COMS21400 : Space Complexity and Beyond

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Outline

Space Complexity

Logarithmic Space

Separations

Randomised Computation

Space complexity for TMs

Definition. Let M be a TM which halts on every input. The **space complexity** of M is $f : \mathbb{N} \to \mathbb{N}$, where $f(n)$ is the maximum number of tape cells that M scans for any input of length n.

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  - $\forall$: for all
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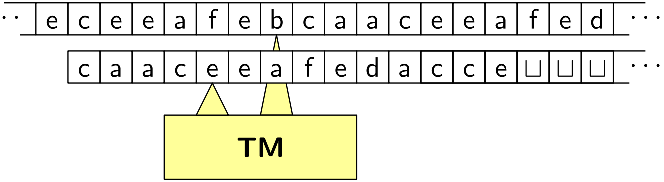

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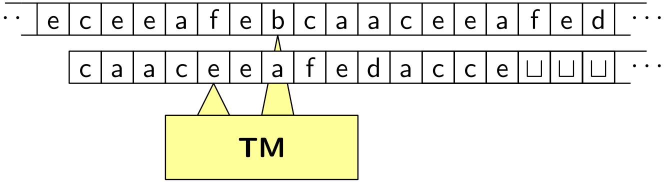

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The **space complexity** is defined by the number of cells scanned on the *work tape only*

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More results.

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$$\boxed{\text{L} \subseteq \text{NL} = \text{co-NL} \subseteq \text{P} \subseteq \text{NP} \subseteq \text{PSPACE} = \text{NPSPACE} \subseteq \text{EXP}}$$

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Fact. Let $Q(x_1, \ldots, x_n)$ have degree $\leq d$ in every variable and Q not identically zero.
Then for any set S of values, with $|S| \geq 2nd$, the number of tuples $(a_1, \ldots, a_n) \in S^n$ s.t. $Q(a_1, \ldots, a_n) = 0$, is at most $\frac{1}{2}|S|^n$.

Checking $Q(x_1, \ldots, x_n)$ is identically zero:

$R =$ "On input Q

1. Choose S with $|S| > 2nd$.
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If Q **is not zero** then R outputs *accept* with probability $\leq 1/(2^k)$

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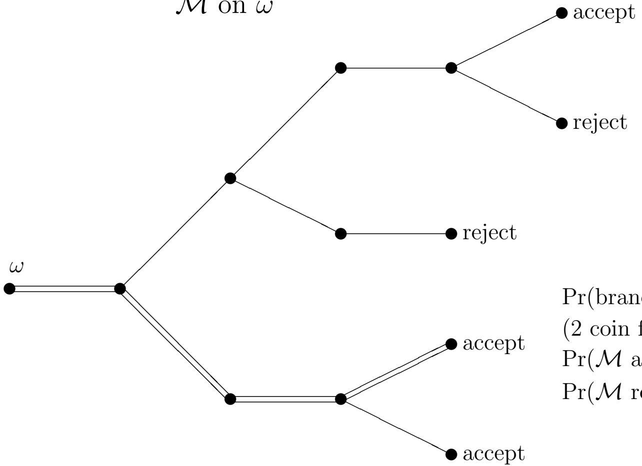
PTMs are real devices, NTMs are not

$\mathcal{M}$ on ω



$\Pr(\text{branch } =\!=\!=) = \frac{1}{4}$
(2 coin flips on this branch)
$\Pr(\mathcal{M} \text{ acc. } \omega) = \frac{1}{8} + \frac{1}{4} + \frac{1}{4} = \frac{5}{8}$
$\Pr(\mathcal{M} \text{ rej. } \omega) = \frac{1}{8} + \frac{1}{4} = \frac{3}{8}$

The class BPP

Definition. A TM M recognises L with **error probability** ε if

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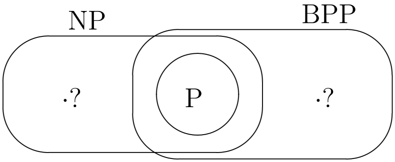

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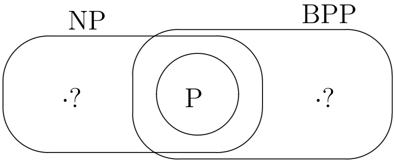

Lemma. *(Amplification)* Let $0 < \varepsilon_1 \leq \varepsilon_2 < 1/2$.
If L can be recognised by a poly-time PTM with error probability ε_2 then it can be recognised by a poly-time PTM with error prob. ε_1.

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Example. The Fermat primality test is given p s.t.

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Thus $\mathit{COMPOSITES} \in \mathrm{RP}$

Fin