Universal Properties of Petri Net Unfoldings

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Abstract

It is an established idea in concurrency theory that every Petri net admits an unfolding semantics. This is a denotational object that represents its domain of possible executions. Unfoldings play an important role in practical analysis and verification.

This paper is concerned with the following well-known problem: while the unfolding resembles a universal construction in the category of Petri nets, it generally fails to satisfy the expected universal property. This is because the unfolding construction overlooks the net's internal symmetries.

There are two solutions: make these symmetries explicit to obtain a weak universal property (one that holds only "up to symmetry"); or break the symmetries by assigning individual identities to components of the net. We review these two solutions and establish, in each case, a universal unfolding of Petri nets to event structures.

This paper demonstrates a 2-categorical approach to Petri net unfoldings. We show that each unfolding semantics determines a 2-categorical relative adjunction involving Petri nets and event structures. Viewed in this way, the above two constructions can be related formally via an appropriate morphism of adjunctions. We exhibit a 2-density property of event structures which implies that unfolding functors are essentially unique.

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1 Introduction

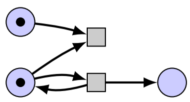
The theory of Petri nets has long helped to analyse the operational behaviour of concurrent systems in various application domains [24, 31, 26]. But this line of research has primarily targeted a restricted class of well-behaved Petri nets satisfying a "safety" condition. Safety is a helpful restriction in practice but it feels mathematically ad-hoc. This paper contributes to the categorical theory of unrestricted (or *unsafe*) Petri nets, pioneered by [22, 5] and continuing to attract interest [16, 3, 18, 2].

Specifically, we consider the unfolding problem for Petri nets, which has attracted attention due to the symmetry issues that arise in the absence of safety (e.g. [23]). We re-examine two approaches, respectively by Hayman–Winskel [16] and by Kock [18], which exemplify the two main solutions to the unfolding problem: embrace the symmetries, or break them.

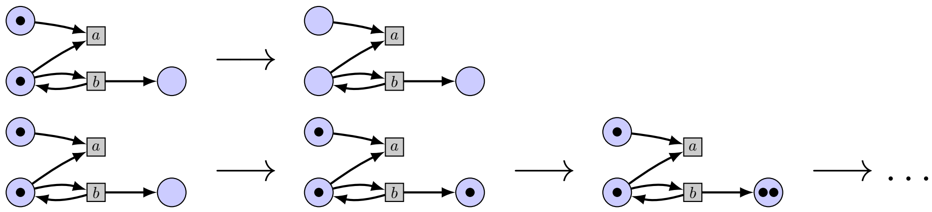
We explore this from a 2-categorical perspective, and show that event structures [25] continue to be an appropriate semantic domain for Petri nets, even in the unsafe setting, provided the symmetries are handled adequately.

1.1 Petri nets and unfoldings

Informally, a Petri net is a graph with two types of nodes, places and transitions, equipped with a collection of symbolic placeholders (tokens) distributed across the places. For example, the following is a Petri net where we have drawn places as circles, transitions as squares, and tokens as bullets:

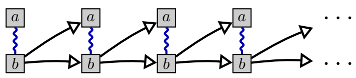


The assignment of tokens to the places of a Petri net is called a marking. This is the only aspect of the net that changes during execution, via the activation (or *firing*) of transitions, consuming tokens and producing new ones. For our example, here are some possible firings:



The dynamic behaviour of a Petri net can be intricate. For instance, the firing of a transition may produce enough tokens to enable other firings; this defines a causality relation between firings. Meanwhile several transitions can compete for the same tokens, which creates conflict and nondeterminism.

To reason about this complex behaviour, including the non-deterministic and concurrent aspects, a denotational semantics is useful. This is what event structures (introduced by Plotkin, Nielsen and Winskel in a seminal 1981 paper about Petri net unfoldings [25]) aspire to provide. The rough idea is to represent all possible sequences of transitions in a single structure that accounts for causal dependency, conflict, and concurrency. The semantics of our example net would be the infinite event structure below, in which an event represents the firing of a single transition (we have added labels to indicate which); an arrow represents a causal dependency; and a wavy blue line indicates a conflict.



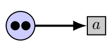
Note that conflict is hereditary, e.g. the first event labelled a is implicitly in conflict with all events labelled b, since no execution can see both events. We will define event structures formally in §3.3. Informally, b can fire repeatedly forever and as soon as a fires the execution must stop.

This kind of semantics is known as an *unfolding* (the cycle in the original Petri net is 'unfolded' to an infinite chain in the event structure). Event structures are closely connected to a special class of acyclic Petri nets known as occurrence nets (§3), thus the terminology 'unfolding' also refers to the process of turning an arbitrary Petri net into an occurrence net.

1.2 Universality issues in the unfolding of Petri nets

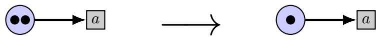
Unfolding semantics can sometimes be ambiguous or non-canonical. The simplest problematic situation is when, at some stage of execution, a transition can choose between several tokens.

A problematic example. Consider the net



whose marking allows for two successive firings. A natural candidate for its unfolding is the event structure

comprising two events without any conflict or dependency. But there is a mismatch: in the event structure, we can distinguish between two single-firing processes, whereas the Petri net has just one possible single-step execution:



Thus the event structure above fails to satisfy the universal property that one might naturally expect of an unfolding: that the executions of a Petri net correspond bijectively with the processes of its unfolding. (We will explain in §3.1 why this is a universal property in the categorical sense. See also [16].)

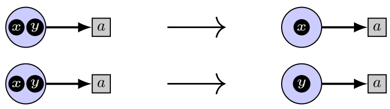
This failure of universality is well-known, and it is common in Petri net theory to impose a further restriction: that every reachable marking has at most one token per place. Petri nets satisfying this condition are traditionally called *safe* nets. Safe nets *do* admit a universal unfolding [25].

In this paper we do not make any safety assumptions and instead consider existing proposals for recovering a universal unfolding in the general case.

Unfolding general Petri nets to event structures. To resolve this issue, two methods have emerged.

The first method consists in enriching the unfolding with additional information expressing that certain processes are 'the same'. This idea culminates in the work of Hayman and Winskel, based on a theory of explicit symmetries [16, 15].

The second proposed method is to modify the very definition of Petri net so that elements (tokens and edges) carry an individual identity. Viewed in this style, the problematic net above admits two processes



distinguished by the specific token that is consumed. This is consistent with the event structure above. This method is formalized using the theory of whole-grain Petri nets developed by Kock [18].

1.3 Objectives and contributions

This paper has two objectives. First, we revisit each method separately and establish in both cases a universal unfolding of Petri nets to event structures. The constructions given by Hayman–Winskel [16] and Kock [18] are currently limited to occurrence nets, which are not adequate as a semantic domain (roughly speaking, because places can be redundant). We will see however that connecting to event structures requires care. The second objective is to connect the two methods, following the intuitive idea that forgetting names induces new symmetries.

Making all of this precise requires some technical constructions. Universal properties are stated in the language of category theory, and in this paper we additionally need some basic

2-category theory. This is essential because the unfolding of Petri nets is universal only in a 2-categorical sense. (We note that 2-categorical methods turn up already in [16, 18, 32]. They are a natural tool for dealing with named elements and symmetries.)

We briefly outline our results and the organization of this paper. In §2, we recall that Petri nets form a category **Petri** and introduce a 2-category $\mathbf{Petri}^{\mathsf{WG}}$ of whole-grain Petri nets (following Kock [18]). We relate them using a functor $\mathsf{Q} : \mathbf{Petri}^{\mathsf{WG}} \to \mathbf{Petri}$ which forgets the names of tokens (and edges) to recover mere multiplicities.

In §3, we introduce the sub-category **Occ** of occurrence nets, which can be defined equivalently either in whole-grain style or in ordinary style. We recall the definition of the category **Ev** of event structures and show that the unfolding of whole-grain Petri nets induces a 2-adjunction between $\mathbf{Petri}^{\mathsf{WG}}$ and **Ev**. This is closely related to the unfolding of whole-grain Petri nets to occurrence nets found in [18] but the link to event structures requires the introduction of new, more general class of morphisms and therefore a new unfolding functor.

In §4, we revisit the unfolding of ordinary Petri nets. Our main contribution is to establish a *relative adjunction* [1] involving the categories **Petri** and **Ev**, and a 2-category **EvSym** of *event structures with symmetry* [32]. Furthermore, we show that the embedding $\mathbf{Ev} \to \mathbf{EvSym}$ is a dense 2-functor, which characterizes the unfolding uniquely up to symmetry.

Finally, in §5, we formally connect the two universal constructions (ordinary and whole-grain) using a morphism of relative pseudo-adjunctions.

Note on terminology and related work. Petri net theory is a vast area and the word 'unfolding' is sometimes used for other purposes. Here we are concerned only with the process of unfolding a Petri net to an occurrence net or an event structure. We also note that the symmetry issues we discuss are similar to issues in the representation of unmarked nets as certain monoidal categories. We refer the reader to [18] for a thorough comparison.

1.4 Elements and multiplicities

An important point for this paper is the distinction between multisets of elements of a set X, and sets of named occurrences of elements of X. This section is just to fix notation for this basic principle.

Multisets. If X is any set, let $\mathcal{M}(X)$ denote the set of multisets of elements of X. Formally these are defined as functions $X \to \mathbb{N}$ with finite support.

Now, for a set S equipped with a function $m : S \to X$, we call $\mathsf{Q}(m)$ the function $X \to \mathbb{N} \cup \{\infty\}$ giving for each $x \in X$ the cardinality of $m^{-1}\{x\}$. Under simple conditions on the fibres of m, we have that $\mathsf{Q}(m) \in \mathcal{M}(X)$. We regard Q as a forgetful operation: while S may contain several named occurrences of an element $x \in X$, in $\mathsf{Q}(m)$ their identity is erased and the copies are indistinguishable.

Multirelations. A span of sets $X \xleftarrow{f} S \xrightarrow{g} Y$ can equivalently be presented by a function $\langle f, g\rangle : S \to X \times Y$. Under similar cardinality conditions on the fibres, via the multiplicity counting-operation Q this induces a *multirelation* $\mathsf{Q}(\langle f, g\rangle)$, seen as element of $\mathcal{M}(X \times Y)$.

Spans can be composed (via pullback) and form a bicategory (e.g. [7]). When infinite coefficients are allowed multirelations are also closed under an infinite form of matrix multiplication, and the multiplicity-count operation is functorial. (The multirelations in this paper are all finitary, and indeed satisfy further conditions to ensure that finiteness is preserved by composition.)

We will overload the variable Q and use it to denote the functor mapping whole-grain Petri nets to ordinary Petri nets. This is appropriate because the functor consists in applying Q to every component.

2 Categories of Petri nets and multiplicity counts

We start by presenting two categories of Petri nets. First we look at the traditional kind of Petri net with the standard notion of morphisms based on multirelations [16]. Then we consider the recent theory of whole-grain Petri nets [18], based on spans instead of multirelations. We will see that the latter is inherently 2-categorical.

The section is organized as follows. In §2.1 we define each kind of net and we see that a whole-grain Petri net induces an ordinary net if individual identities are forgotten. Then in §2.2 we consider morphisms between nets. This gives a category of Petri nets and a 2-category of whole-grain Petri nets, respectively **Petri** and $\mathbf{Petri}^{\mathsf{WG}}$, and a 2-functor $\mathsf{Q} : \mathbf{Petri}^{\mathsf{WG}} \to \mathbf{Petri}$.

2.1 Ordinary Petri nets and whole-grain Petri nets

We recall the classical definition of Petri nets, based on multisets and multirelations.

▶ **Definition 1** (Petri net). *A Petri net P consists of:*

- *a set S of* places *and a set T of transitions;*
- *multirelations* $\mathrm{Pre} : S \nrightarrow T$ *and* $\mathrm{Post} : S \nrightarrow T$ *such that for every $t \in T$ there are finitely many $s \in S$ such that* $\mathrm{Pre}(s,t) > 0$ *and* $\mathrm{Post}(s,t) > 0$*;*
- *a multiset $\mu \in \mathcal{M}(S)$ of places, the* marking*;*

satisfying the following two conditions:

- *The net is* grounded*: for every $t \in T$, there is $s \in S$ such that* $\mathrm{Pre}(s,t) > 0$*.*
- *The net has no* isolated places*: for every $s \in S$, there exists $t \in T$ such that* $\mathrm{Pre}(s,t) > 0$ *or* $\mathrm{Post}(s,t) > 0$*.*

The marking μ specifies the number of tokens in each place at initialisation. The fact that Pre and Post are multirelations indicates that several tokens may pass through a single edge during one step of execution. This multiplicity-based formalism follows the *collective-token philosophy* [28]: the tokens in each place are indistinguishable from each other, have no individual identity, and cannot be individually tracked as they move through the net.

It is common to impose an additional safety condition that forbids having multiple tokens in one place [30] for any reachable marking. We do not need to define this, precisely because this paper is about the universality issues with general "unsafe" Petri nets.

We now look at whole-grain Petri nets [18].

▶ **Definition 2** (Whole-grain Petri net). *A whole-grain Petri net P consists of:*

- *a set S of places and a set T of transitions;*
- *two sets I and O together with spans $S \leftarrow I \rightarrow T$ and $S \leftarrow O \rightarrow T$;*
- *a set M together with a map $M \rightarrow S$, the* marking*;*

such that the functions $I \rightarrow T$ and $O \rightarrow T$ have finite fibres. Additionally, we require that the maps $I \rightarrow T$ and $I + O \rightarrow S$ are surjective.

The surjectivity conditions correspond to the two conditions imposed on ordinary Petri nets [30], as is also the finite fibres hypothesis (that states each place/transition has a finite number of transitions/places connecting to it via I and O). This formalism allows for a much more precise description of the token game where tokens are individually tracked as they are consumed and produced by transitions: this is the *individual-token philosophy.* A basic observation is that we can always forget (or quotient out) the token and edge identities, to recover an ordinary Petri net.

▶ **Proposition 3.** *For a whole-grain Petri net P, with components labelled as in Definition 2, there is an ordinary Petri net $\mathsf{Q}(P)$ with transition set T, place set S, marking $\mu = \mathsf{Q}(M)$, and multirelations* $\mathrm{Pre} = \mathsf{Q}(S \leftarrow I \rightarrow T)$ *and* $\mathrm{Post} = \mathsf{Q}(S \leftarrow O \rightarrow T)$*. This Petri net is denoted* $\mathsf{Q}(P)$.

In other words, $\mathsf{Q}(P)$ is a version of P in which parallel edges are combined into a single edge with multiplicity and the initial marking is reduced to a multiplicity count for each place. The operational behaviour of the nets P and $Q(P)$ correspond; e.g., a transition e is enabled in P if and only if it is enabled as an a transition of $Q(P)$.

▶ Remark 4. Both kinds of Petri nets in this paper are assumed to be grounded and to have no isolated places. These two conditions are common but not systematically imposed, so this deserves a brief comment. In the context of net unfoldings, the grounded-ness condition seems essential, to avoid tokens being uncontrollably produced. However, one could likely do away with the condition on isolated places, at the cost of a slightly more complex 2-category of whole-grain Petri nets (because Theorem 12 below would not hold).

2.2 Morphisms of Petri nets

Maps of Petri nets have a well-established theory; they have a canonical justification as relations preserving the token game [29].

▶ **Definition 5.** *Given* $P = (S, T, \mathrm{Pre}, \mathrm{Post}, M)$ *and* $P' = (S', T', \mathrm{Pre}', \mathrm{Post}', M')$*, a* map of Petri nets $P \to P'$ *consists of a function* $\eta : T \to T'$ *and a multirelation* $\beta : S \nrightarrow S'$ *such that the diagrams*

$$
\begin{array}{ccc}
 & \overset{\mu}{\nearrow} & S \\
1 & & \downarrow \beta \\
 & \underset{\mu'}{\searrow} & S'
\end{array}
\qquad
\begin{array}{ccc}
S & \xrightarrow{\mathrm{Pre}} & T \\
\beta \downarrow & & \downarrow \eta \\
S' & \xrightarrow[\mathrm{Pre}']{} & T'
\end{array}
\qquad
\begin{array}{ccc}
S & \xrightarrow{\mathrm{Post}} & T \\
\beta \downarrow & & \downarrow \eta \\
S' & \xrightarrow[\mathrm{Post}']{} & T'
\end{array}
$$

commute. (The multisets μ and μ' are seen as multirelations from a singleton set, and the function η is seen as a multirelation with $\eta(t, t') = 1$ if $\eta(t) = t'$ and 0 otherwise.)

In a map of Petri nets, the multirelation β can be read as a reverse action on places: every place $s' \in S'$ that admits an edge to a transition $\eta(t)$ must be assigned a multiset of places incoming for t. When β arises from a total function $S' \to S$, say the map (η, β) is a *folding map.* Morphisms of Petri nets compose to form a category that we denote **Petri**.

We now turn to the morphisms of whole-grain Petri nets, closely following Kock [18]. This involves a bit more data (in particular, because in the whole-grain setting, the multirelation β must be replaced by another span) but axioms are stated as simple pullback conditions.

▶ **Definition 6.** *Suppose that* $P = (S, T, I, O, M)$ *and* $P' = (S', T', I', O', M')$ *are whole-grain Petri nets. A* map of (whole-grain) Petri nets $\varphi : P \to P'$ *is a family of five functions* *relating P and P' componentwise such that*

$$\begin{array}{ccccccccc} S & \longleftarrow & I & \longrightarrow & T & \longleftarrow & O & \longrightarrow & S \\ \downarrow & (a) & \downarrow & (b) & \downarrow & (c) & \downarrow & (d) & \downarrow \\ S' & \longleftarrow & I' & \longrightarrow & T' & \longleftarrow & O' & \longrightarrow & S' \end{array} \qquad \begin{array}{ccc} M & \longrightarrow & S \\ \downarrow & (e) & \downarrow \\ M' & \longrightarrow & S' \end{array}$$

commutes. (We keep the individual functions anonymous for clarity; the functions $S \to S'$ are all the same in the diagram.)

The only relevant maps for this paper are those satisfying further conditions, as follows.

- *A map φ is an* étale map *if (b) and (c) are pullback squares and the map $M \to M'$ is an identity function (in particular $M = M'$).*
- *A map φ is a* cabling map *if (a), (d), and (e) are pullback squares and the map $T \to T'$ is an identity function (in particular $T' = T$).*

More informally, étale maps are those preserving the input and output arities of transitions. The pullback condition enforces this via an appropriate isomorphism of fibres. A cabling is in some sense *place-étale*, as it preserves arities of places. (One could relax the identity map axiom to an invertibility condition, but the definition above makes the overall formalism simpler [18].)

▶ Remark 7. Our cabling maps generalise those of Kock [18], who only considered maps between Petri nets having the same initial marking. It is important for our purposes to allow for varying markings (subject to the pullback condition (e)—compare with the first condition in Definition 6) for a proper connection with event structures.

More concretely, the pullback conditions for an étale map assert that a transition and its image must have the same number of incoming and outgoing edges, with the map providing a specific bijection. A cabling map has the analogous property for places, and additionally the two Petri nets must have the same transitions. By combining étale maps and cabling maps we recover a whole-grain version of the maps of ordinary Petri nets in Definition 5.

▶ **Definition 8.** *For whole-grain Petri nets P and P', a* rational map *$P \to P'$ is a span*

$$P \xleftarrow{\varphi} R \xrightarrow{\psi} P'$$

where R is another whole-grain net, φ is a cabling map and ψ is an étale map.

We recover étale maps as the class of rational maps for which $\varphi = \mathrm{id}$. The components of a rational map can be explicitly laid out as

$$\begin{array}{ccccccccc} S_P & \longleftarrow & I_P & \longrightarrow & T_P & \longleftarrow & O_P & \longrightarrow & S_P \\ \uparrow & & \uparrow & & \| & & \uparrow & & \uparrow \\ S_R & \longleftarrow & I_R & \longrightarrow & T_R & \longleftarrow & O_R & \longrightarrow & S_R \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ S_{P'} & \longleftarrow & I_{P'} & \longrightarrow & T_{P'} & \longleftarrow & O_{P'} & \longrightarrow & S_{P'} \end{array}$$

▶ **Proposition 9.** *For $P, P' \in \mathbf{Petri}^{\mathsf{WG}}$, every rational map $P \leftarrow R \to P'$ induces a morphism of ordinary Petri nets $\mathsf{Q}(P) \to \mathsf{Q}(P')$ consisting of the function $T_R \to T'_P$ and the finite multirelation $\mathsf{Q}(S_P \leftarrow S_R \to S_{P'})$.*

Étale maps correspond to folding maps:

▶ **Lemma 10.** *For $P, P' \in \mathbf{Petri}^{\mathsf{WG}}$, if $\psi : P \to P'$ is an étale map (seen as a degenerate rational map), then $\mathsf{Q}(\psi)$ is a folding map.*

Rational maps can be composed componentwise via the standard composition of spans using pullbacks. This relies mostly on elementary properties of pullbacks. One minor technical point is that to preserve the cabling property of the left leg we must assume that pullbacks of identities are identities; it is easy to make a choice of pullbacks in **Set** that satisfies this. There is an identity rational map for every Petri net P given by the identity span $P = P = P$.

Pullbacks are only defined up to isomorphism and so the composition of spans is only associative and unital up to coherent invertible 2-cells. The 2-cells are defined as standard morphisms of spans:

▶ **Definition 11.** *For whole-grain Petri nets P and P' and rational maps as on the left below,*

$$\begin{array}{ccccc} P & \xleftarrow{\varphi} & R & \xrightarrow{\psi} & P' \\ P & \xleftarrow{\varphi'} & R' & \xrightarrow{\psi'} & P' \end{array} \qquad \begin{array}{ccccc} & \overset{\varphi}{\swarrow} & R & \overset{\psi}{\searrow} & \\ P & & \downarrow \alpha & & P' \\ & \underset{\varphi'}{\nwarrow} & R' & \underset{\psi'}{\nearrow} & \end{array}$$

a 2-cell $(\varphi, \psi) \to (\varphi', \psi')$ is a morphism $\alpha : R \to R'$ that makes the above diagrams commute.

Whole-grain Petri nets, rational maps and 2-cells assemble into a bicategory that we denote $\mathbf{Petri}^{\mathsf{WG}}$. That we have a bicategory and not a category is typical for span-like morphisms and this is closely connected to the fact that the whole-grain setting tracks occurrences of elements with individual names, not just multiplicities. Any renaming of elements will give a 2-cell, and in fact *every* 2-cell in this bicategory is a renaming:

▶ **Theorem 12** (Discreteness)**.** *For whole-grain Petri nets P and P' and rational maps $(\varphi, \psi), (\varphi', \psi') : P \to P'$, there is at most one 2-cell $(\varphi, \psi) \to (\varphi', \psi')$ and when it exists it is invertible.*

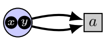
In other words, the bicategory $\mathbf{Petri}^{\mathsf{WG}}$ is locally essentially discrete. This property prevents any coherence issues and greatly eases the proofs of several theorems below.

We have described two categorical models for Petri nets, a bicategory $\mathbf{Petri}^{\mathsf{WG}}$ and a category **Petri**. We now connect them using the multiplicity-counting operation Q (of §1.4), whose action on whole-grain Petri nets extends to a 2-functor $\mathbf{Petri}^{\mathsf{WG}} \to \mathbf{Petri}$ (this is a straightforward verification of the axioms). For this statement to make sense, **Petri** is understood as a locally discrete 2-category.

▶ **Theorem 13.** *There is a 2-functor $\mathsf{Q} : \mathbf{Petri}^{\mathsf{WG}} \to \mathbf{Petri}$ extending the action on objects and morphisms described in Proposition 3 and Proposition 9.*

Proof. It remains only to deal with the 2-cells. A 2-cell $\alpha : R \to R'$ between rational maps $P \leftarrow R \to P'$ and $P \leftarrow R' \to P'$ implies that $T_R = T_{R'}$ and the functions $T_P = T_R \to T'_P$ and $T_P = T_R \to T_{P'}$ are the same. We have seen that α has a component $S_R \to S_{R'}$ which is an isomorphism of spans between $S_P \leftarrow S_R \to S_{P'}$ and $S_P \leftarrow S_{R'} \to S_{P'}$. Necessarily these spans also have the same image under Q. The axioms for a 2-functor follow immediately. ◀

▶ **Example 14.** We emphasize that $\mathsf{Q} : \mathbf{Petri}^{\mathsf{WG}} \to \mathbf{Petri}$ is not an equivalence: the whole-grain Petri net



has four distinct (rational) automorphisms, since the two tokens and the two edges can be permuted, however its image under Q only has an identity automorphism.

3 The token game: paths, occurrence nets, and event structures

The “token game” specifies the operational behaviour of a Petri net. There is only one rule: when a transition is fired, the marking is modified according to the incoming and outgoing edges for that transition. Since there is no restriction on the order of firings, and several transitions may be available at a given point, a Petri net generally admits several possible executions (recall e.g. 1.2), including executions with firings occurring in parallel.

In this section we give a formal notion of execution path (or *process*) for a Petri net, and we then define occurrence nets, which are colimits of paths. We compare the whole-grain view and the traditional view on paths: the latter is based on causal nets [25] and the former uses directed acyclic graphs [18].

3.1 The execution paths of a Petri net

One key idea of [18] is to organize the transitions in an execution path into a directed graph, where a node represents a transition firing and an edge between two firings indicates that a token is produced by one and consumed by the other (via a place). Technically this gives an “open-ended” graph (with dangling edges), because the tokens in the initial marking are not produced by any firings, and the tokens in the final marking are not consumed by any firings. For the following definitions, we temporarily drop the non-isolated condition of Petri net, as it will be convenient to define graphs as executions paths; this is a mild generalisation, as we will see in §3.2, when defining the unfolding, that a non-isolated Petri net will be unfolded into another non-isolated net.

▶ **Definition 15.** *A* grounded, open-ended graph *G consists of a pair of finite sets (N, A) of nodes and arcs, together with partial functions $s : A \rightharpoonup N$ and $t : A \rightharpoonup N$, specifying the source and target nodes of each arc where these are defined, such that t is surjective. The* in-boundary *of G is the subset of A where s is undefined, and the* out-boundary *is the subset of A where t is undefined. These subsets are denoted* $\mathrm{in}(G)$ *and* $\mathrm{out}(G)$*, respectively.*

A (grounded, open-ended) graph $G = (N, A, s, t)$ can be seen as a special kind of whole-grain Petri net $A \hookleftarrow I \xrightarrow{t} N \xleftarrow{s} O \hookrightarrow A$ where I is the complement of $\mathrm{in}(G)$ in A and O is the complement of $\mathrm{out}(G)$ in A. The initial marking is the inclusion map $\mathrm{in}(G) \hookrightarrow A$.

Therefore, a graph G as above gives rise to an ordinary Petri net $\mathsf{Q}(G)$. In this case the operation Q does not discard any information because a graph has at most a single token per place and no parallel edges. A Petri net of this kind is called a *causal net* [25].

▶ **Definition 16.** *For $P \in \mathbf{Petri}^{\mathsf{WG}}$, a* path *of P is a graph G together with an étale map $p : G \to P$. A morphism of paths $(G, p) \to (G', p')$ is defined as an étale map $\psi : G \to G'$ such that $p = p' \circ \psi$. We write $\mathbf{Path}(P)$ for the* category of paths of *P.*

▶ **Definition 17.** *For $P \in \mathbf{Petri}$, a* path *of P is a graph G together with a folding map $p : \mathsf{Q}(G) \to P$. A morphism of paths $(G, p) \to (G', p')$ is defined as an étale map $f : G \to G'$ such that $p = p' \circ \mathsf{Q}(f)$. We write $\mathbf{Path}(P)$ for the* category of paths of *P.*

3.2 Occurrence nets and unfoldings

Occurrence nets are a special class of Petri nets. An occurrence net is intended to represent, in a single *unfolded* net, the full domain of paths for a Petri net $P \in \mathbf{Petri}$. Unfortunately there are difficulties in making this intention precise, essentially because the occurrence net is not always a colimit of paths in the sense one might expect.

We first give an explicit definition of occurrence nets. There are various equivalent definitions; anticipating the connection to event structures we use one based on the causality and conflict relations in a Petri net.

▶ **Definition 18** ([25]). *Let $P = (S, T, \mathrm{Pre}, \mathrm{Post}, \mu)$ be a Petri net with no parallel edges, i.e. the multirelations* Pre *and* Post *are ordinary relations. Say two transitions $t, t' \in T$ are in* immediate conflict*, written $t \,\#_0\, t'$ if there exists a place s with $\mathrm{Pre}(s,t)$ and $\mathrm{Pre}(s,t')$. Define binary relations $<$ and $\#$ on the set $S \uplus T$ as follows:*

- *$<$ is the transitive closure of $\mathrm{Pre} \cup \mathrm{Post}$, so $u < v$ iff there is a non-empty path from u to v in the graph underlying P;*
- *$\#$ is the* hereditary closure *of $\#_0$ under $<$, that is, the smallest relation containing $\#_0$ and such that if $u \# u'$ and $u < v$ then $v \# u'$.*

The Petri net P is an occurrence net *if the relations $<$ and $\#$ are irreflexive; the set $\{u \mid u < v\}$ is finite for every $v \in S \uplus T$; and the multiset μ is a set (at most one token per place) containing precisely the $\leq$-minimal places.*

A whole-grain Petri net P is a *whole-grain occurrence net*[1] if $\mathsf{Q}(P)$ is an occurrence net. We have subcategories $\mathbf{Occ} \hookrightarrow \mathbf{Petri}$ and $\mathbf{Occ}^{\mathsf{WG}} \hookrightarrow \mathbf{Petri}^{\mathsf{WG}}$, and it is not hard to see that the restriction of $\mathsf{Q} : \mathbf{Petri}^{\mathsf{WG}} \to \mathbf{Petri}$ to a functor $\mathbf{Occ}^{\mathsf{WG}} \to \mathbf{Occ}$ is an equivalence of categories, because occurrence nets have no multiplicities.

One benefit of the whole-grain approach is that it is easy to construct an unfolding with the expected universal property. This is the key insight of [18]:

▶ **Theorem 19** (Kock [18]). *Let $\mathbf{Petri}^{\mathsf{WG}}_{\acute{e}t}$ and $\mathbf{Occ}^{\mathsf{WG}}_{\acute{e}t}$ denote the respective wide subcategories on étale morphisms. The inclusion functor $\mathbf{Occ}^{\mathsf{WG}}_{\acute{e}t} \hookrightarrow \mathbf{Petri}^{\mathsf{WG}}_{\acute{e}t}$ has a right adjoint.*

We have extended this to the full categories of rational maps.

▶ **Theorem 20.** *The embedding $\mathbf{Occ}^{\mathsf{WG}} \hookrightarrow \mathbf{Petri}^{\mathsf{WG}}$ has a right adjoint, which we denote $\mathsf{U} : \mathbf{Petri}^{\mathsf{WG}} \to \mathbf{Occ}^{\mathsf{WG}}$.*

Proof. As in [18], for $P \in \mathbf{Petri}^{\mathsf{WG}}$, the occurrence net $\mathsf{U}(P)$ is characterized as a colimit of the projection functor $\mathbf{Path}(P) \to \mathbf{Occ}^{\mathsf{WG}}_{is} : (G, p) \mapsto G$, where $\mathbf{Occ}^{\mathsf{WG}}_{is}$ is the category of (whole-grain) occurrence nets without the "no isolated places" condition. One can show that, since P itself satisfies the condition, so does the colimit. The remaining step is to show that the universal property still holds with respect to rational maps. Full details of this construction will be made available in the long version of this paper. ◀

Although things work smoothly in the whole-grain setting, unfoldings to occurrence nets are more difficult for ordinary Petri nets. This is because, for a net $P \in \mathbf{Petri}$, the projection functor $\mathbf{Path}(P) \to \mathbf{Occ}$ does not admit a colimit in $\mathbf{Occ}$. (The colimit does exist if P is a safe Petri net [16, 30].)

▶ **Example 21.** Figure 1 contrasts the situation in $\mathbf{Petri}^{\mathsf{WG}}$ and $\mathbf{Petri}$ for the problematic example of §1.2, here denoted P. (To read the figure, recall that graphs are special nets (§3.1), although we draw them with nodes and edges.) The symmetries in $\mathbf{Path}(\mathsf{Q}P)$ affect the universality of the potential unfolding of $\mathsf{Q}P$ (which we will call $\mathcal{U}_{\mathsf{Q}P}$ in the next section). The symmetries are not present at all in $\mathbf{Path}(P)$ because morphisms must respect token labels.

[1] Kock calls this an *occurrence hypergraph*; his definition is equivalent but emphasises other aspects [18].

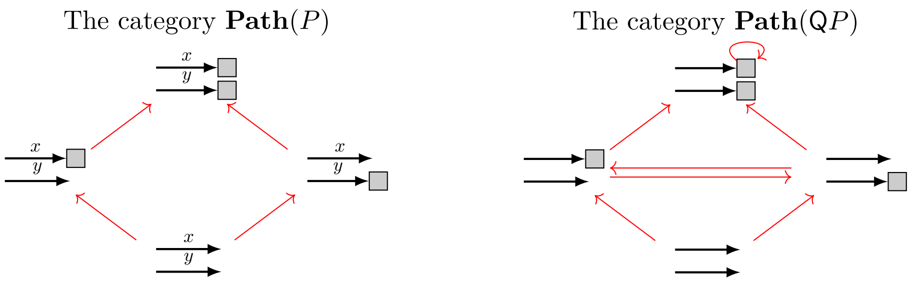


Figure 1 Path categories for the example in §1.2.

One solution, found by Hayman and Winskel, is to enrich occurrence nets with additional structure encoding symmetry [16], to render this problematic automorphism "equivalent" to the identity. With symmetry one can therefore state universal properties "up to symmetry", as will be developed in §4.

3.3 Event structures

Event structures [25] are an abstraction of occurrence nets in which the places have been discarded: an event structure has a single set equipped with a partial order and conflict relation corresponding to the relations on transitions in an occurrence net (Definition 18).

Event structures are more appropriate than occurrence nets as a semantic domain, because the places are 'implementation details' for a Petri net, since execution steps are characterized by transitions.

▶ **Definition 22.** *An* event structure *is a tuple* $(E, \leq, \#)$ *where* $\leq$ *is a partial order on* E*, and* $\#$ *is an irreflexive and symmetric binary relation on* E*, satisfying the axioms below:*

- *For every* $e \in E$ *the set* $\downarrow e := \{e' \in E \mid e' \leq e\}$ *is finite.*
- *For every* $e, e', e'' \in E$*, if* $e \leq_E e'$ *and* $e' \# e''$ *then* $e \# e''$*.*

We now define the execution paths of an event structure:

▶ **Definition 23.** *A finite subset* $X \subseteq E$ *is called* consistent *if no two events in* X *are in conflict. It is called a* configuration *if it is additionally down-closed. The set of consistent subsets of* E *is denoted* $\mathrm{Con}(E)$*, and the set of configurations is called* $\mathscr{C}(E)$*.*

Maps of event structures must faithfully preserve configurations:

▶ **Definition 24.** *If* $(E, \leq_E, \#_E)$ *and* $(E', \leq_{E'}, \#_{E'})$ *are event structures, a function* $f : E \to E'$ *is a* map of event structures *if it satisfies the following axioms:*

- *For every* $x \in \mathscr{C}(E)$*, the direct image* $fx := \{f(e) \mid e \in x\} \in \mathscr{C}(E')$*.*
- *For every* $x \in \mathscr{C}(E)$*, the restriction of* f *to* x *is injective.*

Event structures and maps of event structures form a category **Ev**. Frequently, we will regard **Ev** as a locally discrete 2-category, just like we did for **Petri** and **Occ**.

We now recall (from [25, 30]) the relationship between occurrence nets and event structures. The set of transitions of an occurrence net, equipped with the relations $\leq$ and $\#$ of Definition 18, defines an event structure. This extends to a functor $\mathcal{E} : \mathbf{Occ} \to \mathbf{Ev}$. Conversely, for every event structure E the pre-image $\mathcal{E}^{-1}(E)$ is nonempty. An object of $\mathcal{E}^{-1}(E)$ can be constructed universally so as to give a functor $\mathcal{N} : \mathbf{Ev} \to \mathbf{Occ}$ and an adjunction

$$\mathbf{Occ} \underset{\mathcal{N}}{\overset{\mathcal{E}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\top}}}} \mathbf{Ev}. \tag{1}$$

▶ **Corollary 25.** *There is a natural equivalence of setoids* $\mathbf{Petri}^{\mathsf{WG}}(\mathsf{N}E, P) \simeq \mathbf{Ev}(E, [\![P]\!]^{\mathsf{WG}})$ *for every* $P \in \mathbf{Petri}^{\mathsf{WG}}$ *an* $E \in \mathbf{Ev}$*; in other words, a pseudo-adjunction*

$$\mathbf{Petri}^{\mathsf{WG}} \underset{\mathsf{N}}{\overset{[\![-]\!]^{\mathsf{WG}}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\top}}}} \mathbf{Ev}$$

where $\mathsf{N} = \mathbf{Ev} \xrightarrow{\mathcal{N}} \mathbf{Occ} \simeq \mathbf{Occ}^{\mathsf{WG}} \hookrightarrow \mathbf{Petri}^{\mathsf{WG}}$*,* $[\![-]\!]^{\mathsf{WG}} = \mathbf{Petri}^{\mathsf{WG}} \xrightarrow{\mathsf{U}} \mathbf{Occ}^{\mathsf{WG}} \simeq \mathbf{Occ} \xrightarrow{\mathcal{E}} \mathbf{Ev}$.

(One could also define N explicitly, just by adapting the definition of $\mathcal{N}$ to the whole-grain setting.) In summary, the unfolding semantics of a Petri net is computed in two steps: first as an occurrence net, which combines all paths into a single (generally infinite) net, and then as an event structure. So far we have only done this for whole-grain Petri nets, giving the adjunction in Corollary 25. We now turn to the more complex situation for ordinary nets.

4 Petri net unfoldings via explicit symmetries

In this section we consider the unfolding of ordinary Petri nets. Our main contribution is a universal unfolding semantics in terms of event structures with symmetry.

4.1 Background: Petri net unfoldings to occurrence nets

Unfoldings of Petri nets are often used in practice (e.g. [21]) but it can be difficult to find an explicit construction. We now import a characterization from [16]. To give some intuition for the mutually recursive construction below, the occurrence net $\mathcal{U}_P$ consists of the tokens of P viewed as places, extended in a small-step fashion by considering reachable markings and transition occurrences.

▶ **Proposition 26.** *For every* $P \in \mathbf{Petri}$*, with* $P = (S_0, T_0, \mathrm{Pre}_0, \mathrm{Post}_0, \mu_0)$*, there is a unique occurrence net* $\mathcal{U}_P = (S, T, \mathrm{Pre}, \mathrm{Post}) \in \mathbf{Occ}$ *equipped with a folding map* $\varepsilon_P = (\eta, \beta) : \mathcal{U}_P \to P$*, satisfying the equations below ('*co A*' means that A has no two places related by $<$ or $\#$),*

$$S = \{(s, i) \mid s \in S_0, 0 \leq i < \mu_0(s)\} \cup \{(t, s, i) \mid t \in T, s \in S_0, 0 \leq i < \mathrm{Post}_0(p, \eta(t))\}$$
$$T = \{(A, t) \mid A \subseteq S, t \in T_0, \mathrm{co}\ A, \beta \circ A = \mathrm{Pre}_0(-, t)\}$$

and with $\eta(A, t) = t$*,* $\beta(t, s, i) = \beta(s, i) = s$*,* $\mathrm{Pre}(s, (A, t))$ *iff* $s \in A$*, and* $\mathrm{Post}((A, t), b)$ *iff b is of the form* $((A, t), p, i)$*. (Recall that for an occurrence net the multirelations* Pre *and* Post *are relations, and the marking is uniquely determined.)*

This unfolding fails to satisfy the universal property required for an adjunction, essentially because in unsafe settings the small-step construction produces redundant data. For example, when the above construction is applied to the Petri net in §1.2, two indistinguishable tokens become two separate (distinguishable) places, leading to the symmetry issues already discussed there.

The unfolding does, in fact, satisfy the existence part of a universal property:

▶ **Lemma 27** ([16, 30])**.** *Let* $O \in \mathbf{Occ}$ *and* $P \in \mathbf{Petri}$*. For every map* $(\eta, \beta) : O \to P$ *of Petri nets, there exists a map* $(\eta', \beta') : O \to \mathcal{U}_P$ *such that the diagram*

$$\begin{array}{ccc} O & \xrightarrow{(\eta,\beta)} & P \\ {\scriptstyle (\eta',\beta')}\ \downarrow & \nearrow_{\varepsilon_P} & \\ \mathcal{U}_P & & \end{array}$$

commutes.

The insight of Hayman and Winskel [16] is that by tracking the internal symmetries of $\mathcal{U}_P$ one can prove uniqueness up to a form of symmetry. This can be phrased as a universal property in a weak 2-categorical sense: a pseudo-adjunction.

But adding symmetry to Petri nets is a highly technical endeavour. There are several non-equivalent constructions, requiring nets with multiple markings, and the connection to event structures remains unclear because of a fundamental obstacle [15, §7.1].

Our perspective is that these complications are unnecessary: we bypass Petri nets with symmetries by directly unfolding to event structures with symmetry, which have a much simpler and established theory [32]. As we will see, this method relies on a 2-categorical relative adjunction and a 2-density property.

4.2 The unfolding of a Petri net as an event structure with symmetry

Event structures with symmetry. We first recall event structures with symmetry and the 2-category **EvSym**. This is due to Winskel [32]. Informally, symmetry on an event structure indicates which pairs of executions can be considered equivalent, via a bisimulation relation.

▶ **Definition 28.** *For an event structure E, an isomorphism family is a set $\mathbb{S}$ of bijections $\theta : x \cong y$ between finite configurations of $x, y \in \mathscr{C}(E)$, such that:*

- *$\mathbb{S}$ contains all identity bijections, and is stable under composition and inverse.*
- *For $\theta : x \cong y \in \mathbb{S}$, if $x' \subseteq x$, then the restriction of θ to x' is in $\mathbb{S}$.*
- *For $\theta : x \cong y \in \mathbb{S}$, if $x \subseteq x'$, then there exists an extension $y \subseteq y'$ and a bijection $\theta' : x' \cong y' \in \mathbb{S}$ such that θ' restricts to θ.*

The pair $(E, \mathbb{S})$ is called an event structure with symmetry*.*

We then consider symmetry-preserving maps:

▶ **Definition 29.** *For event structures with symmetry $(E, \mathbb{S})$ and $(E', \mathbb{S}')$, a map $f : E \to E'$* preserves symmetry *if for all $\theta : x \cong y \in \mathbb{S}$, $f\theta : fx \cong fy \in \mathbb{S}'$, where $f\theta : f(e) \mapsto f(\theta(e))$.*

A novelty in the presence of symmetry is the following equivalence relation on maps:

▶ **Definition 30.** *For event structures with symmetry $(E, \mathbb{S})$ and $(E', \mathbb{S}')$ and symmetry-preserving maps $f, g : E \to E'$, say that f and g are* symmetric*, denoted $f \sim g$, if for every $x \in \mathscr{C}(E)$ the bijection $fx \cong gx$ defined by $f(e) \mapsto g(e)$ is in $\mathbb{S}'$.*

Together, event structures with symmetry, symmetry-preserving maps and symmetries of maps form a 2-category **EvSym** [32]. This is a degenerate 2-category in the sense that hom-categories are all *setoids* (sets with an equivalence relation). There is an embedding **Ev** $\to$ **EvSym**. As we will see, the 2-categorical structure makes it possible to consider universal properties "up to symmetry".

The unfolding semantics with symmetry We proceed towards the construction of a 2-functor $[\![-]\!]^{\mathrm{Sym}} : \mathbf{Petri} \to \mathbf{EvSym}$. To build $[\![P]\!]^{\mathrm{Sym}}$ for a Petri net P, we equip the event structure $\mathcal{E}(\mathcal{U}_P)$ with an appropriate isomorphism family, which we define now. The informal idea is that configurations corresponding to the same path in P should be made symmetric.

To make this precise we first need to formally construct the path corresponding to a configuration x of $\mathcal{E}(\mathcal{U}_P)$. Note that x can itself be regarded as an event structure, inheriting the partial order from $\mathcal{E}(\mathcal{U}_P)$, and with no conflict. There is an embedding $x \to \mathcal{E}\mathcal{U}_P$ which, because of the adjunction in (1), corresponds to a morphism $\gamma_x : \mathcal{N}x \to \mathcal{U}_P$ (and the image

$\mathcal{E}(\gamma_x)$ is the inclusion map $x \to \mathcal{U}_P$). Let α_x denote the composite map $\mathcal{N}x \xrightarrow{\gamma_x} \mathcal{U}_P \xrightarrow{\varepsilon_P} P$. The pair $(\mathcal{N}x, \alpha_x)$ is a path of P, because the functor $\mathcal{N}$ turns conflict-free event structures into graphs (see [25]). An isomorphism of event structures $\theta : x \cong y$ between configurations of $\mathcal{E}(\mathcal{U}_P)$ induces a morphism of $\mathcal{U}_P$-paths $\mathcal{N}(\theta) : \mathcal{N}x \cong \mathcal{N}y$. The underlying bijection θ should be a symmetry if the paths $(\mathcal{N}x, \alpha_x)$ and $(\mathcal{N}y, \alpha_y)$ correspond to the same path in P. All of this is summarized below:

▶ **Proposition 31.** *For a Petri net P, the collection*

$$\mathbb{S}_{\mathcal{E}(\mathcal{U}_P)} = \{\theta : x \cong y \mid \theta \in \mathbf{Ev}(x, y) \text{ and } \alpha_y \circ \mathcal{N}(\theta) = \alpha_x\}$$

is an isomorphism family on the event structure $\mathcal{E}(\mathcal{U}_P)$ satisfying the following property: for all $E \in \mathbf{Ev}$ and for all pairs of maps $f_1, f_2 \in \mathbf{EvSym}(E, \mathcal{E}(\mathcal{U}_P))$ such that $f_1 \sim f_2$, $\varepsilon_P \circ \mathcal{N}(f_1) = \varepsilon_P \circ \mathcal{N}(f_2)$.

For $P \in \mathbf{Petri}$, let $[\![P]\!]^{\mathrm{Sym}} \in \mathbf{EvSym}$ denote the pair $(\mathcal{E}(\mathcal{U}_P), \mathbb{S}_{\mathcal{E}(\mathcal{U}_P)})$.

Proof sketch. The axioms for an isomorphism family are easily verified given the "small-step" definition of $\mathcal{U}_P$. For the property, note that this holds essentially by definition when E is a finite event structure with no conflicts. For the general case, we use that any E is a colimit of its configurations and that $\mathcal{N}$, as a left adjoint, preserves colimits. ◀

▶ **Example 32.** We return to the 'problematic' example in §1.2. Applying the above construction, we obtain the same event structure (with two concurrent events), but this time it comes equipped with an isomorphism family that makes the two events symmetric. This is one way to resolve the universality issue by restoring the equivalence, up to symmetry.

In general, in addition to the existence property of Lemma 27, the unfolding satisfies the uniqueness part of a universal property, up to symmetry:

▶ **Lemma 33.** *For $O \in \mathbf{Occ}$ and $P \in \mathbf{Petri}$, if $(\eta, \beta) : O \to P$ is a map of Petri nets and there are two maps $(\eta'_1, \beta'_1), (\eta'_2, \beta'_2) : O \to \mathcal{U}_P$ satisfying the commutative diagram of Lemma 27, then $\eta'_1 \sim \eta'_2$ as maps of event structures with symmetry $J\mathcal{E}O \to [\![P]\!]^{\mathrm{Sym}}$.*

Proof. By assumption the two maps of ocurrence nets are equal when postcomposed by ε_P. The result then follows from our definition of the symmetry in $\mathcal{U}_P$. ◀

▶ **Theorem 34.** *For $E \in \mathbf{Ev}$ and $P \in \mathbf{Petri}$, the function*

$$\mathbf{Ev}(E, \mathcal{E}\mathcal{U}_P) \xrightarrow{\cong} \mathbf{Occ}(\mathcal{N}E, \mathcal{U}_P) \xrightarrow{\varepsilon_P \circ -} \mathbf{Petri}(\mathcal{N}E, P)$$

extends to an equivalence of categories $\mathbf{EvSym}(JE, [\![P]\!]^{\mathrm{Sym}}) \simeq \mathbf{Petri}(\mathcal{N}E, P)$, natural in E.

Proof. First recall that the domain category $\mathbf{EvSym}(JE, [\![P]\!]^{\mathrm{Sym}})$ is a setoid and the codomain $\mathbf{Petri}(\mathcal{N}E, P)$ is a set, i.e. a discrete category. The function extends to a functor, i.e. symmetric pairs have the same image, by Proposition 31. To establish the equivalence it suffices to prove that the function is surjective, which follows from Lemma 27. ◀

This theorem characterizes the natural transformation $\mathbf{EvSym}(J-, [\![P]\!]^{\mathrm{Sym}})$ up to equivalence. Modulo the J, this resembles a characterization of $[\![P]\!]^{\mathrm{Sym}}$ itself by a Yoneda-style argument. We can make this argument precise by showing a density result for J.

4.3 A 2-density result for event structures

We prove a density result for the embedding of event structures into event structures with symmetry. This is a 2-categorical (i.e. **Cat**-enriched) form of density: each event structure with symmetry is a canonical 2-colimit of ordinary event structures. Another way of stating this is as follows:

▶ **Definition 35** (e.g. [17]). *For locally small 2-categories* $\mathbf{D}$ *and* $\mathbf{E}$*, a 2-functor* $J : \mathbf{D} \to \mathbf{E}$ *is* 2-dense *if the 2-functor* $\tilde{J} : \mathbf{E} \to [\mathbf{D}^{\mathrm{op}}, \mathbf{Cat}] : e \mapsto \mathbf{E}(J-, e)$ *is locally an isomorphism of categories, where* $[\mathbf{D}^{\mathrm{op}}, \mathbf{Cat}]$ *is the 2-category of 2-functors, strict natural transformations, and modifications.*

▶ **Theorem 36.** *The embedding 2-functor* $J : \mathbf{Ev} \to \mathbf{EvSym}$ *is 2-dense.*

(We emphasize that 2-density is strictly weaker than density in the 1-categorical sense. In particular the underlying 1-functor $J : \mathbf{Ev} \to \mathbf{EvSym}$ is not dense.)

Proof. For event structures with symmetry A and B, we show that the functor $\tilde{J}_{A,B} : \mathbf{EvSym}(A, B) \to \mathbf{Cat}^{\mathbf{Ev}^{\mathrm{op}}}(\mathbf{EvSym}(J-, A), \mathbf{EvSym}(J-, B))$ is an isomorphism of categories by constructing an inverse. Write $A = (|A|, \mathbb{S}_A)$ and $B = (|B|, \mathbb{S}_B)$.

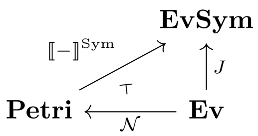
Observe that, for a natural transformation $\alpha : \mathbf{EvSym}(J-, A) \to \mathbf{EvSym}(J-, B)$, applying component $\alpha_{|A|}$ to the 'identity' function $\mathrm{id}_{|A|} : J|A| \to A$ gives a map of event structures with symmetry $\alpha_{|A|}(\mathrm{id}_{|A|}) : J|A| \to B$. We claim that $\alpha_{|A|}(\mathrm{id}_{|A|})$ preserves the symmetry in A. For any bijection $\theta : x \cong y$ in $\mathbb{S}_A$, we have $(\mathrm{id}_{|A|} \circ J\iota_y \circ J\theta) \sim (\mathrm{id}_{|A|} \circ J\iota_x)$, writing $\iota_x : x \to |A|$ and $\iota_y : y \to |A|$ for the inclusion maps (and recalling that $\theta \in \mathbf{Ev}(x, y)$). Since α is a natural transformation and each component preserves the symmetry relation on maps, $\alpha_{|A|}(\mathrm{id}_{|A|}) \circ J\iota_x = \alpha_x(\mathrm{id}_{|A|} \circ J\iota_x) \sim \alpha_x(\mathrm{id}_{|A|} \circ J\iota_y \circ J\theta) = \alpha_{|A|}(\mathrm{id}_{|A|}) \circ J\iota_y \circ J\theta$. In other words, $\alpha_{|A|}(\mathrm{id}_{|A|})\theta \in \mathbb{S}_B$, and we have proven the claim. Therefore, we take $\tilde{J}^{-1}_{A,B}(\alpha) = \alpha_{|A|}(\mathrm{id}_{|A|})$. For functoriality of $J^{-1}_{A,B}$, note that any modification $\alpha \to \beta$ must be unique and gives $\alpha_{|A|}(\mathrm{id}_{|A|}) \sim \beta_{|A|}(\mathrm{id}_{|A|})$. We omit the verification that $\tilde{J}^{-1}_{A,B}$ and $\tilde{J}_{A,B}$ are inverses. ◀

From this we extend the unfolding semantics to maps of Petri nets, up to symmetry.

▶ **Corollary 37.** *The unfolding semantics* $P \mapsto [\![P]\!]^{\mathrm{Sym}}$ *determines a pseudo-functor* $[\![-]\!]^{\mathrm{Sym}} : \mathbf{Petri} \to \mathbf{EvSym}$ *(i.e. composition and identities are only preserved up to symmetry).*

Proof. Every map $(\eta, \beta) : P \to P'$ induces a natural transformation $\mathbf{EvSym}(J-, [\![P]\!]^{\mathrm{Sym}}) \xrightarrow{\simeq} \mathbf{Petri}(\mathcal{N}-, P) \xrightarrow{(\eta,\beta)\circ-} \mathbf{Petri}(\mathcal{N}-, P') \xrightarrow{\simeq} \mathbf{EvSym}(J-, [\![P']\!]^{\mathrm{Sym}})$, which corresponds to a map $[\![P]\!]^{\mathrm{Sym}} \to [\![P']\!]^{\mathrm{Sym}}$ under the density isomorphism. The non-strictness is because pseudo-inverses are only determined up to isomorphism. ◀

It also follows from this construction that the equivalence of categories $\mathbf{Petri}(\mathcal{N}E, P) \simeq \mathbf{EvSym}(JE, [\![P]\!]^{\mathrm{Sym}})$ is natural in P. In other words, we have constructed a J*-relative pseudo-adjunction*:

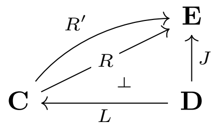


(2)

The 2-functor $[\![-]\!]^{\mathrm{Sym}}$ is determined up to symmetry by the relative 2-adjunction. The general result is as follows:

▶ **Lemma 38.** *Let $L : \mathbf{D} \to \mathbf{C}$ and $J : \mathbf{D} \to \mathbf{E}$ be 2-functors and suppose that J is 2-dense. If L has two J-relative pseudo-adjoint pseudo-functors $R, R' : \mathbf{C} \to \mathbf{E}$, then R and R' are equivalent.*

The following diagram summarizes the situation:



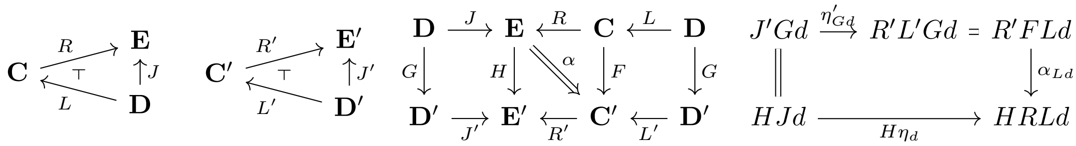
Proof. By the relative pseudo-adjunction property we have for every $c \in \mathbf{C}$ a natural equivalence of pseudo-functors $\mathbf{E}(J-, Rc) \simeq \mathbf{C}(L-, c) \simeq \mathbf{E}(J-, R'c)$, and this is natural in c. By 2-density of J this lifts to an internal equivalence $Rc \simeq Rc'$ in $\mathbf{E}$. Several naturality conditions must be verified but this is straightforward diagram chasing. ◀

Summary of section. We have obtained a new presentation of the unfolding semantics for ordinary Petri nets as a pseudo-functor $[\![-]\!]^{\mathrm{Sym}} : \mathbf{Petri} \to \mathbf{EvSym}$. The universal property of this unfolding is necessarily weaker than in the whole-grain setting, but we have characterized it as a pseudo-adjunction relative to the embedding $\mathbf{Ev} \to \mathbf{EvSym}$. The 2-density of this embedding suffices to characterize the unfolding up to symmetry.

5 Multiplicity count as a morphism of relative adjunctions

In this section, we connect the unfolding in whole-grain style (Corollary 25) and the unfolding in ordinary style (§4). Observe that there are functors connecting the two settings on all sides of the adjunctions: the multiplicity count functor $\mathsf{Q} : \mathbf{Petri}^{\mathsf{WG}} \to \mathbf{Petri}$ and the embedding $\mathbf{Ev} \hookrightarrow \mathbf{EvSym}$. The idea is to assemble them into an appropriate *morphism of adjunctions*.

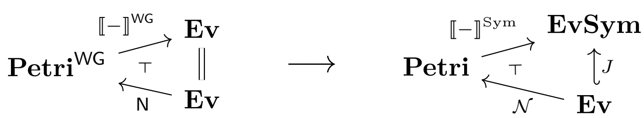
▶ **Definition 39** (Adapted from [1]). *A* right-morphism of relative adjunctions *between relative adjunctions as on the left below consists of functors $F : \mathbf{C} \to \mathbf{C}'$, $G : \mathbf{D} \to \mathbf{D}'$ and $H : \mathbf{E} \to \mathbf{E}'$, and a pseudonatural transformation $\alpha : HR \to R'F$:*



such that the two unlabelled squares in the middle diagram commute, and that the right-most diagram commutes for every $d \in \mathbf{D}$ (η and η' denote the two relative units).

Here we need a slightly more general notion to deal with the pseudo aspects. Since the 2-categories involved here are quite degenerate, there are no coherence axioms at the 2-cell levels, and we only relax the naturality of α:

▶ **Theorem 40.** *There is a (pseudo) right-morphism of relative pseudo-adjunctions*



consisting of the 2-functors $\mathrm{id}_{\mathbf{Ev}} : \mathbf{Ev} \to \mathbf{Ev}$, $\mathsf{Q} : \mathbf{Petri}^{\mathsf{WG}} \to \mathbf{Petri}$, and $J : \mathbf{Ev} \to \mathbf{EvSym}$.

Proof. We define a pseudo-natural transformation $\alpha_P : J[\![P]\!]^{\mathsf{WG}} \to [\![\mathsf{Q}(P)]\!]^{\mathrm{Sym}}$ for $P \in \mathbf{Petri}^{\mathsf{WG}}$ and omit the rest of the details. For $P \in \mathbf{Petri}^{\mathsf{WG}}$, α_P is built from the counit ε_P of the adjunction between $\mathbf{Petri}^{\mathsf{WG}}$ and $\mathbf{Occ}$. Its image $\mathsf{Q}\varepsilon_P : \mathsf{QU}P \to \mathsf{Q}P$ factors through a morphism $g_P : \mathsf{Q}P \to \mathcal{U}_{\mathsf{Q}P}$ by Lemma 27, and we define α_P as $\mathcal{E}g_P$ seen as a morphism $J[\![P]\!]^{\mathsf{WG}} \to [\![\mathsf{Q}P]\!]^{\mathrm{Sym}}$. Pseudonaturality follows from Lemma 33. ◀

6 Conclusion: related work and perspectives

The symmetry problems that arise in the unfolding of unsafe Petri nets are well-known but difficult to explain precisely. We have shown in this paper that event structures with symmetry are an appropriate semantic domain for Petri nets, provided the unfolding is described as a relative adjunction. This method completely bypasses the difficulties of adding symmetry on nets themselves. We will explore connections with other kinds of unfoldings where the symmetry problems also occur. For instance [27, 13, 14] have all (implicitly or explicitly) considered symmetry to describe the complexities of unfolding semantics.

Over the past few years the theory of Petri nets has experienced a new wave of interest, with new contributions on the categorical side ([2, 3, 6, 20, 19]) and new applications to semantics [12, 8] and practical systems modelling [11, 4]. The present work will resonate with this line of work, both through methodology and results. We note that event structures, symmetry, and Petri net unfoldings already have applications to program semantics (e.g. [9, 10]) and this work may lead to new perspectives in that area.

References

1. Nathanael Arkor and Dylan McDermott. The formal theory of relative monads. *Journal of Pure and Applied Algebra*, 228(9):107676, sep 2024. URL: `http://dx.doi.org/10.1016/j.jpaa.2024.107676`, `doi:10.1016/j.jpaa.2024.107676`.
2. John C Baez, Fabrizio Genovese, Jade Master, and Michael Shulman. Categories of nets. In *2021 36th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*, pages 1–13. IEEE, 2021. `doi:10.1109/LICS52264.2021.9470566`.
3. John C Baez and Jade Master. Open petri nets. *Mathematical Structures in Computer Science*, 30(3):314–341, 2020. `doi:10.1017/S0960129520000043`.
4. Kris Brown, Kevin Carlson, Owen Lynch, and Evan Patterson. Catcolab software. `https://catcolab.org/help/credits`. URL: `https://catcolab.org/help/credits`.
5. Roberto Bruni, José Meseguer, Ugo Giovanni Erasmo Montanari, and Vladimiro Sassone. Functorial models for petri nets. *Information and Computation*, 170(2):207–236, 2001. `doi:10.1006/inco.2001.3050`.
6. Benjamin Merlin Bumpus, Sophie Libkind, Jordy Lopez Garcia, Layla Sorkatti, and Samuel Tenka. Additive invariants of open petri nets. *Compositionality*, 7, 2025.
7. Aurelio Carboni, Stefano Kasangian, and Ross Street. Bicategories of spans and relations. *Journal of pure and applied algebra*, 33(3):259–267, 1984.
8. Simon Castellan and Pierre Clairambault. The geometry of causality: Multi-token geometry of interaction and its causal unfolding. *Proc. ACM Program. Lang.*, 7(POPL):689–717, 2023. `doi:10.1145/3571217`.
9. Simon Castellan, Pierre Clairambault, Silvain Rideau, and Glynn Winskel. Games and strategies as event structures. *Logical Methods in Computer Science*, Volume 13, Issue 3, sep 2017. URL: `http://dx.doi.org/10.23638/LMCS-13(3:35)2017`, `doi:10.23638/lmcs-13(3:35)2017`.
10. Simon Castellan, Pierre Clairambault, and Glynn Winskel. The parallel intensionally fully abstract games model of pcf. In *2015 30th Annual ACM/IEEE Symposium on Logic in Computer Science*, pages 232–243. IEEE, 2015. `doi:10.1109/LICS.2015.31`.

11 David Dalrymple. Safeguarded ai: Constructing safety by design. *Advanced Research and Invention Agency. https://www. aria. org. uk/wp-content/uploads/2024/01/ARIA-Safeguarded-AI-Programme-Thesis-V1. pdf*, 2024.

12 Elena Di Lavore, Wilmer Leal, and Valeria de Paiva. Dialectica petri nets. *Fundamenta Informaticae*, 194, 2025. doi:10.46298/fi.13125.

13 Uli Fahrenberg, Christian Johansen, Christopher Trotter, and Krzysztof Ziemianski. Sculptures in concurrency. *Log. Methods Comput. Sci.*, 17(2), apr 2021. URL: https://lmcs.episciences.org/7363, doi:10.23638/lmcs-17(2:5)2021.

14 Eric Goubault and Samuel Mimram. Formal relationships between geometrical and classical models for concurrency. In Lisbeth Fajstrup, Eric Goubault, and Martin Raussen, editors, *Proceedings of the workshop on Geometric and Topological Methods in Computer Science, GETCO 2010, Aalborg, Denmark, January 11-15, 2010*, volume 283 of *Electronic Notes in Theoretical Computer Science*, pages 77–109. Elsevier, jun 2010. doi:10.1016/j.entcs.2012.05.007.

15 Jonathan Hayman and Glynn Winskel. Symmetry in petri nets. *Unpublished manuscript*, 01 2008.

16 Jonathan Hayman and Glynn Winskel. The unfolding of general Petri nets. In Ramesh Hariharan, Madhavan Mukund, and V Vinay, editors, *IARCS Annual Conference on Foundations of Software Technology and Theoretical Computer Science*, volume 2 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 223–234, Dagstuhl, Germany, 2008. Schloss Dagstuhl – Leibniz-Zentrum für Informatik. URL: https://drops.dagstuhl.de/entities/document/10.4230/LIPIcs.FSTTCS.2008.1755, doi:10.4230/LIPIcs.FSTTCS.2008.1755.

17 G. M. Kelly. *Basic Concepts of Enriched Category Theory*, volume 64 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, 1982. Reprinted in: Reprints in Theory and Applications of Categories, No. 10 (2005), pp. 1–136.

18 Joachim Kock. Whole-grain petri nets and processes. *J. ACM*, 70(1), dec 2022. doi:10.1145/3559103.

19 Serge Lechenne, Clovis Eberhart, and Ichiro Hasuo. A compositional framework for petri nets. In Barbara König and Henning Urbat, editors, *Coalgebraic Methods in Computer Science - 17th IFIP WG 1.3 International Workshop, CMCS 2024, Colocated with ETAPS 2024, Luxembourg City, Luxembourg, April 6-7, 2024, Proceedings*, volume 14617 of *Lecture Notes in Computer Science*, pages 174–193, Berlin, Heidelberg, 2024. Springer. URL: https://doi.org/10.1007/978-3-031-66438-0_9, doi:10.1007/978-3-031-66438-0_9.

20 Jade Master and Joe Moeller. Colored petri nets are monoidal double functors. *arXiv preprint arXiv:2510.01946*, 2025.

21 Kenneth L. McMillan. Using unfoldings to avoid the state explosion problem in the verification of asynchronous circuits. In Gregor von Bochmann and David K. Probst, editors, *Computer Aided Verification, Fourth International Workshop, CAV '92, Montreal, Canada, June 29 - July 1, 1992, Proceedings*, volume 663 of *Lecture Notes in Computer Science*, pages 164–177, Berlin, Heidelberg, 1992. Springer. URL: https://doi.org/10.1007/3-540-56496-9_14, doi:10.1007/3-540-56496-9_14.

22 José Meseguer and Ugo Montanari. Petri nets are monoids. *Information and computation*, 88(2):105–155, 1990. doi:10.1016/0890-5401(90)90013-8.

23 José Meseguer, Ugo Montanari, and Vladimiro Sassone. On the semantics of place/transition petri nets. *Mathematical Structures in Computer Science*, 7(4):359–397, 1997. doi:10.1017/S0960129597002314.

24 T. Murata. Petri nets: Properties, analysis and applications. *Proceedings of the IEEE*, 77(4):541–580, 1989. doi:10.1109/5.24143.

25 Mogens Nielsen, Gordon D. Plotkin, and Glynn Winskel. Petri nets, event structures and domains, part I. *Theor. Comput. Sci.*, 13:85–108, 1981. doi:10.1016/0304-3975(81)90112-2.

26 Wolfgang Reisig. *Petri Nets: An Introduction*, volume 4 of *EATCS Monographs on Theoretical Computer Science*. Springer, 1985. doi:10.1007/978-3-642-69968-9.

27 Sam Staton and Glynn Winskel. On the expressivity of symmetry in event structures. In *2010 25th Annual IEEE Symposium on Logic in Computer Science*, pages 392–401, 2010. doi:10.1109/LICS.2010.37.

28 Rob J van Glabbeek and Gordon D Plotkin. Configuration structures, event structures and petri nets. *Theoretical Computer Science*, 410(41):4111–4159, 2009. doi:10.1016/j.tcs.2009.06.014.

29 Glynn Winskel. A new definition of morphism on petri nets. In *Annual Symposium on Theoretical Aspects of Computer Science*, pages 140–150. Springer, 1984. doi:10.1007/3-540-12920-0_13.

30 Glynn Winskel. Event structures. In Wilfried Brauer, Wolfgang Reisig, and Grzegorz Rozenberg, editors, *Petri Nets: Central Models and Their Properties, Advances in Petri Nets 1986, Part II, Proceedings of an Advanced Course, Bad Honnef, Germany, 8-19 September 1986*, volume 255 of *Lecture Notes in Computer Science*, pages 325–392, Berlin, Heidelberg, 1986. Springer. URL: https://doi.org/10.1007/3-540-17906-2_31, doi:10.1007/3-540-17906-2_31.

31 Glynn Winskel. Petri nets, algebras, morphisms, and compositionality. *Information and Computation*, 72(3):197–238, 1987. doi:10.1016/0890-5401(87)90032-0.

32 Glynn Winskel. Event structures with symmetry. In Luca Cardelli, Marcelo Fiore, and Glynn Winskel, editors, *Computation, Meaning, and Logic: Articles dedicated to Gordon Plotkin*, volume 172 of *Electronic Notes in Theoretical Computer Science*, pages 611–652. Elsevier, 2007. Computation, Meaning, and Logic: Articles dedicated to Gordon Plotkin. doi:10.1016/j.entcs.2007.02.022.