reminders, Markov chains (discrete time)

State $x \in E$ is

- **recurrent** if $\mathbb{P}_x(T_x < +\infty) = 1$
- **positive recurrent** if $\mathbb{E}_x(T_x) < +\infty$
- **null recurrent** if $\mathbb{P}_x(T_x < +\infty) = 1$ & $\mathbb{E}_x(T_x) = +\infty$
- **transient** if not recurrent, i.e. $\mathbb{P}_x(T_x < +\infty) < 1$
- d-**periodic** if $d = \mathrm{GCD}(n \geq 0 : p_{xx}^n > 0)$

Reminders, Markov chains (discrete time)

State $x \in E$ is

- **recurrent** if $\mathbb{P}_x(T_x < +\infty) = 1$
- **positive recurrent** if $\mathbb{E}_x(T_x) < +\infty$
- **null recurrent** if $\mathbb{P}_x(T_x < +\infty) = 1$ & $\mathbb{E}_x(T_x) = +\infty$
- **transient** if not recurrent, i.e. $\mathbb{P}_x(T_x < +\infty) < 1$
- d-**periodic** if $d = \mathrm{GCD}(n \geq 0 : p^n_{xx} > 0)$

Markov chain is **irreducible** iff $\forall x, y \in E$,
$\exists n \in \mathbb{N}, x_0^n \in E^{n+1} \mid x_0 = x,\ x_n = y\ \&\ \prod_{i=1}^n p_{x_{i-1}x_i} > 0$

Reminders, Markov chains (discrete time)

State $x \in E$ is

- **recurrent** if $\mathbb{P}_x(T_x < +\infty) = 1$
- **positive recurrent** if $\mathbb{E}_x(T_x) < +\infty$
- **null recurrent** if $\mathbb{P}_x(T_x < +\infty) = 1$ & $\mathbb{E}_x(T_x) = +\infty$
- **transient** if not recurrent, i.e. $\mathbb{P}_x(T_x < +\infty) < 1$
- d-**periodic** if $d = \text{GCD}(n \geq 0 : p_{xx}^n > 0)$

Markov chain is **irreducible** iff $\forall x, y \in E$,
$\exists n \in \mathbb{N}, x_0^n \in E^{n+1} \mid x_0 = x,\ x_n = y \ \& \prod_{i=1}^n p_{x_{i-1}x_i} > 0$

Proposition

For irreducible chain, if one state x is transient (resp. null recurrent, positive recurrent, d-periodic) then all are

Stationary measures

Non-negative measure π on E is **stationary** for P iff
$\forall x \in E, \pi_x = \sum_{y \in E} \pi_y p_{yx}$

Stationary measures

Non-negative measure π on E is **stationary** for P iff
$\forall x \in E, \pi_x = \sum_{y \in E} \pi_y p_{yx}$

Notation: $\mathbb{P}_\nu := \sum_{x \in E} \nu_x \mathbb{P}_x$ chain's distribution when $X_0 \sim \nu$

$\Rightarrow$ For stationary probability distribution π,
$\forall n > 0, \mathbb{P}_\pi(X_n^\infty \in \cdot) = \mathbb{P}_\pi(X_0^\infty \in \cdot)$

Limit theorems 1

Recurrence and stationary measures

Irreducible recurrent chain admits a stationary measure, unique up to multiplicative factor $\forall y \in E,\ \pi_y = \mathbb{E}_x \sum_{n=1}^{T_x} \mathbb{I}_{X_n = y}$

Irreducible chain admits a stationary probability distribution iff it is positive recurrent

Limit theorems 1

Recurrence and stationary measures

Irreducible recurrent chain admits a stationary measure, unique up to multiplicative factor $\forall y \in E, \; \pi_y = \mathbb{E}_x \sum_{n=1}^{T_x} \mathbb{I}_{X_n=y}$

Irreducible chain admits a stationary probability distribution iff it is positive recurrent

Ergodic theorem

Irreducible, positive recurrent chain satisfies almost sure convergence

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(X_n) = \sum_{x\in E} \pi_x f(x)$$

for all π-integrable f, where π = unique stationary distribution

Such chains are called ergodic

Limit theorems 2

Convergence in distribution

Ergodic, aperiodic chain satisfies $\forall x \in E, \lim_{n\to\infty} \mathbb{P}(X_n = x) = \pi_x$ where π: unique stationary distribution

Limit theorems 2

Convergence in distribution

Ergodic, aperiodic chain satisfies $\forall x \in E, \lim_{n\to\infty} \mathbb{P}(X_n = x) = \pi_x$ where π: unique stationary distribution

"Converse"

Irreducible, non-ergodic chain satisfies $\forall x \in E, \lim_{n\to\infty} \mathbb{P}(X_n = x) = 0$

Foster-Lyapunov criterion for ergodicity

Theorem

An irreducible chain such that there exist $V : E \to \mathbb{R}_+$, a finite set $K \subset E$ and $\epsilon, b > 0$ satisfying

$$\mathbb{E}(V(X_{n+1}) - V(X_n)|X_n = x) \leq \begin{cases} -\epsilon, & x \notin K, \\ b - \epsilon, & x \in K, \end{cases}$$

is then ergodic.

Reminders– Markov jump processes (continuous time)

Infinitesimal Generator

$\forall x, y,\ y \neq x \in E$, limits $q_x := \lim_{t\to 0} \frac{1-p_{xx}(t)}{t}$, $q_{xy} = \lim_{t\to 0} \frac{p_{xy}(t)}{t}$ exist in $\mathbb{R}_+$ and satisfy $\sum_{y\neq x} q_{xy} = q_x$

q_{xy}: **Jump rate** from x to y

$Q := \{q_{xy}\}_{x,y\in E}$ where $q_{xx} = -q_x$: **Infinitesimal Generator** of process $\{X_t\}_{t\in\mathbb{R}_+}$

Formally: $Q = \lim_{h\to 0} \frac{1}{h}[P(h) - I]$ where I: identity matrix

Reminders– Markov jump processes (continuous time)

Infinitesimal Generator

$\forall x, y,\ y \neq x \in E$, limits $q_x := \lim_{t \to 0} \frac{1-p_{xx}(t)}{t}$, $q_{xy} = \lim_{t \to 0} \frac{p_{xy}(t)}{t}$ exist in $\mathbb{R}_+$ and satisfy $\sum_{y \neq x} q_{xy} = q_x$

q_{xy}: **Jump rate** from x to y

$Q := \{q_{xy}\}_{x,y \in E}$ where $q_{xx} = -q_x$: **Infinitesimal Generator** of process $\{X_t\}_{t \in \mathbb{R}_+}$

Formally: $Q = \lim_{h \to 0} \frac{1}{h}[P(h) - I]$ where I: identity matrix

Structure of Markov jump processes

Sequence $\{Y_n\}_{n \in \mathbb{N}}$ of visited states: Markov chain with transition matrix $p_{xy} = \mathbb{I}_{x \neq y} \frac{q_{xy}}{q_x}$

Conditionally on $\{Y_n\}_{n \in \mathbb{N}}$, sojourn times $\{\tau_n\}_{n \in \mathbb{N}}$ in successive states Y_n: independent, with distributions $\mathrm{Exp}(q_{Y_n})$

Definitions

- Process $\{X_t\}_{t\in\mathbb{R}_+}$ is **irreducible** (respectively, **irreducible recurrent**) if induced chain $\{Y_n\}_{n\in\mathbb{N}}$ is
- State x is **positive recurrent** if $\mathbb{E}_x(R_x) < +\infty$, where

$$R_x = \inf\{t > \tau_0 : X_t = x\}.$$

- Measure π is **invariant** for process $\{X_t\}_{t\in\mathbb{R}_+}$ if for all $t > 0$, $\pi^T P(t) = \pi^T$, i.e.

$$\forall x \in E, \sum_{y\in E} \pi_y p_{yx}(t) = \pi_x.$$

- Measure π is **stationary** if satisfies **global balance** equations

$$\forall x \in E,\ \underbrace{\pi_x \textstyle\sum_{y\neq x} q_{xy}}_{\text{flow out of } x} = \underbrace{\textstyle\sum_{y\neq x} \pi_y q_{yx}}_{\text{flow into } x}$$

Limit theorems 1

Theorem

For irreducible recurrent $\{X_t\}_{t\in\mathbb{R}_+}$, $\exists$ *invariant measure* π, *unique up to some scalar factor. It can be defined as, for any* $x \in E$*:*

$$\forall y \in E,\ \pi_y = \mathbb{E}_x \int_0^{R_x} \mathbb{I}_{X_t=y} dt,$$

or alternatively with $T_x := \inf\{n > 0 : Y_n = x\}$,

$$\forall y \in E,\ \pi_y = \frac{1}{q_y} \mathbb{E}_x \sum_{n=1}^{T_x} \mathbb{I}_{Y_n=y}.$$

Limit theorems 1

Theorem

For irreducible recurrent $\{X_t\}_{t\in\mathbb{R}_+}$, $\exists$ *invariant measure* π, *unique up to some scalar factor. It can be defined as, for any* $x \in E$:

$$\forall y \in E,\ \pi_y = \mathbb{E}_x \int_0^{R_x} \mathbb{I}_{X_t=y} dt,$$

or alternatively with $T_x := \inf\{n > 0 : Y_n = x\}$,

$$\forall y \in E,\ \pi_y = \frac{1}{q_y} \mathbb{E}_x \sum_{n=1}^{T_x} \mathbb{I}_{Y_n=y}.$$

Corollaries

- $\{\hat{\pi}_y\}$ invariant for $\{Y_n\}_{n\in\mathbb{N}} \Leftrightarrow \{\hat{\pi}_y/q_y\}$ invariant for $\{X_t\}_{t\in\mathbb{R}_+}$.
- For irreducible recurrent $\{X_t\}_{t\in\mathbb{R}_+}$, either all or no state $x \in E$ is positive recurrent.

Limit theorems 2

Theorem

$\{X_t\}_{t\in\mathbb{R}_+}$ *is* **ergodic** *(i.e. irreducible, positive recurrent) iff it is irreducible, non-explosive and such that* $\exists\pi$ *satisfying global balance equations.*
Then π *is also the unique invariant probability distribution.*

Limit theorems 2

Theorem

$\{X_t\}_{t\in\mathbb{R}_+}$ is **ergodic** *(i.e. irreducible, positive recurrent) iff it is irreducible, non-explosive and such that $\exists\pi$ satisfying global balance equations.*
Then π is also the unique invariant probability distribution.

Theorem

For ergodic $\{X_t\}_{t\in\mathbb{R}_+}$ with stationary distribution π, any initial distribution for X_0 and π-integrable f,

$$\text{almost surely} \quad \lim_{t\to\infty}\frac{1}{t}\int_0^t f(X_s)ds = \sum_{x\in E}\pi_x f(x) \quad \text{(ergodic theorem)}$$

and in distribution $X_t \to \pi$ as $t\to\infty$.

Limit theorems 3

Theorem

For irreducible, non-ergodic $\{X_t\}_{t\in\mathbb{R}_+}$, *any initial distribution for* X_0, *then for all* $x \in E$,

$$\lim_{t\to\infty} \mathbb{P}(X_t = x) = 0.$$

Foster-Lyapunov criterion – continuous time

Theorem

Assume (i) Process $\{X_t\}_{t\in\mathbb{R}_+}$ *irreducible non explosive;*

(ii) There is a function $V : E \to \mathbb{R}_+$, *a finite set* $K \subset E$ *and constants* $b, \epsilon > 0$ *such that*

$$\forall x \in E, \sum_{y\neq x} q_{xy}[V(y) - V(x)] \leq -\epsilon + b\mathbb{I}_{x\in K}.$$

Then $\{X_t\}_{t\in\mathbb{R}_+}$ *is ergodic.*

Proof steps

- Induced chain $\{Y_n\}_{n\in\mathbb{N}}$ such that

$$\mathbb{E}(V(Y_{n+1}-V(Y_n)|Y_n=x)\leq -\frac{\epsilon}{q_x}+\frac{b}{q_x}\mathbb{I}_{x\in K}$$

Proof steps

- Induced chain $\{Y_n\}_{n\in\mathbb{N}}$ such that

$$\mathbb{E}(V(Y_{n+1} - V(Y_n)|Y_n = x) \leq -\frac{\epsilon}{q_x} + \frac{b}{q_x}\mathbb{I}_{x\in K}$$

- Hence for $N := \inf\{n > 0 : Y_n \in K\}$,

$$\mathbb{E}[V(Y_N) - V(Y_0)|Y_0 = x] \leq -\epsilon\mathbb{E}\left[\sum_{n=0}^{N-1} \frac{1}{q_{Y_n}}|Y_0 = x\right] + \frac{b}{q_x}$$

Proof steps

- Induced chain $\{Y_n\}_{n\in\mathbb{N}}$ such that

$$\mathbb{E}(V(Y_{n+1} - V(Y_n)|Y_n = x) \leq -\frac{\epsilon}{q_x} + \frac{b}{q_x}\mathbb{I}_{x\in K}$$

- Hence for $N := \inf\{n > 0 : Y_n \in K\}$,

$$\mathbb{E}[V(Y_N) - V(Y_0)|Y_0 = x] \leq -\epsilon\mathbb{E}\left[\sum_{n=0}^{N-1} \frac{1}{q_{Y_n}}|Y_0 = x\right] + \frac{b}{q_x}$$

- Yields, letting $R(K) := \inf\{t > T_1 : X_t \in K\} = T_N$, return time to set K,

$$\mathbb{E}[R(K)|X_0 = x] \leq \frac{1}{\epsilon}\left[V(x) + \frac{b}{q_x}\right]$$

Proof steps

- Induced chain $\{Y_n\}_{n\in\mathbb{N}}$ such that

$$\mathbb{E}(V(Y_{n+1} - V(Y_n)|Y_n = x) \leq -\frac{\epsilon}{q_x} + \frac{b}{q_x}\mathbb{I}_{x\in K}$$

- Hence for $N := \inf\{n > 0 : Y_n \in K\}$,

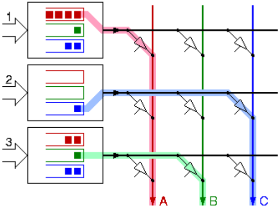
$$\mathbb{E}[V(Y_N) - V(Y_0)|Y_0 = x] \leq -\epsilon\mathbb{E}\left[\sum_{n=0}^{N-1} \frac{1}{q_{Y_n}}|Y_0 = x\right] + \frac{b}{q_x}$$

- Yields, letting $R(K) := \inf\{t > T_1 : X_t \in K\} = T_N$, return time to set K,

$$\mathbb{E}[R(K)|X_0 = x] \leq \frac{1}{\epsilon}\left[V(x) + \frac{b}{q_x}\right]$$

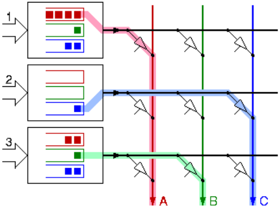
- Implies, reasoning on chain $\{Z_n\}_{n\in\mathbb{N}}$ of visits of $\{Y_n\}_{n\in\mathbb{N}}$ to set K, that $\mathbb{E}_x(R_x) < +\infty$ for all $x \in K$, hence ergodicity

Scheduling in cross-bar switches



- Switch with N input and N output ports
- Time slot n: $A_n(i,j)$ packets arrive at input port i, destined to port j
- Transmission: permutation $\sigma_n \in \mathcal{S}_n$, symmetric group, matches input port i with output port $\sigma_n(i)$

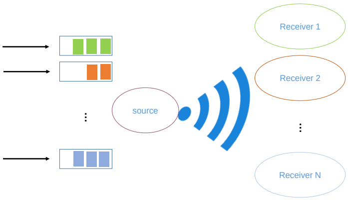
Scheduling in cross-bar switches



- Switch with N input and N output ports
- Time slot n: $A_n(i,j)$ packets arrive at input port i, destined to port j
- Transmission: permutation $\sigma_n \in \mathcal{S}_n$, symmetric group, matches input port i with output port $\sigma_n(i)$

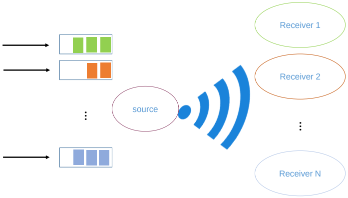
$\Rightarrow$ How to choose σ_n to ensure ergodicity, i.e. stationary regime instead of queue blowup?

Scheduling downlink wireless transmissions



- Wireless source to send packets to wireless receivers
- Time slot n: $A_n(r)$ packets arrive at source for receiver r
- Wireless medium conditions change in each slot n: $S_n(r) =$ number of packets that could be sent to receiver r if it was chosen then

Scheduling downlink wireless transmissions



- Wireless source to send packets to wireless receivers
- Time slot n: $A_n(r)$ packets arrive at source for receiver r
- Wireless medium conditions change in each slot n: $S_n(r) =$ number of packets that could be sent to receiver r if it was chosen then

$\Rightarrow$ How to choose which receiver to schedule based on queue lengths (backlogs) and medium condition to ensure ergodicity, i.e. stationary regime instead of queue blowup?

Max-Weight scheduling

- Traffic types $r \in \mathcal{R}$, i.i.d. arrivals: $A_n(r) \in \mathbb{N}$ in slot n
- i.i.d. set $\mathcal{S}_n \subset [s_{max}]^{\mathcal{R}}$ of feasible services in slot n
- $X_n(r)$: backlog of type r requests at end of slot n

Max-Weight scheduling

- Traffic types $r \in \mathcal{R}$, i.i.d. arrivals: $A_n(r) \in \mathbb{N}$ in slot n
- i.i.d. set $\mathcal{S}_n \subset [s_{max}]^{\mathcal{R}}$ of feasible services in slot n
- $X_n(r)$: backlog of type r requests at end of slot n
- Evolution equation $X_{n+1}(r) = (X_n(r) - s_n(r))^+ + A_{n+1}(r)$, where $s_n \in \mathcal{S}_n$

Max-Weight scheduling

- Traffic types $r \in \mathcal{R}$, i.i.d. arrivals: $A_n(r) \in \mathbb{N}$ in slot n
- i.i.d. set $\mathcal{S}_n \subset [s_{max}]^{\mathcal{R}}$ of feasible services in slot n
- $X_n(r)$: backlog of type r requests at end of slot n
- Evolution equation $X_{n+1}(r) = (X_n(r) - s_n(r))^+ + A_{n+1}(r)$, where $s_n \in \mathcal{S}_n$
- (w, α)-Max-weight scheduling rule for for $w_r, \alpha > 0$:

  Choose $s_n \in \text{Argmax}_{s \in \mathcal{S}_n} \left\{ \sum_{r \in \mathcal{R}} w_r X_n(r)^\alpha s(r) \right\}$

Max-Weight scheduling: ergodicity properties

- Assume (to ensure irreducibility) $\mathbb{P}(A_n(r) = 0) \in]0, 1[$, $\mathbb{P}(\exists s \in \mathcal{S}_n(r) : s(r) > 0) > 0$

- Let **schedulable region** $\mathcal{C}$ be set of vectors $x \in \mathbb{R}_+^{\mathcal{R}}$ such that $\exists z(\mathcal{S}) \in \text{env}(\mathcal{S}) : \forall r \in \mathcal{R}, x_r \leq \sum_{\mathcal{S} \subset [s_{max}]^{\mathcal{R}}} \mathbb{P}(\mathcal{S}_n = \mathcal{S}) z_r(\mathcal{S})$ where $\text{env}(\mathcal{S})$: convex hull of set $\mathcal{S}$

- Let $\rho_r := \mathbb{E}(A_n(r))$

Max-Weight scheduling: ergodicity properties

- Assume (to ensure irreducibility) $\mathbb{P}(A_n(r) = 0) \in]0,1[$, $\mathbb{P}(\exists s \in \mathcal{S}_n(r) : s(r) > 0) > 0$
- Let **schedulable region** $\mathcal{C}$ be set of vectors $x \in \mathbb{R}_+^{\mathcal{R}}$ such that
$$\exists z(\mathcal{S}) \in \mathrm{env}(\mathcal{S}) : \ \forall r \in \mathcal{R}, x_r \leq \sum_{\mathcal{S} \subset [s_{max}]^{\mathcal{R}}} \mathbb{P}(\mathcal{S}_n = \mathcal{S}) z_r(\mathcal{S})$$
where $\mathrm{env}(\mathcal{S})$: convex hull of set $\mathcal{S}$
- Let $\rho_r := \mathbb{E}(A_n(r))$

Theorem

If $\mathbb{E}A_n(r)^{1+\alpha} < +\infty$ and for some $\epsilon > 0$, $(\rho_r + \epsilon)_{r \in \mathcal{R}} \in \mathcal{C}$, then process $\{X_n\}_{n \in \mathbb{N}}$ is ergodic.
Conversely, if $\rho \notin \mathcal{C}$, then for any strategy (max-weight or other), process $\{X_n\}_{n \in \mathbb{N}}$ is transient.

Comments

- Maximizes set of offered loads ρ for which ergodicity holds (for ρ on frontier of $\mathcal{C}$, chain at best null-recurrent)

Comments

- Maximizes set of offered loads ρ for which ergodicity holds (for ρ on frontier of $\mathcal{C}$, chain at best null-recurrent)
- Does not require explicit learning of either ρ (statistics of request arrivals) or $\mathcal{S}_n$ (statistics of time varying capacity)

Comments

- Maximizes set of offered loads ρ for which ergodicity holds (for ρ on frontier of $\mathcal{C}$, chain at best null-recurrent)

- Does not require explicit learning of either ρ (statistics of request arrivals) or $\mathcal{S}_n$ (statistics of time varying capacity)

- Switch scheduling: convex enveloppe of permutation matrices $M_\sigma = (\mathbb{I}_{j=\sigma(i)})_{i,j\in[N]}$ = Doubly stochastic matrices, i.e. $M \in \mathbb{R}_+^{N\times N}$ such that

$$\forall i \in [N], \sum_{j\in[N]} M_{ij} = 1 = \sum_{j\in[N]} M_{ji}$$

(Birkhoff-von Neumann theorem)

Comments

- Maximizes set of offered loads ρ for which ergodicity holds (for ρ on frontier of $\mathcal{C}$, chain at best null-recurrent)

- Does not require explicit learning of either ρ (statistics of request arrivals) or $\mathcal{S}_n$ (statistics of time varying capacity)

- Switch scheduling: convex enveloppe of permutation matrices $M_\sigma = (\mathbb{I}_{j=\sigma(i)})_{i,j\in[N]}$ = Doubly stochastic matrices, i.e. $M \in \mathbb{R}_+^{N\times N}$ such that

$$\forall i \in [N], \sum_{j\in[N]} M_{ij} = 1 = \sum_{j\in[N]} M_{ji}$$

(Birkhoff-von Neumann theorem)
Hence switch process ergodic if and only if

$$\forall i \in [N], \sum_{j\in[N]} \mathbb{E}(A(i,j)) < 1 \& \sum_{j\in[N]} \mathbb{E}(A(j,i)) < 1.$$

Proof elements

- Ergodicity: Use Foster's criterion with Lyapunov function $V(X) = \sum_{r \in \mathcal{R}} w_r \frac{X_r^{1+\alpha}}{1+\alpha}$

Proof elements

- Ergodicity: Use Foster's criterion with Lyapunov function $V(X) = \sum_{r \in \mathcal{R}} w_r \frac{X_r^{1+\alpha}}{1+\alpha}$
- Transience: for $\rho \notin \mathcal{C}$, use convex separation theorem:

$$\exists b \in \mathbb{R}^{\mathcal{R}}, \delta > 0 : \forall x \in \mathcal{C}, \sum_{r \in \mathcal{R}} b_r \rho_r \geq \delta + \sum_{r \in \mathcal{R}} b_r x_r.$$

From monotonicity of $\mathcal{C}$, can choose $b_r \geq 0, \; r \in \mathcal{R}$

Proof elements

- Ergodicity: Use Foster's criterion with Lyapunov function $V(X) = \sum_{r\in\mathcal{R}} w_r \frac{X_r^{1+\alpha}}{1+\alpha}$
- Transience: for $\rho \notin \mathcal{C}$, use convex separation theorem:

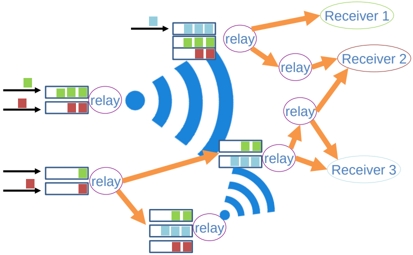
$$\exists b \in \mathbb{R}^{\mathcal{R}}, \delta > 0 : \forall x \in \mathcal{C}, \sum_{r\in\mathcal{R}} b_r \rho_r \geq \delta + \sum_{r\in\mathcal{R}} b_r x_r.$$

From monotonicity of $\mathcal{C}$, can choose $b_r \geq 0,\ r \in \mathcal{R}$
$\Rightarrow$ Lower bound:

$$\begin{aligned}\sum_r b_r X_n(r) &\geq \sum_{m=1}^{n}\sum_{r\in\mathcal{R}} b_r A_m(r) - \sum_{m=1}^{n}\sum_{r\in\mathcal{R}} s_n(r)\\ &\geq n\left[\sum_{r\in\mathcal{R}} b_r\left(\rho_r - \sum_{\mathcal{S}\subset[s_{max}]^{\mathcal{R}}} \mathbb{P}(\mathcal{S}_n = \mathcal{S}) z_r(\mathcal{S})\right)\right] + o(n)\\ &\geq n\delta + o(n),\end{aligned}$$

by law of large numbers and convex separation result. Hence almost surely $\lim_{n\to\infty}\sup_{r\in\mathcal{R}} X_n(r) = +\infty$

multi-hop, multipath networks



- Several traffic types, packets from each type: may be created at several network locations
- Each network location: may choose which traffic type to forward, and to which neighbor to forward it (interferences may constrain decisions at distinct locations)
- Each created packet replicated at only one location if still present; disappears when reaches its destination

Max-weight backpressure: setup

- Data types $r \in \mathcal{R}$, i.i.d. arrivals $A_n(r)$ in slot n . Also, let $\mathcal{R}' := \mathcal{R} \cup \{ext\}$
- Set of potential transmissions per time slot: $\mathcal{S} \subset [s_{max}]^{\mathcal{R} \times \mathcal{R}'}$
- $X_n(r)$: backlog of type r-data in time slot n

Max-weight backpressure: setup

- Data types $r \in \mathcal{R}$, i.i.d. arrivals $A_n(r)$ in slot n . Also, let $\mathcal{R}' := \mathcal{R} \cup \{ext\}$
- Set of potential transmissions per time slot: $\mathcal{S} \subset [s_{max}]^{\mathcal{R} \times \mathcal{R}'}$
- $X_n(r)$: backlog of type r-data in time slot n
- Evolution equation

$$X_{n+1}(r) = X_n(r) + \sum_{r' \in \mathcal{R}} s'_n(r', r) - \sum_{r' \in \mathcal{R}'} s'_n(r, r') + A_{n+1}(r),$$

where $\{s'_n(r, r')\}_{(r,r') \in \mathcal{R} \times \mathcal{R}'}$: $s'_n(r, r') \leq s_n(r, r')$ for some $s_n \in \mathcal{S}$, and:

$$X_n(r) - \sum_{r' \in \mathcal{R}'} s'_n(r, r') = \left(X_n(r) - \sum_{r' \in \mathcal{R}'} s_n(r, r') \right)^+$$

Max-weight backpressure: policy

- (w, α)-max-weight backpresssure policy, for $w_r > 0, \alpha > 0$, selects $s_n \in \mathcal{S}$ achieving

$$\mathrm{Max}_{s \in \mathcal{S}} \left\{ \sum_{(r,r') \in \mathcal{R} \times \mathcal{R}'} s(r,r')[w_r X_n(r)^\alpha - w_{r'} X_n(r')^\alpha] \right\}$$

Max-weight backpressure: policy

- (w,α)-max-weight backpresssure policy, for $w_r > 0, \alpha > 0$, selects $s_n \in \mathcal{S}$ achieving

$$\mathrm{Max}_{s\in\mathcal{S}} \left\{ \sum_{(r,r')\in\mathcal{R}\times\mathcal{R}'} s(r,r')[w_r X_n(r)^\alpha - w_{r'} X_n(r')^\alpha] \right\}$$

Backpressure from r to r': $w_r X_n(r)^\alpha - w_{r'} X_n(r')^\alpha$.
Schedule transfers $r \to r'$ only if backpressure positive.
By convention, $X_n(ext) = 0$.

Max-weight backpressure: policy

- (w,α)-max-weight backpresssure policy, for $w_r > 0, \alpha > 0$, selects $s_n \in \mathcal{S}$ achieving

$$\mathrm{Max}_{s\in\mathcal{S}}\left\{\sum_{(r,r')\in\mathcal{R}\times\mathcal{R}'} s(r,r')[w_r X_n(r)^\alpha - w_{r'} X_n(r')^\alpha]\right\}$$

Backpressure from r to r': $w_r X_n(r)^\alpha - w_{r'} X_n(r')^\alpha$.
Schedule transfers $r \to r'$ only if backpressure positive.
By convention, $X_n(ext) = 0$.

- **Schedulable region** $\mathcal{C}$ = set of vectors $x \in \mathbb{R}_+^{\mathcal{R}}$ such that

$$\exists c \in \mathrm{env}(\mathcal{S}) : \ \forall r \in \mathcal{R}, x_r + \sum_{r'\in\mathcal{R}} c(r',r) \le \sum_{r'\in\mathcal{R}'} c(r,r').$$

Ergodicity properties

Denote $\rho_r := \mathbb{E}(A_n(r))$. Then

Theorem

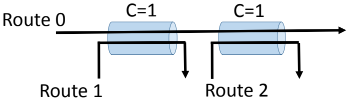
If $\{X_n\}_{n\in\mathbb{N}}$ is irreducible, $\mathbb{E}A_n(r)^{1+\alpha} < +\infty$ and for some $\epsilon > 0, (\rho_r + \epsilon)_{r\in\mathcal{R}} \in \mathcal{C}$, then $\{X_n\}_{n\in\mathbb{N}}$ is ergodic.
Conversely, if $\rho \notin \mathcal{C}$, then for any strategy (max-weight backpressure or other) $\{X_n\}_{n\in\mathbb{N}}$ is transient.

Proof elements: parallel proof for Max-weight, showing ergodicity with same Lyapunov function $V(x) = \sum_r w_r \frac{x(r)^{1+\alpha}}{1+\alpha}$

Comments

- Enjoys same optimal ergodicity properties as Max-weight, in multi-hop setting with varieties of network paths to choose from
- No need to explicitly estimate traffic parameters
- Extends to case of i.i.d., rather than constant sets $\mathcal{S}_n$ of feasible transmissions
- Proposed in '93 as a practical way to schedule transmissions in wireless networks (Tassiulas-Ephremides), and as an algorithm to determine approximate solutions to multicommodity flow problems (Awerbuch-Leighton). Max-weight special case rediscovered later for switches

Internet flow control



- Network links $\ell \in \mathcal{L}$ with capacity C_ℓ
- $X_t(r)$ transmissions of type $r \in \mathcal{R}$, use links $\ell \in r$
- Each gets allocation $\lambda_r \geq 0$, solving

$$\begin{array}{ll} \text{Max} & \sum_{r \in \mathcal{R}} X_t(r) U_r(\lambda_r) \\ \text{such that} & \forall \ell \in \mathcal{L},\ \sum_{r \ni \ell} X_t(r) \lambda_r \leq C_\ell \end{array}$$

- Utility function $U_r(\lambda) = w_r \frac{\lambda^{1-\alpha}}{1-\alpha}$ if $\alpha \neq 1$, $w_r \log(\lambda)$ for $\alpha = 1$ $\rightarrow (w, \alpha)$-fairness (TCP: approximately $w_r = 1/T_r^2, \alpha = 2$)

Flow dynamics

- Requests for type r-transmissions arrive at (Poisson) rate ν_r
- Volume to be served: Exp(μ_r). Denote $\rho_r := \nu_r/\mu_r$
- Schedulable region $\mathcal{C}$: $x \in \mathbb{R}_+^{\mathcal{R}}$ such that $\forall \ell \in \mathcal{L}, \sum_{r \ni \ell} x_r \leq C_\ell$

Flow dynamics

- Requests for type r-transmissions arrive at (Poisson) rate ν_r
- Volume to be served: $\text{Exp}(\mu_r)$. Denote $\rho_r := \nu_r/\mu_r$
- Schedulable region $\mathcal{C}$: $x \in \mathbb{R}_+^{\mathcal{R}}$ such that $\forall \ell \in \mathcal{L}, \sum_{r \ni \ell} x_r \leq C_\ell$

Theorem

For positive (w, α) and (w, α)-fair sharing, if for some $\epsilon > 0$, $(1+\epsilon)\rho \in \mathcal{C}$, then process $\{X_t\}_{t \in \mathbb{R}_+}$ is ergodic.
Conversely, if $\rho \notin \mathcal{C}$, for any feasible bandwidth allocation policy ((w, α)-fair or otherwise), process $\{X_t\}_{t \in \mathbb{R}_+}$ is transient.

Proof elements: Foster's criterion in continuous time

- Take Lyapunov function $V(x) := \sum_{r\in\mathcal{R}} \frac{1}{\mu_r} \int_0^{x_r} U_r'\left(\frac{\rho_r}{x}\right) dx$
  "Drift" of Lyapunov function:

$$\begin{aligned}
\Delta \quad & := \sum_{r\in\mathcal{R}} \nu_r [V(x+e_r) - V(x)] + \mu_r x_r \lambda_r [V(x-e_r) - V(x)] \\
& \approx \sum_{r\in\mathcal{R}} (\nu_r - \mu_r x_r \lambda_r) \frac{\partial}{\partial x_r} V(x) \\
& = \sum_{r\in\mathcal{R}} (\rho_r - x_r \lambda_r) U_r'\left(\frac{\rho_r}{x_r}\right) \\
& = \sum_{r\in\mathcal{R}} (\rho_r - x_r \lambda_r) w_r \left(\frac{\rho_r}{x_r}\right)^{-\alpha}
\end{aligned}$$

Proof elements: Foster's criterion in continuous time

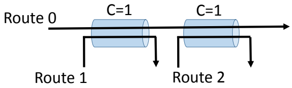
- Take Lyapunov function $V(x) := \sum_{r\in\mathcal{R}} \frac{1}{\mu_r}\int_0^{x_r} U_r'\left(\frac{\rho_r}{x}\right)dx$
  "Drift" of Lyapunov function:

$$\begin{aligned}\Delta \quad &:= \sum_{r\in\mathcal{R}} \nu_r[V(x+e_r)-V(x)] + \mu_r x_r \lambda_r [V(x-e_r)-V(x)]\\ &\approx \sum_{r\in\mathcal{R}}(\nu_r - \mu_r x_r \lambda_r)\frac{\partial}{\partial x_r}V(x)\\ &= \sum_{r\in\mathcal{R}}(\rho_r - x_r\lambda_r)U_r'\left(\frac{\rho_r}{x_r}\right)\\ &= \sum_{r\in\mathcal{R}}(\rho_r - x_r\lambda_r)w_r\left(\frac{\rho_r}{x_r}\right)^{-\alpha}\end{aligned}$$

- $(1+\epsilon)\rho\in\mathcal{C} \Rightarrow$ allocations $\tilde\lambda_r = (1+\epsilon)\rho_r/x_r$ feasible
- Rates λ_r maximize $F(\lambda) := \sum_r w_r x_r \frac{\lambda_r^{1-\alpha}}{1-\alpha}$ over feasible allocations
- Hence concave function $t\in[0,1]\to F(t\lambda+(1-t)\tilde\lambda)$ maximal at $t=1$

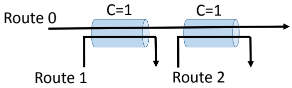
$$\begin{aligned}&\Rightarrow \sum_{r\in\mathcal{R}}[\lambda_r - \tilde\lambda_r]w_r x_r \tilde\lambda_r^{-\alpha}\geq 0\\ &\Leftrightarrow \sum_{r\in\mathcal{R}} w_r[x_r\lambda_r - (1+\epsilon)\rho_r]\left(\frac{\rho_r}{x_r}\right)^{-\alpha}\geq 0\\ &\Rightarrow \Delta \leq -\epsilon\sum_{r\in\mathcal{R}} w_r\rho_r^{1-\alpha}x_r^{\alpha}\end{aligned}$$

A suboptimal (unfair) allocation



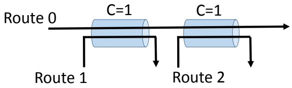
- Two-link network: ergodic under fair allocations if $\rho_0 + \rho_1 < 1, \rho_0 + \rho_2 < 1$

A suboptimal (unfair) allocation



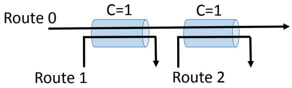
- Two-link network: ergodic under fair allocations if $\rho_0 + \rho_1 < 1, \rho_0 + \rho_2 < 1$
- Alternative allocation: give capacity 1 to types 1 and 2 if $x_1 + x_2 \geq 1$; give capacity 1 to type 0 only if $x_1 + x_2 = 0$

A suboptimal (unfair) allocation



- Two-link network: ergodic under fair allocations if $\rho_0 + \rho_1 < 1, \rho_0 + \rho_2 < 1$
- Alternative allocation: give capacity 1 to types 1 and 2 if $x_1 + x_2 \geq 1$; give capacity 1 to type 0 only if $x_1 + x_2 = 0$
  $\Rightarrow$ New condition for ergodicity: $\rho_0 < (1 - \rho_1)(1 - \rho_2)$
  e.g. Network unstable for $\rho_i \equiv 2/5, i = 0, 1, 2$

A suboptimal (unfair) allocation



- Two-link network: ergodic under fair allocations if $\rho_0 + \rho_1 < 1, \rho_0 + \rho_2 < 1$
- Alternative allocation: give capacity 1 to types 1 and 2 if $x_1 + x_2 \geq 1$; give capacity 1 to type 0 only if $x_1 + x_2 = 0$
  $\Rightarrow$ New condition for ergodicity: $\rho_0 < (1 - \rho_1)(1 - \rho_2)$
  e.g. Network unstable for $\rho_i \equiv 2/5, i = 0, 1, 2$
- Not an unrealistic allocation: results from differentiated service with priority to packets on short routes in network's routers and rate reduction by reactive control (TCP) at sender for longer route

Takeaway messages

- Ergodicity can be established with Foster's criterion and adequate Lyapunov function even when stationary distribution not known explicitly
- Several models for which **schedulable region** characterizes set of traffic parameters (loads per class) which make system ergodic, and for which known simple policy achieves ergodicity whenever possible with no explicit inference of traffic parameters
- Even though ergodicity a "first order" property (saying delays stay finite, not their magnitude), can yield useful insights, e.g. potential problems due to prioritizing packet service in Internet routers