Submitted to MFPS 2026

Concurrent Strategies as Street Fibrations

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Abstract

Recently the theory of event structures has been used as a foundation for an expressive model of concurrent games and strategies. In this paper we revisit the problem of adding symmetry to concurrent games and strategies. Symmetry is essential for many applications but a source of mathematical complexity, in part because strategies with symmetry can be considered up to several notions of equivalence.

Our first contribution is a complete characterization of two classes of strategies for which composition admits an identity. For these two classes the identity laws hold, respectively, up to a 'weak' and a 'strong' notion of equivalence. We show that the weak class of strategies corresponds precisely to those inducing Street fibrations over the game. We show how the established 'thin' approach to symmetry also fits in this general framework.

A second contribution is a formal connection between games and higher-categorical profunctors. This follows a long tradition of relating games and relational models, but here additionally includes a very general model of symmetries. Our characterization of strategies as Street fibrations makes the connection to profunctors significantly easier to establish.

Keywords: Event structures, game semantics, fibrations, profunctors

1 Introduction

Event structures [32,40] are a simple mathematical model for concurrency: a set of computational events, constrained by causality and conflict relations, describe all possible behaviours of a system of concurrent processes.

More recently event structures have provided the foundation for an expressive theory of games and strategies with concurrent features [36]. In that context, the 'events' are occurrences of moves in (two-player) games, and a strategy is a concurrent process describing the behaviour of one player as it interacts with the opponent over the course of the game. These concurrent games and strategies form an expressive model of program interaction, relying on the mathematics of event structures. This model already admits several extensions [17,43,2,35] and is intimately connected to other instances of concurrency in game semantics [20,31,1,21,30].

Applications of concurrent games to programming language semantics (e.g., [12,9]) have required a generalization of the model in which event structures are equipped with 'symmetry', an equivalence relation

⋆ We are grateful for discussions with Pierre Clairambault and Simon Castellan on the topic of this paper.

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that expresses when executions are essentially the same [41]. This generalization is essential, for instance to support monads and comonads, but greatly complicates the mathematical theory, in particular because symmetry requires a new form of identity strategy, a new composition operation, and offers a choice of possible notions of equivalence of strategies [11].

In this paper we revisit concurrent games with symmetry with hindsight, making connections to established categorical structures for managing symmetries. We construct a new model of games and strategies with symmetry, arguing that it is the 'maximally general' model of concurrent games with symmetry. We show that it embeds all previous constructions. Furthermore we give a new characterization of 'weak' strategies as Street fibrations (§1.1), and establish a formal connection to pseudo-profunctors (§1.2).

1.1 The problem of characterizing strategies

Concurrent strategies interact and compose (see §5), but it is a technical challenge to show that this composition obeys the expected laws of composition. In particular, the existence of an identity strategy (the 'copycat') satisfying the left and right unit laws is not granted: one must first impose certain conditions, leading to a characterization and definition of strategies in this broader context.

In [11] a set of conditions is shown to be sufficient for a strategy with symmetry to satisfy the unit laws up to a notion of equivalence of strategies (called *strong equivalence*, §5.1). This result falls short of a characterization: these sufficient conditions are in fact not all necessary. In this paper we remedy this and give a complete characterization of strong strategies (those satisfying the laws up to strong equivalence).

We additionally consider a weaker notion of equivalence of strategies, perhaps more natural from a categorical point of view (called *weak equivalence*). Strong equivalence implies weak equivalence, and therefore the class of 'weak' strategies (those satisfying the unit laws up to weak equivalence) is necessarily larger. Our main result (Theorem 6.15) shows that weak strategies also admit a characterization, namely as forming Street fibrations over a particular category induced by the game. Street fibrations are a weakening of the usual definition of fibrations, due to Ross Street [37], which has the advantage of being stable under equivalence of categories. (We give precise definitions in §4.) This result extends a similar characterization theorem obtained for strategies without symmetry [42].

1.2 Strategies and profunctors

It is well-known that games and strategies share many features with models consisting of relations or 'generalized' relations (matrices, spans, profunctors, a.k.a. *cartesian bicategories* [7]). There are various ways to make this connection precise [3,4,5,38,18] but typically one obtains an oplax functor structure between two models. The oplaxness is because despite the similarities there is a key conceptual difference: the composition of strategies is 'interactive' and not 'relational', the latter being less constrained.

This perspective helps to guide our analysis of symmetry in concurrent strategies. In this paper we describe an oplax functor from our model of games and concurrent strategies with symmetry to a model of small categories and setoid-valued (pseudo) profunctors (Theorem 7.2).

These two models are instances of 3-dimensional categories. The 3-cells are necessary because the 2-cells themselves exhibit symmetry, but we will see that their structure is highly degenerate and so 3-dimensional issues of coherence do not arise. To explain this degeneracy, we make precise (in §2) the role of setoids (i.e. degenerate groupoids) in the theory of event structures with symmetry.

Our characterization of strategies as Street fibrations makes the oplax functor especially easy to describe, via the well-known categorical equivalence between fibred and indexed structures. The representation of strategies as profunctors is designed to abstract away the complexities of interaction whilst maintaining a similar description level for symmetries. The oplax functor is one way to explain the mechanisms for symmetry in concurrent games in terms of more established categorical structures.

1.3 Summary of contributions and outline of the paper

Our main contributions are Theorems 6.13 and 6.15, which characterize weak and strong strategies respectively, and Theorem 7.2, which establishes the connection to profunctors.

To state and prove these results we develop some background. In §2 we give some background on event structures with symmetry, in §3 we enrich this with polarity and define the 'Scott category' of configurations. In §4 we give background on categorical fibrations, proving results relevant to our purposes, and define the profunctor model. In §5 we finally move to the theory of games and strategies; §6 contains the characterization results and §7 the connection to profunctors. Finally §8 makes a connection to thin concurrent games, a popular approach to symmetry.

Preliminaries on event structures.

An *event structure* consists of a set of events E with a partial order relation $\leq$ of *causal dependency* and a nonempty set Con of *consistent* finite subsets of E satisfying four axioms: (1) *finite causes*: for all $e \in E$, the set $\{e' \in E \mid e' \leq e\}$ is finite; (2) for all $e \in E$, the singleton $\{e\}$ is consistent; (3) if $X \in \mathrm{Con}$ and $Y \subseteq X$ then $Y \in \mathrm{Con}$; (4) if $e \leq e'$ where $e' \in X \in \mathrm{Con}$, then $X \cup \{e\} \in \mathrm{Con}$.

A *(finite) configuration* of $(E, \leq, \mathrm{Con})$ is a finite, consistent, and down-closed subset of E. The set of all such configurations is denoted $\mathscr{C}(E)$. We typically use E as shorthand for $(E, \leq, \mathrm{Con})$, when no confusion arises. A *(total) map of event structures* $(E, \leq_E, \mathrm{Con}_E) \to (D, \leq_D, \mathrm{Con}_D)$ is a function $f : E \to D$ satisfying two axioms:

- *configurations are preserved by direct image*: for all $x \in \mathscr{C}(E)$, $fx \in \mathscr{C}(D)$;
- *local injectivity*: the restriction of f to any configuration $x \in \mathscr{C}(E)$ is injective.

Informally, possible executions of E are all represented via f as possible executions of D. Indeed for every configuration $x \in \mathscr{C}(E)$, there is a bijection between x and its direct image: $f_{|x} : x \xrightarrow{\sim} fx$ which is order-reflecting ($f_{|x}^{-1}$ is always monotone). In general, a map $f : E \to D$ need not preserve causal dependency; when it does it is called *rigid*. Equivalently, the map f is rigid iff when $x \in \mathscr{C}(E)$ and $y \in \mathscr{C}(D)$ and $y \subseteq f(x)$ then there is $z \in \mathscr{C}(E)$, necessarily unique, for which $z \subseteq x$ and $fz = y$ [41].

In the spectrum of possible models for concurrency, event structures are central in that they relate to other models via adjunctions [44,22]. They provide a concise, abstract alternative to representations as sets of traces, but at the same time they are more concrete and operational than categorical or topological models for concurrency. Maintaining this compromise is sometimes challenging but holds us close to operational significance and relations with other models.

2 Symmetry in event structures and the role of setoids

We review the basic elements of symmetry in event structures [41] and prove a number of new results. One main goal of this section is to point out the special role played by setoids. (We will see below in Lemma 4.2 that a map of event structures with symmetry induces a setoid structure on each fibre.)

Definition 2.1 An *event structure with symmetry* consists of an event structure E equipped with a family $\widetilde{E} = \{\widetilde{E}[x, y]\}_{x,y \in \mathscr{C}(E)}$, where each $\widetilde{E}[x, y]$ is a set of bijections $x \xrightarrow{\sim} y$, satisfying the following axioms:

- *Congruence*: The family $\widetilde{E}$ contains all identity bijections, and it is closed under composition of bijections, inverses, and restriction of a bijection to a sub-configuration.
- *Bisimulation*: for every $\theta \in \widetilde{E}[x, y]$, if $x \subseteq x' \in \mathscr{C}(E)$, then there exists $y' \in \mathscr{C}(E)$ and $\theta' \in \widetilde{E}[x', y']$ such that $\theta'_{|x} = \theta$.

Remark 2.2 Event structures with symmetry arise from the combination of two abstract ideas: internal equivalence relations as spans in a category (e.g. [24]), and bisimulation as open maps [25]. Indeed, to give an event structure with symmetry $(E, \widetilde{E})$ is to give a jointly monic span $E \xleftarrow{l} R \xrightarrow{r} E$ satisfying the axioms for an internal equivalence relation in the category of event structures and in which the maps l and r are open. (*Open* means for l that it is rigid and if $x \in \mathscr{C}(R)$ and $y \in \mathscr{C}(E)$ with $lx \subseteq y$ then there is $z \in \mathscr{C}(R)$ such that $x \subseteq z$ and $lz = y$.) The notions of maps and homotopy we give below, as well as games with symmetry, could all be developed in this style.

Definition 2.3 A *map of event structures with symmetry* $f : (E, \widetilde{E}) \to (D, \widetilde{D})$ is a map of event structures

$f : E \to D$ such that, for every $\theta \in \widetilde{E}[x, y]$, the composite below lies in $\widetilde{D}[fx, fy]$:

$$f\theta \quad := \quad fx \xrightarrow[\sim]{f_{|x}^{-1}} x \xrightarrow[\sim]{\theta} y \xrightarrow[\sim]{f_{|y}} fy.$$

Symmetry gives a notion of 2-cell between maps of event structures. For maps $f, g : (E, \widetilde{E}) \to (D, \widetilde{D})$ of event structures with symmetry, a *homotopy* $f \sim g$ consists of a symmetry bijection $\theta_x : fx \xrightarrow[\sim]{} gx$ for every $x \in \mathscr{C}(E)$, such that

$$\begin{array}{ccc} & & fx \\ & \overset{f_{|x}}{\nearrow_{\sim}} & \\ x & & \wr\big\downarrow \theta_x \\ & \underset{g_{|x}}{\searrow^{\sim}} & \\ & & gx \end{array}$$

commutes. It is immediate that if such a homotopy exists then it must be unique, with $\theta_x = g_{|x} \circ f_{|x}^{-1}$, and so $\sim$ is just a relation on maps. It is an equivalence relation, since there are identity homotopies and homotopies can be composed, and moreover it is preserved by composition of maps. Therefore:

Proposition 2.4 *There is a 2-category* **EvSym** *of event structures with symmetry, maps between them, and homotopies as 2-cells.* [3]

Each hom-category $\mathbf{EvSym}(\mathcal{E}, \mathcal{D})$ is a groupoid with at most one element in each hom-set. This is equivalently presented as a *setoid*: a set with an equivalence relation. (In fact one could regard the 2-category **EvSym** simply as a **Setoid**-enriched category, where **Setoid** denotes the cartesian monoidal category of setoids and equivalence-preserving functions.)

There is an internal notion of equivalence in **EvSym**: for event structures with symmetry $\mathcal{E}$ and $\mathcal{D}$, an *equivalence* $\mathcal{E} \simeq \mathcal{D}$ is a pair of maps $f : \mathcal{E} \to \mathcal{D}$ and $g : \mathcal{D} \to \mathcal{E}$ such that $f \circ g \sim \mathrm{id}_{\mathcal{D}}$ and $\mathrm{id}_{\mathcal{E}} \sim g \circ f$.

Definition 2.5 The *category of configurations* $\mathscr{C}(\mathcal{E})$ of an event structure with symmetry $\mathcal{E} = (E, \widetilde{E})$ is the sub-category of **Set** with object set $\mathscr{C}(E)$ and morphisms generated by inclusion maps $x \hookrightarrow y$ (where $x \subseteq y$) and symmetry bijections $x \xrightarrow[\sim]{} y$.

By the properties of symmetry, one can show that a function $x \to y$ is a morphism in $\mathscr{C}(\mathcal{E})$ if and only if it can be factored as $x \xrightarrow[\sim]{} z \hookrightarrow y$ for some configuration $z \in \mathscr{C}(E)$, and moreover this factorization is unique. We now discuss two important operations on event structures with symmetry.

2.1 *Synchronization via pseudo-pullbacks*

A *pseudo-pullback* (or *homotopy pullback*) is a strict 2-limit which represents a form of synchronization up to symmetry. In **EvSym**, where 2-cells are an equivalence relation, the pseudo-pullback of a cospan $E \xrightarrow{f} D \xleftarrow{f'} E'$ consists of an object P with projections $p : P \to E$ and $p' : P \to E'$ satisfying $fp \sim f'p'$, having the following universal property: for every other $E \xleftarrow{q} Q \xrightarrow{q'} E'$ such that $fq \sim f'q'$, there is a unique $h : Q \to P$ such that $q = ph$ and $q' = p'h$.

The 2-category **EvSym** has all pseudo-pullbacks [41]. For a cospan $E \xrightarrow{f} D \xleftarrow{f'} E'$, we say $x \in \mathscr{C}(E)$ and $x' \in \mathscr{C}(E')$ are *synchronizable via* $\theta : fx \xrightarrow[\sim]{} f'x'$ if there is a partial order on the graph of the bijection θ such that the two bijections $x \xrightarrow[\sim]{} \mathrm{graph}(\theta) \xleftarrow{\simeq} x'$ are monotone. [4] We can identify the configurations of the pseudo-pullback P with the sets $\mathrm{graph}(\theta)$ for tuples (x, x', θ) where x and x' are synchronizable via θ. The symmetry $\widetilde{P}$ is characterized as follows: for synchronizable pairs (x, x', θ) and (y, y', ξ), a bijection $\varphi : \mathrm{graph}(\theta) \xrightarrow[\sim]{} \mathrm{graph}(\xi)$ is in the symmetry on P if the dashed bijections below are in the symmetries

[3] In fact, **EvSym** has the structure of a homotopy category with path and cylinder objects [11].

[4] One can always construct a pre-order on $\mathrm{graph}(\theta)$ induced by the orders on x and x'. The non-automatic property is antisymmetry, which corresponds to the absence of deadlocks (or causal loops) in the synchronization of x and x'.

of E and E'.

$$\begin{array}{ccccc} x & \xrightarrow{\sim} & \mathrm{graph}(\theta) & \xleftarrow{\sim} & x' \\ \wr\downarrow & & \varphi\downarrow\wr & & \downarrow\wr \\ y & \xrightarrow{\sim} & \mathrm{graph}(\xi) & \xleftarrow{\sim} & y' \end{array}$$

2.2 Hiding

If $(E, \leq, \mathrm{Con})$ is an event structure and $V \subseteq E$ is a subset of events, we can restrict E to V. Let $E \downarrow V$ denote the event structure with events V, partial order restricted from E, and consistent subsets the restriction of Con to subsets of V. For $x \in \mathscr{C}(E \downarrow V)$ we can always complete x to a configuration $[x]$ of E, defined as the minimal $y \in \mathscr{C}(E)$ such that $y \cap V = x$. Concretely, $[x] = \{e \in E \mid \exists e' \in x.\ e \leq e'\}$.

Now let $\widetilde{E}$ be a symmetry on E. If V is closed under symmetry (in the sense that for all $\theta \in \widetilde{E}[x, y]$, if $e \in x \cap V$ then $\theta(e) \in V$) then $E \downarrow V$ has a symmetry consisting of all bijections $\varphi : x \xrightarrow{\sim} y$ for which there exists $\psi : [x] \xrightarrow{\sim} [y]$ such that the diagram below commutes.

$$\begin{array}{ccc} x & \hookrightarrow & [x] \\ \phi\downarrow & & \downarrow\psi \\ y & \hookrightarrow & [y] \end{array}$$

3 Polarity and the Scott category of configurations

Game semantics begins with the assignment of a polarity to each event, positive when the event represents a move of Player, and negative for a move of Opponent.

Definition 3.1 An *event structure with symmetry and polarity* is an event structure with symmetry $\mathcal{E} = (E, \widetilde{E})$ equipped with a labelling function $\mathrm{pol} : E \to \{\boxminus, \boxplus\}$, such that every bijection in $\widetilde{E}$ preserves the polarity of events.

We denote by $\mathbf{EvSym}_\mathrm{p}$ the 2-category of event structures with symmetry and polarity, with maps required to preserve polarity and homotopy 2-cells defined as in **EvSym**.

We have seen that an event structure with symmetry has a category of configurations, a subcategory of **Set** containing symmetry bijections and inclusion maps. We now generalize this to the setting with polarities in a way that turns out to be useful for characterizing strategies. First we fix some terminology: for $x, y \in \mathscr{C}(E)$ and $x \subseteq y$, say the inclusion map $x \hookrightarrow y$ is *positive* (denoted $x \xhookrightarrow{\boxplus} y$) when all events in $y \setminus x$ have positive polarity, and *negative* ($x \xhookrightarrow{\boxminus} y$) when they are all negative. Let **pSet** denote the category of sets and partial functions.

Definition 3.2 If $\mathcal{E}$ is an event structure with polarity and symmetry, the *Scott category* of $\mathcal{E}$ is the subcategory $\mathscr{C}(\mathcal{E})$ of **pSet** with as objects the elements of $\mathscr{C}(E)$, and morphisms generated by:

- symmetry bijections $x \xrightarrow{\sim} y$;
- positive inclusions maps $x \xhookrightarrow{\boxplus} y$, where $x \subseteq y$ and $y \setminus x$ only contains positive events; and
- negative *reverse* inclusions maps $y \xhookleftarrow{\boxminus} x$, represented by partial functions $y \rightharpoonup x$ acting as identity on x and undefined otherwise.

Remarkably, one can show that every morphism $y \rightharpoonup x$ in the Scott category $\mathscr{C}(E)$ admits a unique factorization of the form $y \xhookleftarrow{\boxminus} w \xrightarrow{\sim} u \xhookrightarrow{\boxplus} x$ [11]. We will make extensive use of this fact.

Remark 3.3 One can see an ordinary event structure with symmetry (without polarity) as one with polarity having only positive events. This gives an embedding $\mathbf{EvSym} \to \mathbf{EvSym}_\mathrm{p}$. This way, the category of configurations of Definition 2.5 becomes a special case of the Scott category.

The definition of the Scott category already appears in [11]. The novelty in this paper consists in using the Scott category in a characterization of strategies. This simplifies, and generalizes, the results of [11]. The terminology "Scott" is because of an analogy with the order on functions in domain theory; see [42] for further discussion.

We now extend the Scott category construction $\mathscr{C}(-)$ to a 2-functor $\mathbf{EvSym}_{\mathrm{p}} \to \mathbf{Cat}$ acting on maps and homotopies. Given a map $f : \mathcal{E} \to \mathcal{D}$, recalling that every morphism in $\mathscr{C}(\mathcal{E})$ must be of the form $x \overset{\boxminus}{\hookleftarrow} z \xrightarrow[\sim]{\theta} w \overset{\boxplus}{\hookrightarrow} y$, we can apply f to each component to obtain $fx \overset{\boxminus}{\hookleftarrow} fz \xrightarrow[\sim]{f\theta} fw \overset{\boxplus}{\hookrightarrow} fy$ in $\mathscr{C}(\mathcal{D})$. If $g : \mathcal{E} \to \mathcal{D}$ is another map with $f \sim g$, then by definition we have a symmetry $\theta_x : fx \xrightarrow[\sim]{} gx$ for every $x \in \mathscr{C}(\mathcal{E})$ and so a transformation $\mathscr{C}(f) \cong \mathscr{C}(g)$.

Proposition 3.4 *This definition of $\mathscr{C}(-)$ determines a 2-functor, which is locally full and faithful. In other words, there is exactly one natural transformation $\mathscr{C}(f) \Rightarrow \mathscr{C}(g)$ if $f \sim g$, and no such transformations otherwise.*

Proof. We omit the proof of functoriality. For local fullness and faithfullness, suppose $\varphi : \mathscr{C}(f) \Rightarrow \mathscr{C}(g)$ is a natural transformation. We show that for every $x \in \mathscr{C}(E)$ the function $\varphi_x : fx \to gx$ is a symmetry bijection. By the representation of morphisms in $\mathscr{C}(\mathcal{D})$ we have that

$$\varphi_x \quad = \quad fx \overset{\boxminus}{\hookleftarrow} z \xrightarrow[\sim]{\theta} w \overset{\boxplus}{\hookrightarrow} gx.$$

By the bisimulation property of θ, there must be $t \in \mathscr{C}(\mathcal{D})$ such that $w \overset{\boxminus}{\hookrightarrow} t$ and $t \xrightarrow[\sim]{} fx$. But fx and gx must have the same number of events of each polarity, so we must have $w = t = gx$ and $fx = z$, thus $\varphi_x = \theta : fx \xrightarrow[\sim]{} gx$.

Now we show that φ_x must be the specific bijection $f(e) \mapsto g(e)$, rather than another symmetry. This is trivially true if $x = \emptyset$, and by naturality if φ_x satisfies the property and $x \longrightarrow\!\!\!\subset^{e} y$ then φ_y also satisfies the property. The result follows because for every configuration there must exist at least one $\longrightarrow\!\!\!\subset$-chain. □

Point of notation. In the rest of the paper all event structures are equipped with symmetry. So we use traditional letters $(E, A, B, \dots)$ to denote them, keeping the symmetries $(\widetilde{E}, \widetilde{A}, \widetilde{B}, \dots)$ implicit. Accordingly, $\mathscr{C}(E)$ always refers to the Scott category $\mathscr{C}(E, \widetilde{E})$; this should cause no confusion.

4 Fibrations, Street fibrations, and Setoid-valued profunctors

The goal of this paper is to study maps of event structures with symmetry and polarity (called 'pre-strategies' from §5) through the fibrational properties of the induced functors on Scott categories.

Definition 4.1 Let $p : \mathcal{C} \to \mathcal{D}$ be any functor and let $D \in \mathcal{D}$. The *fibre* of p over D is the category $p^{-1}(D)$ consisting of the objects of $\mathcal{C}$ that are mapped to D and the morphisms of $\mathcal{C}$ that are mapped to id_D.

The *essential fibre* of p over D is the category $p^{-1}_{\sim}(D)$ defined to have as objects pairs $(C \in \mathcal{C}, \varphi : pC \xrightarrow[\sim]{} D)$ and morphisms $(C, \varphi) \to (C', \varphi')$ the $f \in \mathcal{C}(C, C')$ such that $\varphi' \circ pf = \varphi$.

The theory of fibrations specifies conditions on F under which the assignments $D \mapsto p^{-1}(D)$ and $D \mapsto p^{-1}_{\sim}(D)$ give pseudofunctors $\mathcal{D}^{\mathrm{op}} \to \mathbf{Cat}$.

The next lemma, a key observation for this paper, shows that for a functor of the form $\mathscr{C}(f) : \mathscr{C}(E) \to \mathscr{C}(D)$ between Scott categories, all fibres and essential fibres are setoids.

Lemma 4.2 *Let $f : E \to D$ be a map of event structures with symmetry and polarity and consider the induced functor $\mathscr{C}(f)$. For all $x \in \mathscr{C}(D)$, both the fibre and essential fibres over x are setoids.*

Proof. For any morphism $\alpha : y \rightharpoonup z$ in $\mathscr{C}(E)$ the diagram

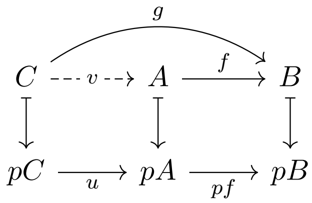
$$\begin{array}{ccc} y & \xrightharpoonup{\ \alpha\ } & z \\ {\scriptstyle f_{|y}}\downarrow & & \downarrow{\scriptstyle f_{|z}} \\ fy & \underset{f\alpha}{\rightharpoonup} & fz \end{array}$$

commutes. If α lies in the fibre over x, then by definition we have $fy = fz = x$ and $f\alpha = \mathrm{id}_x$, therefore the commutative diagram implies $\alpha = f_{|z}^{-1} \circ f_{|y}$. Similar proof for the essential fibre. □

4.1 Rudiments of fibrations and setoids

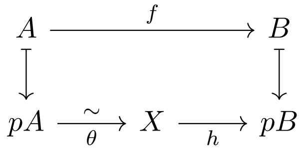
We recall the general notions in the next definition, see e.g. [24, B1.3] for a textbook account. The situation will be greatly simplified when fibres are setoids (Lemma 4.5).

Definition 4.3 For a functor $p : \mathcal{C} \to \mathcal{D}$, a morphism $f : A \to B$ in $\mathcal{C}$ is called *cartesian* if for every $g : C \to B$ and $u : pC \to pA$ such that $pg = pf \circ u$, there is a unique map $v : C \to A$ such that $pv = u$ and $f \circ v = g$:



We say that p is a *fibration* if for every $B \in \mathcal{C}$ and $h : X \to pB$ in $\mathcal{D}$ there is a cartesian morphism $f : A \to B$ with $pf = h$. The morphism f is called a *cartesian lift* of h at B.

We say that p is a *Street fibration* if the morphism $h : X \to pB$ merely has a cartesian *pseudo-lift* at B: a cartesian morphism $f : A \to B$ together with an isomorphism $\theta : pA \xrightarrow{\sim} X$ such that $pf = h \circ \theta$, as pictured below.



(Fibrations are sometimes called *Grothendieck fibrations*, to distinguish them from Street fibrations.)

In general, there may be many possible lifts of a morphism $h : X \to pB$ at an object B, but a cartesian lift satisfies a universal property which relates it to every other lift. Cartesian lifts are *essentially unique*: if $f : A \to B$ and $f' : A' \to B$ are cartesian lifts of $h : X \to pB$ then there a unique isomorphism $\varphi : A \to A'$ such that $p\varphi = \mathrm{id}_X$ and $f = f' \circ \varphi$. Likewise for pseudo lifts: if $(f : A \to B, \theta : pA \xrightarrow{\sim} X)$ and $(f' : A' \to B, \theta' : pA' \xrightarrow{\sim} X)$ are cartesian pseudo-lifts of h, then there is a unique isomorphism $\psi : A \to A'$ such that $p\psi = \theta'^{-1} \circ \theta$ and $f = f' \circ \psi$.

We now consider the special case in which a functor p has essentially unique lifts:

Definition 4.4 A functor $p : \mathcal{C} \to \mathcal{D}$ has *essentially unique lifts* when the following condition holds: for every $B \in \mathcal{C}$ and $h : X \to pB$ in $\mathcal{D}$, if $f : A \to B$ and $f' : A' \to B$ satisfy $pf = pf' = h$ then there is a unique isomorphism $\varphi : A \to A'$ such that $p\varphi = \mathrm{id}_X$ and $f' = f \circ \varphi$.

Similarly, we say that $p : \mathcal{C} \to \mathcal{D}$ has *essentially unique pseudo-lifts* when the following condition holds: for every $B \in \mathcal{C}$ and $h : X \to pB$ in $\mathcal{D}$, if $(f : A \to B, \theta : pA \xrightarrow{\sim} X)$ and $(f' : A' \to B, \theta' : pA' \to X)$ satisfy $pf = \theta \circ h$ and $pf' = \theta' \circ h$ then there is a unique isomorphism $\varphi : A \to A'$ such that $\theta' = \theta \circ p\varphi$ and $f' = f \circ \varphi$.

When lifts are essentially unique they are automatically cartesian, the functor is automatically a fibration, and all fibres are setoids; and similarly for pseudo-lifts.

Lemma 4.5 *The following are equivalent for a functor $p : \mathcal{C} \to \mathcal{D}$.*

- *p has essentially unique lifts (resp. pseudo-lift).*
- *p is a fibration (resp. Street fibration) and every fibre (resp. essential fibre) is a setoid.*

When these conditions are satisfied, we call the functor p a fibration in setoids (resp. a Street fibration in setoids).

Proof. Elementary verification. □

We now discuss the relationship of fibrations and Street fibrations in setoids with *indexed* setoids.

Definition 4.6 A fibration-in-setoids $p : \mathcal{C} \to \mathcal{D}$ is *cloven* when it is equipped with a choice of lift $f^*C \to C$ for every $C \in \mathcal{C}$ and $f : D \to pC$. Similarly, a Street fibration-in-setoids is *cloven* when equipped with a choice of pseudo-lifts.

Assuming the axiom of choice, such a choice of lifts can always be made for a fibration. We consider the following 2-categories for a small category $\mathcal{D}$:

- **SetoidSFib**($\mathcal{D}$) is the 2-category with objects pairs $(\mathcal{C}, p : \mathcal{C} \to \mathcal{D})$ where $\mathcal{C}$ is a small category and p is a Street fibration-in-setoids. The morphisms $(\mathcal{C}, p) \to (\mathcal{C}', p')$ are pairs consisting of a functor $F : \mathcal{C} \to \mathcal{C}'$ and a natural isomorphism $\varphi : p \Rightarrow p' \circ F$, and the 2-cells $(F, \varphi) \to (F, \varphi)$.
- **SetoidFib**($\mathcal{D}$) is the sub-2-category of **SetoidSFib**($\mathcal{D}$) consisting of cloven fibrations-in-setoids, strict morphisms (with $p = p' \circ F$ and $\varphi = \mathrm{id}$), and all 2-cells between them.
- $[\mathcal{D}^{\mathrm{op}}, \mathbf{Setoid}]$ is the 2-category of pseudo-functors $\mathcal{D}^{\mathrm{op}} \to \mathbf{Setoid}$. Morphisms are pseudo-natural transformations (with the naturality square commuting up to equivalence), and 2-cells are component-wise equivalence of pseudo natural transformations.

The fibres of a cloven fibration, and the essential fibres of a cloven Street fibration, can be presented functorially. Indeed there are 2-functors

$$\begin{aligned} \mathbf{fibres} &: \mathbf{SetoidFib}(\mathcal{D}) \longrightarrow [\mathcal{D}^{\mathrm{op}}, \mathbf{Setoid}] \\ \mathbf{ess\text{-}fibres} &: \mathbf{SetoidSFib}(\mathcal{D}) \longrightarrow [\mathcal{D}^{\mathrm{op}}, \mathbf{Setoid}] \end{aligned}$$

(Via the Grothendieck construction [24, B1.3], both are actually equivalences of 2-categories; **fibres** is even a strict equivalence.)

4.2 A 3-dimensional profunctor model

We introduce a model of profunctors valued in setoids. Although setoids are categorically equivalent to sets, this level of structure is important for us because it contains just the right amount of information to model the symmetries of event structures.

The profunctor model is formally a tricategory, but the structure at level 3 is very simple because the 3-cells encode an equivalence relation on the 2-cells. In particular there are no three-dimensional coherence axioms to verify. (The same will apply to games and strategies in §5.)

Definition 4.7 The tricategory **Setoid-Prof** is given by the following components.

- Objects are small categories $(\mathcal{C}, \mathcal{D}, \ldots)$.
- For objects $\mathcal{C}$ and $\mathcal{D}$, **Setoid-Prof**$[\mathcal{C}, \mathcal{D}]$ is the 2-category $[\mathcal{D}^{\mathrm{op}} \times \mathcal{C}, \mathbf{Setoid}]$ of pseudo-functors, pseudo natural transformations, and equivalences. We write $P : \mathcal{C} \mathrel{\rightarrow\!\!\!\!\!+} \mathcal{D}$ if $P \in \mathbf{Setoid\text{-}Prof}[\mathcal{C}, \mathcal{D}]$.
- The identity on $\mathcal{C}$ is the hom-functor $\mathcal{C}(-, =) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$ post-composed with the embedding $\mathbf{Set} \to \mathbf{Setoid}$ that regards a set as a discrete setoid.
- The composition of $P : \mathcal{C} \mathrel{\rightarrow\!\!\!\!\!+} \mathcal{D}$ and $Q : \mathcal{D} \mathrel{\rightarrow\!\!\!\!\!+} \mathcal{E}$ is the profunctor $Q \odot P : \mathcal{E}^{\mathrm{op}} \times \mathcal{D} \to \mathbf{Setoid}$ whose action on objects is given by

$$(Q \odot P)(E, C) = \int^{D \in \mathcal{D}} P(D, C) \times Q(E, D)$$

where the integral sign denotes a pseudo-coend in **Setoid**, constructed as the set $\sum_{D \in \mathcal{D}} P(D, C) \times Q(E, D)$ with the smallest equivalence relation $\sim$ such that

$$\begin{aligned} &(D, p, q) \sim (D, p', q) && \text{when } p \sim p' \text{ in } P(D, C), \\ &(D, p, q) \sim (D, p, q') && \text{when } q \sim q' \text{ in } Q(E, D), \\ &(D, P(\alpha, C)(p'), q) \sim (D', p', P(E, \alpha)(q)) && \text{for all } \alpha \in \mathcal{D}(D', D), \\ &&& \quad p' \in P(D', C), q \in Q(E, D). \end{aligned}$$

We omit the action of the functor $Q \circ P$ on morphisms, as well as the horizontal composition of 2-cells and 3-cells.

pseudo-profunctors of this kind are not new to this paper. See the references [27,14] for fully-detailed presentations, albeit in greater generality.

Remark 4.8 We emphasize that **Setoid-Prof** is not a fully weak tricategory. The hom-2-categories are all strict and so the "vertical" composition of 2-cells and 3-cells is strictly associative and unital. However, the tricategory is not fully strict because the composition of profunctors is only associative and unital up to equivalence, where an *equivalence* is a pair of 2-cells which are inverses of each other up to a 3-cell.

5 Games and strategies with symmetry

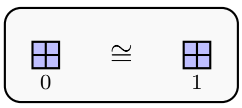
In this section we consider games and strategies as event structures with symmetry. We define identity strategies and the composition operation. These notions already appear in [11], but we generalize the model of [11] to also include 'weak' strategies, that satisfy the laws of composition only up to weak equivalence.

5.1 Games and pre-strategies

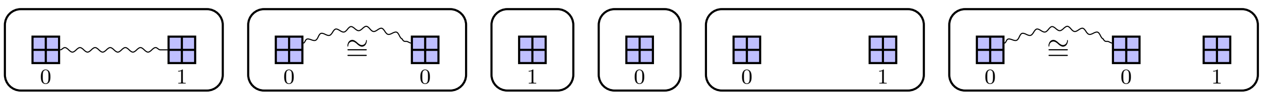
The basic setup is simple to describe: a *game* is an event structure with symmetry and polarity, and a *pre-strategy* on a game A is an event structure with symmetry and polarity S equipped with a map $\sigma : S \to A$. We will later define a strategy as a pre-strategy for which composition works well; see Definition 5.4.

All reasoning about strategies and pre-strategies will be up to equivalence, where equivalence is defined as follows. For pre-strategies $\sigma : S \to A$ and $\sigma' : S' \to A$, a *map of pre-strategies* $\sigma \to \sigma'$ is a map $f : S \to S'$ such that $\sigma' \circ f \sim \sigma$. The map f is called *strict* if $\sigma' \circ f = \sigma$. A *weak equivalence* (or just *equivalence*) $\sigma \simeq \sigma'$ is a pair of maps $f : \sigma \to \sigma'$ and $g : \sigma' \to \sigma$ which form an equivalence $S \simeq S'$ in **EvSym**. It is a *strong equivalence* if f and g are strict maps (in which case we write $\sigma \simeq_{\mathrm{s}} \sigma'$), an *isomorphism* if f and g are inverses of each other, and a *strong isomorphism* if they are both strict and inverses.

Example 5.1 To illustrate, consider a simple game A consisting of two positive, symmetric, and consistent events. We might draw A as:



where '≅' indicates a symmetry between singleton configurations. The following are six possible pre-strategies S on this game. In each case the map $S \to A$ is the only function preserving the labels. Recall that a squiggly line between events indicates *conflict* (i.e. any set containing both is inconsistent):



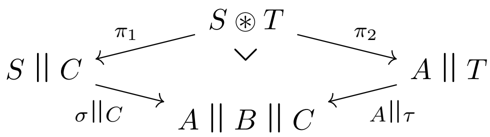
(a) (b) (c) (d) (e) (f)

The pre-strategies (b) and (d) are strongly equivalent to each other and weakly equivalent to (c), while (e) and (f) are strongly equivalent. (No other equivalences hold.)

5.2 The composition of pre-strategies

For games A and B, a pre-strategy *from A to B* is a pre-strategy on the game $A^\perp \parallel B$, where $(-)^\perp$ inverts the polarity of every event, and $\parallel$ is the parallel composition operator on event structures with symmetry. (The latter is defined simply as a disjoint union of events with all structure inherited component-wise.) For strategies of the form $\sigma : S \to A^\perp \parallel B$, we sometimes keep S implicit and write $\sigma : A \mathrel{\rightarrow\!\!\!\!\!+} B$.

Pre-strategies $\sigma : S \to A^\perp \parallel B$ and $\tau : T \to B^\perp \parallel C$ are composed in two steps. The first step is to describe the interaction of σ and τ via the homotopy pullback



in **EvSym**, where the maps $S \to A \parallel B$ and $T \to B \parallel C$ are the morphisms underlying the two pre-strategies. The event structure with symmetry $S \circledast T$ (called the *interaction* of σ and τ) supports two equivalent maps $S \circledast T \to A \parallel B \parallel C$. For preciseness, let $\sigma \circledast \tau$ denote the left way around the diagram.

The second step is to "hide" the events of B, by restricting $S \circledast T$ to the subset of events mapped to $A \parallel C$. This gives an event structure with symmetry denoted $S \odot T$. There is an unambiguous way to add polarities back to obtain a pre-strategy $\sigma \odot \tau : S \odot T \to A^\perp \parallel C$, called the *composition* of σ and τ.

This composition is associative up to equivalence, and functorial with respect to maps of strategies. In particular, weak equivalences $\sigma \simeq \sigma'$ and $\tau \simeq \tau'$ induce a weak equivalence $\tau \odot \sigma \simeq \tau' \odot \sigma'$; and the same holds if $\simeq$ is replaced with $\simeq_s$. (For proofs of these facts see [8,11].)

5.3 Copycat

Every game A admits a "copycat" strategy $A \twoheadrightarrow A$, to serve as the identity morphism. This will be denoted $\text{æ}_A : \mathbb{C}_A \to A^\perp \parallel A$. Both $\mathbb{C}_A$ and æ_A are characterized by the next lemma, which we have formulated in such a way that the connection with the profunctor model of Section 4.2 is manifest. (The explicit construction of copycat can be found in [11]; it is not required for this paper.)

Recall that, for a category $\mathcal{C}$, the *twisted arrow category* $\mathrm{Tw}(\mathcal{C})$ is obtained as the category of elements of the hom-functor $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$. That is, objects are arrows $(f : c \to c')$ in $\mathcal{C}$ and morphisms $(f_1 : c_1 \to c_1') \to (f_2 : c_2 \to c_2')$ are pairs of morphisms $g : c_2 \to c_1$ and $g' : c_1' \to c_2'$ such that $f_2 = g' \circ f_1 \circ g$.

Lemma 5.2 (Configurations of copycat, [11]) *For a game A, there exists an event structure with symmetry and polarity $\mathbb{C}_A$ such that the category $\mathscr{C}(\mathbb{C}_A)$ is isomorphic to the category $\mathrm{Tw}(\mathscr{C}(A))$. Furthermore there is a map of event structures with symmetry $\text{æ}_A : \mathbb{C}_A \to A^\perp \parallel A$, corresponding (via $\mathscr{C}(-)$ and the mediating isomorphisms) to the canonical projection $\mathrm{Tw}(\mathscr{C}(A)) \to \mathscr{C}(A)^{\mathrm{op}} \times \mathscr{C}(A)$.*

Lemma 5.3 (Composition with copycat, adapted from [11]) *For every pre-strategy $\sigma : A \twoheadrightarrow B$ there are strict maps of strategies $\lambda_\sigma : \sigma \to \text{æ}_B \odot \sigma$ and $\rho_\sigma : \sigma \to \sigma \odot \text{æ}_A$, natural in σ.*

In the next section, we will study in detail what happens when a pre-strategy is composed with copycat, and in particular we will recall the concrete constructions of λ_σ and ρ_σ.

5.4 Strategies

Definition 5.4 A *(weak) strategy from A to B* is a pre-strategy $\sigma : S \to A^\perp \parallel B$ for which the canonical maps λ_σ and ρ_σ are weak equivalences of pre-strategies. Say σ is a *strong strategy* if they are strong equivalences. [5]

It follows directly from this definition (and the associativity of composition) that strategies are closed under weak equivalence and under composition. We organize all of this data into a tricategory of games and strategies. The level of strictness is the same as for **Setoid-Prof**: the composition of 2-cells is strictly associative and unital, and the 3-cells are just a congruence relation on the 2-cells. We emphasize that nonetheless a three-dimensional structure is needed, because copycat is only an identity up to equivalence of strategies, and the definition of (both weak and strong) equivalence relies on the 3-cells.

[5] In this paper by 'strategy' we always mean weak strategy, unless strong is specified. This emphasizes that they are the more general notion. The reference [11] instead uses strong strategies as default, specifying 'weak' otherwise.

Definition 5.5 The tricategory **Strat** is given by the following components.

- Objects are games $(A, B, \dots)$.
- For objects A and B, $\mathbf{Strat}[A, B]$ is the 2-category whose objects are strategies from A to B; morphisms are maps of strategies; and 2-cells are homotopies between maps.
- The identity on A is the copycat strategy $\mathfrak{C}_A$.
- The composition 2-functor $\mathbf{Strat}[B, C] \times \mathbf{Strat}[A, B] \to \mathbf{Strat}[A, C]$ extends the composition $\odot$ to maps of strategies and their equivalences, via the universal property of pseudo-pullbacks.

There is a sub-tricategory consisting of games, strong strategies, strict maps between them, and homotopies. This sub-model, which we denote by $\mathbf{Strat}_{\mathrm{strong}}$, is the '$\sim$-bicategory' developed in [11].

Remark 5.6 Tricategories are generally hard to construct because of the many coherence axioms, but in this case all axioms involving equations of 3-cells hold automatically because the structure is thin. We must only check that the axioms of a bicategory hold up to the equivalence relation given by the 3-cells. This is much simpler and follows from our description of composition using pullback and hiding. (We emphasize that these axioms do *not* hold up to equality, and so a three-dimensional structure is really needed.)

In summary, we have presented a compositional model of games and strategies up to equivalence. This is in some sense the most general compositional model based on **EvSym**, because by definition strategies are pre-strategies for which composition works well. Our goal is to provide a direct characterization.

6 Characterizations of weak strategies and strong strategies

The central result of this section is that weak strategies correspond to Street fibrations. To establish this, we revisit the possible lifting properties that pre-strategies may satisfy (§6.1). We then introduce an important 'collapse' construction on strategies (§6.2) which allows us to characterize first the strong strategies (§6.3) and then the weak ones (§6.4).

Remark 6.1 Throughout this section we consider pre-strategies on a game A, rather than $A^\perp \parallel B$. We regard them as morphisms $\varnothing \rightarrow\!\!\!\!+\!\!\!\!\rightarrow A$ in **Strat** and study the post-composition with copycat on A. All our definitions and results apply equally to pre-strategies $\sigma : A \rightarrow\!\!\!\!+\!\!\!\!\rightarrow B$ with completely analogous arguments. Note in particular that $\mathfrak{C}_B \odot \sigma \odot \mathfrak{C}_A \simeq_{\mathrm{s}} \mathfrak{C}_{A^\perp \parallel B} \circ \sigma$. We stick to the special case for readability.

6.1 Lifting properties for pre-strategies

The following properties will play a role in the technical development. Pseudo-receptivity is new to this paper.

Definition 6.2 A pre-strategy $\sigma : S \to A$ is said to be:

- *innocent* (a.k.a. *courteous* in [31,10,13]) if, for every $x \in \mathscr{C}(S)$, every positive inclusion $y \overset{\boxplus}{\hookrightarrow} \sigma x$ in $\mathscr{C}(A)$ has a (necessarily unique, by local injectivity) strict lift;
- *receptive* if, for every $x \in \mathscr{C}(S)$, there exists a unique (strict) lift of every negative inclusion $y \overset{\boxminus}{\hookleftarrow} \sigma x$ in $\mathscr{C}(A)$.
- *pseudo-receptive* if, for every $x \in \mathscr{C}(S)$, every negative inclusion $y \overset{\boxminus}{\hookleftarrow} \sigma x$ in $\mathscr{C}(A)$ has an essentially unique pseudo-lift;

The condition known as *strong receptivity* in [11] is the conjunction of receptivity and pseudo-receptivity, which are independent conditions.

Remark 6.3 One could equivalently state innocence in terms of *immediate causality*: for every $s, s' \in S$, if s has positive polarity and $s \rightarrowtriangle s'$ then $\sigma s \rightarrowtriangle \sigma s'$, where $\rightarrowtriangle$ means $<$ with no events in between. For a proof of equivalence see [42]. (We also note an unfortunate terminology clash: innocence is not related to the notion of innocent strategy in Hyland-Ong game semantics [23].)

The following lemma will be important in our proofs:

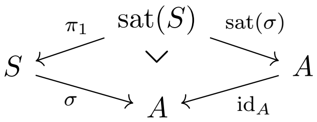
Lemma 6.4 *For a pseudo-receptive pre-strategy $\sigma : S \to A$, the following* $-$-innocence *condition holds: if $s, s \in S$ with $s \rightarrowtriangle s'$ and both s, s' are negative, then $\sigma s \rightarrowtriangle \sigma s'$.*

Recall that in an event structure the notation $x \,{-\!\!\subset}^{e}\, y$ means that $e \notin y$ and $y = x \cup \{e\}$.

Proof. We sketch a proof of the lemma. Necessarily $x \,{-\!\!\subset}^{s}\, x_1 \,{-\!\!\subset}^{s'}\, x'$ for some $x, x_1, x' \in \mathscr{C}(S)$. Suppose for a contradiction that $s \rightarrowtriangle s'$ and σs and $\sigma s'$ are concurrent, so both $\sigma x \,{-\!\!\subset}^{\sigma s}\, \sigma x_1 \,{-\!\!\subset}^{\sigma s'}\, \sigma x'$ and also $\sigma x \,{-\!\!\subset}^{\sigma s'}\, y \,{-\!\!\subset}^{\sigma s}\, \sigma x'$ for some $y \in \mathscr{C}(A)$. By applying pseudo-receptivity to the inclusions $\sigma x \hookrightarrow y \hookrightarrow \sigma x'$ we obtain $x \,{-\!\!\subset}^{t'}\, x_2 \,{-\!\!\subset}^{t}\, x''$ and a symmetry $\theta : \sigma x'' \cong \theta x'$ where θ fixes σx and acts by $\sigma t' \mapsto \sigma s'$ and $\sigma t \mapsto \sigma s$. We now have two pseudo-lifts of the inclusion $\sigma x \hookrightarrow \sigma x'$, namely $x \hookrightarrow x'$ and $x \hookrightarrow x''$, so there must be a symmetry $\varphi : x \cong x'$ which maps to θ. Necessarily φ acts by $s \mapsto t$ and $s' \mapsto t'$. Since $s \rightarrowtriangle s'$ we must have $t \rightarrowtriangle t'$ which is a contradiction as t' comes before t in the chain $x \,{-\!\!\subset}^{t'}\, x_2 \,{-\!\!\subset}^{t}\, x''$. □

Next we consider a lifting condition for symmetries, also imported from [11].

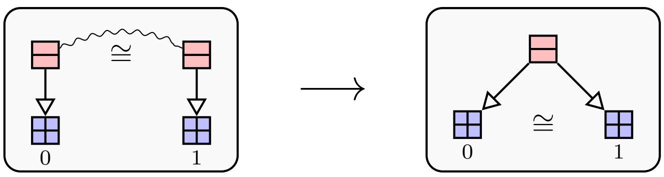
Definition 6.5 If $\sigma : S \to A$ is a pre-strategy, its *saturation* is the pre-strategy $\mathrm{sat}(\sigma) : \mathrm{sat}(S) \to A$ obtained as the right projection in the pseudo-pullback below.



The saturation process can be understood as freely adding to $\mathscr{C}(S)$ a lift for every isomorphism in $\mathscr{C}(A)$, in a way that is compatible with the causal structure of S (in the sense that this free completion is performed internally in **EvSym**). Indeed, configurations of $\mathrm{sat}(S)$ corresponds to pairs of a configuration x of S and a symmetry $\sigma x \cong y$ and such a pair is mapped to y under $\mathrm{sat}(\sigma)$. There is a canonical, strict map of strategies $\eta_\sigma : \sigma \to \mathrm{sat}(\sigma)$ of pre-strategies such that $\eta_\sigma x$ is the configuration corresponding to the pair $(x, \mathrm{id}_{\sigma x})$. (The existence of η_σ also follows directly from the universal property of the pseudo-pullback.)

A pre-strategy is said to be *saturated* if η_σ is a strong equivalence of pre-strategies, i.e. it admits a pseudo-inverse $\mathrm{act} : \mathrm{sat}(\sigma) \to \sigma$ which is a strict map. The existence of act makes $\mathscr{C}(\sigma)$ an *isofibration* but the converse does not hold: some isofibrations are not saturated because the isomorphism lifts do not organize themselves into a map of event structures. Here is an example.

Example 6.6 The following pre-strategy determines a Grothendieck fibration (so in particular an isofibration) between the induced Scott categories, but an easy argument shows that it is not saturated.



In general, note that the saturation of a pre-strategy is itself saturated: this follows from elementary reasoning with pseudo-pullbacks. We shall make use of the fact that any pre-strategy is weakly equivalent to its saturation. This observation appears to be new to this paper.

Lemma 6.7 *For any pre-strategy $\sigma : S \to A$, σ and $\mathrm{sat}(\sigma)$ are weakly equivalent via the maps η_σ and π_1.*

Proof. Elementary reasoning using the universal property of the pseudo-pullback. □

Next we state the key result of [11], that gives sufficient conditions on a pre-strategy for the unit laws to hold *up to strong equivalence.*

Theorem 6.8 ([11]) *Let $\sigma : S \to A$ be a pre-strategy which is saturated, innocent, and strongly receptive (i.e. both receptive and pseudo-receptive). Then σ is a strong strategy, i.e. the map $\lambda_\sigma : \sigma \to \text{Ϫ}_A \odot \sigma$ is a strong equivalence of pre-strategies.*

These conditions are sufficient but not necessary. Below we will give a complete characterization of strong strategies, via a slightly weaker set of conditions (Theorem 6.13 in §6.3).

Lemma 6.9 *The lifting conditions of Definition 6.2 satisfy the stability properties we list below.*

(i) *Innocent pre-strategies are closed under weak equivalence.*

(ii) *Pseudo-receptive pre-strategies are closed under weak equivalence.*

(iii) *Saturated pre-strategies are closed under strong equivalence, but not weak equivalence.*

6.2 The collapse of a pre-strategy

Our first step is to describe an operation on pre-strategies which collapses 'redundant' lifts of negative inclusions. This is an essential construction for this paper.

Theorem 6.10 *Let $\sigma : S \to A$ be an innocent and pseudo-receptive pre-strategy. There is a pre-strategy $collapse(\sigma) : collapse(S) \to A$, strongly equivalent to σ, which satisfies the uniqueness part of the receptivity condition.*

We construct $collapse(\sigma)$ using the machinery of stable families (Appendix A), convenient for constructing event structures and symmetries. A map $\sigma \to collapse(\sigma)$ is obtained easily and canonically, but our construction of the pseudo-inverse $collapse(\sigma) \to \sigma$ requires the axiom of choice as we make an essential use of Zorn's lemma. All details can be found in Appendix C.

Corollary 6.11 *Suppose $\sigma : S \to A$ is an innocent, pseudo-receptive strategy which satisfies the existence part of the receptivity condition. Then $collapse(\sigma)$ is innocent, pseudo-receptive, and receptive.*

Proof. Innocence and pseudo-receptivity are preserved by strong equivalence (Lemma 6.9), and both the existence and uniqueness parts of receptivity hold for $collapse(\sigma)$. □

6.3 A characterization of strong strategies

Lemma 6.12 *If a pre-strategy $\sigma : S \to A$ is pseudo-receptive then $sat(\sigma)$ satisfies the existence part of receptivity.*

Proof. The map $sat(\sigma)$ sends a configuration $w \in \mathscr{C}(sat(S))$, corresponding to a pair $(x \in \mathscr{C}(S), \sigma x \overset{\theta}{\underset{\sim}{\longrightarrow}} y)$, to y. If $y \overset{\boxminus}{\hookrightarrow} y'$ in $\mathscr{C}(A)$, then by the properties of symmetry there is y'' such that

$$\begin{array}{ccc} y'' & \underset{\sim}{\longrightarrow} & y' \\ {\scriptstyle\boxminus}\uparrow & & {\scriptstyle\boxminus}\uparrow \\ \sigma x & \underset{\sim}{\longrightarrow} & y \end{array}$$

commutes. Assuming σ is pseudo-receptive, there is $x' \in \mathscr{C}(S)$ with $x \overset{\boxminus}{\hookrightarrow} x'$ with a symmetry $\sigma x' \underset{\sim}{\longrightarrow} y''$ making the following commute:

$$\begin{array}{ccccc} \sigma x' & \underset{\sim}{\longrightarrow} & y'' & \underset{\sim}{\longrightarrow} & y' \\ & \nwarrow{\scriptstyle\boxminus} & {\scriptstyle\boxminus}\uparrow & & {\scriptstyle\boxminus}\uparrow \\ & & \sigma x & \underset{\sim}{\longrightarrow} & y\,. \end{array}$$

The pair $(x', \sigma x' \underset{\sim}{\longrightarrow} y')$ corresponds to a configuration w' of $\mathrm{sat}(S)$ which is sent to y' under $sat(\sigma)$. Now, by a straightforward characterization of the inclusion order in $\mathscr{C}(\mathrm{sat}(S))$, it holds that $w \subseteq w'$ and $sat(\sigma)(w') = y'$, showing the existence part of receptivity for $sat(\sigma)$. □

Theorem 6.13 *A pre-strategy $\sigma : S \to A$ is a strong strategy if and only if it is saturated, innocent, and pseudo-receptive.*

Proof. (Only if) If $\sigma : S \to A$ is a strong strategy then by definition it is strongly equivalent to $\text{æ}_A \odot \sigma$, which is saturated, innocent, and strongly receptive (i.e. receptive and pseudo-receptive) by Lemma 9 of [11]. The conclusion follows from Lemma 6.9. (If) Assume σ is innocent, pseudo-receptive and saturated. Since σ is saturated, σ and $sat(\sigma)$ are strongly equivalent, and therefore $\mathrm{sat}(\sigma)$ is also innocent and pseudo-receptive. Hence, by Theorem 6.10, $collapse(\mathrm{sat}(\sigma))$ and $\mathrm{sat}(\sigma)$ are strongly equivalent, and so $\sigma \simeq_{\mathrm{s}} collapse(\mathrm{sat}(\sigma))$. Since strong strategies are closed under strong equivalence, it now suffices to show that $collapse(\mathrm{sat}(\sigma))$ is a strong strategy. The argument is as follows: since σ is pseudo-receptive, $\mathrm{sat}(\sigma)$ satisfies the existence part of receptivity by Lemma 6.12, and therefore by Corollary 6.11 the pre-strategy $collapse(sat(\sigma))$ is innocent, pseudo-receptive, and receptive; so by Theorem 6.8 it is a strong strategy. □

6.4 A characterization of weak strategies

Theorem 6.14 *A pre-strategy $\sigma : S \to A$ is a weak strategy if and only if it is innocent and pseudo-receptive.*

Proof. (Only if) If $\sigma : S \to A$ is a weak strategy then by definition it is weakly equivalent to $\text{æ}_A \odot \sigma$, which is saturated, innocent, and strongly receptive (i.e. receptive and pseudo-receptive) by Lemma 9 of [11]. Innocence and pseudo-receptivity can be transported to σ, by Lemma 6.9. (If) Recall σ is weakly equivalent to $\mathrm{sat}(\sigma)$. The latter is saturated, innocent, and pseudo-receptive, and therefore a strong strategy by Theorem 6.13. Since σ is weakly equivalent to a strong strategy it is itself a weak strategy. □

It remains to show that innocence and pseudo-receptivity together correspond to the Street fibration condition. The proof involves reasoning about pseudo-lifts for maps in **EvSym**, see Appendix B.

Theorem 6.15 (Strategies as Street fibrations) *A pre-strategy $\sigma : S \to A$ is a weak strategy if and only if $\mathscr{C}(\sigma) : \mathscr{C}(S) \to \mathscr{C}(A)$ is a Street fibration.*

7 Strategies as Setoid-profunctors

Our characterization of strategies as Street fibrations in setoids allows for an easy connection with setoid-valued profunctors. In this section we formalize this as an oplax functor of tricategories $\|-\| : \mathbf{Strat} \to \mathbf{Setoid\text{-}Prof}$. The 'oplaxness' means that there is a non-invertible comparison map $\|\tau \odot \sigma\| \to \|\tau\| \circ \|\sigma\|$ because viewing strategies as profunctors forgets important causal structure affecting their composition.

The components of the functor $\|-\|$ are straightforward to describe. On objects, we define $\|A\|$ to be the Scott category $\mathscr{C}(A)$. Then we turn any strategy $\sigma : S \to A^\perp \parallel B$ into a setoid-valued profunctor $\|\sigma\|_{A,B} : \mathscr{C}(A) \nrightarrow \mathscr{C}(B)$ by letting $\|\sigma\|_{A,B}(x, y)$ be the essential fibre in S over the configuration $x \parallel y \in \mathscr{C}(A^\perp \parallel B)$. This is always a setoid (Lemma 4.2).

To define this more formally, first note the following property of the Scott category construction:

Lemma 7.1 *For games A and B, there is an isomorphism of categories $\mathscr{C}(A^\perp \parallel B) \cong \mathscr{C}(B) \times \mathscr{C}(A)^{\mathrm{op}}$.*

From this we derive an isomorphism $\mathbf{SetoidSFib}(\mathscr{C}(A^\perp \parallel B)) \to \mathbf{SetoidSFib}(\mathscr{C}(B) \times \mathscr{C}(A)^{\mathrm{op}})$ of 2-categories, whose action on a Street fibration is defined by post-composition. Additionally, since every strategy is a Street fibration, there is an embedding $\mathbf{Strat}[A, B] \hookrightarrow \mathbf{SetoidSFib}(\mathscr{C}(A^\perp \parallel B))$ and recall from §4 the 2-functor $\textbf{ess-fibres} : \mathbf{SetoidSFib}(\mathscr{C}(B) \times \mathscr{C}(A)^{\mathrm{op}}) \to [\mathscr{C}(B)^{\mathrm{op}} \times \mathscr{C}(A), \mathbf{Setoid}]$. We now assemble all this data.

Theorem 7.2 *There is an oplax functor of tricategories $\|-\| : \mathbf{Strat} \to \mathbf{Setoid\text{-}Prof}$.*

Proof. We only give the key steps of the construction. The action on $\|-\|$ on hom-2-categories is given by a family of 2-functors $\|-\|_{A,B}$, for games A and B, defined by

$$\mathbf{Strat}[A, B] \hookrightarrow \mathbf{SetoidSFib}(\mathscr{C}(A^\perp \parallel B)) \to \mathbf{SetoidSFib}(\mathscr{C}(B) \times \mathscr{C}(A)^{\mathrm{op}}) \to \mathbf{Setoid\text{-}Prof}[\|A\|, \|B\|].$$

For a functor of tricategories we need transformations relating identities and composition in each model. For identities, Lemma 5.2 characterizes the configurations of copycat in terms of the twisted arrow category, which corresponds to the identity profunctor (in particular our oplax functor is *normal*). For composition, we use that every configuration of a composite strategy $T \odot S$ can be projected to a pair of 'matching' configurations from S and T. This is extended in a straightforward way to account for essential fibres and symmetries.

Finally we verify that the axioms of pseudo-functors between bicategories hold, up to symmetry. This is done by repeated use of the universal property of the composition of profunctors; see [18] for a similar bicategorical result. There are no further axioms to verify on the 3-cells because the structure is degenerate and all necessary equations hold automatically. □

8 Thin games

In this section we review an established approach to symmetry in concurrent games based on 'thin' concurrent games and strategies [13,12]. The theory of thin games, primarily developed by Clairambault and collaborators, has been very successful in applications of concurrent games to semantics [19,16].

We show that **Strat** embeds the bicategory **Thin** of thin concurrent games. This result can be seen as challenging the common view that there are two separate approaches to symmetry in game semantics: the thin approach (e.g. [28,13,34]) and the saturated approach (e.g. [11,29]). We show that the model of weak strategies as Street fibrations provides a general mathematical universe in which the two approaches can be formally connected. We only give the key definitions, but see [13] for a full account.

Definition 8.1 A *thin game* is an event structure with symmetry $\mathcal{A} = (A, \widetilde{A})$ equipped with two subfamilies of $\widetilde{A}$, denoted $\widetilde{A}_+$ and $\widetilde{A}_-$, both satisfying the conditions of event structures with symmetry (Def. 2.1) and subject to the following axioms:

- If $\theta \in \widetilde{A}_+[x, y] \cap \widetilde{A}_-[x, y]$, then $x = y$ and $\theta = \mathrm{id}_x$.
- If $\theta \in \widetilde{A}_+[x, y]$, if $x \overset{\boxplus}{\hookrightarrow} x'$ and $\theta' \in \widetilde{A}[x', y']$ with $\theta'|_x = \theta$ then $\theta' \in \widetilde{A}_+[x, y]$.
- If $\theta \in \widetilde{A}_-[x, y]$, if $x \overset{\boxplus}{\hookrightarrow} x'$ and $\theta' \in \widetilde{A}[x', y']$ with $\theta'|_x = \theta$ then $\theta' \in \widetilde{A}_-[x, y]$.

A *thin strategy* on a thin game $(\mathcal{A}, \widetilde{A}_+, \widetilde{A}_-)$ is a pre-strategy $\sigma : \mathcal{S} \to \mathcal{A}$ (in the sense of §5.1, i.e. $\mathcal{S} = (S, \widetilde{S})$ is an ordinary event structure with symmetry) which is innocent and strongly receptive, and additionally satisfies the following condition: for all $\theta \in \widetilde{S}$, if $\sigma\theta \in \widetilde{A}_+$ then $\theta = \mathrm{id}_x$ for some $x \in \mathscr{C}(S)$.

Parallel composition extends to thin games componentwise, and the operation $(-)^\perp$ acts on thin games by additionally reverting the roles of the positive and negative symmetries.

Thin games and strategies are very constrained. The main advantage of these stronger axioms is that they enable simpler notions of identities and composition, with the associativity and unit laws holding up to isomorphism rather than equivalence.

Formally, the identity on a thin game A is a 'thin copycat' strategy $\text{\c{c}}^t_A : \mathbb{C}^t_A \to A^\perp \parallel A$. It is (weakly) equivalent to the copycat $\mathbb{C}_A$ from §5.3, but the latter is not a thin strategy in general. (The strategy $\mathbb{C}^t_A$ is in fact much simpler than $\mathbb{C}_A$, but sadly not well-defined when A is an arbitrary game, see [13, A.4].) Thin strategies can be composed using a composition operation $\odot^t$ defined in the same way as $\odot$ but using 1-categorical pullbacks rather than pseudo-pullbacks. (The operation $\odot^t$ is simpler than $\odot$ but not well-defined for arbitrary non-thin strategies, as **EvSym** does not have enough pullbacks.)

Altogether one gets a 2-dimensional model:

Proposition 8.2 ([33,15]) *There is a bicategory* **Thin** *of thin games, thin strategies between them as morphisms, and (weak) isomorphisms of strategies as 2-cells.*

Theorem 8.3 *There is a pseudo-functor of tricategories* J : **Thin** $\to$ **Strat**, *where the bicategory* **Thin** *is regarded as having only identity 3-cells.*

Proof. We give only a sketch of the construction. For a thin game $\underline{\mathcal{A}} = (\mathcal{A}, \widetilde{A}_+, \widetilde{A}_-)$ we define the plain game $J\underline{\mathcal{A}} = \mathcal{A}$. A thin strategy is in particular a strategy in the sense of **Strat** and indeed this extends to an embedding $\mathbf{Thin}[\underline{\mathcal{A}}, \underline{\mathcal{B}}] \to \mathbf{Strat}[J\underline{\mathcal{A}}, J\underline{\mathcal{B}}]$.

We must then relate identities and compositions. We have already stated that $\mathbb{C}_A$ and $\mathbb{C}^t_A$ are equivalent strategies. To show that $\odot$ and $\odot^t$ are also equivalent, one direction uses the universal property of the pseudo-pullback and the other that the pullback of thin strategies is in fact a bipullback [13]. □

The functor J : **Thin** → **Strat** appears to forget important structure, namely the positive and negative sub-families. But, as we show next, this structure is 'property-like': if a game admits a thin game structure then that structure is essentially unique from the point of view of the bicategory **Thin**. The result is new to this paper.

Proposition 8.4 *Let $\underline{A} = (\mathcal{A}, \widetilde{A}_+, \widetilde{A}_-)$ and $\underline{A}' = (\mathcal{A}, \widetilde{A}'_+, \widetilde{A}'_-)$ be thin games with the same underlying event structure with symmetry and polarity $\mathcal{A}$. Then there is an internal equivalence $\underline{A} \simeq \underline{A}'$ in the bicategory* **Thin***.*

Proof. An equivalence in **Thin** is given by a pair of strategies that are each other's pseudo-inverse. The key observation is that the thin copycat strategy $\mathbb{C}^t_A \to A^\perp \parallel A$ can simultaneously serve as a strategy $\underline{A} \twoheadrightarrow\!\!\!\!\!+ \; \underline{A}'$ and a strategy $\underline{A}' \twoheadrightarrow\!\!\!\!\!+ \; \underline{A}$, in addition to its usual role as identity strategy on $\underline{A}$ and $\underline{A}'$. That this forms an equivalence follows quickly from the fact that copycat is an identity strategy. The only minor difficulty is in verifying the thinness axiom for copycat in each case. This follows from the observation that any bijection in the intersection of $\widetilde{A}_+$ and $\widetilde{A}'_-$ (similarly $\widetilde{A}_-$ and $\widetilde{A}'_+$) must be an identity bijection. This can be shown by induction on the size of the bijection. We omit the details. □

Remark 8.5 The pseudo-functor J is identity on morphisms and higher-cells, and Proposition 8.4 makes it 'essentially injective on objects' in a bicategorical sense. It would seem that J can thus be regarded as a higher-categorical embedding, in a sense that is consistent with existing 1-categorical notions of property-like structure and embeddings [26,6]. We content ourselves with this informal observation, as more work is required for a formal statement: it is still unclear whether J reflects equivalences in an appropriate sense and higher-categorical generalizations can be subtle.

9 Conclusion

We have presented a very general model of concurrent games on event structures with symmetry, in which strategies satisfy the laws of composition up to a weak notion of equivalence. This model brings together several existing approaches to symmetry [11,12].

Importantly, we have characterized weak strategies as Street fibrations. This gives them a canonical status and provides a mathematically straightforward connection with Setoid-valued profunctors. This reinforces our view that weak notions of equivalences must be taken seriously in models of this kind.

An interesting problem is that of characterizing strong strategies in a similar style. It is known that all strong strategies define Grothendieck fibrations, but Example 6.6 shows that any characterization necessarily relies on a stronger condition. Therefore the problem remains open.

This work brings us closer to a formalization of concurrent games in a type theory (or proof assistant) that supports and encourages reasoning up to homotopy. We will pursue this in further work.

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A Stable families, event structures, and symmetry

Stable families.

Some proofs in §C rely on stable families [40], the key results on which are presented here.

Definition A.1 A *stable family* is a nonempty family of finite sets $\mathcal{F}$, which is:

- *Complete*: if $Z \subseteq_{\text{fin}} \mathcal{F}$ is *compatible*, meaning that there exists $x \in \mathcal{F}$ with $\bigcup Z \subseteq x$, then $\bigcup Z \in \mathcal{F}$;
- *Stable:* if $Z \subseteq_{\text{fin}} \mathcal{F}$ is compatible and nonempty, then $\bigcap Z \in \mathcal{F}$;
- *Coincidence-free:* for all $x \in \mathcal{F}$, $e, e' \in x$ with $e \neq e'$,

$$\exists x_0 \in \mathcal{F}.\ x_0 \subseteq x \ \&\ (e \in x_0 \iff e' \notin x_0)\,.$$

We call elements of $\mathcal{F}$ its *configurations* and $\bigcup \mathcal{F}$ its *events*.

Definition A.2 A *(total) map* $f : \mathcal{F} \to \mathcal{G}$ between stable families $\mathcal{F}$ and $\mathcal{G}$ is a function f from the events of $\mathcal{F}$ to those of $\mathcal{G}$ such that for all $x \in \mathcal{F}$ its direct image $fx \in \mathcal{G}$ and if $e, e' \in x$ and $f(e) = f(e')$ then $e = e'$.

The choice of map ensures an "inclusion" functor from the category of event structures $\mathbf{E}$ to that of stable families; the finite configurations of an event structure form a stable family.

Stable families and event structures.

The embedding functor from event structures to stable families has a right adjoint Pr giving a coreflection (an adjunction with unit an isomorphism). We now recall the construction of this adjoint. Let x be a configuration of a stable family $\mathcal{F}$. Define the *prime* configuration of e in x by

$$[e]_x =_{\text{def}} \bigcap\{y \in \mathcal{F} \mid e \in y \ \&\ y \subseteq x\}\ .$$

By coincidence-freeness, the function $top : \mathscr{C}(\mathrm{Pr}(\mathcal{F})) \to \mathcal{F}$ which takes a prime configuration $[e]_x$ to e is well-defined; it is the counit of the adjunction [39,40].

Theorem A.3 *For $\mathcal{F}$ a stable family, $\mathrm{Pr}(\mathcal{F}) =_{\text{def}} (P, \mathrm{Con}, \leq)$ is an event structure where*

- $P = \{[e]_x \mid e \in x \ \&\ x \in \mathcal{F}\}$,
- $Z \in \mathrm{Con}$ *iff* $Z \subseteq P \ \&\ \bigcup Z \in \mathcal{F}$, *and*
- $p \leq p'$ *iff* $p, p' \in P \ \&\ p \subseteq p'$.

There is an order-isomorphism $\theta : (\mathscr{C}(\mathrm{Pr}(\mathcal{F})), \subseteq) \cong (\mathcal{F}, \subseteq)$ where $\theta(y) =_{\text{def}} top\, y = \bigcup y$ for $y \in \mathscr{C}(\mathrm{Pr}(\mathcal{F}))$; its mutual inverse is φ where $\varphi(x) = \{[e]_x \mid e \in x\}$ for $x \in \mathcal{F}$.

Symmetry as spans of open maps.

In the main body of the paper we have presented symmetry on an event structure E as a family of isomorphisms $\widetilde{E}[x, y]$ for configurations $x, y \in \mathscr{C}(E)$. But in fact (as we alluded to in Remark 2.2) the structure of a symmetry on E can equivalently be presented as a separate event structure $\widetilde{E}$, together with a span of jointly monic *open maps*

$$E \xleftarrow{\ l\ } \widetilde{E} \xrightarrow{\ r\ } E$$

forming an internal equivalence relation in the category $\mathbf{E}$. Each configuration $w \in \mathscr{C}(\widetilde{E})$ corresponds to an element $\widetilde{E}[lw, rw]$ in the above presentation.

In the other direction, one constructs the event structure $\widetilde{E}$ by observing that it forms a stable family (regarding each bijection as a set of pairs) and applying the functor Pr.

The above equivalence extends to morphisms of event structures with symmetry. Indeed maps $f : (E, \widetilde{E}) \to (D, \widetilde{D})$ of event structures with symmetry (in the sense of §2) are equivalently presented as pairs of maps $f : E \to D$ and $\widetilde{f} : \widetilde{E} \to \widetilde{D}$ of ordinary event structures, appropriately commuting with the spans of open maps. Both perspectives are helpful and we use the notation $\widetilde{f}$ in proofs of §C.

B Proof of Theorem 6.15

We restate the theorem and give the full proof.

Theorem B.1 (Strategies as Street fibrations) *A pre-strategy $\sigma : S \to A$ is a weak strategy if and only if $\mathscr{C}(\sigma) : \mathscr{C}(S) \to \mathscr{C}(A)$ is a Street fibration.*

Proof. (If) Pseudo-receptivity is immediate because $\mathscr{C}(\sigma)$ is a Street fibration. We check innocence. Supposing $y \overset{\boxplus}{\hookrightarrow} \sigma x$, by assumption it has a pseudo-lift, i.e. a morphism $z \to x$ in $\mathscr{C}(S)$ together with a symmetry $\sigma z \xrightarrow[\sim]{\theta} y$ such that the morphism $z \to x$ maps to $\sigma z \xrightarrow[\sim]{\theta} y \overset{\boxplus}{\hookrightarrow} \sigma x$. It is clear that in this case the $\supseteq^-$ part of the morphism $z \to x$ must be trivial, so it is of the form $z \xrightarrow[\sim]{\phi} u \overset{\boxplus}{\hookrightarrow} x$ for a unique ϕ and u. Thus $\sigma z \xrightarrow[\sim]{\sigma\phi} \sigma u \overset{\boxplus}{\hookrightarrow} \sigma x$. By uniqueness of factorization of morphisms of the Scott category, we must therefore have $\sigma u = y$ and $\sigma(u \overset{\boxplus}{\hookrightarrow} x) = (y \overset{\boxplus}{\hookrightarrow} \sigma x)$.

(Only if) We must show that for every $x \in \mathscr{C}(S)$, every morphism

$$z \overset{\boxminus}{\hookleftarrow} t \xrightarrow[\sim]{} u \overset{\boxplus}{\hookrightarrow} \sigma x \tag{B.1}$$

has an essentially unique pseudo-lift.

Existence. Using the bisimulation property of symmetry, there exists w such that the morphism in (B.1) can be written as a composite

$$z \xrightarrow[\sim]{} w \overset{\boxminus}{\hookleftarrow} u \overset{\boxplus}{\hookrightarrow} \sigma x.$$

Combining our two assumptions on σ (pseudo-receptivity and innocence), we can find t and v in $\mathscr{C}(S)$ such that $t \overset{\boxminus}{\hookleftarrow} v \overset{\boxplus}{\hookrightarrow} x$, and a symmetry $\sigma t \xrightarrow[\sim]{} w$, such that

$$\sigma(t \overset{\boxminus}{\hookleftarrow} v \overset{\boxplus}{\hookrightarrow} x) \quad = \quad \sigma t \xrightarrow[\sim]{} w \overset{\boxminus}{\hookleftarrow} u \overset{\boxplus}{\hookrightarrow} \sigma x.$$

The morphism

$$t \overset{\boxminus}{\hookleftarrow} v \overset{\boxplus}{\hookrightarrow} x$$

together with the symmetry $\sigma t \xrightarrow[\sim]{} w \xrightarrow[\sim]{} z$ is therefore a pseudo-lift of $z \xrightarrow[\sim]{} w \overset{\boxminus}{\hookleftarrow} u \overset{\boxplus}{\hookrightarrow} \sigma x$ and thus of the morphism in (B.1).

Essential uniqueness. Suppose that there are two pseudo-lifts of the morphism in (B.1), i.e. two morphisms

$$v' \overset{\boxminus}{\hookleftarrow} w'_0 \xrightarrow[\sim]{} u'_0 \overset{\boxplus}{\hookrightarrow} x \tag{B.2}$$

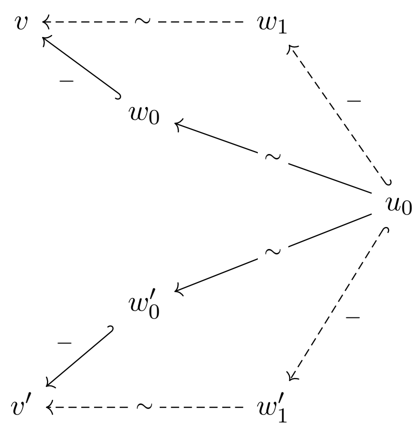
$$v \overset{\boxminus}{\hookleftarrow} w_0 \xrightarrow[\sim]{} u_0 \overset{\boxplus}{\hookrightarrow} x \tag{B.3}$$

in $\mathscr{C}(S)$, respectively mapping to

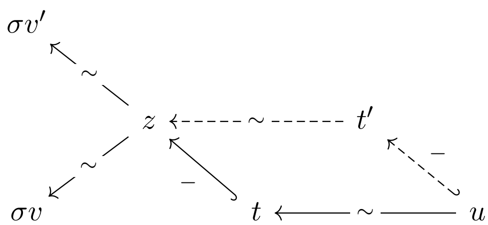
$$\sigma v' \xrightarrow[\sim]{} z \overset{\boxminus}{\hookleftarrow} t \xrightarrow[\sim]{} u \overset{\boxplus}{\hookrightarrow} \sigma x \qquad \text{and} \qquad \sigma v \xrightarrow[\sim]{} z \overset{\boxminus}{\hookleftarrow} t \xrightarrow[\sim]{} u \overset{\boxplus}{\hookrightarrow} \sigma x$$

for given symmetries $\sigma v \xrightarrow[\sim]{} z$ and $\sigma v' \xrightarrow[\sim]{} z$.

It must be the case that the sub-configurations $u_0, u'_0 \hookrightarrow x$ are the same, since there is a unique strict lift of $u \overset{\boxplus}{\hookrightarrow} \sigma x$ by assumption. Then, use the bisimulation property of symmetry to rewrite the two morphisms $v' \overset{\boxminus}{\hookleftarrow} w'_0 \xrightarrow[\sim]{} u_0$ and $v \overset{\boxminus}{\hookleftarrow} w_0 \xrightarrow[\sim]{} u_0$ as follows:



Similarly rewrite the morphism $z \overset{\boxminus}{\hookleftarrow} t \xrightarrow[\sim]{} u$ via some t' yielding the following situation:



At this point we have constructed two pseudo-lifts of the morphism $t' \overset{\boxminus}{\hookleftarrow} u$, namely $v \xrightarrow[\sim]{} w_1 \overset{\boxminus}{\hookleftarrow} u_0$ and $v' \xrightarrow[\sim]{} w'_1 \overset{\boxminus}{\hookleftarrow} u_0$. The assumption that pseudo-lifts of negative reverse inclusions are essentially unique yields a unique iso $v \xrightarrow[\sim]{} v'$ between the pseudo-lifts in (B.2) and (B.3), which concludes the proof. □

C The collapse of a pre-strategy

Preliminary terminology and notation.

Recall that a pre-strategy $\sigma : S \to A$ is $-$-*innocent* iff whenever $s' \twoheadrightarrow s$ with $\mathrm{pol}(s) = -$ in S then $\sigma(s') \twoheadrightarrow \sigma(s)$ in A. The proofs below often hinge on $-$-innocence, which is implied by pseudo-receptivity.

Next a pre-strategy $\sigma : S \to A$ is *collapsed* iff $x \subseteq^- x_1, x_2$ in $\mathscr{C}(S)$ and $\sigma x_1 = \sigma x_2$ in $\mathscr{C}(A)$ implies $x_1 = x_2$. Provided a pre-strategy $\sigma : S \to A$ is pseudo-receptive we will show how to construct an equivalent collapsed pre-strategy $\mathit{collapse}(\sigma)$, the objective of this subsection.

For a configuration x of any event structure with polarity E we write x^+ and x^- for the subsets of positive and negative events, respectively (these are not generally configurations themselves). Similarly, let E^- and E^+ denote the corresponding subsets of events in E. And, when E has symmetry and $\theta : x \xrightarrow{\sim}$ is a symmetry bijection, we denote by $\theta^+ : x^+ \cong y^+$ and $\theta^- : x^- \cong y^-$ the respective sub-bijections (of sets).

Finally, for an event e of an event structure E, we let $[e]$ denote its down-closure $\{e' \in E \mid e' \leq e\}$. This is the smallest configuration of E containing e.

Definition C.1 Let $\sigma : S \to A$ be a $-$-innocent pre-strategy with essentially unique $\subseteq^-$-lifts. (In particular these assumptions hold if σ is pseudo-receptive.) W.l.o.g. assume that S^+ and A^- are disjoint.

Taking

$$\mathcal{S} =_{\mathrm{def}} \{x^+ \cup (\sigma x)^- \mid x \in \mathscr{C}(S)\}\,.$$

defines a stable family; this only depends on the $-$-innocence of σ —see Lemma C.4 below. (Noting that $(\sigma x)^- = \sigma(x^-)$ we shall simply write σx^- from now on.)

It is easy to see that for any $x \in \mathscr{C}(S)$ there is a bijection

$$x \cong x^+ \cup \sigma x^- \tag{C.1}$$

taking $s \in x$ to s if s is positive and to $\sigma(s)$ if s is negative. Equip $\mathcal{S}$ with a symmetry $\widetilde{\mathcal{S}}$, the family comprising all composite bijections

$$x^+ \cup \sigma x^- \cong x \cong_S^\theta y \cong y^+ \cup \sigma y^-$$

for $x \cong_S^\theta y$ in the isomorphism family of S, using the bijections (C.1). Using that σ is both $-$-innocent and pseudo-receptive, we can show $\widetilde{\mathcal{S}}$ is an isomorphism family—Lemma C.4 below.

Define $\mathit{collapse}(S)$ to be the event structure $\mathrm{Pr}(\mathcal{S})$ with the polarity of $p \in \mathrm{Pr}(\mathcal{S})$ the same as the polarity of $\mathit{top}(p)$. Its symmetry $\widetilde{\mathit{collapse}}(S) = \mathrm{Pr}(\widetilde{\mathcal{S}})$ where $\widetilde{\mathcal{S}}$ is the stable family $\widetilde{\mathcal{S}}$ which comprises all composite bijections

$$x^+ \cup \sigma x^- \cong x \cong^\theta y \cong y^+ \cup \sigma y^-\,,$$

where $x \cong_S^\theta y$. The family $\widetilde{\mathcal{S}}$ is shown to be a stable family below in Lemma C.4.

The prime configurations of $\mathcal{S}$ are precisely the configurations $[s]^+ \cup \sigma[s]^-$ for $s \in S$ —see Lemma C.4 below. (By definition, the prime configurations are the events of $\mathit{collapse}(S)$.)

Define the function $\mathit{collapse}(\sigma) : \mathit{collapse}(S) \to A$ to take a prime configuration $[s]^+ \cup \sigma[s]^-$, where $s \in S$, to $\sigma(s)$.

Definition C.2 Under the assumptions on σ above, define a function $g : S \to \mathit{collapse}(S)$ as the map taking $s \in S$ to the prime configuration $[s]^+ \cup \sigma[s]^-$.

To justify the claims in the definition of *collapse* above we first need a proposition on a consequence of $-$-innocence.

Proposition C.3 *Let $\sigma : S \to A$ be a $-$-innocent pre-strategy. Suppose $x_0 \subseteq x$ in $\mathscr{C}(S)$ and*

$$\sigma x_0 \subseteq^- y \subseteq \sigma x$$

in $\mathscr{C}(A)$. Then there is a (necessarily unique) $z \in \mathscr{C}(S)$ such that

$$x_0 \subseteq^- z \subseteq x$$

with $\sigma z = y$.

Proof. Assume $x_0 \subseteq x$ in $\mathscr{C}(S)$ and $\sigma x_0 \subseteq^- y \subseteq \sigma x$ in $\mathscr{C}(A)$. Define the set

$$z =_{\text{def}} \{s \in x \mid \sigma(s) \in y\}\,.$$

Clearly $x_0 \subseteq^- z \subseteq x$ and $\sigma z = y$. It remains to show z a configuration of S. As $z \subseteq x$ the set z is clearly consistent. So for z to be a configuration of S it suffices to show it down-closed. And for this it suffices to show for all $s \in z \setminus x_0$ if $s' \rightarrowtriangle_S s$ then $s' \in z$ —this property already holds of all events in x_0 as it is a configuration. Suppose $s \in z \setminus x_0$. Then s is negative, so as σ is $-$-innocent, $\sigma(s') \rightarrowtriangle_A \sigma(s)$. As $\sigma(s) \in y$, and y is down-closed, $\sigma(s') \in y$ making $s' \in z$ —and z down-closed so a configuration. □

Lemma C.4 *Above, in Definition C.1, on the assumption that the pre-strategy $\sigma : S \to A$ is $-$-innocent,*

(A) the family $\mathcal{S}$ is stable.

Moreover, on the further assumption that σ is pseudo-receptive,

(B) the family $\widetilde{\mathcal{S}}$ is an isomorphism family, alternatively described by $\widetilde{\mathcal{S}} = \{\theta^+ \cup \widetilde{\sigma}\theta^- \mid \theta \in \widetilde{S}\}$;

(C) the prime configurations of $\mathcal{S}$ are precisely those of the form $[s]^+ \cup \sigma[s]^-$, for $s \in S$; and the functions collapse(σ) and g are maps of esp's for which collapse$(\sigma)g = \sigma$, such that

(D) if $z \subseteq^- z_1, z_2$ and collapse$(\sigma)z_1 =$ collapse$(\sigma)z_1$ then $z_1 = z_2$, for all $z, z_1, z_2 \in \mathscr{C}(collapse(S))$; and

(E) $gx_1 = gx_2 \implies x_1 \cong_S x_2$, for all $x_1, x_2 \in \mathscr{C}(S)$.

Proof.
(A). We show $\mathcal{S}$ is a stable family:

(i) if $u \uparrow v$ ($\uparrow$ denotes binary compatibility: u and v admit an upper bound in the family) in $\mathcal{S}$ then $u \cup v \in \mathcal{S}$,

(ii) if $u \uparrow v$ in $\mathcal{S}$ then $u \cap v \in \mathcal{S}$,

(iii) if e_1, e_2 are distinct events in $u \in \mathcal{S}$ then there is $v \in \mathcal{S}$ with $v \subseteq u$ and $(e_1 \in v \iff e_2 \notin v)$.

i) Suppose $x_1, x_2, x \in \mathscr{C}(S)$ with

$$x_1^+ \cup \sigma x_1^-,\ x_2^+ \cup \sigma x_2^- \subseteq x^+ \cup \sigma x^-$$

in $\mathcal{S}$. Then, $[x_1^+ \cup x_2^+] \subseteq x$ in $\mathscr{C}(S)$ and

$$\sigma[x_1^+ \cup x_2^+] \subseteq^- \sigma x_1 \cup \sigma x_2 \subseteq \sigma x$$

in $\mathscr{C}(A)$. By Proposition C.3, there is $z \in \mathscr{C}(S)$ with $\sigma z = \sigma x_1 \cup \sigma x_2$ and

$$[x_1^+ \cup x_2^+] \subseteq^- z \subseteq x\,.$$

Then, $z^+ \cup \sigma z^- \in \mathcal{S}$ with

$$z^+ \cup \sigma z^- = (x_1^+ \cup x_2^+) \cup (\sigma x_1 \cup \sigma x_2)^-\,,$$

ensuring 1).

ii) Suppose $x_1, x_2, x \in \mathscr{C}(S)$ with

$$x_1^+ \cup \sigma x_1^-,\ x_2^+ \cup \sigma x_2^- \subseteq x^+ \cup \sigma x^-$$

in $\mathcal{S}$.

We first claim there is $y \in \mathscr{C}(A)$, necessarily unique, such that

$$\sigma([x_1^+] \cap [x_2^+]) \subseteq^- y \subseteq^+ \sigma x_1 \cap \sigma x_2\,.$$

To see this suppose $a \in \sigma x_1 \cap \sigma x_2$ has negative polarity. Then if $a' < a$ with a' positive there is $s \in [x_1^+] \cap [x_2^+]$ for which $\sigma(s) = a'$. Thus, $a' \in \sigma([x_1^+] \cap [x_2^+])$. Hence we achieve the claim by taking

$$y =_{\text{def}} \sigma([x_1^+] \cap [x_2^+]) \cup \{a \in \sigma x_1 \cap \sigma x_2 \mid a \text{ is negative}\}\,.$$

Now

$$\sigma([x_1^+] \cap [x_2^+]) \subseteq^- y \subseteq \sigma x\,.$$

By Proposition C.3, there is $z \in \mathscr{C}(S)$ with $\sigma z = y$ and

$$[x_1^+] \cap [x_2^+] \subseteq^- z \subseteq x\,.$$

We have

$$z^+ \cup \sigma z^- = (x_1^+ \cap x_2^+) \cup (\sigma x_1^- \cap \sigma x_2^-)\,,$$

as required for 2).

iii) Suppose $e_1, e_2 \in x^+ \cup \sigma x^-$ are distinct in the configuration of $\mathcal{S}$ obtained from $x \in \mathscr{C}(S)$. As remarked in Definition C.1, there is a bijection $x \cong x^+ \cup \sigma x^-$. So in all cases there is a subconfiguration v of x such that $v^+ \cup \sigma v^-$ separates them. For instance, assuming $e_1 = s_1 \in x^+$ and $e_2 = \sigma(s_2) \in \sigma x^-$ we can take v to be a subconfiguration of x containing one but not the other of s_1 and s_2.

We have shown $\mathcal{S}$ is a stable family and so determines an event structure *collapse*$(\mathcal{S}) =_{\text{def}} \Pr(\mathcal{S})$.

Now further assume that σ is pseudo-receptive. We first show that, for $x_1, x_2 \in \mathscr{C}(S)$,

$$\text{if } x_1^+ \cup \sigma x_1^- = x_2^+ \cup \sigma x_2^- \text{ then } x_1^+ = x_2^+ \;\&\; \sigma x_1 = \sigma x_2 \;\&\; \exists\theta.\ x_1 \cong_S^\theta x_2 \;\&\; \mathrm{id}_{[x_1^+]} \subseteq \theta \;\&\; \tilde{\sigma}\theta = \mathrm{id}_{\sigma x_1}$$

Assuming $x_1, x_2 \in \mathscr{C}(S)$ and $x_1^+ \cup \sigma x_1^- = x_2^+ \cup \sigma x_2^-$ we see that $x_1^+ = x_2^+$ and $\sigma x_1^- = \sigma x_2^-$. It follows directly that $\sigma x_1 = \sigma x_2$. As $x_1^+ = x_2^+$, their down-closures, the configurations $[x_1^+]$ and $[x_2^+]$ are equal. Hence,

$$\sigma[x_1^+] = \sigma[x_2^+] \subseteq^- \sigma x_1 = \sigma x_2\,.$$

As σ has essentially unique $\subseteq^-$-lifts, we obtain $x_1 \cong_S^\theta x_1$ as above.

(B). By definition, $\widetilde{\mathcal{S}}$ comprises all bijections

$$\theta' : x^+ \cup \sigma x^- \cong x \cong_S^\theta y \cong y^+ \cup \sigma y^-$$

as θ ranges over the isomorphism family of S. It can be checked that $\theta' = \theta^+ \cup \tilde{\sigma}\theta^-$, making

$$\widetilde{\mathcal{S}} = \{\theta^+ \cup \tilde{\sigma}\theta^- \mid \exists x, y.\ x \cong_S^\theta y\}\,.$$

We prove that $\widetilde{\mathcal{S}}$ inherits the properties required of an isomorphism family from the isomorphism family of S. It is easy to see reflexivity and symmetry of $\widetilde{\mathcal{S}}$. To check transitivity, suppose

$$\theta' : x^+ \cup \sigma x^- \cong x \cong_S^\theta y \cong y^+ \cup \sigma y^- \;\&\; \phi' : z^+ \cup \sigma z^- \cong y \cong_S^\phi w \cong w^+ \cup \sigma w^- \;\& y^+ \cup \sigma y^- = z^+ \cup \sigma z^-\,.$$

By (1) above, there is $y \cong_S^\psi z$ which restricts to the identity on $y^+ = z^+$ and induces the identity between σy^- and σz^-. The composite bijection

$$x^+ \cup \sigma x^- \cong x \cong_S^{\phi\psi\theta} w \cong w^+ \cup \sigma w^-\,,$$

is the composition $\phi'\theta'$.

We turn to the extension and restriction properties required of an isomorphism family. Here it will be useful to observe a necessary and sufficient condition for one bijection to extend another in $\widetilde{\mathcal{S}}$. **Observation:** Let

$$\theta' : x_1^+ \cup \sigma x_1^- \cong x_1 \cong_S^\theta y_1 \cong y_1^+ \cup \sigma y_1^- \text{ and}$$
$$\phi' : x^+ \cup \sigma x^- \cong x \cong_S^\phi y \cong y^+ \cup \sigma y^- \text{ and}$$

be bijections in $\widetilde{\mathcal{S}}$. The bijection θ' extends the bijection ϕ', *i.e.* $\phi' \subseteq \theta'$, iff both the following commute,

$$\begin{array}{ccc} x_1 & \cong_S^\theta & y_1 \\ \cup\!\shortmid & & \cup\!\shortmid \\ [x^+] & \cong_S^\phi & [y^+] \end{array} \quad \text{and} \quad \begin{array}{ccc} \sigma x_1 & \cong_A^{\widetilde{\sigma}\theta} & \sigma y_1 \\ \cup\!\shortmid & & \cup\!\shortmid \\ \sigma x & \cong_A^{\widetilde{\sigma}\phi} & \sigma y\,. \end{array}$$

To prove the observation consider the two cases: events in x^+ and events in σx^-.

We resume the proof that $\widetilde{\mathcal{S}}$ is an isomorphism family. Suppose

$$x^+ \cup \sigma x^- \cong x \cong_S^\theta y \cong y^+ \cup \sigma y^- \ \& \ v^+ \cup \sigma v^- \subseteq x + \cup \sigma x^-\,,$$

for configurations $x, y, v \in \mathscr{C}(S)$. As $[v^+] \subseteq^- v$,

$$\sigma[v^+] \subseteq^- \sigma v \subseteq \sigma x\,.$$

Clearly $[v^+] \subseteq x$. So, by Proposition C.3, there is a unique $x_1 \in \mathscr{C}(S)$ such that

$$[v^+] \subseteq^- x_1 \subseteq x \quad \& \quad \sigma x_1 = \sigma v\,.$$

Therefore $x_1^+ = v^+$ which with $\sigma x_1 = \sigma v$ implies

$$x_1^+ \cup \sigma x_1^- = v^+ \cup \sigma v^-\,.$$

Now we have

$$x \cong_S y \quad \& \quad x_1 \subseteq x\,.$$

From the isomorphism family of S, we obtain (a unique) $y_1 \in \mathscr{C}(S)$ with $x_1 \cong_S y_1$ a restriction of $x \cong_S y$,*i.e.*

$$\begin{array}{ccc} x & \cong_S & y \\ \cup\!\shortmid & & \cup\!\shortmid \\ x_1 & \cong_S & y_1 \end{array} \tag{2}$$

commutes. As required, we obtain subconfiguration $y_1^+ \cup \sigma y_1^-$ of $y^+ \cup \sigma y^-$ and a bijection

$$v^+ \cup \sigma v^- = x_1^+ \cup \sigma x_1^- \cong x_1 \cong_S y_1 \cong y_1^+ \cup \sigma y_1^-\,,$$

in $\widetilde{\mathcal{S}}$ which is a restriction of the original bijection $x^+ \cup \sigma x^- \cong y^+ \cup \sigma y^-$. To show this, the Observation says we need

$$\begin{array}{ccc} x & \cong_S & y \\ \cup\!\shortmid & & \cup\!\shortmid \\ [v^+] & \cong_S & [y_1^+] \end{array} \quad \text{and} \quad \begin{array}{ccc} \sigma x & \cong_A & \sigma y \\ \cup\!\shortmid & & \cup\!\shortmid \\ \sigma v & \cong_A & \sigma y_1 \end{array}$$

commute. But this follows from the commuting diagram (2) on recalling that $v^+ = x_1^+$ and $\sigma v = \sigma x_1$.

For the remaining condition, suppose

$$x^+ \cup \sigma x^- \cong x \cong_S^\theta y \cong y^+ \cup \sigma y^- \ \& \ x^+ \cup \sigma x^- \subseteq v^+ \cup \sigma v^-\,,$$

for configurations $x, y, v \in \mathscr{C}(S)$. This clearly entails $\sigma x \subseteq \sigma v$.

It is easy to check that the bijection $x \cong_S^\theta y$ restricts to make the diagram

$$\begin{array}{ccc} x & \cong_S^\theta & y \\ \cup\!\mid & & \cup\!\mid \\ [x^+] & \cong_S & [y^+] \end{array}$$

commute in the category of partial bijections. In particular, $[x^+] \cong_S^\theta [y^+]$ with $[x^+] \subseteq v$ so there is $w \in \mathscr{C}(S)$ and a commuting diagram

$$\begin{array}{ccc} v & \cong_S & w \\ \cup\!\mid & & \cup\!\mid \\ [x^+] & \cong_S & [y^+]\,. \end{array} \tag{3}$$

As $\sigma x \subseteq \sigma v$ we have

$$\sigma[x^+] \subseteq^- \sigma x \subseteq \sigma v\,,$$

and by the above a commuting diagram,

$$\begin{array}{ccc} \sigma v & \cong_A & \sigma w \\ \cup\!\mid & & \\ \sigma x & & \cup\!\mid \\ \cup\!\mid & & \\ \sigma[x^+] & \cong_A & \sigma[y^+]\,. \end{array}$$

As $\cong_A$ forms an isomorphism family, this determines a unique $z \in \mathscr{C}(A)$ for which

$$\begin{array}{ccc} \sigma v & \cong_A & \sigma w \\ \cup\!\mid & & \cup\!\mid \\ \sigma x & \cong_A & z \\ \cup\!\mid & & \cup\!\mid \\ \sigma[x^+] & \cong_A & \sigma[y^+]\,. \end{array} \tag{4}$$

commutes. As $[y^+] \subseteq w$, by Proposition C.3, there is a unique $w_1 \in \mathscr{C}(S)$ such that

$$[y^+] \subseteq^- w_1 \subseteq w \quad \& \quad \sigma w_1 = z\,.$$

This makes $[y^+] \subseteq^- w_1$ an $\subseteq^-$-lift of $\sigma[y^+] \subseteq^- z$. We also have

$$\begin{array}{ccc} \sigma x & \cong_S & \sigma y \\ \cup\!\mid & & \cup\!\mid \\ \sigma[x^+] & \cong_S & \sigma[y^+]\,. \end{array}$$

commutes. This provides another (essential) lift $[y^+] \subseteq^- y$, this time with $\sigma y \cong_A z$. As σ is pseudo-receptive (part (ii)) we obtain the commuting diagram

$$\begin{array}{ccc} w_1 & \cong_S & y \\ \cup\!\mid & \nearrow & \\ [y^+]\,, & & \end{array}$$

which, by the isomorphism family of S, we can extend to

$$\begin{array}{ccc} w & \cong_S & y_1 \\ \cup\!\mid & & \cup\!\mid \\ w_1 & \cong_S & y \\ \cup\!\mid & \nearrow & \\ [y^+]\,, & & \end{array} \tag{5}$$

for some $y_1 \in \mathscr{C}(S)$. Because $y \subseteq y_1$ we get $y^+ \cup \sigma y^- \subseteq y_1^+ \cup \sigma y_1^-$. Also, the bijection

$$v^+ \cup \sigma v^- \cong v \cong_S w \cong_S y_1 \cong y_1^+ \cup \sigma y_1^-$$

in $\widetilde{\mathcal{S}}$ extends the original bijection $x^+ \cup \sigma x^- \cong y^+ \cup \sigma y^-$ in $\widetilde{\mathcal{S}}$. To see this, use the Observation above. Combine the commuting diagrams (3) and (5), to obtain that

$$\begin{array}{ccc} v & \cong_S & y_1 \\ \cup\!\mid & & \cup\!\mid \\ [x^+] & \cong_S & [y^+] \end{array}$$

commutes. From (4), noting $z = \sigma w_1$, combined with the application of σ to the top commuting square of (5) we obtain that

$$\begin{array}{ccc} \sigma v & \cong_A & \sigma y_1 \\ \cup\!\mid & & \cup\!\mid \\ \sigma x & \cong_A & \sigma y \end{array}$$

commutes.

(C). We show that the functions *collapse*(σ) and g are maps of esp's.

First, a characterization of the primes of $\mathcal{S}$, which constitute the events of *collapse*(S). Let $x \in \mathscr{C}(S)$. From (1) above, independently of the choice of x, the sub-configurations of $x^+ \cup \sigma x^-$ are exactly $y^+ \cup \sigma y^-$, where y is a sub-configuration of x. It follows that prime configurations of $\mathcal{S}$ always take the form $[s]^+ \cup \sigma[s]^-$, for some $s \in S$.

The function *collapse*(σ) : *collapse*$(S) \to A$ takes a prime $[s]^+ \cup \sigma[s]^-$ to $\sigma(s)$. A typical configuration of *collapse*(S) is a set

$$\{p \in \mathrm{Pr}(\mathcal{S}) \mid p \subseteq x^+ \cup \sigma x^-\}\,,$$

for some $x \in \mathscr{C}(S)$. Hence *collapse*(σ) takes it to σx in a locally injective way.

It is easy to see that $g : S \to$ *collapse*(S) is a map. It takes a configuration $x \in \mathscr{C}(S)$ to the set of prime sub-configurations of $x^+ \cup \sigma x^-$; it is locally injective because of the bijection $x \cong x^+ \cup \sigma x^-$. In addition,

$$\mathit{collapse}(\sigma) g(s) = \mathit{collapse}(\sigma)([s]^+ \cup \sigma[s]^-) = \sigma(s)\,,$$

for all $s \in S$. Hence, *collapse*$(\sigma) \circ g = \sigma$.

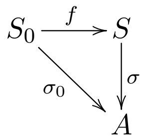
(D). Assume $z, z_1, z_2 \in \mathscr{C}(\mathit{collapse}(S))$ with $z \subseteq^- z_1$ and $z \subseteq^- z_2$. Suppose *collapse*$(\sigma)z_1 =$ *collapse*$(\sigma)z_2$. Then $\bigcup z_1 = x_1^+ \cup \sigma x_1^-$ and $\bigcup z_2 = x_2^+ \cup \sigma x_2^-$, for some $x_1, x_2 \in \mathscr{C}(S)$. From $z \subseteq^- z_1, z_2$, we see that $x_1^+ = x_2^+$ and, from *collapse*$(\sigma)z_1 =$ *collapse*$(\sigma)z_2$, that $\sigma x_1 = \sigma x_2$. Therefore $z_1 = z_2$.

(E). Finally we show that $g x_1 = g x_2 \implies x_1 \cong_S x_2$, for all $x_1, x_2 \in \mathscr{C}(S)$. Supposing $g x_1 = g x_2$ we also have $x_1^+ \cup \sigma x_1^- = \bigcup g x_1 = \bigcup g x_2 = x_2^+ \cup \sigma x_2^-$ which implies $x_1 \cong_S x_2$, by (1) above. □

Lemma C.5 *For σ any pseudo-receptive (and therefore $-$-innocent) pre-strategy, $g : \sigma \simeq_{\mathrm{s}}$ collapse(σ).*

Proof. Recall the map $g : S \to$ *collapse*(S) which takes $s \in S$ to the prime configuration $[s]^+ \cup \sigma[s]^-$.

We produce a converse map $f : \mathit{collapse}(S) \to S$ via Zorn's Lemma. By Zorn's Lemma there is a $\subseteq$-maximal map $f : S_0 \to S$ such that $gf = \mathrm{id}_{S_0}$ and



commutes, with a rigid embedding $S_0 \hookrightarrow \mathit{collapse}(S)$ s.t. σ_0 is the restriction of $\mathit{collapse}(\sigma)$ to S_0, *i.e.* σ_0 is the composition

$$\sigma_0 : S_0 \hookrightarrow \mathit{collapse}(S) \xrightarrow{\mathit{collapse}(\sigma)} A\,.$$

(Clearly the empty event structure embeds rigidly in $\mathit{collapse}(S)$ and satisfies the above constraints; Zorn's Lemma ensures this can be extended maximally.)

We show that $\sigma_0 = \mathit{collapse}(\sigma)$. Suppose otherwise, that $S_0 \neq \mathit{collapse}(S)$. Then, there is $p \in \mathrm{Pr}(\mathcal{S})$ such that $p \notin S_0$ yet $[p) \in \mathscr{C}(S_0)$. By assumption, $gf[p) = [p)$.

In Lemma C.4, we saw that a prime configuration p of $\mathcal{S}$, *i.e.* an event of $\mathit{collapse}(S)$, has the form

$$p = [s]^+ \cup \sigma[s]^-\,,$$

for some $s \in S$. We have $g[s) = [p)$ so $g[s) = gf[p)$. By Lemma C.4, it follows that $[s) \cong_S f[p)$. Because $[s) \overset{s}{\longrightarrow\!\!\subset} [s]$ we also have $f[p) \overset{s_0}{\longrightarrow\!\!\subset} [s_0]$ with $[s] \cong_S [s_0]$ extending $[s) \cong_S f[p)$.

Extend S_0 to $S_0' = S_0 \cup \{p\}$, a larger substructure of $\mathit{collapse}(S)$. The map f extends to a map $f' : S_0' \to S$ which acts so $f'(p) = s_0$, and the function σ_0 to $\sigma_0' : S_0' \to A$ so $\sigma_0'(p) = \sigma(s_0)$. Clearly we still have $\sigma f' = \sigma_0'$. With the new value p, we maintain $gf' = \mathrm{id}_{S_0'}$, as we have

$$gf'(p) = g(s) = [s]^+ \cup \sigma[s]^- = p\,.$$

This contradicts the maximality of σ_0, unless $S_0 = \mathit{collapse}(S)$ as we require.

To conclude we show f and g form an equivalence

$$f : \mathit{collapse}(\sigma) \simeq_{\mathrm{s}} \sigma : g\,.$$

Certainly $gf = \mathrm{id}_{\mathit{collapse}(S)}$. Let $x \in \mathscr{C}(S)$. We have

$$g(fg\,x) = gfg\,x = \mathrm{id}_{\mathit{collapse}(S)}g\,x = g\,x\,,$$

whence

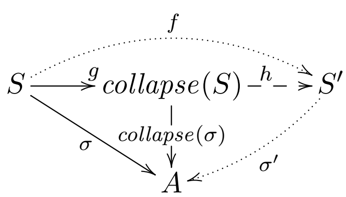
$$fg\,x \cong_S x\,,$$

using Lemma C.4. □

Universality of the collapse.

Although we have not used it directly, note the following universal property.

Corollary C.6 *(universal characterization) Let $\sigma : S \to A$ and $\sigma' : S' \to A$ be pre-strategies where σ is $-$-innocent and pseudo-receptive and σ' is collapsed. Then for an map of pre-strategies $f : \sigma \to \sigma'$ there is a unique map of pre-strategies $h : \mathit{collapse}(\sigma) \to \sigma'$ such that $f = hg$:*

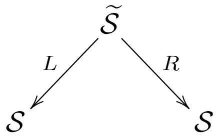


Amongst pre-strategies in A which are $-$-innocent and pseudo-receptive, collapse is left adjoint to the inclusion of the subcategory of collapsed pre-strategies in the category of all pre-strategies.

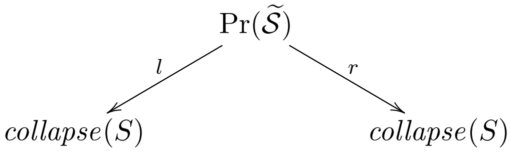
If $\sigma : S \to A$ is $-$-innocent and pseudo-receptive then $\widetilde{collapse}(S) \cong collapse(\widetilde{S})$ and $\widetilde{collapse}(\sigma) \cong collapse(\widetilde{\sigma})$.

Proof. (Sketched) To verify the universal characterization, the map h must take an event $[s]^+ \cup \sigma[s]^-$ of $collapse(S)$, where $s \in S$, to $f(s)$. The claimed adjunction follows directly from the universality. Via the adjunction *collapse* extends to a functor.

An event of $\widetilde{\mathcal{S}}$ is a pair of events of $\mathcal{S}$. Define $L : \widetilde{\mathcal{S}} \to \mathcal{S}$ and $R : \widetilde{\mathcal{S}} \to \mathcal{S}$ to be the left and right projections on events. That $\widetilde{\mathcal{S}}$ satisfies the axioms of an isomorphism family expresses that $L, R : \widetilde{\mathcal{S}} \to \mathcal{S}$ forms a symmetry in the category of stable families:



Applying the right adjoint Pr to the inclusion functor of (families of configurations of) event structures to stable families, we obtain the symmetry



where we have written $l =_{\mathrm{def}} \mathrm{Pr}(L)$ and $r =_{\mathrm{def}} \mathrm{Pr}(R)$. By definition, $\widetilde{collapse}(S) = \mathrm{Pr}(\widetilde{\mathcal{S}})$.

Recall by Lemma C.4,

$$\widetilde{\mathcal{S}} = \{\theta^+ \cup \widetilde{\sigma}\theta^- \mid \exists x, y.\ x \cong^{\theta}_{S} y\}\,.$$

There is an $\subseteq$-isomorphism between finite configurations of $\widetilde{S}$, with symmetry maps $l_S, r_S : \widetilde{S} \to S$, and the isomorphism family of S: under the isomorphism

$$\mathscr{C}(\widetilde{S}) \cong \{\theta \mid \exists x, y.\ x \cong^{\theta}_{S} y\}$$

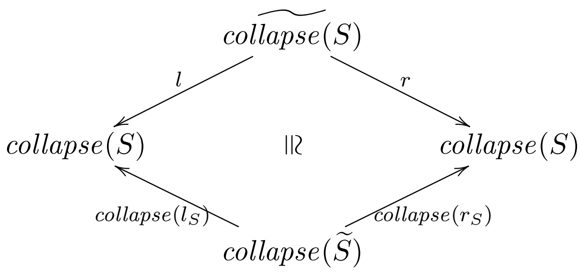
$z \in \mathscr{C}(\widetilde{S})$ is taken to the bijection $l_S z \cong z \cong r_S z$. This isomorphism induces an $\subseteq$-isomorphism

$$\widetilde{\mathcal{S}} \cong \{z^+ \cup \widetilde{\sigma} z^- \mid z \in \mathscr{C}(\widetilde{S})\}\,.$$

Whence

$$\begin{aligned}\widetilde{collapse}(S) =_{\mathrm{def}} \mathrm{Pr}(\widetilde{\mathcal{S}}) &\cong \mathrm{Pr}(\{z^+ \cup \widetilde{\sigma} z^- \mid z \in \mathscr{C}(\widetilde{S})\}) \\ &=: collapse(\widetilde{S})\,.\end{aligned}$$

The isomorphism respects the left and right symmetry maps:



commutes. It follows that

$$\widetilde{collapse}(\sigma) \cong collapse(\widetilde{\sigma})\,.$$

□