Cryptography – BCS 3
Public-Key Cryptography – Cyclic Group and Elliptic Curves

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Agenda

1. Cyclic Groups
2. Elliptic Curves

Cyclic Groups

Definition

Let N be a positive integer. Then $\mathbb{Z}/N\mathbb{Z} = \mathbb{Z}_N$ is a group under addition mod N, and $\mathbb{Z}_N^*$ is a group under the multiplication mod N

Definition (Cyclic Group)

Let G be a finite group of order n, with identity element e. G is cyclic if there exists an element of order n, called generator of G. A cyclic group is abelian. If x is a generator, $G = \{e, x, \ldots, x^{n-1}\}$

Definition

The group $\mathbb{Z}_N^*$ is cyclic when N is a prime.

Some computations...

In any group G, we can define an exponentiation operation :

- if $i = 0$, then a^i is defined to be 1
- if $i > 0$, then $a^i = a \cdot a \cdot \ldots \cdot a$ ($i - 1$ times)
- if $i < 0$, then $a^i = a^{-1} \cdot a^{-1} \cdot \ldots \cdot a^{-1}$ ($i - 1$ times)

For all $a \in G$ and all $i, j \in \mathbb{Z}$:

- $a^{i+j} = a^i \cdot a^j$
- $(a^i)^j = a^{ij}$
- $a^{-1} = (a^i)^{-1} = (a^{-1})^i$

Some relations to know to compute

Definition

The order of a group is its size.

Fact

- If G is a group and $m = |G|$ its order :
  - $a^m = 1$ for all $a \in G$
  - $a^i = a^{i \bmod m}$ for all $a \in G$ and $i \in \mathbb{Z}$
- Example : In $\mathbb{Z}_{21}^* = \{1, 2, 4, 5, 8, 10, 11, 13, 16, 17, 19, 20\}$ under the operation of multiplication modulo 21. $m = 12$.

$$5^{86} \bmod 21 = 5^{86 \bmod 12} \bmod 21 = 5^{2 \bmod 12} \bmod 21$$
$$= 25 \bmod 21 = 4$$

Subgroups and examples

- If G is a group, $S \subseteq G$ is a subgroup if it is a group under the same operation as that under which G is a group
- If we already know that G is a group, there is a simple way to test whether S is a subgroup :
  - it is one if and only if $x \cdot y^{-1} \in S$ for all $x, y \in S$
  - y^{-1} is the inverse of y in G
- Fact : Let G be a group and let S be a subgroup of G. Then, the order of S divides the order of G.

$(\mathbb{Z}/p\mathbb{Z})^*$ for a prime p is cyclic of order $p-1$

- E.g. $p = 11$, the subgroups are $S_1 = \{1\}$, $S_2 = \{-1, 1\}$, $S_5 = \{1, 4, 5, 9, 3\}$, and $S_{10} = \{1, 2, 4, 8, 5, 10, 9, 7, 3, 6\}$.
- For each divisors of $p - 1 = 10$, a subgroup of that order exists
- 2 is a generator, as well as 2^k for $\gcd(k, 10) = 1$

Cyclic Groups and generators

- If $g \in G$ is any member of the group, the order of g is defined to be the least positive integer n st $g^n = 1$ We let $\langle g \rangle = \{g^i | i \in \mathbb{Z}\} = \{g^0, g^1, \ldots, g^{n-1}\}$ denote the set of group elements generated by g. This is a subgroup of order n.
- An element g of the group is called a generator of G if $\langle g \rangle = G$ or, equivalently, it its order is $m = |G|$
- A group is cyclic if it contains a generator
- If g is a generator of G, then for every $a \in G$, there is a unique integer $i \in \mathbb{Z}_m$ s.t. $g^i = a$. This i is called the discrete logarithm of a to base g, and we denote it by $\mathrm{DLOG}_{G,g}(a)$.
- $\mathrm{DLOG}_{G,g}(a)$ is a function that maps G to $\mathbb{Z}_m$, and moreover this function is a bijection.
- The function $\mathbb{Z}_m$ to G defined by $i \mapsto g^i$ is called the discrete exponentiation function

Choosing cyclic group and generators

- The discrete log function is conjectured to be one-way (hard-to-compute) for some cyclic groups G. Due to this fact, we often seek cyclic groups.
- Examples of cyclic groups :
  1. $\mathbb{Z}_p^*$ for a prime p
  2. a group of prime order
- Finding generators : How to chose a candidate and test it ?
- Let G be a cyclic group and let $m = |G|$. Let $p_1^{\alpha_1} \ldots p_n^{\alpha_n}$ be the prime factorization of m and let $m_i = m/p_i$ for $i = 1, \ldots, n$. Then, $g \in G$ is a generator of G iff for all $i = 1, \ldots, n : g^{m_i} \neq 1$.
- IfG is a cyclic group of order m, and g a generator of G : $\mathrm{Gen}(G) = \{g^i | i \in \mathbb{Z}_m^*\}$ and $|\mathrm{Gen}(G)| = \varphi(m)$.

Example : determine all the generators of $\mathbb{Z}_{11}^*$

- Its size is $m = \varphi(11) = 10$ and the prime factorization of $10 = 2 \cdot 5$. Thus the test for whether a given $a \in \mathbb{Z}_{11}^*$ is a generator is that $a^2 \neq 1 \bmod 11$ and $a^5 \neq 1 \bmod 11$.
- $\mathrm{Gen}(\mathbb{Z}_{11}^*) = \{2, 6, 7, 8\}$
- Double-checking : $|\mathbb{Z}_{11}^*| = 10$, $\mathbb{Z}_{10}^* = \{1, 3, 7, 9\}$

$$\{2^i \in G | i \in \mathbb{Z}_{10}^*\} = \{2^1, 2^3, 2^7, 2^9 \bmod 11\} = \{2, 6, 7, 8\}$$

a	1	2	3	4	5	6	7	8	9	10
$a^2 \bmod 11$	1	4	9	5	3	3	5	9	4	1
$a^5 \bmod 11$	1	10	1	1	1	10	10	10	1	10

Algorithm for finding a generator

- Most common choice of a group in crypto is $\mathbb{Z}_p^*$ for a prime p
- Idea : Pick a random element and test it. Choose p s.t. the prime factorization of the order of the group $(p-1)$ is known. E.g., chose a prime p s.t. $p = 2q+1$ for some prime q
- The probability that an iteration of the algorithm is successful : $\frac{|\mathrm{Gen}(\mathbb{Z}_p^*)|}{|\mathbb{Z}_p^*|-2} = \frac{\varphi(p-1)}{p-3} = \frac{\varphi(2q)}{2q-2} = \frac{q-1}{2q-2} = \frac{1}{2}$

Algorithm 1 Finding a generator

```
q = (p − 1)/2 ; found ← false
while found ≠ true do
    g ← Z_p^* \ {1, p − 1}
    if (g^2 mod p ≠ 1) && (g^q mod p ≠ 1) then
        found ← true
    end if
end whilereturn g
```

Squares mod n

Quadratic Residue mod n

- Def : an element a mod n is a quadratic residue mod n if there exists b with $a = b^2 \bmod n$
- other elements are called *non-quadratic residue*
- $1, 4, 9, 5, 3$ are square mod 11
- other values $2, 3, 6, 7, 8, 10$ are not square mod 11

a	1	2	3	4	5	6	7	8	9	10
a^2 mod 11	1	4	9	5	3	3	5	9	4	1

Elliptic Curves

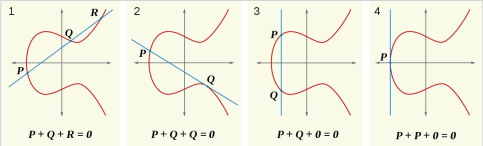
Elliptic curve is a group for the addition

- an elliptic curve $\mathcal{E}$ is a set of points satisfying the equation $y^2 = x^3 + ax + b$ over $\mathbb{Z}/p\mathbb{Z}$
- These points form a group with an additive notation
- This group is not cyclic, but from one element we can define a cyclic group
- From one point G, the $\langle G \rangle$ is the group generated by the point G with the addition (defined in the next slide)
- Specific point at *infinity* : ∞ identity element for this group
- From G, we can define $k \times G = G + G + \ldots + G$ $(k-1)$ times
- If the order of the group $\langle G \rangle$ (number of different points) is prime, it is difficult to invert the scalar multiplication operation

Elliptic Curve point multiplication : Double-and-Add

Algorithm 2 Double-and-add $d = d_0 + 2d_1 + 2^2 d_2 + \ldots + 2^m d_m$

```
N ← P
Q ← ∞
for i from 0 to m do
    if d_i = 1 then
        Q ← point − add(Q, N)
    end if
    N ← point − double(n)
end forreturn Q = dP
```



El Gamal Encryption – ECIES

Informal description

1. Alice gets Bob's public key, g^x. He knows his own private key x
2. Alice generates a fresh, ephemeral value y, and g^y (public)
3. Alice computes c from m, the symmetric encryption of m with key k (authenticated encryption scheme) : $c = E(k; m)$
4. Alice sends the public ephemeral g^y and the ciphertext c
5. Bob, knowing x and g^y, $k = KDF(g^{xy})$ and recovers m from c

Common Parameters

- Key Derivation Function) : HMAC-SHA-1-80 with 80-bit
- symmetric encryption scheme AES-GCM noted E
- elliptic curve parameters : $\langle G \rangle$ of order n, ∞ infinity
- Bob's PK : $K_B = k_B G$, $k_B \in [1, n-1]$ random private key
- optional shared information : S_1 and S_2

El Gamal Encryption – ECIES

Encryption : To encrypt a message m Alice :

- generates a random $r \in [1, n-1]$ and computes $R = rG$
- derives a shared secret $S = P_x$, $P = (P_x, P_y) = rK_B \neq \infty)$
- uses a KDF to derive symmetric encryption and MAC keys : $k_E \| k_M = KDF(S \| S_1)$
- encrypts the message : $c = E(k_E; m)$
- computes the tag of c and S_2 : $d = MAC(k_M; c \| S_2)$
- output $R \| c \| d$

Decryption : To decrypt the ciphertext $R \| c \| d$

1. derives the shared secret : $S = P_x$ with $P = (P_x, P_y) = k_B R$: $P = k_B R = k_B rG = rk_B G = rK_B$, or output failed if $P = \infty$
2. $k_E \| k_M = KDF(S \| S_1)$; output failed if $d \neq MAC(k_M; c \| S_2)$
3. uses symmetric encryption scheme to decrypt $m = E^{-1}(k_E; c)$

ECDSA Elliptic Curve Digital Signature Algorithm

Signature process : d_A private key

1. $e = SHA - 2(m)$, convert it to an integer.
2. Let z be the L_n leftmost bits of e where L_n the bit length of n.
3. Choose a **cryptographically secure random** $k \in [1, n-1]$
4. Compute the curve point $(x_1, y_1) = k \times G$
5. Compute $r = x_1 \bmod n$. If $r = 0$, go to step 3.
6. Compute $s = k^{-1}(z + rd_A) \bmod n$. If $s = 0$, go to step 3.
7. The signature is (r, s). ($(r, -s \bmod n)$ is also valid)

ECDSA

Verification : $Q_A = d_A G$ public key

1. Check $Q_A \neq \infty$, $Q_A \in \mathcal{E}$, $n \times Q_A = \infty$
2. $r, s \in [1, n-1]$, if not, return invalid
3. Compute $e = SHA-2(m)$ and z the L_n leftmost bits of e
4. Compute $u_1 = zs^{-1} \bmod n$ and $u_2 = rs^{-1} \bmod n$
5. $(x_1, y_1) = u_1 \times G + u_2 \times Q_A$. If $(x_1, y_1) = \infty$, Return Invalid.
6. Return Valid if $r = x_1 \bmod n$, invalid otherwise

- Do not use twice the same k (PlayStation 3)
- if the first significant bits of k are known, it is possible to recover the secret key !
- Check that $C = u_1 \times G + u_2 \times Q_A = k \times G$

Sage and Elliptic Curve

- Weierstrass equation : $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$
- sage : EllipticCurve([0,0,1,-1,0]) : Elliptic Curve defined by $y^2 + y = x^3 - x$ over Rational Field
- Elliptic curves over $\mathbb{Z}/N\mathbb{Z}$ with N prime are of type "elliptic curve over a finite field" :
- sage : F=Zmod(95) ; EllipticCurve(F, [2,3]) : $y^2 = x^3 + 2x + 3$ over Ring of integers modulo 95
- definition of point : sage P = E(-1,1)
- group order : P.order()